YuMi Deadly Maths

Problem Solving

Prep to Year 9

Prepared by the YuMi Deadly Centre
Queensland University of Technology
Kelvin Grove, Queensland, 4059

QUT

YuMiDeadly
Growing community through education

Prep to Year 9: Supplementary Resource 2 – **Problem Solving**

YuMi Deadly Maths Program
Supplementary Resource 2

Prep to Year 9

Problem Solving

Version 2, October 2017

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Queensland University of Technology
Kelvin Grove, Queensland, 4059

research.qut.edu.au/ydc

ACKNOWLEDGEMENT

The YuMi Deadly Centre acknowledges the traditional owners and custodians of the lands in which the mathematics ideas for this resource were developed, refined and presented in professional development sessions.

YUMI DEADLY CENTRE

The YuMi Deadly Centre is a Research Centre within the Faculty of Education at QUT which aims to improve the mathematics learning and the employment and life chances of Aboriginal and Torres Strait Islander and low SES students at early childhood, primary and secondary levels, in vocational education and training courses, and through a focus on community within school and neighbourhood.

"YuMi" is a Torres Strait Islander Creole word meaning "you and me" but is used here with permission from the Torres Strait Islanders' Regional Education Council to mean working together as a community for the betterment of education for all. "Deadly" is an Aboriginal word used widely across Australia to mean smart in terms of being the best one can be in learning and life.

YuMi Deadly Centre's motif was developed by Blacklines to depict learning, empowerment, and growth within country/community. The three key elements are the individual (represented by the inner seed), the community (represented by the leaf), and the journey/pathway of learning (represented by the curved line which winds around and up through the leaf). As such, the motif illustrates YDC's vision: *Growing community through education*.

The YuMi Deadly Centre can be contacted at ydc@qut.edu.au. Our website is research.qut.edu.au/ydc.

ABOUT YUMI DEADLY MATHS AND THIS RESOURCE

From 2000–09, researchers who are now part of the YuMi Deadly Centre (**YDC**) collaborated with principals and teachers predominantly from Aboriginal and Torres Strait Islander schools and occasionally from low socio-economic status (SES) schools in a series of small projects to enhance student learning of mathematics. These projects tended to focus on a particular mathematics strand (e.g. whole-number numeration, operations, algebra, measurement) or on a particular part of schooling (e.g. middle school teachers, teacher aides, parents). They resulted in the development of specialist materials but not a complete mathematics program (these specialist materials can be accessed via the YDC website, research.qut.edu.au/ydc).

In October 2009, YDC received funding from the Queensland Department of Education and Training through the Indigenous Schooling Support Unit, Central-Southern Queensland, to develop a train-the-trainer project, called the **Teaching Indigenous Mathematics Education** or **TIME** project. The aim of the project was to enhance the capacity of schools in Central and Southern Queensland Indigenous and low SES communities to teach mathematics effectively to their students. The project focused on Years P to 3 in 2010, Years 4 to 7 in 2011 and Years 7 to 9 in 2012, covering all mathematics strands in the Australian Curriculum: Number and Algebra, Measurement and Geometry, and Probability and Statistics. The work of the TIME project across these three years enabled YDC to develop a cohesive mathematics pedagogical framework, **YuMi Deadly Maths**, that covers all strands of the *Australian Curriculum: Mathematics* and now underpins all YDC projects.

YuMi Deadly Maths (**YDM**) is designed to enhance mathematics learning outcomes, improve participation in higher mathematics subjects and tertiary courses, and improve employment and life chances. YDM is unique in its focus on creativity, structure and culture with regard to mathematics and on whole-of-school change with regard to implementation. It aims for the highest level of mathematics understanding and deep learning, through activity that engages students and involves teachers, parents and community. With a focus on big ideas, an emphasis on connecting mathematics topics, and a pedagogy that starts and finishes with students' reality, it is effective for all students. It works successfully in different schools/communities as it is not a scripted program and encourages teachers to take account of the particular needs of their students. Being a train-the-trainer model, it can also offer long-term sustainability for schools.

YDC believes that changing mathematics pedagogy will not improve mathematics learning unless accompanied by a whole-of-school program to challenge attendance and behaviour, encourage pride and self-belief, instil high expectations, and build local leadership and community involvement. YDC has been strongly influenced by the philosophy of the Stronger Smarter Institute (C. Sarra, 2003) which states that any school has the potential to rise to the challenge of successfully teaching their students. YDM is applicable to all schools and has extensive application to classrooms with high numbers of at-risk students. This is because the mathematics teaching and learning, school change and leadership, and contextualisation and cultural empowerment ideas that are advocated by YDC represent the best practice for **all** students.

This resource supplements the series of books that describe the YDM approach and pedagogical framework for Prep (Foundation) to Year 9. and focuses on an aspect of mathematics teaching and learning that is part of all topics, namely **problem solving**. It discusses problem solving, identifies six components of problem-solving skill (metacognition, thinking skills, plans, strategies, content knowledge, and affective traits). It describes these components and how they relate to problem types. It proposes a scope and sequence to teach the components across Prep to Year 9. *Problem solving* is the second of three supplementary YDM resource books. The others are *Big Ideas of Mathematics* and *Literacy in Mathematics*.

YDM underlies three types of projects available to schools: (a) general training in the YDM pedagogy (through a variety of projects titled with YDM acronym); (b) Accelerated Inclusive Mathematics (AIM) training in remedial pedagogy to accelerate learning (through AIM projects, XLR8 projects and AIM Early Understandings projects); and (c) Mathematicians in Training Initiative (MITI) training in enrichment/extension pedagogy to build deep learning of powerful maths and increase participation in advanced Years 11–12 and tertiary maths subjects.

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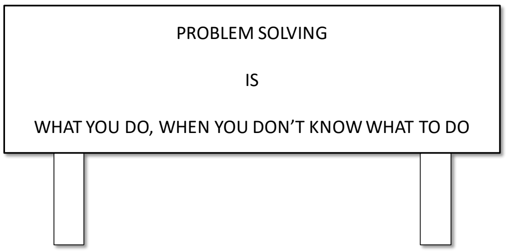
1 Summary

This section summarises the major ideas in this resource with regard to developing a problem-solving program.

1.1 Overview of resource

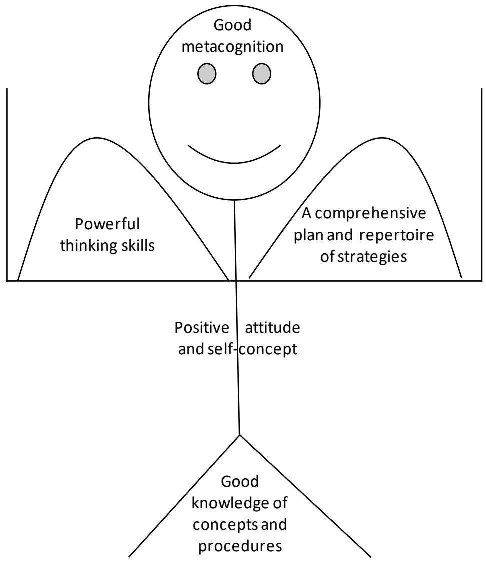
The main ideas underpinning this resource are listed below.



1. Whether an activity is a problem depends on the knowledge and capacity of the solver because it requires the solver to not know what to do.

2. Problem types follow a continuum from routine (e.g. word problems) that have a domain of knowledge to creative (e.g. matchstick puzzles) that have no domain of knowledge.

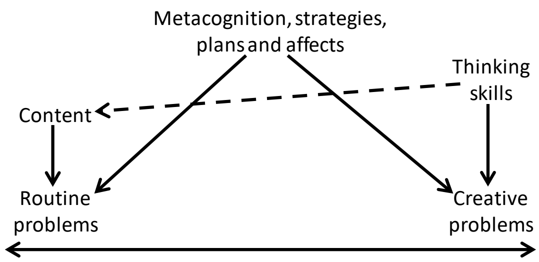
3. Expert problem solvers have six characteristics, illustrated in diagram on right:
   - thinking skills (e.g. visual, logical, creative)
   - plans (Polya's four stages – SEE, PLAN, DO, CHECK);
   - strategies (e.g. restate the problem, act it out, find another solution, look for a pattern, work backwards)
   - metacognition (e.g. monitoring, planning);
   - content (rich schema and automaticity of the basics); and
   - affects (e.g. motivation, resilience, self-concept).



The expert problem solver

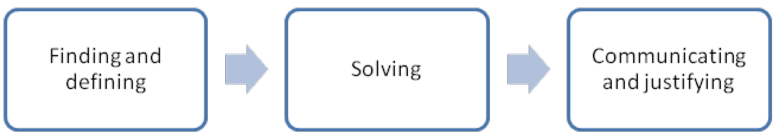
4. Plans provide a framework for metacognition, Strategies relate to thinking skills. There are five types of problem-solving strategies.

5. Content as rich schema has four attributes: it defines, connects, covers applications, and remembers. Automaticity allows students to draw on basic content without impacting on thinking; it has no cognitive load.



6. Routine problems are best solved by content and creative problems by thinking skills as in the diagram on right. Metacognition, plans, strategies and affects assist all types of problems. Thinking skills help content form in a way that assists problem solving.

7. Problem solving is made up of three stages as in the diagram on right.



8. A good problem-solving program spreads the components over the Year levels.

1.2 Program

Program for teaching mathematics problem solving

COMPONENTS	EARLY PRIMARY YEARS P–2	MIDDLE PRIMARY YEARS 3–4	UPPER PRIMARY YEARS 5–6	JUNIOR SECONDARY YEARS 7–9
Stages	Focus on solving and begin characterising problem types	Continue solving (extend to all types) Begin finding or posing simple problems Develop communication and justification		Continue finding, posing and solving Begin to formalise communication
Metacognition	Introduce planning, monitoring and checking	Introduce overseeing, evaluating and making decisions	<u>Major focus</u> Teach awareness and conscious control of thinking	Focus on evaluation and conscious judgement Continue previous work
Thinking skills	<u>Major focus</u> Directly teach skills • *Logical thinking* • *Visual thinking* • *Patterning*	Consolidate thinking skills by applying them to problem-solving and learning situations. Directly teach new skills • *Creative and flexible thinking* • *Evaluative thinking and decision making*		
Plan	Develop procedures for comprehending, planning and checking	<u>Major focus</u> Develop Polya's four stages	Directly teach Polya's approach with poster Discuss and solve problems in terms of the stages	
Strategies	Develop core strategies • *Reread the question* • *Identify given and wanted *** • *Act the problem out* • *Check your solution* • *Look for a pattern*	Continue to develop strategies • *Identify given, needed and wanted *** • *Restate problem in own words* • *Make a model* • *Make a drawing, diagram, graph* • *Find another way to solve it* • *Construct a table* • *Identify sub-goal*	Continue introducing new strategies Begin relating to Polya • *Write a number sentence* • *Select appropriate notation* • *Study the solution process* • *Find another solution* • *Guess and check* • *Work backwards* • *Solve a simpler problem*	<u>Major focus</u> Systematically develop a repertoire of strategies related to Polya's four stages • *Account for all possibilities* • *Change your point of view* • *Check for hidden assumptions* • *Generalise*
Content	Build well-sequenced and clearly understood collections of basic concepts, principles and strategies Understanding to pre-empt later work		Systematically connect topics into a comprehensive scheme Move from additive to multiplative thinking and specific to general	
Affects	Interest and motivation Willingness to take risks	Attitude and perseverance	Self-concept, self-image Attribution	Continued engagement and maintenance of resilience

Note: ** strategy is given in two steps

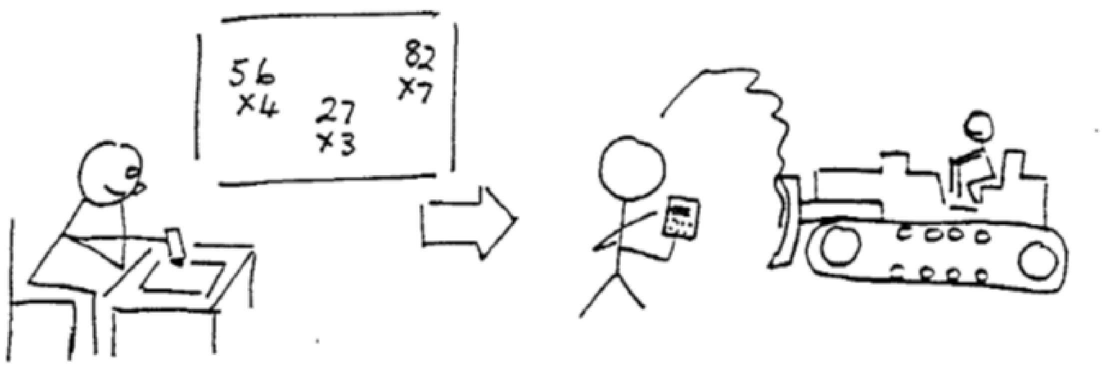
2 Overview of Problem Solving

2.1 The importance of problem solving

The ability to solve problems has always been seen as one of the major objectives in the teaching of mathematics. This focus on problem solving in mathematics has emerged from many factors but it is important here to highlight four.

The first of these is the rapid and continuing change in the functioning of society through the growth in technology. Change is now the vocational norm. Therefore students leaving school to enter employment have a greater need for mathematical abilities which facilitate the learning of new skills, rather than preserving existing skills. The ability to solve problems and transfer knowledge is increasing in importance for these students. In the past, recall of stipulated skills would have sufficed.

The type of technological change has consequences for problem solving. The advent of calculators and computers has reduced the need for proficiency in computation and, at the same time, the power of calculators and computers in accessing and manipulating information has increased the need for thinking. Calculators and computers will supply the answer to the computation but not assist in determining which computation to use and when to use it. They leave us free to use our thinking abilities to tackle real problems.



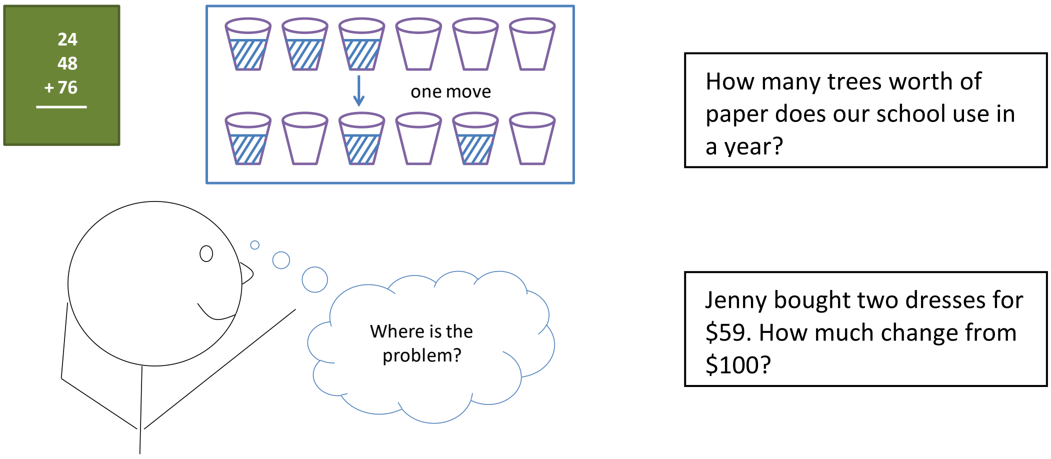
The second factor has been the failure of “back to basics” movements to meet the needs of modern students. Strong emphasis on drill and practice of unnecessary computations detracts from problem-solving ability without increasing computational ability.

The third factor has been research developments which have highlighted the crucial role that problem solving plays in learning and development. As well, analysis of the problem-solving process has offered insights into what would make an effective teaching program for problem solving.

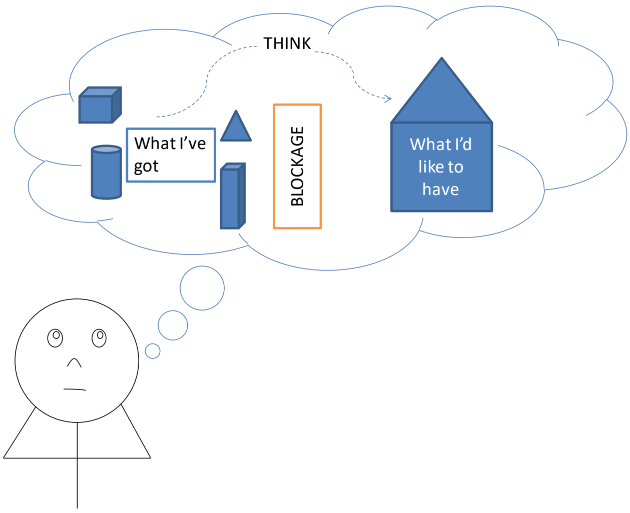
The fourth factor has been a realisation that problem solving is at the basis of mathematics itself. The discipline of mathematics emerged from humankind’s attempts to solve problems. Information and techniques useful in a variety of problems were assembled. These included number properties, operation concepts and the algorithms. However, the reason for the existence of this knowledge was to facilitate problem solving and the application of this knowledge to problems remained the crucial part of mathematics. In fact, the importance of mathematics in our culture is because it assists people to overcome their difficulties. Modern society requires people with mathematical training, particularly with the growth of technology. Such training has to include problem solving, if it is to be mathematically relevant.

2.2 Definition of problem solving

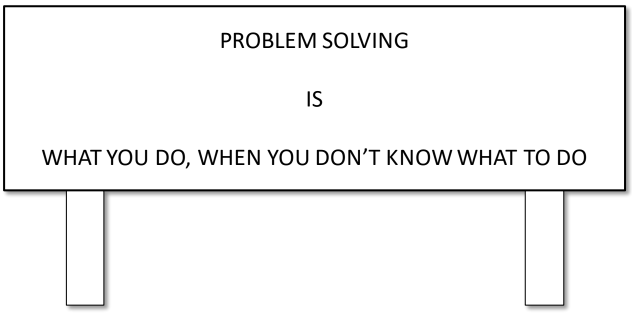
A wide diversity of activity is called problem solving. This varies from straightforward practice exercises to long-term investigations.



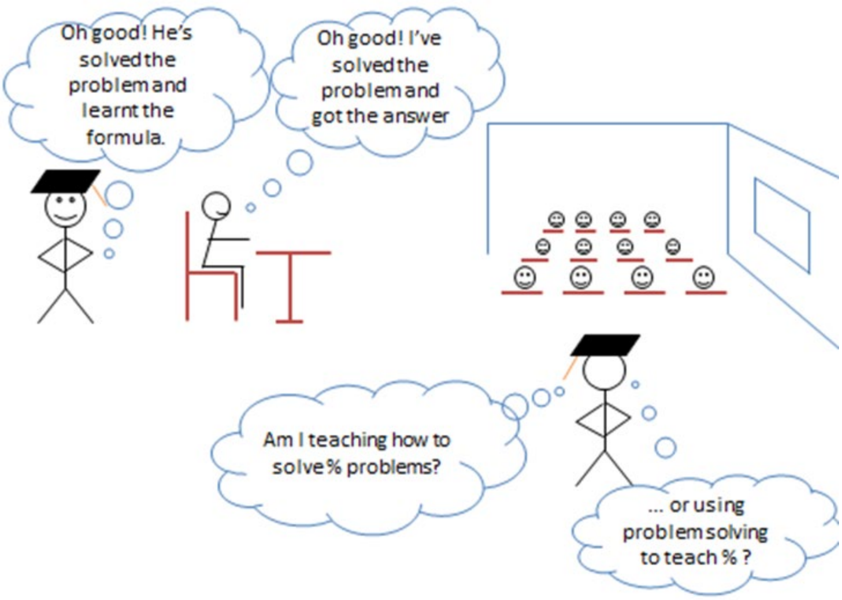
In this resource, a mathematics problem for person is considered to be a situation that: (a) involves an initial state (information provided) and a goal state (solution); (b) requires mathematics; (c) the person wants to find a solution (that is, the person accepts the problem); and (d) there is some **blockage** for that person between the initial and goal states (that is, the person must do some thinking to reach the solution).



Problem solving in mathematics can then be defined as the process of finding a solution in situations where the series of actions that need to be performed to reach that solution are not immediately known (this is put succinctly in the figure below as "problem solving is what you do when you don't know what to do"). This definition makes determining whether a problem exists **dependent on the solver**. A particular task may well be a problem for one student yet may be a simple, straightforward practice exercise for another, or what was a problem for a student last year is no longer a problem for him/her this year.



This lack of clarity is exacerbated by the two roles problem solving has in mathematical instruction. First, problem solving is a **topic** within mathematics for which specific content can be listed for study. For example, word problems or percentage problems could be part of the specific content of a mathematics course, as could problem-solving strategies like “make a drawing, diagram or graph”. Second, problem solving is an **approach to teaching** that can be the basis for all instruction in mathematics. Students can learn new mathematical knowledge and skills by attempting to solve problems where, in the process of finding solutions, students encounter the new knowledge and skills. For example, the formula for the area of a rectangle can be introduced by investigating relationships between length, width and the number of squares encompassed by rectangles drawn on graph paper.



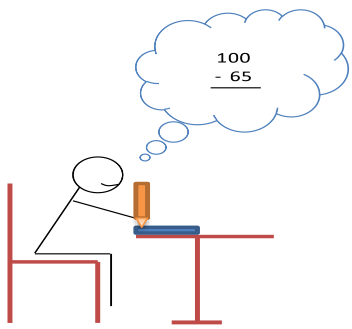
This resource is about how to teach problem solving but, once started on this path, the students’ problem-solving skills can be used for other instruction. In other words, problem solving can be an end in itself, but can also be the means to an end. In fact, both roles of problem solving can be welded together – solving problems can be the best way to teach problem solving.

2.3 Types of problems

Of basic importance to our understanding of how to teach problem solving is that we can identify a continuum of problem types – namely, **routine** to **creative** (or non-routine).

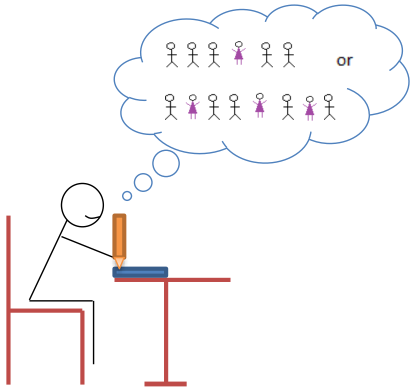
Routine problems are based primarily on content (that is, they have an underlying domain of knowledge). They are usually the more common type of problem presented to students, especially in the early years. They are designed to reinforce students’ understanding of particular concepts as well as to provide practice in applying particular skills. While these problems do require some interpretation on the part of the student, they can be solved by the application of previously learnt content, for example:

John had $1.00 to spend at the shop.
He bought a drink for 65c.
How much money did he have left?



As we move along the continuum, problems incorporate both content (knowledge) and problem-solving processes (skills). They are an extension of routine problems where students must transform and/or restructure previously acquired knowledge in order to solve the problem. Problems of this nature are usually introduced to students in the upper primary years.

Creative problems are at the other end of the spectrum; they do not depend on a domain of knowledge known to the solver and they focus on the processes of problem solving. Students are confronted with something they do not recognise and to which they cannot apply previously taught mathematical knowledge. Creative problems require thinking, and there is often more than one possible solution. For example:



Three boys are in front of Jane in a line.
Two boys are behind Jane.
How many children are in the line?

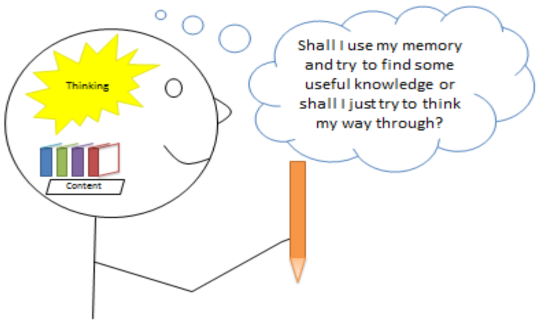
This creative problem requires thinking, and the realisation that there is no information on the number of girls in the line, and no limit on the number of boys.

In some mathematical problems, the student is provided with all of the information required to solve the problem in the text of the problem. It is generally expected that students would be able to solve the problem relatively quickly (for example, within the space of one lesson). There is usually a finite number of acceptable answers (often only one correct answer), and the answer is either correct or incorrect. These kinds of problems are used in supervised examinations, usually in short response or multiple choice formats. They are generally what people have in mind when they talk of mathematical problem solving. Most of the examples in this resource are of this type.

However, other mathematical problems are open ended. The question is often very general in nature, with many different acceptable responses (note the change in language: we talk of student "responses" or "solutions", rather than "answers"). The student may not be provided with all of the information required to solve the problem, requiring them to conduct research and/or collect data. It may take several days for students to prepare their response to the problem. The responses are usually complex, requiring mathematical arguments, examples, and extended pieces of writing. Words such as "investigate", "research", "analyse", "explain", "why", and "report on" are frequently used in the text of these problems. They are often called investigations. Investigations are discussed in more detail section 6.3 of this resource.

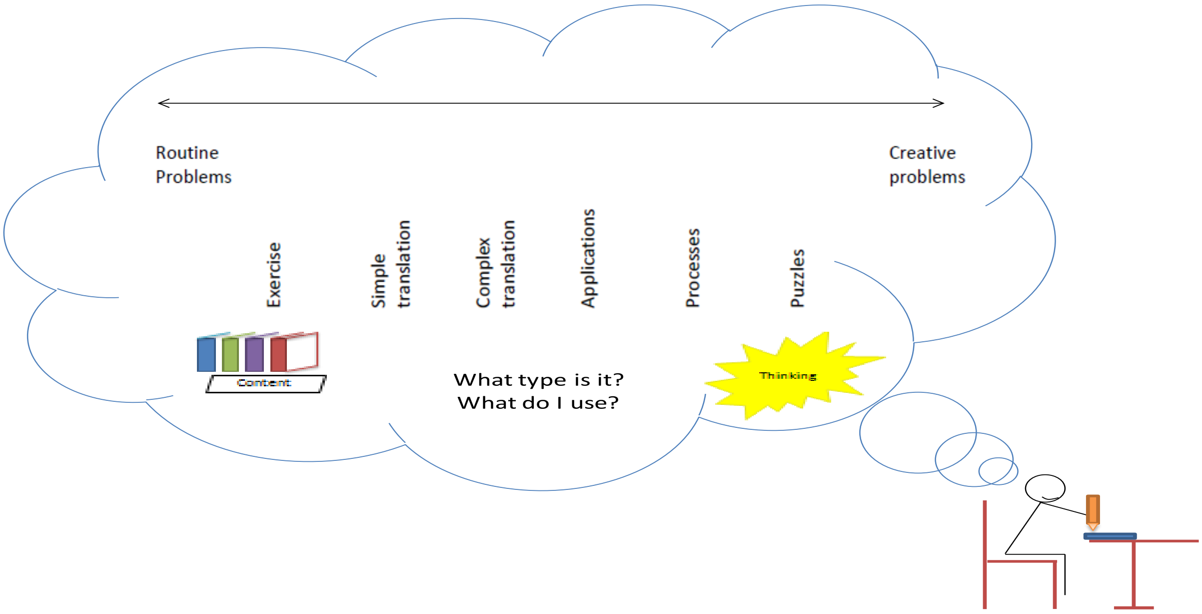
Why is this distinction so important?

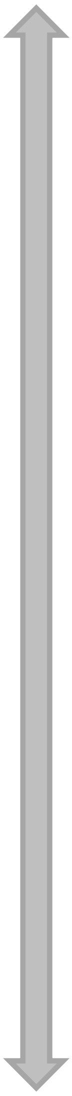
This can perhaps be best explained by considering what happens when students attempt a task which for them is a problem. This means that they do not have the specific knowledge that enable them to know what to do immediately. Therefore, their options for solving the problem are twofold: they can somehow acquire the required knowledge **or** they can think their way around it using problem-solving processes. To acquire specific knowledge, they could reformulate the problem to allow existing knowledge to be transformed to meet the problem's needs or they could research or develop the knowledge for themselves (a much more difficult task). Thinking around their lack of knowledge usually involves breaking the problem into parts or considering the problem from an different perspective.



Although this is somewhat dependent on the characteristics of the solver, it is the contention of this resource that some problems are most efficiently solved by focusing on the underlying content. These are the routine problems. Other problems require a focus on thinking or the processes of problem solving. These problems do not depend on content that can be usefully acquired to solve them. These are the creative problems.

In their book *Teaching problem solving: What, why and how*, Charles and Lester (1982) identify six potential types of problems which can be placed on the continuum of routine to creative, summarised in the diagram below and in the table on the next page.



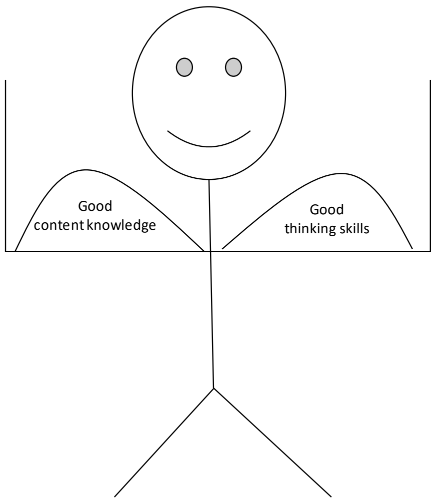


Type of Problem	Example	Comments
ROUTINE		
1. Exercises	$8\sqrt{496}$	These are problems only for those who have not been taught the procedures. They provide practice in using algorithms and maintaining basic facts.
2. Simple translation	*Frank has 11 pencils. John has 4 more pencils than Frank. How many pencils does John have?*	These require only the appropriate mathematical concept or procedure to be identified and applied, in this case, translating the problem into a single number sentence. They reinforce the understanding of the concept or procedure and maintain computational proficiency.
3. Complex translation	*Frank buys 5 radios at $23 each and a CD player at $58. How much change did he get from $300?*	This is similar to a simple translation problem but it involves at least two steps. These types of problems provide the same experience as simple translation problems but involve more than one translation and possibly more than one concept or procedure.
4. Application	*How many metres of steel wire are needed to make one million paper clips?*	These are real-world or realistic situations in which mathematical skills, facts, concepts and procedures may be used to reach a solution. Problems of this type often involve more than mathematics – the students must undertake investigations and to obtain, organise, summarise and represent data. These problems make students aware of the value of mathematics in everyday situations.
5. Process	*Fifteen people meet for the first time and shake hands with each other. How many handshakes?*	These are problems which cannot rely on rehearsed procedures. They require the use of processes and strategies to think through and represent the problem. They require the students to play around with ideas and develop a plan of attack (students have to conjecture, test their conjectures, play hunches, draw diagrams, etc.). They serve to develop processes and strategies for understanding, planning and solving problems as well as evaluating attempts at solutions.

6. Puzzles	*Without lifting your pen, draw four straight line segments that pass through all 9 dots in a 3 × 3 array.*	Puzzles require people to think in unusual or original ways. They often have no apparent use of mathematical concepts and procedures. They allow students to engage in enriching recreational mathematics and they point out the importance of flexibility in thinking.
CREATIVE		

2.4 The requirements of problem solving

This resource is built on the assumption that students should be prepared to be able to solve all types of problems – that **both routine and creative problems are part of mathematics and mathematics learning**. This requires both comprehensive **content knowledge** and good **thinking skills**. Students need to have knowledge of concepts, procedures and thinking skills that are as complete, powerful and varied as can be developed (as in the diagram, "The good problem solver", on right).



The good problem solver

It is, of course, more complex than this. Automated (memorised) knowledge is useful in problem situations without significantly affecting thinking. Secondly, knowledge stored in an organised and connected manner is more easily retrieved for use with problems than discrete pigeon-holed facts. Thirdly, knowledge can be classified as generic or specific, with general knowledge being more widely applicable across problem types. Most knowledge that students acquire and use is specific to the context of particular problems. For example, knowledge of percentages is directly applicable to the following problem:

Stellar clothing is having a 25% discount sale. Jan is changing the ticket prices. The dress has a price of $60. What sale price should Jan put on it?

There is other, more general, knowledge which will assist the correct application of percentage knowledge. For example, constructing a table containing the information given in the problem, such as the one below, is of great benefit. Furthermore, students who take the time to correctly read and understand the problem before they try to solve it and who check their work afterwards are more successful than students who do not.

× 0.75 or × $^3/_4$

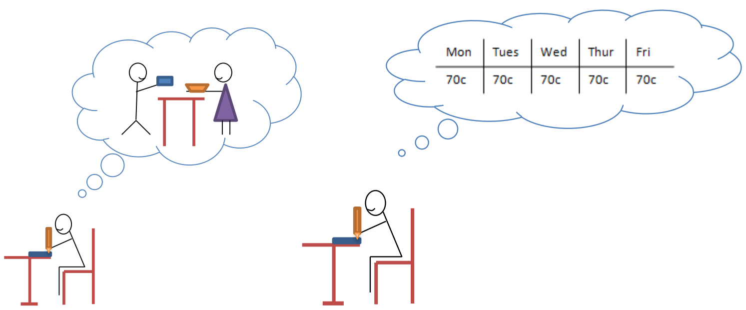
Percentage	Price
100%	$60
100% – 25% = 75%	??

× 0.75 or × $^3/_4$

Knowledge of procedures that enable problem solving to be planned, such as "read, do, check", and that supply a framework for solving any problem are called heuristic plans or general heuristics. We shall call them **plans**. Knowledge of procedures such as "construct a table" which are applicable across a variety of problems and point the general direction of the answer are called **strategies** (see Chapter 5). A complete, well-integrated plan and a wide repertoire of strategies enhance problem-solving performance.

However, difficulties still emerge even with these plans and strategies. For example, Year 4 students were using a drawing to solve the problem below:

John buys a pie each day for 70c. How much change from $10 will John have at the end of a week?

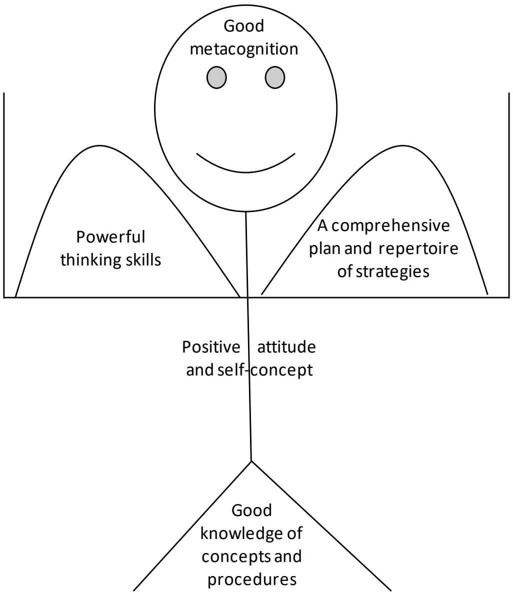


What drawing is useful? When do we use the drawing? Once we have it, what operation is appropriate? Are we getting anywhere this way? To develop and choose the appropriate drawing requires good visual thinking. Visual, logical, patterning, evaluating, creative and flexible thinking is required for problem solving as is decision making. These are called **thinking skills**.

Furthermore, all the previous abilities are a waste if their use cannot be planned, monitored and evaluated. Such control or awareness of thinking that allows students to oversee that thinking is called metacognition, executive control and executive processes. We shall call them **metacognition**.

We are still not complete. Problem solving, as we previously defined, is "what we do when we don't know what to do". Situations of uncertainty are stressful for many students. Students need to be interested and motivated just to start problems. Problems often take time to solve because the answer is not immediately obvious. Several attempts may be required. Students have to be able to persevere and some attempts may be unsuccessful. Students have to accept occasional failure. This takes a good self-concept. Of course, problems are inherently interesting, which is in their favour, but traits such as attitude, motivation, perseverance and positive self-concept are crucial in problem solving. We shall call these **affective traits**.

In summary, a good problem requires all the above: mathematics knowledge, plans, strategies, metacognition, thinking skills, and affective traits (as for "The expert problem solver" in the diagram on right).



The expert problem solver

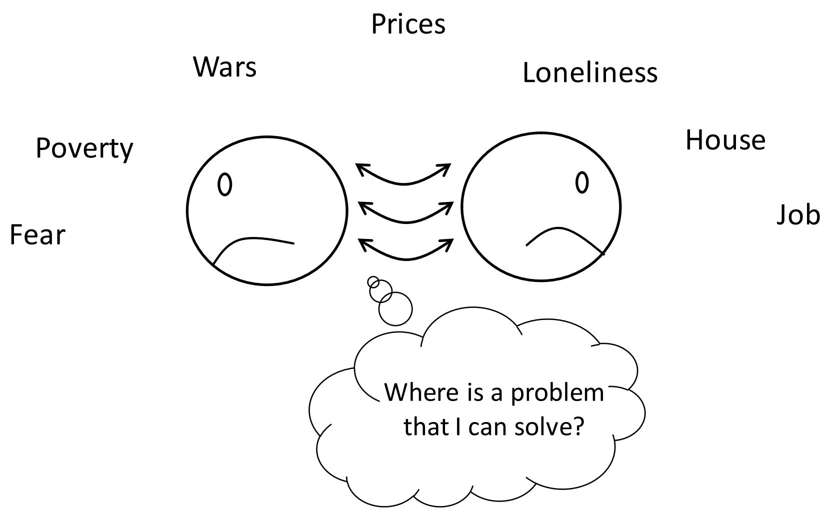
These are summarised below.

1. Knowledge – specific mathematical rules, procedures and concepts stored in memory, in an integrated and connected form with the important aspects automated.
2. Plans – a framework to guide the attack on the problem encompassing understanding the problem, planning and completing a solution and evaluating that solution.
3. Strategies – general procedures useful across many problems.
4. Thinking skills – the abilities to think logically and consistently, to think visually, to think flexibly, to think creatively, to find and continue patterns and to make decisions.
5. Metacognition – the abilities to oversee, monitor and evaluate thinking, to plan and check and, in the final analysis, to be aware of their own thinking.
6. Affective traits – attitude, motivation, perseverance, willingness to take risks, self-concept, attribution, and so on.

As may be evident in the description above, there is a connection, which will be taken up in later chapters, between metacognition and plans and between thinking skills and strategies.

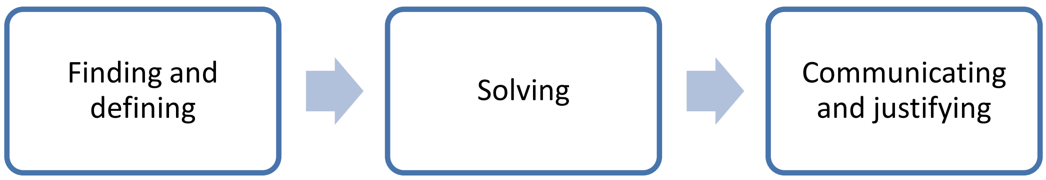
2.5 Widening the focus of problem solving

All types of problems should be part of mathematics because our major aim should be to prepare students to solve problems in their everyday life. There are differences between everyday real-life problem solving and classroom problem solving. The major part of these differences is in the way the problem is presented. Classroom problems come all nicely laid out for solving, with few irrelevancies. Real-life problems are messy: confused information, full of irrelevancies, and embedded in other problems. The first step in real problem solving is to **find and formulate** the problem and to determine whether it is a problem, the solution of which is worthwhile finding.



Humans are social beings. Real problems affect more than one person. **Communicating** the solution to others is part of the problem-solving process. Many problems, particularly real problems, have more than one solution or one path to solution and it is not evident what the correct answer will be. **Justifying** and defending a particular solution is, therefore, part of the problem-solving process as is promoting that solution as a worthwhile activity in the face of other conflicting needs.

Thus, there are three parts or stages to problem solving as in the diagram below



The three stages that make up the complete problem-solving process.

The first of these, **finding and defining**, is best taught by **problem posing**. This means that you ask students to make up problems instead of solving them. This is particularly effective for word problems. If you arrange students in groups and ask each group to compose as many different problems as they can think of for 8 + 5 = 13. The following are possible:

Joining	8 children are playing on the oval and 5 more children join them, how many are playing on the oval now?
Comparing	8 children are playing on Oval 1, 5 more children are playing on Oval 2 than on Oval 1, how many children are playing on Oval 2?
Inaction	8 boys and 5 girls are playing on the oval, how many children were playing on the oval?
Inverse	Some children are playing on the oval, after 5 leave to go home, there are 8 remaining. How many children were playing on the oval at first?

This also works at the creative end of problem solving. Students can make up problems and puzzles for each other.

The second of the stages, **solving**, has been well covered in this section. However, it should be noted that problem solving, for most people, is a messy, idiosyncratic, doodling, acting out, and making lists type of activity. The answer comes but it would be hard for someone else to follow the reasons for the answer or the working towards the answer. Trying to neaten this process up can often destroy it.

The third of the stages, **communicating and justifying**, is the neat section. Here the solver rewrites their solution so it can be followed by others, following the protocols laid down by that topic (usually the instruction "show all working").

This means that problem solving, and communicating and justifying, follow each other and must work together. They must not be confused by the teacher but often are, and can have detrimental effects on students. For example, a class was set the problem:

You buy a hat for $23 and a t-shirt for $38, how much change from $100?

A student started working. She added 23 and 38 to get 61 and then subtracted the 61 from 100 to get the correct answer of $39. However, the student set out the problem as below:

$$\begin{array}{r} 23 \\ \underline{+\,38} \\ 61 \\ \underline{-\,100} \\ 39 \end{array}$$

The teacher was rushed, looked at what the student had done and marked it wrong. The student was perplexed; she scratched her head, looked all confused and, in the end, redid her answer as below.

$$\begin{array}{r} 23 \\ 38 \\ \underline{+\,100} \\ 161 \end{array}$$

The teacher had mixed up solving and communicating – she should have said "your answer is right but now communicate it to me correctly". By marking the communication wrong with no explanation, the student went from the correct result with poor communication, to an incorrect result (in fact, a process that was illogical) with good communication.

So, do not confuse solving and communicating – problem solving is messy and disorganised while an account of how a problem is solved has to be neat and well organised. Like the process of writing, it is never achieved in one step. These two stages need to be carefully delineated in instruction and evaluation.

3 Metacognition, Plans and Affective Traits

This section describes the characteristics of an expert problem solver: Metacognition and thinking skills, plans, strategies, and affective traits.

3.1 Metacognition and thinking skills

This subsection looks at metacognition and thinking skills, describing the ideas within them and providing examples of their use.

3.1.1 Metacognition

Metacognition is a term from the study of psychology that means the awareness and understanding of one's own thought processes. So, in the context of problem solving, understanding when to make decisions, and which decisions to make, is a result of metacognition. Metacognitive processes control the problem-solving process by continually overseeing, monitoring and evaluating. They coordinate planning and checking. In particular, problem-solving performance is enhanced when the problem solver can think about, and consciously control, these overseeing and monitoring functions. This is because metacognitive processes coordinate both the application of specific knowledge and the problem-solving process.

We identify the following processes as important.

1. **Overseeing and monitoring.** This is the ability to continually monitor what is occurring in the attempts to solve problems – to have part of the thinking tracking what is, and has been, done. Expert problem solvers constantly monitor their thinking to ensure that their work contributes to the overall goal.
2. **Checking and evaluating**. This is the ability to continually check how thinking is going, and to evaluate if progress is satisfactory. Recovering from a false start is important. When a method does not seem to be working, many students abandon it completely, often erasing or crossing out the first attempt. Trying to 'debug' an initial good idea to find and remove faults can be a good idea. Checking needs to apply to the selection of method(s) as well as the process (e.g. the arithmetic). A false start can also provide valuable information to guide the direction of the next attempt.
3. **Planning and predicting**. This is the ability to continually look ahead, to plan and to predict what is possible. Novice problem solvers spend a lot of time executing their plans, but not enough time developing them. While they may be able to select a successful approach, often they cannot explain why they did so.
4. **Decision making**. This is the ability to make decisions about new directions. This is a particularly important trait in problem solving as students have to make decisions on what directions and strategies to try. If a method does not seem to be working, students need to decide whether to abandon, 'debug', and/or to modify the approach. Decision making can only be learnt by giving students classroom experience in making their own decisions.
5. **Being aware of and controlling**. This is being conscious of what you are thinking and being able to report on, and discuss, that thinking. It is also consciously controlling the direction of this thinking. These are important skills to encourage reflection.

Unconscious metacognitive processes are thought to exist from early childhood. Awareness of these processes, and conscious control over them, are believed to emerge in middle childhood. It is believed that metacognition can be enhanced by interaction with others with more expertise. This is particularly so for conscious control, where the learner moves from regulation by others to conscious self-regulation.

There are group techniques to enhance metacognition. The best of these are summarised below.

1. **Reporting**. Students report on their problem solving by stating what and how they thought as well as the answer (using the third stage of problem solving to improve metacognition). An extension of this is to have groups solve the problems but have a ‘reporter’ chosen by lot, who has to explain the thinking and justify the solution. Each student takes a turn to be the reporter.
2. **Reciprocal teaching**. This is giving students the opportunities to act as teacher in the classroom, leading others’ learning (summarising, questioning, clarifying, predicting and checking). One way to do this is to lead a student through a problem and then, later or next day, to let them work with a small group leading the group to solve the problem.
3. **Think–pair–share**. Students are organised to solve a problem on their own (think), to discuss their solution with a partner (pair) and then to come to one solution in a group of four (two pairs) through discussion and negotiation (share). It requires students to explain and justify their answers and describe their thinking
4. **Social-interaction roles**. Students solve problems in groups where members are given roles. An effective and simple way is to form groups of three: one is designated leader (makes decision if there are differences of opinion in the group), one is the checker (continuously checks that direction is the best to go), and the third is recorder/reporter (the only one with pen/paper to record findings, and explains the group’s solutions). Roles are changed for each problem. De Bono’s thinking hats are another example of social interaction roles (de Bono, 1985).

Finally, as decision making is so important, it is useful to prepare students for problem solving by allowing them to make decisions in other areas of mathematics; for example: (a) discussing examples with students to encourage them to determine what information will be needed for the example; (b) giving students examples and letting them choose the ones they answer (e.g. here are 20 questions, do any 10 you like); and (c) posing problems with more than one answer or solution method and asking students to find all the answers and methods.

3.1.2 Thinking skills

Thinking skills are the mental processes that are used to make sense of experience. They enable the incorporation of each new experience into a larger schema ng of "how things are". This develops the ability to examine an issue in several different ways. The following seem to be most important in mathematical problem solving:

1. **Visual thinking** in the context of mathematics draws upon students’ ability to use physical models, manipulatives, and diagrams to analyse problems and to form useful mental images of the mathematical concepts and processes. It helps students to organise and retain information, thus providing them with a powerful tool for solving problems. It also empowers students to find a variety of ways that seem natural and sensible to them to explore the concepts and solve the problem. For example, in geometry, students can apply visual thinking to distinguish one shape from another, transform shapes (flipping–sliding–turning), exploring symmetry recognise similar and congruent shapes even if in different orientations, explore symmetry and tessellations, draw shapes, and dissect more complex shapes and solids. The visual thinking can be assisted by the use of physical objects such as wooden cubes. These skills assist students to visualise situations and mentally manipulate images to solve a variety of spatial problems. Activities where students have to select a shape to fit a gap are excellent for developing visual thinking, for example, tangrams, pentonimoes and soma cubes.
2. **Logical thinking** is one of the global big ideas of mathematics. Being able to interpret facts and relationships and organise them in a logical sequence. Logical thinking is not only a valuable skill in problem solving but in most other mathematical tasks as well. It is related to the verbal logical thinking and thus has a relationship with language in problem solving.

   It is too easy to assume that students are natural logical thinkers, for rarely do mathematics programs for the early years incorporate specific activities to help develop logical thinking. The topic “sets and logic”, which has for several years adorned many mathematics curricula, has usually been a mask for a variety of set symbols. The logic component has rarely surfaced. Consequently, many students are ill-equipped for problem solving in the middle and upper grades; irrespective of the heuristics and strategies they might be taught. Students in early grades need to be particularly exposed to logical thinking activities.

3. **Evaluative thinking** is being able to assess whether the activity is productive in moving towards a solution and whether conclusions are accurate and make sense in the context of the problem. It includes checking, learning from previous errors, and looking for other options (solutions and solution methods).
4. **Patterning thinking** is being able to recognise patterns and identify relationships. It is at the core of thinking mathematically and solving problems. Patterning skills can be developed through activities with beads on string, attribute blocks, coloured cubes and a variety of other concrete and pictorial aids. Patterns can be followed, described and invented. In the secondary years, patterns lead to generalised and abstract processes such as algebra.
5. **Creative thinking** is having a open-minded approach to the solving of a problem; being able to generate a number of different solutions to a problem. Creative thinking is not in the exclusive domain of the language arts. It plays an important role in mathematical problem solving and should be encouraged through activities which direct students to think of different ways of doing things.
6. **Flexible thinking** is being able to change one's point of view; being able to a view a problem from more than one perspective. As students who experience considerable "traditional" mathematics teaching often have difficulty in making decisions about what to do next, activities that allow students to make such decisions will help them become more flexible in their approach to problem solving. Sometimes called lateral thinking, flexible thinking works well with creative thinking.

3.2 Plans

A good plan of attack (or plan as we will call it) assists in metacognition. Such a plan (or "rule of thumb") is also known as an heuristic. A powerful heuristic plan from which to attack the problem can assist when, in a problem situation, a learner's content knowledge appears, initially, to be inadequate. This is particularly so when the plan is linked with a strong repertoire of strategies (see Chapter 5).

The best plan is, or is based on, **Polya's four stages** (Polya, 1957). This heuristic plan is most widely advocated as facilitating problem solving. It has four stages as below.

	STAGE	**QUESTIONS THAT COULD BE ASKED**
1.	**SEE** Identifying and defining the problem. Interpreting the problem to understand what is given, what is relevant, and what is required.	What is the problem? What is happening? What are we asked to do? What do we have to find? What are we told? What do we want? What is the important data? What can be ignored?
2.	**PLAN** Select the appropriate technique. Determine a plan of action. Design a way to tackle the problem.	What do we know? Do we need extra information? What do we need to solve the problem? How can we obtain more information or data? What should we use? What is a good plan?
3.	**DO** Implement the plan, by undertake one or more steps to reach the solution.	Carry out the chosen plan. Revise if necessary. Does the plan work? If not, return to the SEE and PLAN stages.

	STAGE	QUESTIONS THAT COULD BE ASKED
4.	**CHECK** Look back at what has been done. Check the solution and the procedure. Reflect on what has been done. Evaluate.	Does the solution answer the question posed in the problem text? Does the solution meet all the problem conditions? Does it make sense? Is it reasonable? If the answer is not exact, is it close enough? Is it the only solution? Could we have used another method to reach it? Can we generalise it?

Problem types at different ends of the continuum have **differing requirements** with regard to plans and strategies. With word problems (simple and complex translation problems), the main difficulty lies in the SEE section (and, to some extent, the PLAN section) of Polya's four stages. Puzzles, on the other hand, have their difficulties in PLAN. They require strategies that are related to creative and flexible thinking. Students who experience difficulty with word problems (a difficulty that is not computation) mostly cannot get started.

Where students are unable to correctly respond to a worded mathematics problem, Newman (1983) proposed five questions that could be used to diagnose the nature of the error.

QUESTION	DIAGNOSIS
Read the question aloud to me.	Difficulty with basic reading (SEE)
What does the question ask you to do?	Unable to understand what has been read (SEE)
How are you going to find the answer?	Unable to determine the process(es) needed to solve the problem (PLAN)
Show me how to get the answer. As you do it, talk aloud about what you are doing.	Unable to apply the process(es) needed to solve the problem (DO)
What does your answer mean?	Unable to interpret the answer in the context of the problem (CHECK)

Asking these five questions, in order, every time that a student needs assistance ensures that the assistance is targeted appropriately. Students should be helped to overcome only the diagnosed weakness and then be allowed to continue with the solution independently.

The **way to teach Polya's four stages is to do it directly**. Students can learn a lot from teacher modelling. This is particularly important in the SEE, PLAN and CHECK stages, which are often mental processes and may not be written down. Teachers should adopt a 'think aloud' strategy while solving problems to share their thinking with students.

There are two approaches to teaching Polya's four stages. The first method is to put the four stages up as a **poster** and then teach to it following the four stages. Give a problem, then: (a) have the class discuss the problem in terms of SEE (b) when this is finished, have the class discuss a PLAN to solve the problem; (c) then let the class DO this plan; and (d) finally bring the students back together to discuss CHECK.

The second method, which follows from the first, is then to use Polya's four stages in problems as a basis for **hints for students**, particularly for those having difficulty with a particular problem. For example, problems could be written on index cards (left below). Then, hints could be written on the reverse of the cards (right below). Two examples are given below.

PROBLEM

I spent $47.75 on groceries. I gave the shop assistant $50. She made a mistake and returned $7.75 in change.
If I am honest, how much money should I give back to her?

HINTS

SEE:	*How much change did I receive from the $50?*
PLAN:	*What kind of problem is it? +, −, ×, ÷ ?* *How much should I have received?*
DO:	*Use a calculator.*
CHECK:	*Estimating, does the shop assistant get more or less than $5.00 back? Does your*

PROBLEM

Suppose you jogged 8km every day.
How many years would it take to jog over 100,000 km?

HINTS

SEE:	*How many days in a year?*
PLAN:	*What kind of problem is it? +, −, ×, ÷ ?* *How many km would you go in a year?* *How does the distance in a year relate to the question?*
DO:	*Use a calculator.*
CHECK:	*Over 30 years but less than 40 years.*

3.3 Affective traits

In addition to the meta-cognitive and planning requirements of problem solving ability, we must also consider affective factors. The following four affective traits appear to be crucial: (a) Attitude, motivation and engagement; (b) Self-concept, confidence and actualisation; (c) Attribution; and (d) Perseverance and resilience. Each of these will be considered in order. They will be described and then teaching hints given as to how to make them positive to mathematical problem solving.

1. Attitude, motivation and engagement

Attitude to mathematics is affected by the students' beliefs and feelings about mathematics and problem solving (e.g. fear, dislike, hatred, confidence, enjoyment, love) and their relationship to mathematics and problem solving (e.g. have they previously been successful at mathematics?).

Motivation towards mathematics. This is the students' interest in finding a solution to a "problem". It affects the ways that students choose to behave and their self-confidence, ability to overcome obstacles and challenges, and capacity to recover from setbacks. Student motivation determines whether or not students engage in a particular pursuit and is affected by their beliefs about what is important. Motivation can have a large effect on learning. As Sullivan (2011) states: *… low-achieving students are particularly at risk in so far as their inappropriate motivation may inhibit their learning opportunities.*

Engagement with mathematics. This is the extent and willingness to which students undertake and endeavour to complete mathematical activities. It is based on thoughts, behaviours and actions and incorporates behavioural, cognitive and affective engagement as follows.

a) **Behavioural engagement** encompasses the idea of active participation and involvement in academic and social activities, and is considered crucial for the achievement of positive academic outcomes.

b) **Cognitive engagement** involves the idea of investment, recognition of the value of problem solving and a willingness to go beyond the minimum requirements.

c) **Affective engagement** includes students' reactions to the school environment, influencing their willingness to work at school.

It is believed that engagement occurs in mathematical problem solving when students enjoy and value learning; see it as relevant in their own lives (now and in the future); and see connections between what they learn of it at school and the mathematics they use outside school.

Teaching hints. The following are hints that educators' believe will improve attitude, motivation and engagement. With this set of affects, improvement comes from making problem solving interesting and providing the student with situations where there is support and they can achieve success.

- finding ways to encourage memories of positive experiences in solving problems;
- being aware of what motivates students to become engaged in mathematics so they maintain engagement, even when faced with difficult mathematical problems;
- ensuring students see setbacks as temporary and specific, using task management strategies, such as breaking the problems down into achievable chunks;
- providing tasks at an appropriate level of challenge to prevent boredom, but not beyond their ability, that are considered by students to be purposeful (so they hold their interest);
- utilising a variety of rich and challenging tasks that allow students time and opportunities to make decisions; have many possible solutions; engage students in productive exploration; and use a variety of forms of representation;
- increasing the students' sense of control over their learning;
- building on what students know, mathematically and experientially; and
- providing a positive and caring learning environment in the classroom and at home, understanding that the first step to problem solving success is a positive attitude.

2. Self -confidence and self-efficacy

There are a variety of "self-something" affects. We shall label this section with two of them, self-confidence and self-efficacy, recognising that self-esteem is closely related to self-confidence and self-actualization links with self-efficacy.

Self confidence and self-esteem with respect to mathematics. Self-confidence is the feeling of trust in one's own abilities, qualities, and judgement with respect to mathematical problem solving, a measure of belief in one's ability to solve problems. Self-esteem represents reflects an overall subjective emotional evaluation of one's own worth. It is a judgment of oneself as well as an attitude toward the **self.**

When confronted with a problem and they do not understand or where they cannot easily see an approach that might work, students with low self-confidence or self-esteem may panic, giving up before they start. Excessive stress can then further reduce performance in problem solving.

Low self-confidence and esteem with respect to problem solving is often a trait of low achieving students – they come to believe that they cannot solve problems. Of course, low self-confidence and esteem can be the cause of low achievement as well as its consequence.

Overconfidence can also impede successful problem solving. A flippant approach or the attitude that 'this is easy' may result in the failure to see the complexities in a problem and the selection of an inappropriate strategy. It can also lead students to overlook important detail.

Self-efficacy and self-actualization with respect to mathematics. Self-efficacy refers to belief in one's capabilities to learn or perform in problem solving. It influences academic motivation, learning, and achievement and is one of the key issues for teaching problem solving. It refers to belief in one's capacity to execute the behaviours necessary to produce specific performance attainments (Bandura, 1997). Self-efficacy reflects a confidence in the ability to control motivation, behaviour, and social environment. **Self-actualization** arises from the theory of hierarchy of needs created by Maslow (Maslow, 1999). It represents the fulfilment of the highest level of need (as defined by Maslow) when one's full potential is realised

Teaching hints. Positive self-confidence, esteem and efficacy (and even actualization) in problem solving are built around giving students the opportunity to experience success. This improves all the "selfs". However, it may not be easy because success often requires effort in addition to well-chosen problems. Providing students with a range of 'getting started' strategies can enhance self-confidence as they are able to engage with the problem.

Students should know they have the respect of teachers but need encouragement to learn that challenge and effort enhance self-esteem and are not threats.

It is important to realise that teaching requires that that students have a positive self-concept that will allow them to persevere long enough at a problem to overcome the immediate challenge they confront (at least). Many students avoid risk taking and do not persevere with the challenges that are required in order to complete the problem. Teachers can sometimes meet the challenge of dealing with students who have given up by reducing the demand of the problem, but this can become evident to the students.

Teaching ideas should include:

- selection of problems and activities that built positive attitude, motivation and engagement;
- carefully designed active instruction accompanied by high expectations;
- selecting problems that are achievable so that students do not experience repeated failure; and
- organising lessons so that students feel that they have some control over their learning (this can be as little as choosing which problems they will do) and using this to build self-belief.

3. Attribution

Attribution covers what students perceive are the causes of success and failure. This affect is particularly important. There are four types of causes – ability, effort, task difficulty, and luck. Two of these are stable (ability and task difficulty) and two are within students' control (ability and effort).

Attribution is considered to be **positive** when students' causes of success are stable and within the student's control, (e.g. ability), and their causes of failure are unstable and beyond their control (e.g. luck). Attribution is considered to be **negative** when students' causes of success are unstable and outside their control (e.g. luck) and causes for failure are stable (e.g. ability). Negative attribution is damaging. Attributing failure to low ability leads to the expectation that failure will continue in the future, which, in turn, may cause an individual to stop working at a task. Because having high ability is often strongly valued, a perceived lack of ability frequently leads to negative emotions and reduction in self-esteem.

The damaging aspect of attribution is exacerbated by it being hard to change. Success alone will not work because, as a negative attribution, students will attribute their success to luck or task difficulty and still expect failure as the normal situation. Students need success and a program to convince them that this success is due to factors within their control such as ability and effort.

Stability and locus of control are important. Stability is important for expectancy of success. Attributing failure to lack of effort (because of fatigue) means failure can be changed because removal of fatigue and increased effort can turn things around. Attributing failure to ability is destructive as it cannot be changed. Attributing failure to task difficulty (if it is outside of the students' control) can lead students to give up.

Negative attribution can lead to three worse conditions:

(a) **Performance avoidance** where failure is so expected that students invest little effort in trying to do the mathematics and are then able to use lack of effort as an excuse;

(b) **Learned helplessness** where the student feels that nothing can be done and rely on the teacher to show them how to do everything – this is the result of consistent negative attribution over an extended period time – students feel that they are hopeless at mathematics and they are helpless – it takes a long time to turn around; and

(c) **Mathaphobia** where learned helplessness has reached the point that the students will stop working in mathematics lessons and do nothing.

Teacher expectations are also important because they act as a self-fulfilling prophecy. If teachers think students can learn (independent of whether they are or not), the students achieve well. If teachers think that particular students will experience difficulty in learning, then those students do so. Some teachers respond to students experiencing difficulty by providing easier problems for them, thus reinforcing low achievement. If teacher treatment is consistent over time, and if students do not actively resist or change it, it is likely to negatively affect motivation, self-concept and attribution.

Teaching. Some specific teaching ideas include:

- have high expectations and give **specific** feedback – with the feedback focusing on changing of attribution beliefs from negative to positive; explain clearly and in some detail how performance can be improved as this helps build adaptive attributional beliefs;
- demonstrate interest in your students as individuals, and acknowledge that they may have had previous negative mathematics experiences;
- maintain a carefully chosen program that enables success but continually attributes the students' success to positive things that are within their control;
- allow and/or require students to provide evidence of corrective action, such finding textual examples that correctly demonstrate how the type of problem is solved;
- capture student imaginations - select problems that show the extended values of mathematics in ways students find inspiring.

4. Perseverance and resilience

Perseverance is having the confidence and sheer doggedness to keep trying to solve a problem when most people would give up. It is persistence. Sometimes perseverance can enable a lower ability student to do better than high ability students. Expert problem solvers know that challenging mathematical problems require an investment of time and that there may be a substantial period of exploration and several unsuccessful attempts before finding a strategy that works (Carlson 1999). However, persistence must be combined with appropriate monitoring behaviours to identify, for example, when a strategy is leading towards a successful solution and when it is unproductive and alternative strategies need to be explored.

Resilience is the strength of character to keep trying to solve new problems despite a lack of success in earlier problems. Such learners approach mathematical problems with confidence, persistence and a willingness to discuss, reflect and research. Students need resilience to struggle through problems when solutions are hard to find, to deal with barriers and misunderstandings and to keep working on relevant ideas. Building resilient children not only increases their ability to solve unfamiliar problems, it also builds adolescent wellbeing, which can help reduce the risk of depression.

It is important not to prevent students from gaining resilience. Students who see themselves as being good at problem solving when at school may not develop resilience if every time they get stuck on a problem, their teacher gives them the solution.

Wigley (1992) argues that resilient students:

(a) have the skills they need to decide what a problem is asking of them and to function mathematically in the world beyond school, and the willingness to continue their mathematical development as and when needed;

(b) know that, if they think hard, talk to others, read about mathematical ideas and reflect on the information gained, they will be able to make headway with seemingly difficult ideas and problems;

(c) are less dependent on external rewards, persist even when facing challenges, and are not vulnerable when extrinsic motivational rewards stop and the problem becomes challenging;

(d) have a reflective and thoughtful stance towards mathematical problem solving and persevere when faced with difficulties because they know that the more they work at a problem the more successful they will be; and

(e) adapt positively to the difficulties presented by mathematical problems.

Teaching ideas. The focus of teaching for perseverance and resilience is to enable a positive adaptive stance to problem solving that will allow learning to continue despite barriers and difficulties as in the following ideas.

- Allow students to reformulate problems with which they are presented. Students generally pose problems at the level with which they are most comfortable. The teacher's challenge is how to move them into a less comfortable stage by proposing more complex tasks, possibly with other students.
- Refrain from telling students how to solve a problem. Give students time to struggle through a problem. Students who solve difficult problems on their own, without the help of other students or teachers, often gain a better understanding of the mathematical concepts than their classmates. If the student requires help, assist with the part of the problem that is causing the difficulty, not with the entire problem.
- Give problem posing activities. They require students to ask themselves questions, which assists students in exploring the potential of many tasks. This capacity to pose questions is one of the goals of mathematics teaching, and has important elements of strategic competence.
- Encourage collaborative working where students support one another in problem solving. This can help change mindsets and overcome current negative attitudes. Support students to know when and where to put in their learning effort.
- Allow students to see that teachers do not always immediately find the correct approach to an unfamiliar problem. Teacher modelling of perseverance that requires several attempts to solve a problem is a powerful learning tool.

Encouraging students to see that problem solving takes effort but that it will result in improvement. Discuss with students resilience and what it can offer.

4 Content and Decoding

In this section the two other components of the expert problem solver are examined, namely content knowledge and the ability to decode the information provided in the problem text.

4.1 Content knowledge

Studies of experts have shown that they have two aspects to their content knowledge: knowledge of the domain structured into rich schemas; and a good recall of basic knowledge. An idea is a rich schema if it has four attributes: (a) the knowledge completely defines the idea; (b) the knowledge connects the idea to all the other related ideas; (c) the knowledge provides all the applications of the idea; and (d) the knowledge contains all the experiences with that idea. Thus expert problem solvers can recall basic knowledge and have knowledge that defines, connects, applies, and remembers.

A frustration for teachers is that, in problem solving, students often do not use the mathematics that they have been taught. This may be because students do not understand the mathematics well. To use knowledge in problem solving it needs to be understood, assimilated and mastered. The strongest test of understanding is to be able to apply the knowledge in a different context. It is often some years before students feel confident to apply new knowledge to solve problems. Another reason for students' failure to use their mathematical knowledge is that they may not know how to apply the theory in a practical way.

1. Defining knowledge

Expert problem solvers in a particular domain have that domain completely defined. This means that they can recognise the idea, say addition, anywhere. For example, they can recognise all the following as forms of addition:

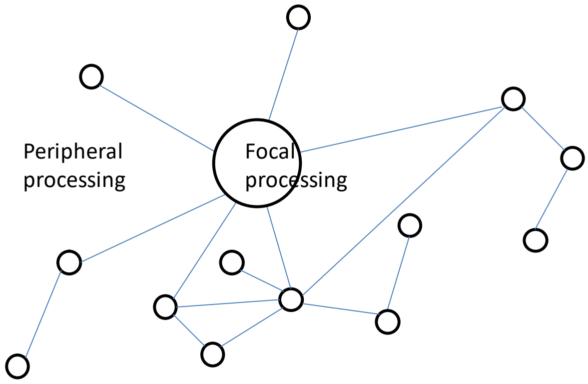
- Joining – *If there are three cars in a car park and four more cars arrive in the car park, how many cars are in the car park altogether?*
- Comparison – *If three cars are parked in the first row of a car park and four more cars are parked in the second row than in the first row, how many cars are parked in the second row?*
- Inverse – *After four cars were driven out of a car park, three cars remained. How many cars were in the car park at the start?*
- Inaction – *There were three Toyotas and four Fords in the car park. How many cars were in the car park altogether?*

This means that students have to be given definitions that cover all ways in which an idea can be represented. This also includes all models, which for addition, are such as those below:

- Set – *There were four logs in the trailer, three more were added, how many logs in the trailer?*
- Number line – *The 3 m log was place end to end with the 4m log, what was the distance covered by both logs?*

2. Connected knowledge

Problems occur when the task appears not to be solvable by the knowledge that is being focused on (called focal processing). This lack of success results in the focus changing to the periphery (called peripheral processing). When knowledge is highly connected, there are many chances for this peripheral processing to find knowledge that will lead to a solution – the processing follows the connections. If knowledge is unconnected, there is little peripheral processing available and little chance of a solution. Expert problem solvers have been found to have highly connected knowledge. In pictorial terms, their knowledge resembles a collection of nodes joined by many connections.



For the example of addition, rich schema would result in connections to number, counting, all the methods of addition, subtraction (as the inverse – what undoes the adding), multiplication (through repeated addition), and all situations where adding is used other than in number (predominantly measurement and algebra, but also statistics and probability and some geometry).

3. Applications knowledge

Rich schema has knowledge of the applications for the idea under consideration. For addition, this would involve all the forms of addition listed in the defining knowledge section above and ways to use the idea in applications. This would include:

- **Part-part-total** – Seeing addition as two parts forming a total and being able to use a P-P-T to solve problems (e.g. in the inverse problem above, the 4 taken out is a part, the 3 left is a part, while the number at the start is a total, so regardless of taking away in language and action, as a problem where you know the parts and want the total, it is addition);
- **Transformation** – Seeing an addition example such as 3 + 4 = 7 as a change, as shown in the first example on the right, with the inverse problem resulting in the change becoming subtraction. In the second example on the right, the answer requires the change to be reversed meaning the solution requires addition; and

$$3 \xrightarrow{+4} 7$$

$$? \xrightarrow{-4} 3$$

- **The difference between addition and multiplication** – Both addition and multiplication require sets of objects to be joined. With addition there can be any number of sets, the sets can be of any size, and each number describes the quantity of objects in the sets. In multiplication, all sets have to be the same size as each other and one of the two numbers refers to the quantity of objects in the sets and the other number refers to the quantity of sets.

4. Experience knowledge

Rich schema include the experience of the solver so that he or she can use situations in the past to help with present problems. Students often have little memory of their past mathematics experiences, preferring to forget them. If one uses the evaluative strategies to analyse past experiences, and remembers this as part of the experience, it is even stronger. Teachers can assist students to draw on their experience by asking the question "Have you done anything like this before?"

5. Automated knowledge

Knowledge that has been learnt until it can be immediately recalled without hesitation (e.g. rote learnt basic facts) is called automated. Automated knowledge has little to no cognitive load. This means that it is available for problem solving without taking thinking power away from the task of solving the problem. This is an important reason for encouraging the memorisation of basic facts. The Australian Curriculum includes memorisation of basic facts in the achievement strand of procedural fluency.

Memorisation of approaches to, or solutions for, standard problem types has limited value. However automated knowledge can enable a problem solver to: (a) use the “solve a simpler problem” strategy (try for solution quickly with small numbers); (b) quickly try a series of estimates to get a feel for the size of the answer; and (c) assess whether the result makes sense.

However, it is important that memorisation of basic facts is used to build on an understanding of the underlying concepts, principles and strategies, not as an alternative to understanding. The strategies, described in Chapter 5, enable the learner to get answers (albeit slowly) and, with practice the learner may speed up the strategy application until it is automated. Drill without understanding or problem solving strategies may not be as effective.

4.2 Decoding

Decoding the text of worded problems is challenging. Information is presented in words, but there may also be symbols, tables and visual images. Studies have found that up to 64% of errors in mathematical problem solving can be attributed to errors in the interpretation of the problem text (Cummins, Kintsch, Reusser, & Weimer, 1988; De Corte & Verschaffel, 1985). The ideas in this section are drawn from Resource 3 on Literacy in Mathematics.

4.2.1 Reading

Reading is critical to the decoding of a problem text. If students struggle to make meaning of a problem text, they may have problems with reading more generally. Students can be asked to read the text out loud, allowing the teacher to check for reading problems. If the student is able to read the text, they may not understand all of the vocabulary. The teacher could follow up by asking the student the meaning of any words or expressions that might cause difficulty.

Mathematical problem texts should be read at least three times: twice at the beginning and once at the end.

Scanning: When first encountering a problem text, it should be scanned (read quickly) to gain an overall feel for the problem. This provides a context a closer interpretation of the problem. However, when scanning, it is easy to overlook or misread a word and obtain the wrong impression, as in the following example:

Anne left home with $100 to go shopping. She spent all but $1. How much did she have left?

In the scanning stage, students might overlook the crucial word “but” and believe that Anne spent $1. In this case “but” is crucial to the meaning of the problem. Once the text is accurately decoded, the solution is easy. So scanning must be followed by close reading.

Close reading is the second stage of reading a problem text, where the details are gathered from the text. It examines each word and phrase to extract all relevant information.

There are four ways of extracting information from a problem text (Raphael, 1982, 1986):

Right there: the information needed to answer the question can be found “right there” in the same sentence;

Think and search: finding the relevant information and relating the various pieces of information to each other requires a search of the whole problem text;

Author and you: information drawn from the problem text must be combined with students' own factual or procedural knowledge; and

On my own: a response to the problem requires students to draw on their own knowledge.

Questions that students should ask of themselves during the close reading phase include:

- What facts am I given in the text (often easily identified as numbers)?
- What am I asked to do (it can help to look for the question mark)?
- What additional facts do I know that are relevant (author and you approach)?
- What information is not important?
- Can I summarise the information in a table or diagram?

Consider the following examples:

1. *There was some money in the box. John took away $8. This left $14. How much was in the box?*

Some words can confuse students as this example shows. Focusing on "took away" may result in thinking the problem is solved by subtraction:

2. *The deacon said, "The trip to the Gold Coast takes me one hour and twenty minutes". "That's strange", said the minister, "It only takes me 80 minutes". How much faster is the minister?*

Close reading is all that is needed here (and the recognition that 1h 20 min is 80 min). However, some students may not know what a deacon or a minister is and may be distracted by this irrelevant information. Students from lower socio-economic backgrounds are more likely to be confused by irrelevant contextual information (Cooper & Dunne 2000).

If the problem involves multiple steps, it can be difficult to process all of the information at once. Students may have to read the text more slowly, or several times, highlighting or underlining important sections. It may help to break the text up into meaningful chunks and phrases. The burden on short term memory may result in earlier information being forgotten by the time the student reaches the end of the problem (Barton & Heidema, 2002). In such cases a framework for getting started with maths problems (Appendix B) may assist students in sorting through the information in the text.

Final reading: Once a solution to the problem has been found (that is, at the end), the text should be reread to check that:

- the proposed solution fully answers the question, that is, that all steps have been completed (it is easy to leave out a final step when the problem requires many steps);
- the answer is reasonable in the context of the problem (for example, if the problem asked about the age of a person, an answer of 563 would clearly be incorrect);
- an appropriate degree of accuracy has been provided (for example, money should be rounded to the nearest dollar (whole number) or cent (hundredth);
- the answer uses the correct units (if the question asked for the number of metres, providing an answer in centimetres would be incorrect);
- the answer is presented in the required manner (for example, if the question asks that the response be presented in a table).

4.2.2 Classifying information

The information in a problem text can be classified as: (a) "given" – the information we work on and start from; (b) "wanted" – the required answer(s); (c) "needed" – anything we have to work out on the way to the answer; and (d) "not needed" – any unnecessary information ("red herrings"). Formally identifying information in this way can be useful exercise as this example shows.

> *There were 900 cases at the depot. Three large trucks each removed 126 cases. They were followed by five small trucks which each removed 57 cases. How many bottles were left at the depot after the large trucks had been loaded?*

The **given** information was depot (900 cases), the number of trucks (3 large and 5 small), and the capacity of each truck (large 126 cases and small 57 cases). The **wanted** information was the amount left after the large trucks had been loaded. The **needed** information was the total number of cases on the *large* trucks (3 × 126 = 378 cases). The **not needed** (irrelevant) information was the total number of cases on the *small* trucks (5 × 57 = 285 cases). Thus the number of cases left was 900 – 378 = 522 cases.

To assist in identifying the different types of information, students can annotate the text by circling, crossing out, and underlining. For example, *s*tudents could circle the "given" information, ~~cross out~~ the "not needed" (irrelevant) information, underline the words that say what is "wanted", and *write down* what is "needed". If the problem is posed verbally, students should be encouraged to make notes, classifying and summarising the "given", "needed" and "wanted" information.

In the example above, the "given" information is entirely numeric and the question mark indicated the "wanted" information. The strategy of locating the "given" information by looking for the numbers and the "wanted" information by looking for the question mark works in these, and many other mathematics, problems. However, it has little to do with logic. More capable and older students should be exposed to problems that do not contain numbers or question marks.

This example is also unusual because it includes unnecessary information. Many students consider that if they have not used all of the information provided, they must have made an error. For this reason, students should frequently be exposed to questions that include irrelevant information.

Students should also be encouraged to see that contextual information is often irrelevant. In this example, it did not matter *what* was being removed or *how* they were removed, provided that there were two different methods or removal. What mattered were the **mathematical relationships** between the facts. Asking students to write their own problem using the same mathematical relationships in a different context, and comparing with their classmates' problems can help to illustrate this idea.

If students struggle to develop a plan for solving a problem, it could be that they did not identify all of the required information while reading the problem. Worded problems in mathematics are often lexically dense (packing a lot of information into few words) making the decoding task challenging. For example, students might:

- overlook important words, for example, prepositions such as *to, from, above, below*, or words that indicate sequencing such as *initially*, *then*, *after;*
- dismiss some information as being irrelevant, for example, information such as *the older child, named Tom, likes ice cream* may convey important details about number (*older* means exactly two), gender (male), or a comparison of ages (not twins), despite the object of the sentence (ice cream) being irrelevant to the solution of the problem;
- fail to notice that they need to supply information not provided in the problem text, for example, the number of days in March, or the number of metres in kilometre.

To understand what a problem text is asking students to do, they must know the meaning of key task words such as *identify, demonstrate, explain, predict*.

In this section, two of the components of the expert problem solver have been examined, namely content knowledge and the ability to decode the information provided in the problem text. The next section examines the strategies that can be used to solve problems.

5 Strategies

The major part of any problem solving is a program to teach a rich repertoire of strategies. We recommend the following strategies (names for strategies taken from Meiring (1980). The following table relates the strategies to the thinking skills.

Logical thinking	• Reread the question (looking for keywords) • Identify given, wanted and needed information • Restate the problem in your own words • Write a number sentence
Visual thinking	• Act the problem out • Make a model of what happens in the problem • Make a drawing, diagram or graph • Select appropriate notation to picture the problem
Evaluative thinking	• Check your solution • Find another way to solve it • Study the solution process • Find another solution • Generalise
Patterning thinking	• Look for a pattern • Construct a table • Account for all possibilities (systematically) • Eliminate possibilities
Creative/Flexible thinking	• Identify a sub-goal (or break the problem into parts) • Guess and check (trial and error) • Work backwards • Solve a simpler problem (includes use simpler numbers) • Change your point of view • Check for hidden assumptions

Each of these strategies are described and illustrated with an example(s), under the five different thinking skills.

The reason for the examples is that the advocated way to **teach a problem-solving strategy** is to: (a) first, teach it directly with an example; (b) second, give a few examples all of which will use that strategy; and (c) finally, provide a mixture of problems only some of which will use that strategy. This is called the **match–mismatch** teaching approach.

5.1 Logical strategies

1. Reread the question

Students need to read the problem carefully to understand what it is saying. Reading strategies for problem texts have already been discussed at length in the previous chapter: twice at the beginning of the problem and once at the end. However, if after the initial scanning and then close reading of the problem, the student has not been able to extract the required information, he/she will need to continue reading the problem text. At this stage, it could be helpful to break the text up into parts and, where needed, to take notes.

Consider the following problem:

Fiona has two identical family meat pies. She cuts the first pie equally into four large slices and the other pie equally into eight small slices, so that a large slice weighs 40 grams more than a small slice. What is the mass of one whole pie?

The problem text contains 44 words, spread over five main ideas. However, these five ideas contain nine relevant mathematical facts. The first step is to break the problem into parts.

The problem contains a lot of information, beyond the short term memory capacity of most people. Summarising the facts in a list, a diagram, or a table as shown below, is essential.

SECTION OF PROBLEM TEXT	RELEVANT MATHEMATICAL FACTS
Fiona has <u>two</u> <u>identical</u> family meat pies.	• 2 pies • Pies are identical (that is, the same size and mass)
She cuts the first pie <u>equally</u> into <u>four</u> large slices.	• 4 large slices • Large slices are all equal in mass
And then (she then cuts – implied) the other pie <u>equally</u> into <u>eight</u> small slices,	• 8 small slices • Small slices are all equal in mass
so that a large slice weighs <u>40</u> grams <u>more than</u> a small slice.	• L = S + 40 • Note that this does not mean L = 40, which is what would occur if the words "more than a small slice" are overlooked
What is the mass of <u>one</u> whole pie?	• Mass of 1 pie

A similar problem is:

Lucy and Joanne both bought some sausages at the shop. Joanne bought half the quantity of sausages that Lucy bought. The total cost of Lucy's and Joanne's sausages was $15.00. How much did Lucy's sausages cost?

2. Identify given, wanted and needed information

This is also a decoding strategy and has been discussed in the previous chapter. It proposed that the information in a problem is classified as: (a) "given" – the information we work on and start from; (b) "wanted" – the required answer(s); (c) "needed" – anything we have to work out on the way to the answer; and (d) "not needed" – any unnecessary information ("red herrings"). It also showed how the problem text could be annotated to classify the types of information provided.

Formally identifying information in this way can be useful exercise as this example shows.

Two friends who really loved mathematics, Justin and Stephen, meet after not seeing each other for many years. They have the following conversation:

Justin: Are you married? Do you have any children? How many? How old are they?

Stephen: Yes, I am married I have three children and the product of their ages is 72.

Justin: (After some thinking) I cannot figure out their ages. You have not given me enough clues.

Stephen: Right! What if I told you that the sum of their ages is the same as the number of your address when we were at school?

Stephen: (After doing some more thinking) I still can't figure out their ages. I need another hint.

Justin: Well I can also tell you that the oldest is a girl.

Stephen: Aha! Now I can, without any doubt figure out the ages of your children.

What are the ages of Justin's children? …..their ages can only be natural numbers.

The **given** mathematical information appears to be that there are three children, the product of their ages is 72, and that the sum of the ages is the same as the unknown (to us) house number. These facts could be circled. The **wanted** information was the age of each child – this should be underlined. Since there are many combinations of three numbers that multiply to give 72, in order to eliminate some options the **needed** information is the sum of those three ages. The **not needed** (irrelevant) information appears to be the rest of the conversation. However, there is insufficient information to solve the problem. It is necessary to revisit the problem text. There are two other relevant pieces of information:

- First, Stephen could not solve the problem after being given the information about his address. Although we do not know the address, Stephen did and if it was insufficient, then there must have been several combinations of the factors of 72 with the same sum.
- Second, the sentence that the oldest child is a girl, in apparently focussing on the gender, conceals the important fact that the first-born child is not the same age as the other two children. As this was sufficient for Stephen to solve the problem, then excluding factor combinations where the two highest numbers were the same must have resulted in only one remaining option.

The problem becomes much easier if it is restated as:

Find all the combinations of three natural numbers with a product of 72.
Eliminate all factor combinations that have unique sum.
Select the option where the two highest factors are different.

This is a good example of how restating the problem in your own words simplifies the problem – the next strategy to be discussed.

Another example of this strategy (identify given, wanted, and needed information) is:

Jan is saving to buy a car. She has $5 822 in the savings bank and $2 683 in the building society. Her Lotto syndicate of 9 people has just won $45 684. When she her share of the winnings is in the bank, what is the total of her savings?

3. Restate the problem in your own words

Students need to internalise in their own words (and understandings) what it is that the problem wants them to do. A major source of failure in problem solving is spending too little time in understanding and defining the problem. A good problem solver rarely answers the problem as it is stated. They rework it in their own words (paraphrase) so that it is more straightforward. This process occurs in stages and may require more than one reading and a lot of rethinking. Teachers can assist students by asking the question "tell me what the problem means". An example is:

Frank is buying CDs (compact disks). He bought as many CDs as each CD cost him in dollars. He spent $196. How much did each CD cost?

Restating, the problem could become

The price of the CDs and the number of CDs were the same, and the total cost of $196.

or (even better)

What number multiplied by itself equals 196?

The answer is then straightforward to achieve with a calculator or by guess and check. Another example of the write this example more simply strategy is:

Lindy had $467 in the bank. This was 8 times less money than Jake. How much did Jake have?

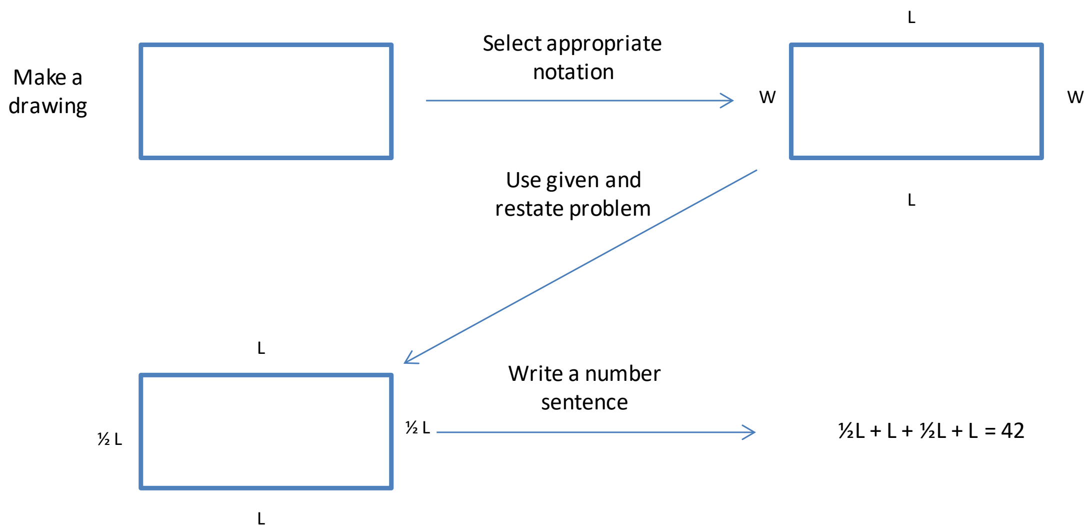
4. Write a number sentence

Write a number sentence (or write an open sentence) is an extension of 2 and 3 above. It requires the problem to be converted to one or more mathematical relationships, usually expressed as equations or formulae. It is a very powerful method (if students can do it). It is a higher order strategy because many others are needed to make it possible. As well as following on from 'given, wanted and needed' and 'restate the problem', it also may require students to 'select appropriate notation' and 'make a drawing'. After this they may be able to perceive a relationship between given and wanted that can be represented by a number sentence. In this way, students can change a problem from reality to a mathematical situation where a lot of learnt skills can come into play. A substantial proportion of the problem-solving process has been completed and the solution can now be found by computational means. Thus, this strategy is **the end product of other strategies** and may require a level of knowledge or the ability to think abstractly that is difficult for many students to achieve. It must be used with care. Yet many teachers see it as the beginning point of problem solving. An example is:

An ant travels 42 cm around a rectangle. The rectangle is twice as long as it is wide. How long is it?

The problem can be changed to an equation as follows, and we only have to solve the equation:

$$\tfrac{1}{2}L + L + \tfrac{1}{2}L + L = 42 \rightarrow 3L = 42 \rightarrow L = 14 \text{ cm}$$



Another example where number sentences (in this case two sentences) are useful is:

A container with 12 blocks has a mass of 224 g. A container with 8 blocks has a mass of 192 g. What is the mass of the container?

5.2 Visual strategies

1. Act the problem out

Often students have to visualise what is happening in a problem. To make this easier, they can gather actual materials and act out what has to be done. The physical involvement helps them to understand the problem better and the action makes the problem more concrete and they are able to discover relationships that may lead to the solution. An example is:

Can you make change for $1.00 with 6 coins, 7 coins, and 8 coins?

The "act it out" way of solving the problem is to gather a collection of coins – 1 cent, 2 cent, 5 cent, 10 cent, 20 cent and 50 cent pieces – and then try to form $1.00 with 6 coins, 7 coins and 8 coins. A further example:

16 heads and 54 legs, how many emus and how many wombats?

An excellent response from a primary-level student was: "I imagined all the wombats standing on their two back legs with their two front legs in the air. There would be 16 × 2 = 32 legs on the ground. That leaves 54 – 32 = 22 legs in the air. So there must have been 22 ÷ 2 = 11 wombats and 16 – 11 = 5 emus." This method of solution involved acting it out mentally, or visualising.

Another example of this solution method is:

> *A salesperson bought a car for $600 and sold it for $800. Later they bought it back for $700 and sold it again for $900. How much profit did they make?*

2. Make a model

Often acting out the problem with actual materials is inconvenient and impossible, but students can make a model to help visualise the problem. Easily accessible objects can represent harder to obtain or awkward to use real objects in the problem. Materials should be chosen so that they represent the problem accurately and enable the model to be related to the problem. An example is:

> *Each day during the week, Mum sent Joe to school with three biscuits. On Saturday there were 20 biscuits left. How many biscuits did Mum bake on the previous Sunday?*

Represent cakes with counters. Three counters for each of Monday, Tuesday, Wednesday, Thursday and Friday plus twenty cakes on Saturday gives thirty five counters. Thus 35 cakes were baked on the previous Sunday. This problem could also be solved using the *work backwards* strategy (see below).

The emus and wombats problem from the previous section can also be solved using a model:

> *16 heads and 54 legs, how many emus and how many wombats?*

Students could use counters to help to visualise this problem, as in the previous example. However, older mathematicians might make an *algebraic* model with two equations: $4w + 2e = 54$ and $w + e = 16$ (where w and e are wombats and emus, respectively), and then solve them to find the answer.

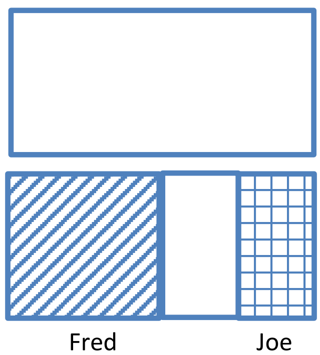
A further example is:

> *9 people on 23 wheels, how many bicycles and how many tricycles?*

3. Make a drawing, diagram or graph

Many problems have so much information that it is difficult to organise it to see relationships. Tables can make relationships apparent but making a drawing, diagram or graph may be more effective. If fact this strategy is one of the most powerful available. The drawings, diagrams and graphs act like an adjunct to memory allowing it to handle complex problems. An example is:

> *Fred has a large lawn mower and can mow a rectangular park in one hour. Joe has a smaller mower and takes two hours. If they both mow the park together, how long will it take?*



Many people incorrectly think that this is a problem about averages and give an answer of 1½ hours. However, if a drawing of the park is made, then a drawing can be made (on right) of where the mowing would have reached, say, after ½ an hour. This shows that Fred mows twice as much as Joe, so at the finish Fred would have mowed $^2/_3$ of the lawn and Joe $^1/_3$. Since Fred can mow the lot in 1 hour, he mows $^2/_3$ in $^2/_3$ of an hour (40 minutes) which is the answer. Another example is:

> *Jack set his watch at 9:15 am. At 2:45 pm he noticed his watch had lost 11 minutes. How much time would his watch have lost by 3:15 am the next day?*

4. Select appropriate notation

The power of mathematics lies often in the wide descriptive powers of symbols. If a word problem can be expressed in symbols there may be a rule that can be applied to solve it or a way of using symbols that can be used to reach a solution. The most appropriate and powerful symbols in many cases are not necessarily numbers but lines, dots, boxes, etc. For this reason, this strategy is placed after 'draw a diagram'. An example is:

> *Each day of the week Judy bought a pie for $1:50. Her mother gave her $20 to spend on lunches. How much change did she left after two weeks?*

Some students will draw a child buying a pie over a counter. But this does not help to solve the problem. A diagram which does help requires notation, like in a calendar, for each week day and a circle for each pie, as below. Then, it can be easily seen that the amount of money spent in one week is 5 x $1.50 and in two weeks would be 10 x $1.50 or $15, leaving $5 change.

Week 1 $20	Mon	Tue	Wed	Thu	Fri
	$1.50	$1.50	$1.50	$1.50	$1.50

Further examples are:

> *The bush walking club hiked 23 km on Monday and 27 km on Tuesday. Their three-day walk was 78 km. How far did they have to walk on Wednesday?*

> *The James family (Mother, Father, Bill and Brett) stayed in a motel for 6 nights. The room cost $112 per night. Each morning they all had breakfast at $14.50 each. What was their total bill?*

Some educators argue that there is no appropriate notation for drawing a diagram that will really help students solve a problem like that below. Are they right?

> *Nancy had 30 bananas. She had 5 times as many bananas as oranges. How many oranges did she have?*

5.3 Evaluative strategies

1. Check the solution

This is so simple but nearly all problem solvers make careless errors when they don't do it. An example is:

> *Five consecutive numbers added gives 150. What are the numbers?*

A "guess and check" method may get this right but a more high-powered number sentence approach is better, for example:

$$N + (N + 1) + (N + 2) + (N + 3) + (N + 4) = 150$$
$$5N + 10 = 150$$
$$5N = 140$$
$$N = 28$$

This may lead to an answer of N = 28 (not giving the 5 numbers as required). Checking and/or a third reading of the question will help prevent such errors.

2. Find another way to solve it (use a different method)

In real life the same task is seldom resolved the same way. Different and better solutions are developed. The same can be done with problems. For example, the previous example can be solved by "guess and check" or using a number sentence. Solving a problem using a different method is a very powerful way of checking that the answer is correct.

3. Study the solution process (looking back)

Often when solving problems (particularly when doing one step at a time) students can't see the forest for the trees. Overall, what was the strategy? Once the problem has been solved it is possible to look back to consider this and we always should. This is how students learn strategies. Teachers should encourage students to go through their problem solutions – explain them to themselves and, if they can, to others – identifying the successful strategies so they can be used again.

4. Find another solution

Mathematics teaching often directs students to expect one and only one correct solution. In real life this is often not the case and students must learn to find **all** solutions. An example is:

You have $1 made up of seven coins. What are the coins?

"Guess and check" may give a solution, but the student may stop there. A table and organised thinking is necessary to find **all** the solutions (and to be convinced there are no more).

5. Generalise

Students may need to generalise from simpler cases to solve problems, but when they have solved them, generalising the solution is fruitful practice of problem solving. An example is:

I have three weights. I can weigh any whole number mass from 1 to 13 kg with them (on my balance). What weights do I have?

Answer (by guess and check?) is 1, 3, 9kg. One can expand the problem to:

What if I have 4 weights? How can I get to weigh the most number of possibilities?

This forces the student to consider what the numbers 1, 3 and 9 have in common (they are cubic numbers) and why do they work as far as 13 (because 1 + 3 + 9 = 13). This may point to the next weight being the next cubic number. 27 (that is, weights of 1, 3, 9, 27) to give all masses to 1 + 3 + 9 + 27 = 40. The student can now check if these four weights do, in fact, give all masses to 40.

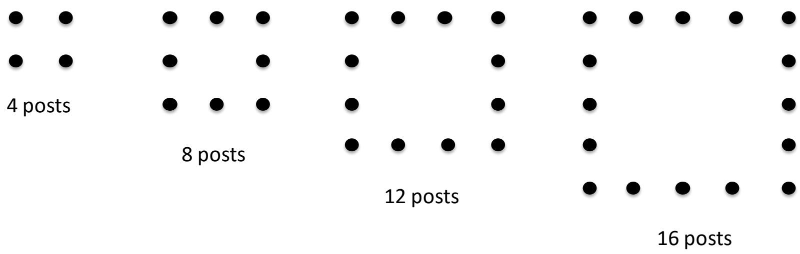
An interesting question is why do the cubic numbers have this property? If this can be explained (proved), then we could use this generalised method of solving the problem to a mass of any size.

5.4 Patterning strategies

1. Look for a pattern

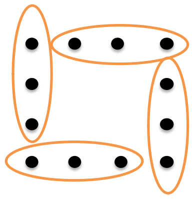
This strategy involves active search as well as passive observation. The pattern has to be identified and continued. It may also be necessary to find the original data that gives the problem. For example:

A farmer was fencing square paddocks. For the small holding pens, he used posts as below



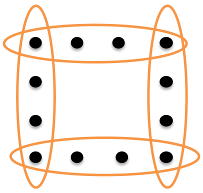
How many posts did the farmer use for square paddocks which had (a) 10 posts along each side? (b) 100 posts along each side? and (c) 1000 posts along each side?

Looking at the drawings, we see that 2 on each side uses 4 posts, 3 on each side uses 8 posts and 4 on each side uses 12 posts. Continuing this pattern, 10 on each side would use 36 posts, 100 on each side would use 396 posts and 100 on each side would use 3996 posts. However, this pattern can be seen in more than one way. However, in every case, the answer is the same and the general approach of "look for a pattern" is successful.



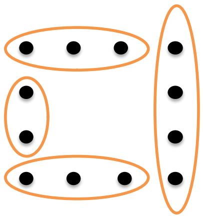
4 on each side is 4 × (4 – 1) = 4 × 3 = 12

100 on each side is 4 × (100 – 1) = 4 × 99 = 396



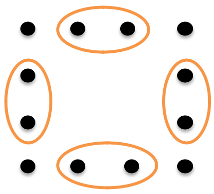
4 on each side is 4 × 4 – 4 = 12

100 on each side is 4 × 100 – 4 = 396



4 on each side is 4 + 3 + 3 + 2 = 12

100 on each side is 100 + 99 + 99 + 98 = 396



4 on each side is 4 × (4 – 2) + 4 = 14 × 2 + 4 = 12

100 on each side is 4 × (100 – 2) + 4 = 4 × 98 + 4 = 396

Other examples are:

> *A pair of rabbits when born, can give birth to another pair of rabbits at the end of their second month and continue to do this at the end of each month from then on. How many rabbits will exist, if we assume none die and all breed correctly, after 18 months, if we start at the beginning with a pair of baby rabbits?*
>
> *Kirra bought a car for $15,000. She asked the seller if she could pay for the house gradually over the next four months. The seller said that he would accept weekly payments. He suggested that she pay 1c immediately, 2c next week, 4c the week after, and continue doubling the amount each week for the next four months (21 weeks in total). Should Kirra accept this deal? Why/why not?*

2. Construct a table

Tables are particularly useful in problem solving. They can be used to:

- record and organise data
- summarise data
- compare data
- identify a pattern (relationship) between two or more variables
- record the steps of an iterative (repeated) process

For example:

Peter's grandmother gave him a bank account containing $500.00 as a reward for improving in mathematics. The bank will deposit 5% interest into the account at the end of each year. Peter decides not withdraw any money from the account until he leaves school in six years' time. How much can he expect to have in the account at the end of five years?

This question requires the students make an interest calculation six times (assuming that they are unaware of the compound interest formula). A good way of organising this iterative approach is to construct a table with six rows and column headings such as: year; starting amount (principal); interest; and closing amount.

Questions which require students to collect data, for example, by questioning, measuring or observing, become easier if the student uses one or more tables to record, organise and summarise the data. When looking for a pattern or relationship, it is much easier to discover it when the information is in tabular form.

The handshake problem is another example where constructing a table will assist in solving the problem.

In a room full of people, everyone shakes hands. How many handshakes are there in total?

An open ended and abstract problem of this kind will challenge most students. To make it more manageable, students may need to develop some basic assumptions such as: you don't shake hands with yourself; and you only shake hands once with each student. In problems of this kind, it can be helpful to construct a table to explore what happens with particular values (that is, solve a simpler problem):

Number of people	Number of handshakes
1	0
2	1
3	3
4	6
5	10
6	15
7	21
etc	etc

Students may eventually recognise that as a new person enters the room, the new person must shake hands with everyone that is already in the room. So, if there are n people in the room, the pattern is:

$$1 + 2 + 3 + 4 + 5 + \dots + n - 1$$

Older students might be able to generalise this pattern to $\frac{n(n-1)}{2}$ (hint: start by adding the first term to the last, the second term with the second last, etc).

3. Account for all possibilities

A strategy that often goes hand in hand with constructing a table is that to account for all the possibilities. This strategy does not mean to examine all possibilities, but rather that we account for them in some **systematic** way. An example is:

Are there any rectangles whose sides are whole numbers for which their perimeter equals their area?

Draw up a table under headings. Then systematically, holding the width at 1 unit, then at 2 units, then at 3 units and so on, vary the length and calculate the perimeter and area until all cases are found. (These are length 4, width 4, and length 6, and width 3).

LENGTH	WIDTH	PERIMETER	AREA

Other similar examples are:

5 litre and 8 litre buckets of ice cream cost $4.58 and $6.50 respectively. What do we buy to get 30 litres at the lowest cost?

At twelve o'clock the hands on the clock face overlap. At what other time(s) does this occur?

The lucky draw at the fete sells tickets with numbers between 000 and 999. If the digits add to 14, a prize is won. How many prizes winning tickets are there in the roll of tickets?

3. Eliminate possibilities

In this strategy students use a process of elimination until they find the correct answer. It can be used in situations where there is a limited set of possible answers and a set of criteria that the answer must meet. An example is:

> *In the game of Rugby League, a team can score five points for a try, three points for a subsequent conversion and three points for a penalty kick or a drop goal. If a team does not score any conversions, what scores* ***cannot*** *be achieved if the total team score was less than 20 points?*

In this problem, students understand that there is a finite set of possible answers: 1, 2, 3, …., 19. Students should work through each criterion to find the solution. Any multiple of five would be a possible score of the game. If the team only scored tries, they could score 5, 10, or 15. Therefore, all multiples of five should be eliminated. Any multiple of eight could be score for converted tries: 8 or 16. Therefore, all multiples of eight should be eliminated. If a team scored some converted and some unconverted tries, combinations of five and eight also need to be eliminated: 5 + 8 = 13, 5 + 5 + 8 = 18. So, after eliminating these scores, we have:

1, 2, 3, 4, ~~5~~, 6, 7, ~~8~~, 9, ~~10~~, 11, 12, ~~13~~, 14, 15, ~~16~~, 17, ~~18~~, 19

So, the following scores could not be the score of the game: 1, 2, 3, 4, 6, 7, 9, 11, 12, 14, 15, 17, 19

> *Sue, Freda and Michelle had breakfast together. Each chose a different item: toast, cereal with milk, and a fruit salad. Sue sat next to the person who ate the toast. Freda is on a diet and is not eating bread and dairy products. Who ate which breakfast?*

To solve this problem, use a table to organize the information (below). As Sue did not eat the toast, put a cross in Sue's column next to toast. With the second clue, you know that Freda did not eat the toast because she is not eating bread products. Also she did not eat the cereal because she is avoiding milk products. So, Freda must have eaten the fruit salad. Put a cross in Calvin's columns for toast and cereal, and a tick for fruit salad. Now, put crosses next to fruit salad for Sue and Michelle because they did not eat the fruit salad. Look at the table, two of the choices have crosses in Sue's column. She must have eaten cereal for breakfast. Then put crosses in Michelle's column for cereal, and you find that she must have eaten the toast.

	Sue	Freda	Michelle
Toast	✗	✗	✓
Cereal with milk	✓	✗	✗
Fruit salad	✗	✓	✗

Here are two similar problems:

> *Abby, Brian, and Chris live in the same street. Two of them live on the right side of the street. One house is painted green, another has a driveway, and a third house is made of brick. The brick house is on the left side of the street. Brian has a red car, which is parked in his driveway. Abby lives across the street from Chris. Which house does Josh live in?*

> *Five sisters all have their birthday in a different month and each on a different day of the week. Paula was born in March but not on Saturday. Abigail's birthday was not on Friday or Wednesday. The girl whose birthday is on Monday was born earlier in the year than Brenda and Mary. Tara wasn't born in February and her birthday was on the weekend. Mary was not born in December nor was her birthday*

on a weekday. The girl whose birthday was in June was born on Sunday. Tara was born before Brenda, whose birthday wasn't on Friday. Mary wasn't born in July. Find the month and day of the week each sister's birthday falls.

Eliminating possibilities is an important strategy in the case of multiple choice questions. If there are a small number of options (commonly four or five), it can be easier to eliminate those options that are clearly incorrect than it is to calculate the correct answer. Estimation may also be sufficient to eliminate the incorrect responses.

5.5 Creative and flexible strategies

1. **Identify a sub-goal** or **Break the problem into parts**

This is a way of making the problem smaller so that we can get started. Some problems require a series of steps to be performed before they can be solved. Other problems have specific solutions that seem out of reach yet we intuitively feel that certain steps will produce an answer or provide the material to find the answer. So identifying a sub-goal (the first step, say) may make the solution more apparent. Students can then move onto the next sub-goal. In other words, the problem has been broken into parts that can be followed in sequence. An example is:

Ancient Egyptians represented fractions only with unit numerators. So $^{1}/_{2}$, $^{1}/_{3}$, $^{1}/_{4}$ can be used but not $^{2}/_{3}$. To represent non-unit fractions, Egyptians make a sum of different unit fractions, for example $^{2}/_{3} = ^{1}/_{2} + ^{1}/_{6}$ and $^{3}/_{4} = ^{1}/_{2} + ^{1}/_{4}$. It should be noted that a denominator cannot be represented twice (e.g. $^{2}/_{3} = ^{1}/_{3} + ^{1}/_{3}$ was not allowed). How would an ancient Egyptian express $^{5}/_{8}$?

The answer to this problem appears difficult as there is so much information to digest. But looking at the $^{5}/_{8}$ makes us wonder if $^{1}/_{8}$ could not be used. So we make a sub-goal of using $^{1}/_{8}$, that is, $^{5}/_{8} = ? + ^{1}/_{8}$. *We work out what ? is, that is, $^{5}/_{8} = ? + ^{1}/_{8} = ^{4}/_{8} + ^{1}/_{8}$, Since, $^{4}/_{8} = ^{1}/_{2}$, we have $^{5}/_{8} = ½ + ^{1}/_{8}$. Yes!*

Other examples are:

Express $^{3}/_{7}$ as an Egyptian fraction.

Find the missing numbers in the array below

+				
		46		39
23			47	
	75			59
		54	66	

Breaking the problem into parts can be an important strategy in the case of multi-step problems (see the next section).

2. Solve a simpler (or similar) problem

In many problem-solving situations, information from simple cases (or similar cases) can be used to give the answer in a complex problem. The need to solve a simpler problem will not usually be mentioned in the statement of the problem. So, students always need to think of looking at simpler cases is a powerful way of getting insight into how to solve a complex problem.

For word problems, the most common use of this strategy is in replacing larger numbers with smaller ones to identify what operations to use (this particular approach is sometimes called "calculator codes" because it gives a code for putting the numbers into a calculator). An example is:

Jane bought her car for $13 687. This was $4 987 more than Jenny paid for her car? How much did Jenny pay?

The strategy is to replace the numbers with smaller ones. For example: Jane bought her car for $8. This is $3 more than Jenny paid for her car? How much did Jenny pay? The answer is $8 – $3 = $5. Therefore, returning to the original problem, the answer can be found by replacing the smaller numbers with the original numbers, 13 687 – 4 968, and then using a calculator to find the answer 8 719. A further example is:

John's car uses an average of 11.8 litres per 100 km, John travels 3,827 km, how much will the petrol cost at 149.7 cents per litre?

This strategy is also useful for problems like the following:

How thick is a sheet of paper?

To solve it, look at something simpler, like measuring the thickness of 500 sheets and then dividing the answer by 500 to find the thickness of one sheet. It should be noted that although this increases the number, the measurement task is simpler.

3. Guess and check

This is a little appreciated but very effective strategy (as long as the guesses are "educated"). The key element to its success is the "check". This strategy can be useful when students are really blocked (the strategy is sometimes called **trial and error** or **guess and improvement**). An example is:

A T-shirt is $22 and shorts are $28 each. Robert spent $272 on 11 pieces of clothing. How many T-shirts and how many shorts did he buy?

There are 11 items so try 5 T-shirts. 5 T-shirts at $22 each and 6 shorts at $28 cents each is $278. This is too high so reduce the number of shorts, e.g. 6 T-shirts 5 shorts. This gives the correct amount. Other examples are:

The car and the boat cost $18,264. The car costs $5,796 more than the boat. How much does the car cost?

The book was open. There were two page numbers showing. Their product was 7482. What were the page numbers?

Students need well developed organisational skills and persistence for this strategy to be successful. They must write down their trials, so that they can learn from them. Too often students do not record or obliterate unsuccessful trials. Consequently, they do not have the opportunity to check if an error was the cause of the unsuccessful trial and may even lead them to repeat unsuccessful trials.

4. Work backwards

This strategy has very wide implications for process problems. It allows for creative problems such as the following to be easily solved.

697 people enter a knockout tennis game. How many games are needed to find a winner?

Most students attempt to solve this by drawing a way that there can be games with winners moving on (with early ones having forfeits if there is not enough pairs). This is quite complex. However, the problem becomes easy if work backwards – do not think of winners, think of losers. If at the end, there is one winner, this means 696 losers and so 696 games.

It is important in word problems that provide details of the finish and want the start. In these situations, it is sometimes easier to reverse things and work backwards. An example is:

Joanne had some money. She spent $27 on a shirt and $49 on jeans. She had $54 left. How much did she start with?

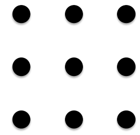
Start at the end and work backwards: after buying the shirt and jeans, $54; after buying the shirt and before buying the jeans, $54 + 49 = $103; and before buying the shirt, $103 + 17 = $130. A further example is:

Add the first 100 odd numbers.

Five trucks entered the depot. They all carried 47 crates except for the last which had 49. "This is great", said the foreman, "before you came we did not have enough crates!" There were now 568 crates in the depot. How many crates were in the depot before the trucks arrived?

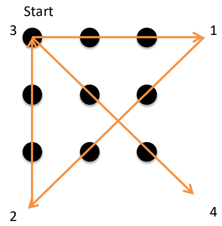
5. Change your point of view and **Check for hidden assumptions**

A problem can be made more difficult by our perception of it, by the perspective we take. It is too easy to take an incorrect and non-productive perspective of a problem. So when stuck, students should consciously redefine the problem, trying to take a new view. An example of this is:



Without lifting your pen draw 4 straight line segments to pass through all dots.

This is impossible if the lines are confined to the region surrounded by the dots. As soon as students realise you can go outside 9 dots, the solution is easy.



If students cannot get out of the mental set that it is stopping them from solving the problem by changing their point of view, it could be that they have made a fundamental assumption that makes it impossible to change their view enough. They have to find this assumption and remove it.

Example:

3 friends pay $30 for a pizza. The manager realises that they have been over-charged and gives the waiter $5 to take back to the friends. The waiter, realising that he cannot divide $5 easily between three people, decides to keep $2, and returns $3 to the friends. Each of the friends paid $9, for a total of $27. The $2 that the waiter kept makes $29. What happened to the remaining $1?

The answer is to question the assumption in the question that the $2 should be added to the $27. If we analyse where the money went, for example, in a table as shown below, we can show that the $2 was already counted in the $27, so it cannot be added to it!

Manager	Waiter	Back to the friends as change	
$25	$2	$3	adds to $30

This strategy is different to the *Restate the problem* strategy. It requires a shifting of focus on the problem components, a detailed look at what the problem says and does not say, and any assumptions made in the problem text or by the problem solver.

Other examples are:

A father and son were driving home. The car crashed and the father died instantly. His son was injured and rushed by ambulance to the hospital. The doctor on duty came in and took one look at him and said "I can't operate! This is my son!" What is going on here?

Fred had to drive 400 km at an average speed of 80 km/h. With his car engine overheating, he travelled the first 200 km at the average speed of only 40 km/h. How fast must he travel the second 200 km to make his target?

Make a calendar out of two cubes by writing numbers on their faces so that all dates in a month can be seen. What numbers must be on the faces of each cube?

5.6 Teaching tips

Although these strategies are presented separately, a single problem may require the application of several strategies to reach a solution. Expert problem solvers have a rich repertoire of problem solving strategies and look for opportunities to apply them to the problem at hand. In many cases the selection of a strategy appears to be unconscious, but analysis of experts' approach to solving a problem shows that they rely on the strategies discussed above.

Because a single problem may require the application of several strategies, it is not usually effective to teach the problem solving strategies individually. A more effective approach is to discuss them with students in the reflection stage. After a solution has been reached, there is scope for productive discussions about the strategies that were used, the alternative strategies that might have been possible, and what clues were in the problem that led to the choice of strategies.

6 *Multi-Step Problems and Investigations*

6.1 Word problems

This section looks at word problems, how to determine which operations are required, how to get started with a simple structure to help with solutions, and how to make the problems more challenging.

6.1.1 Deciding which operation

Of particular interest to most schools is students' ability to work with word problems, that is, to translate a real world story into a mathematical process, usually involving arithmetic operations.

Clues from the type of mathematical objects (numbers)

One way of determining the arithmetic operation is to consider the numbers, or mathematical objects, that are involved. This is best done in three steps.

1. Decide if the operation is addition/subtraction or multiplication/division. Arithmetic operations such as addition, subtraction, multiplication and division are *binary* operations. That means that two inputs form one output. In some cases there appear to be more than two inputs, for example, 2 + 3 + 4 = 9, but this is really a case of repeated addition: 2 + 3 = 5 and then 5 + 4 = 9. Thus, binary operations have three parts, two of which are usually known (inputs) and one unknown (output). The relationship between the three numbers can help to identify which type of operation applies.

   If all three numbers refer to the objects being considered, for example, 2 fish, 3 fish and 5 fish, then the operation is addition/subtraction. If one of the numbers describes how many groups, repeats, lots there are, while the other two numbers refer to objects, it is multiplication/division, for example 20 fish, 5 fish and 4 groups. Multiplication/division also has equal sized groups while addition/subtraction need not.

2. Deciding if the operation is addition or subtraction. If the operation is addition/subtraction, look at the problem in terms of part-part-total (P-P-T). If the parts are known and the total is wanted, the operation is addition; if the total and one part are known and the other part is wanted, it is subtraction.

   As is shown below in the examples, addition can be in action and in words either joining or separating, it depends on the parts (P) and the total (T). Similarly, subtraction can be either separation and joining – again, it depends on the parts and the total.

   Examples:

Addition	**Joining**	**Separation**
3 + 5 = ? P & P known, T unknown	Joe caught 3 fish in the morning and 5 fish in the afternoon. How many fish did he catch?	Joe caught some fish. He gave away 5 fish and this left 3 fish. How many fish did he catch in total?
Subtraction	**Separation**	**Joining**
7 – 4 = ? one P & T known, other P is unknown	Joe caught 7 fish and gave away 4. How many fish did he keep?	Joe caught 4 fish in the morning and some more in the afternoon. He caught 7 fish overall. How many fish did he catch in the afternoon?

3. Deciding if the operation is multiplication or division. If the problem is multiplication/division, look at the problem in terms of factor-factor-product (F-F-P). If the factors are known and the product is wanted, the operation is multiplication; if the product and one factor are known and the other factor is wanted, it is division.

   As is shown below in the examples, multiplication can be in action and in words either combining equal groups or partitioning into equal groups, it depends on the factors (F) and the product (P). Similarly, division can be either combining equal groups or partitioning into equal groups – again, it depends on the factors and products.

   Examples:

Multiplication	**Combining**	**Partitioning**
3 x 7 = ? Fs are known, P is unknown	3 men went out fishing. Each man caught 7 fish. How many fish were caught in total?	7 men went out fishing and all caught the same number of fish. In total, they caught 21 fish. How many fish did each man catch?
Division	**Combining**	**Partitioning**
21 ÷ 7 = ? one P and F known, other F unknown	Joe caught 21 fish and shared them equally amongst his 7 friends. How many did each friend get?	Joe caught some fish. He divided them equally between the 3 families. Each family got 7 fish. How many fish did Joe catch?

Clues from the text

Often the text of a worded problem provides clues as to the operation needed. Some examples are provided in this section.

Words for operations:

Several words can imply particular arithmetic operations. For example, "together" usually implies addition, "reduction" implies subtraction, "times" suggests multiplication and "shared" implies division. A list of such textual clues is in Appendix A. These words need to be used with caution. Some words are not what they seem. There are no hard and fast rules.

When the words "how many" or "how much" are followed by a comparative form of an adjective, such as "how much further" or "how many more", they imply the inverse operation. This is demonstrated in the following examples.

This simple problem is solved by addition:

> *Peter has $2 and Mary has $3.* ***How much*** *do Peter and Mary have together?*

However, by modifying the question so that the comparative form of the adjective "more" is used after the words "how much", subtraction is required:

> *Peter has $2 and Mary has $3.* ***How much more*** *than Peter does Mary have?*

In the next example the word "times" indicates multiplication:

> *The house is five* ***times*** *the length of the bedroom. If the bedroom is three metres long, how long is the house?*

However, using the comparative adjective "longer" after the words "how many times" changes the operation to one of division:

The house is five times the length of the bedroom and ten times that of the bathroom. ***How many times longer*** *is the bedroom than the bathroom?*

In other words, the use of a comparative form of adjective after the words "how much" or "how many" results in the inverse operation.

Units of measurement

Sometimes the units of measurement for the given, wanted and needed data can suggest the operation(s) to be used. For example:

Peter travels 9 km in 3 hours. What is his average speed?

In this problem, speed would be measured in kilometres per hour (where "per" implies division). Since the number of hours and distance travelled in kilometres are given in the problem, then to produce an answer in kilometres per hour, the units needed for the answer suggest that the distance must be divided by the time.

In another example, area is measured in square units (cm^2, m^2, etc). As multiplication is used to square a number, to find the area, it is necessary to multiply (not add) the length by the width. This can be a useful tip for those students who regularly confuse area and perimeter.

Type of number

The problem text can provide clues as to the type of number required in the answer. If the answer is a whole number (discrete data), words such as how many (suggesting counting), fewer, fewest are used. On the other hand, words that imply that the answer could be a fraction (continuous data), include words such as how much, less, or least.

As mentioned earlier, it is difficult to present textual clues as hard and fast rules, as there can be many exceptions. However, the question "Are there any words in this problem that give you a clue about what to do?" can be a useful prompt for teachers to use with a student struggling to make a start on a problem.

6.1.2 Getting started in word problem solving

Word problems are generally translation problems. The major difficulty with these types of problems is getting started. One way to get started is to focus on three strategies "Make a drawing, diagram or graph", "Identity given, needed and wanted", and "Restate the problem in your own words". These can be combined into a template or framework that has been found useful for word problems.

A copy of the framework is in Appendix B. It has the following parts: (a) PROBLEM – where the word problem is written; (b) DRAWING – where a drawing that helps solve the problem is given; (c) GIVEN, etc – where list what is given, what is needed and not needed and what is wanted; (d) STORY – where the students rewrite the problem in their own words and in a form that makes the problem more straight forward and easier to understand; and (e) WORKING – where students puts all their working out when solving the problem. The idea is to get the students to do all three strategies before doing the WORKING and solving the problem. It also helps the students to consider the problem in a step-wise fashion, rather than holistically, which can be overwhelming for some students. Discuss what students have written before moving onto the next part.

6.2 Ways of increasing the difficulty of word problems

Once you have started on word problems with simple stories, word problems can be modified to make them more complex (i.e. moved from a simple to a complex translation problem). This approach allows the teacher to adapt questions in textbooks and on worksheets to suit the students. Some of the ways of making a problem more complex are shown below.

The examples below assume that we start with a simple translation problem, such as*: John had $343. He spent $176 on a television. How much money does he have left? The examples below show how that simple problem can be made more challenging.* They explain how this basic question can be modified.

(a) Increase size of numbers – John had $9 500. He spent $6 799 on a car. How much does he have left?

(b) Use decimal and fraction numbers – John had $343.72. He spent $179.98 on a television. How much money does he have left? or John had $343.72. He spent one third on a television. How much money does he have left?

(c) Change the context of problem to an unfamiliar environment – John was buying gold futures. Nixon East gave him a profit of $343 yet on Raisin Sands he lost $176. What was his total profit?

(d) Change the language and meaning – a bicycle cost $343. This was $176 more than a scooter. How much does the scooter cost?

(e) Change conditions – John had $343. John bought a television. He had $167 left. How much did the television cost?

(f) Give numbers in different order to how they are implemented- John spent $176 on a television. Before the purchase he had $343. How much money does he have now?

(g) Introduce some more numbers – a 42 cm television cost $343. This was $176 more than the 32 cm television. How much does the smaller television cost?

(h) Increase the number of steps – John had $343. He spent $176 on a television and $27 on headphones. How much money does he have left?

(i) Involve more than one operation – John had $343. He bought 6 pairs of jeans, each costing $27. How much money does he have left?

(j) Give too much information - John had $343. The television was $176 and the headphones were $39. John bought the television. How much money does he have left?

(k) Do not give enough information – John had $343. He bought a television. How much money does he have left?

(l) Change some combination of the above – John was buying gold futures. He lost $13 on each of 5 Minnepa shares. He lost $86 on a Raisin Sands, share. He made a profit on a Nixon East share. Overall he made a profit of $151 on the 7 shares. What was the profit he made on the Nixon East share?

(m) Absurd problem – John had $343. The television cost $674. How much money does he have left?

(n) Make the problem open ended – John wanted to buy a car. He had a part time job where he earned $135 each week. What would you recommend that John do?

6.3 Multi-step problems

In multi-step mathematical problems, two or more steps are needed to find the information required to answer the question being asked. The trick is to identify which operations are needed and the order in which they must be applied.

Multi-step problems often contain more information than single-step problems. This makes the task of identifying all of the information in the question more challenging. In the previous chapters, a strategy was proposed for classifying the information in the problem text: (a) “given” – the information we work on and start from; (b) “wanted” – the required answer(s); (c) “needed” – anything we have to work out on the way to the answer; and (d) “not needed” – any unnecessary information (“red herrings”). In multi-step problems, the third category (“needed”) is important.

6.3.1 "Needed" information

The difference between single-step and multi-step problems can be illustrated using some examples:

> *Steve planted 128 tomatoes in rows in his vegetable garden. 20 of the plants produced yellow tomatoes. All the other plants produced red tomatoes. How many of the plants produced red tomatoes?*

The information in the problem text can be summarised as follows:

- 128 plants — given
- 20 yellow plants — given
- All the others are red — given
- Number of red — wanted

In this problem all of the information to find the number of red tomato plants is provided. All that is needed is to subtract 20 from 128. This is a one step problem, since only one calculation is required. Note also that there was no information in the "needed" category.

If the problem is changed slightly, two steps are required:

> *Steve planted 128 tomatoes in rows in his vegetable garden. In the four middle rows he planted five plants that produced yellow tomatoes. All the other plants produced red tomatoes. How many of the plants produced red tomatoes?*

The information provided now increases:

- 128 tomato plants — given
- 4 rows with yellow tomato plants — given
- 5 yellow tomato plants in each of these rows — given
- Rows with yellow tomato plants are in the middle — not needed
- All the other plants have red tomatoes — given
- Number of red tomato plants — wanted

To find out how many plants produce red tomatoes, we need to know how many yellow tomato plants there are. As this information is not provided in the question, we must calculate it from the "given" information. So we add to our list of information:

- Number of plants with yellow tomatoes — needed

Since the number of yellow tomato plants is needed to find the answer, that calculation is the first step: 4 × 5 = 20. To answer the question ("wanted" information), the second step involves subtraction: 128 – 20 = 108. So 108 of the tomato plants will produce red tomatoes.

6.3.2 Breaking the problem into parts

Note that in the previous example, the **break the problem into parts** strategy was used. This is a particularly important strategy for multi-step problems.

6.3.3 Meta-cognitive skills

In multi-step problems, there is a danger in stopping after completing one step. The question asks about the number of **red** tomatoes, but the first step gives the number of **yellow** tomatoes. Students must use their metacognitive skills to monitor their progress in solving the problem and to decide after each step if they have obtained the "wanted" information.

6.3.4 Increasing the amount of "needed" information

By adjusting the problem further, three steps are needed:

Steve planted eight rows of tomato plants in his vegetable garden. Each row contained 16 plants. In each of the four middle rows he included five plants that produced yellow tomatoes. All the other plants produced red tomatoes. How many of the plants produced red tomatoes?

Information:

- 8 rows of plants — given
- 16 plants in each row — given
- 5 yellow tomato plants in 4 of the rows — given
- Rows with yellow tomato plants are in the middle — not needed
- All the others plants produce red tomatoes — given
- Number of red tomato plants — wanted

To find out how many plants produce red tomatoes, we need to know how many plants there are in total and how many plants with yellow tomatoes there are. Neither of these facts are given in the question. It is necessary to add to the list of information:

- Number of plants — needed
- Number of plants with yellow tomatoes — needed

In this question, it does not matter which order the two "needed" facts are obtained:

- Number of plants — $8 \times 16 = 128$
- Number of plants with yellow tomatoes — $4 \times 5 = 20$

The third and final step is as in the previous question: 128 – 20 = 108.

6.3.5 Sequencing of steps

We can modify the question further to add in even more steps:

Steve planted eight rows of tomato plants in his vegetable garden. Each row contained 16 plants. In each of the four middle rows he planted five plants that produced yellow tomatoes. All the other plants produced red tomatoes. Each plant is expected to produce 15 tomatoes. How many red tomatoes can Steve expect from his vegetable garden?

Note that the final question (the "wanted" information) has changed compared to the earlier versions of this problem. Now, the information becomes:

Information:

- 8 rows of plants — given
- 16 plants in each row — given
- 5 plants with yellow tomatoes in 4 of the rows — given
- Rows with yellow tomato plants are in the middle — not needed
- All other plants produce red tomatoes — given
- Each plant produces 15 tomatoes — given
- Number of red tomatoes — wanted

To find out how many **tomatoes** are expected to be red, the "needed" information becomes:

- Total number of plants — needed
- Number of plants with yellow tomatoes — needed
- Number of plants with red tomatoes — needed

In this instance, the sequencing of the steps is also important. To find the number of **red tomatoes**, the number of **plants with red tomatoes** is required, which in turn, requires the number of plants with yellow tomatoes and the total number of plants. The **working backwards** strategy assists in identifying the sequence in which the steps must be undertaken.

6.3.6 Summary

Although the steps in this problem are individually straightforward, the number of steps increases the cognitive load. As the human short term memory cannot generally cope with more than seven ideas, multiple step problems usually require written notes to keep track of them all. This contrasts with single step problems that are often within the capacity of the short term memory.

To summarise the ideas developed in the example above:

- In single-step problems all of the information to solve the problem is usually given in the problem text. All that is required of the student is to determine the operation required to solve the problem. This can often be done mentally (that is, without written notes).
- Multi-step problem texts usually contain more information than single-step problem texts. Hence, the task of identifying all of the information in the question is more challenging.
- In multi-step problems some of the information needed to solve the problem is missing. Students need to identify the nature of the missing information and a strategy to obtain the "needed" information from the "given" information. This requires them to **break the problem into parts**.
- In multi-step problems, there is a danger in stopping before the question has been fully answered. Students must use their metacognitive skills to monitor their progress in solving the problem and to decide after each step if they have reached the solution.
- In multi-step problems, the sequencing of the steps may be important. The **working backwards** strategy assists in identifying the sequence in which the steps must be undertaken.
- In multi-step problems, students usually have to make written notes to keep track of the various steps undertaken to reach a solution.

To conclude, multi-step problem are more complex than single-step ones, for several reasons: a) the additional information given in the problem text that must be read and extracted; b) the additional information that is not provided, but nevertheless is needed to reach a solution, must be identified; c) the problem generally involves additional calculations, which must be identified and undertaken; and d) the metacognitive tasks of monitoring progress towards answering the question are more demanding.

6.4 Open ended tasks

6.4.1 Investigations and inquiries

Some mathematical problems are open ended. The question is often very general in nature, with many different acceptable responses (note the change in language: we talk of student "responses" or "solutions", rather than "answers"). Open-ended investigations are activities in which students take the initiative in finding the answers to problems. The student may not be provided with all of the information required to solve the problem. The problems require some kind of investigation in order to locate the information that will lead to the answers, requiring them to conduct research and/or collect data. So, open-ended tasks may require the student to plan a course of action, undertake research, collect data, organise and interpret the data, and reach a conclusion through the application of logical thinking. This process must then be communicated to others in some form. The responses are usually complex, requiring mathematical arguments, examples, and extended writing.

Words such as "investigate", "research", "analyse", "explain", "why", and "report on" are frequently used in the text of these problems. They are often referred to as investigations or inquiries. It may take several days for students to prepare their responses to a problem.

Mathematical investigations provide opportunities for:

- language development;
- working cooperatively;
- concrete experiences of natural phenomena;
- stimulating curiosity and creativity;
- motivation and enjoyment of mathematics;
- developing investigation and problem-solving skills;
- using mathematical technology;
- experiencing and developing an understanding of the nature of mathematics; and
- conceptual development.

Teachers of younger students tend to place more emphasis on the opportunities in the first half of the list and teachers of older students often place more emphasis on those in the second half of the list.

The Australian Curriculum: Mathematics has four proficiencies that many consider to be at the heart of learning mathematics. Three of these (Reasoning, Problem solving and Understanding) are developed through the process of mathematical inquiry or investigation. The processes of investigation or inquiry can be applied to all content strands in the curriculum. They also contribute to the development of General Capabilities such as literacy, numeracy, critical and creative thinking, ITC competence, and personal and social competence.

In the classroom, a question that is robust and fruitful enough to drive a mathematical inquiry or investigation should generate a need to know in students and stimulate additional questions. The initial question may come from any source: the student, the teacher, the teaching materials, the internet or some other source. In fact, some of the best investigations arise out of unplanned teaching moments. The teacher plays a critical role in guiding the identification of questions, particularly when they come from students. While inquiries should evolve from questions that are meaningful and relevant to students, they also must be able to be answered from their own observations and the mathematical information that they can obtain from reliable sources. The knowledge and procedures students use to answer the questions must be accessible and manageable, as well as appropriate to the students' developmental level. Skilful teachers help students focus their questions so that they can experience both interesting and productive investigations.

Mathematical investigations and inquiries are an ideal vehicle for learning in a way that is both culturally rich and academically rigorous, building on experiences that have family and/or community significance. They provide opportunities for students to acquire relevant knowledge and skills, whether contextual or mathematical, and to act as mentors for their peers. Ideally, students should be involved in deciding how they will present their findings.

It is rare for an investigation or inquiry to be limited to one area of mathematics. More commonly, they link across topics, connecting several big ideas. This helps students to develop an appreciation of the structure of mathematics.

Investigations and inquiries provide opportunities for the assessment of student work, both formative and summative.

An example of an open-ended investigation or inquiry is to ask students to examine their own mobile phone usage and locate the best plan (pre- or post-paid) for that level of usage. The YDM MITI program is based on open-ended investigations and inquiries.

More than traditional problem solving

Although often associated with problem solving, open-ended, inquiry-based investigations involve more than traditional problem solving. Specifically, an investigation differs from traditional problem solving in several respects.

- It goes beyond traditional problem solving where the key data are presented "up front" and students choose a strategy to produce a single, usually brief, response. In contrast, investigations embed the information within the problem context for students to discover as they work with the problem.
- The problems are often multidisciplinary, relating to learning areas beyond mathematics.
- The strategies and processes used in the investigation (such as constructing, describing, explaining, predicting, and representing, together with organising, coordinating, quantifying, and transforming data) tend to differ from those used for traditional problem solving.
- An investigation often involves iterative cycles where ideas are tested, retested and revised.
- Investigations are multifaceted, with the final product employing a variety of representational formats including text, graphs, tables, diagrams, spreadsheets, and oral reports. There may be several acceptable approaches, representations and solutions.
- Investigations can result in mathematical models that can deal with more than one instance: they are reusable, shareable, and/or modifiable.
- Investigations are multifaceted, with the final product employing a variety of representational formats including text, graphs, tables, diagrams, spreadsheets, and oral reports.

In a study of 83 tasks that were presented as mathematical modelling investigations in some popular USA mathematics textbooks, the vast majority of the student activities involved performing operations and interpreting results (Meyer, 2015). This means that many tasks, despite being labelled as mathematical modelling, involved little more than traditional problem solving methods. In our experience, the approach of Australian textbooks is similar.

Differentiation

One advantage of inquiry as a learning experience is that students are able to engage with a task in a way that suits their learning style and at a level appropriate to their mathematical development. Some students may not be able to progress beyond collecting data and showing relationships between variables graphically. However, that graph may be sufficient for them to obtain results and make predictions. Other, more capable, students may be able to demonstrate a more sophisticated understanding of the relevant mathematical relations. Teachers should have pre-prepared prompts to extend and enrich an investigative task to a level appropriate for each student.

6.4.2 Mathematical modelling

Mathematical modelling is a specific form of mathematical inquiry. The skills of modelling should start to be developed in the junior secondary school.

Models are conceptual processes used to construct, describe, explain and/or predict the behaviour of complex systems (English, 2008). Mathematical models differ from other categories of models mainly because they focus on the structural, or mathematical, characteristics of the systems they describe. They are underpinned by quantitative processes (such as counting, measuring, calculating, graphing, inferring, extrapolating, abstracting), although they may also include qualitative methods (including describing and explaining). However, a central part of mathematical modelling process is extending, generalising and abstracting from specific examples to "build" the model. As modelling starts with a small number of specific examples to develop an approach that can apply to a wider range of circumstances, it is a form of inductive reasoning.

Mathematical models can be represented using language (written and spoken), symbols, visual images/graphics (both computer- and paper-based), or experience-based metaphors. They generally have two parts: a conceptual system for describing or explaining the relevant mathematical objects and the relations between them; and procedures for generating useful outputs (Lesh & Harel, 2003). Mathematical modelling is the process of developing, evaluating, modifying, and applying mathematical models. It provides learning opportunities that encourage the development of a broad range of problem solving skills. It is not just reaching the goal that is important, but also the interpretation of the goal, the information provided, and the possible steps to a solution. Modelling allows students to develop useful skills such as interpreting, thinking, communicating, justifying, revising, refining, and extending that can be applied more generally (Doyle, 2006; English, 2008).

Mathematical modelling is an important part of learning mathematics and other learning areas. According to English (2008), "an appreciation and understanding of the world as comprising interlocked complex systems is critical for all citizens in making effective decisions about their lives as both individuals and as community members." (p. 139). She continues, "Mathematical modelling is foundational to modern scientific research, such as biotechnology, aeronautical engineering, and informatics" (p. 140).

Mathematical modelling is a form of investigation. However it differs in the following ways:

- students may make initial assumptions to reduce the complexity of the real world, yet produce models yielding results that are useful approximations of the real world;
- a mathematical model can deal with more than one instance: it is reusable, shareable, and/or modifiable;
- numerical values produced by a model The results of the model are usually predictions or estimates; and
- it is usually clear why the model is needed, providing students with a way of evaluating its effectiveness.

The process of mathematical modelling

The mathematical modelling process starts with a model-eliciting problem (usually grounded in reality) that requires a model to describe, explain, or predict the behaviour of a system or process. There are six actions that are required to complete a modelling task, described below.

1. Identifying the variables: These are the elements or features relevant to the situation to be modelled. Research and data collection may be necessary.
2. Formulating a model from the variables: As a system can include a large number of variables, formulating a model can be a complex process. Reducing the number of variables by making certain assumptions can simplify the situation without detracting from the usefulness of the model in explaining or predicting reality. Simpler problems can be explored to find relations and patterns that can be refined, extended, generalised and abstracted back to the original problem. As patterns are most easily observed visually, developing a model may require the translation of data into other representational forms, particularly graphs. The generalised situation is mathematised into a useable form, for example an equation or graph.
3. Performing operations using the model: This may require interpolation or extrapolation from a graph, or calculating the outputs of a function or formula. It could also include the manufacture of an item using a 3D printer.
4. Interpreting the results: The results produced by the model are related back to the original context, to acquire a real-world meaning.
5. Validating the mathematical model: This is necessary because the modelling process is a trade-off between simplicity and accuracy. The challenge is to find a model where the real-world situation is simplified through the use of assumptions that remove complexity, but which nevertheless yields results that approximate reality. Models can be validated by comparing their outputs with known values. The impact of any assumptions can be judged by examining the effect of varying those assumptions; ideally, changes to the assumptions make little difference to the model's outputs.

6. Reporting on conclusions: This requires student to share their findings with others, either in spoken form (oral or multi-media presentations) or written form (as descriptions, explanations or predictions).

The use of mathematical technology can assist in many of these steps.

Mathematical modelling tasks do not need to focus equally on all six parts of the modelling process, provided that the open-ended nature of the tasks is not jeopardised. For example:

- Designing the markings for a school car park so that that largest number of vehicles can accommodated would require students to focus on identifying, and collecting data on, several variables, such as the size of the car park; the desired width and length of the car bays; whether angle or parallel parking is better and, in the case of angle parking, the best angle to use; and the space required for manoeuvring (entering and leaving the car park, driving around the car park, reversing in and out of parking bays). The challenge in this task is identifying the variables and collecting the data. The model is a relatively straightforward application of area (if no angle parking) or standard trigonometry (if angle parking is possible).
- Developing a model for predicting the day of the week that your birthday will fall in any year, taking account of leap years. The challenge in this problem is developing the model that will make the predictions.
- Investigating the relationship between length and volume using an open box constructed by cutting squares from the corners of a single sheet of rectangular paper and folding up the edges of the paper to form the sides of the box. By changing the size of the cut-out squares, boxes of differing volumes are created. The task is to determine the size of the cut-out square that maximises the volume of the box. The mathematical model is a relatively simple application of the volume formula. The challenge in this example is exploring the pattern of outputs from the model.

Other tasks could provide the model and ask students to evaluate and refine it, for example the construction of alternate models and the use of more abstract representations (model exploration activities), or the use of generalised, abstract models in new situations (model adaptation activities).

The teacher can select the approach that best suits the students, their level of mathematical development, and the lesson objectives.

6.4.3 Why use open ended tasks

The use of age-appropriate investigation, inquiry and mathematical modelling tasks in the junior secondary years has a number of benefits. Open-ended tasks: (a) have clear connections to reality; (b) provide engaging, kinaesthetic approaches to learning mathematics; (c) encourage student creativity and independence; (d) show mathematics to be a body of interconnected knowledge rather than a series of discrete topics; (e) can pre-empt some of the ideas that students will encounter in senior mathematics; and (f) develop the skills of modelling that are needed for success in senior mathematics, the sciences and economics.

7 Problem-Solving Program

This section completes this resource on problem solving by looking at how problem solving can be taught and discussing how to set up a program to teach the components of problem solving across Years P–9.

7.1 Teaching problem solving

What is required to teach problem solving can be put simply – take all the students through the three stages of the problem-solving process, ensuring that you develop appropriate content, strategies, plans, thinking skills and metacognition as you go, and ensuring that you maintain a positive classroom atmosphere to build resilient affects. This may appear to be a mammoth task. It includes deep learning of the existing mathematical curriculum plus affects, strategies, plans, thinking skills and metacognition. However, it is a task not beyond teachers, because, in many ways, it is just making explicit what good teachers have always done. Furthermore, most students come to school as excellent problem solvers. Schooling seems to diminish problem solving, not enhance it. Thus, teachers can do a lot of good by simply not getting in the way of their students' self-learnt problem-solving abilities.

In summary, effective teaching of problem solving will be facilitated by the following techniques.

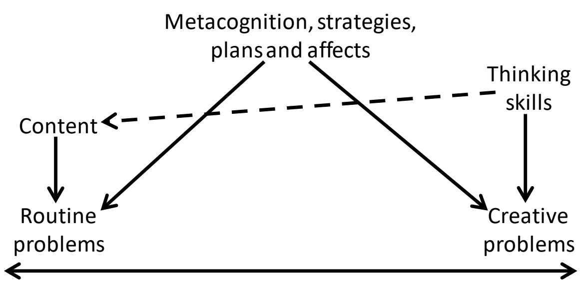
1. Develop the three stages (finding, solving and communicating) as separate skills before requiring students to do all three in problem-solving situations. Then interrelate the stages and components of problem solving as students move through the year levels.

2. Focus on the physical environment of the classroom and your own teaching methodology or approach to teaching as much as the content to be covered. A warm pleasant room decorated with students' work with materials easily at hand and with chairs around tables, is a much more conducive atmosphere for problem solving than students sitting singly in rows. A questioning, encouraging, enquiring approach to teaching facilitates student participation in problem solving more than teacher-led traditional instruction.

3. Balance the following: (a) teaching activities which directly teach metacognition, thinking skills, plans, strategies and content; (b) solving and discussing problems to indirectly develop strategies, skills, plans and processes, with solving problems as practice for the total problem solving process. Do not partition problem solving separately from the remainder of mathematics instruction, integrate problem solving with everyday teaching.

4. Do not focus on the correct solution but on discussion on how the solution was found, and do not let evaluation procedures harm the problem solving program. Adopt the role of facilitator and companion, not the role of expert and leader and be a problem solver yourself. Students should be encouraged to reflect on their solutions, considering the choice of strategies, decisions made, and the mathematical ideas that worked and did not work. Many students thank that there is only one right way to solve the problem. Discussion of alternative approaches builds resourcefulness and flexibility of thinking.

5. Cater for routine and creative problems differently as shown to the right; routine problems are best solved by content and creative problems by thinking skills. However good thinking skills lead to good learning of mathematics, so use thinking skills to appropriately structure the content in memory for routine problems. This means that thinking skills are useful for all problems though, for some problems, its impact may be indirect.



6. Include problem solving and the teaching of problem solving all the time – even in lessons where it does not appear to be relevant (e.g. even skill practice could involve students choosing which examples to do). Problem solving is a practical art, like playing a sport or a musical instrument. To improve, students need opportunities for frequent and regular practice of challenging problems. They should include problems with few clues as to how to solve them so that students have to draw on the full range of their mathematical knowledge. The selection of techniques should not be triggered by a chapter heading, a standard context or too much teacher assistance.
7. Allow time for open-ended investigation, inquiry and/or modelling tasks. They may appear to take valuable class time, but if chosen carefully to suit the learning objectives, they provide as many learning opportunities as other pedagogical approaches.

We need to stress that problem solving is more than just a topic in mathematics; it requires something different from teachers. It cannot be divorced from the concepts, skills and attitudes on which it is based and from the methods to teach those concepts, skills and attitudes. For problem solving, teachers may have to re-think their whole mathematics program right down to their fundamental beliefs about what mathematics is and how students learn.

7.2 Planning a school program

For this resource, planning a school problem-solving program will be built around the three stages (finding, solving and communicating) and the six components of the expert problem solver, namely, metacognition, thinking skills, plans, strategies, content and affects. The idea is to work out what of each of these components should be taught when. The Australian Curriculum in Mathematics identifies "Problem Solving" as one of the proficiency strands that apply from the Foundation year to Year 10, stating that;

> Students develop the ability to make choices, interpret, formulate, model and investigate problem situations, and communicate solutions effectively. Students formulate and solve problems when they use mathematics to represent unfamiliar or meaningful situations, when they design investigations and plan their approaches, when they apply their existing strategies to seek solutions, and when they verify that their answers are reasonable. (ACARA, 2015, 'Mathematics content structure' page)

However, the curriculum documents do not assist in dividing the components into Year levels from which we could start, nor is there a strong and consistent history of approaching problem solving by segmenting the skills into particular year levels. Indeed, there is evidence that gifted students in lower year levels approached mathematical problem solving in a similar manner to students of average ability in higher year levels (Threllfall, 2008). This suggests that mathematical problem solving ability should be viewed as a continuum along which students move according to their ability, rather than the stage of their education.

Thus, a problem-solving program should not be about individual Year levels but around clusters of levels, namely, Early Primary Years P–2, middle primary Years 3–4, upper primary Years 5–6, and junior secondary Years 7–9. This will enable a more flexible program with wriggle room for teachers and for schools.

Taking each component in turn, we will look at where they may fit best.

1. Metacognition – There is some findings that argue that awareness of metacognitive processes is a middle-upper primary activity with metacognitive skill available to younger students but not with awareness.
2. Thinking skills – the common understanding is that thinking skills are a major focus of early childhood and should be emphasised in the early primary years with consolidation and extension of skills in the later year levels.
3. Plans – The general approach is that, although planning and checking should start early, the direct teaching of a formal plan such as Polya's four stages should not start until the middle primary years.

4. Strategies – It is anticipated that strategies will be taught across Years P–9 with the logical and visual early, the evaluative and patterning in the middle and the creative and flexible near the end. Of course, this does not apply to all strategies in a cluster. For instance, “act it out” is a very early strategy while “select appropriate notation” is a later strategy, yet both are visual strategies. However, against this, most of the creative/flexible strategies such as “work backwards”, “select a sub-goal”, “change your point of view” and “check for hidden assumptions” are obviously late as they require students to restructure what they are thinking. Avoid making the strategy explicit to students too early (for example, do *not* start a lesson with “Today we are going to learn about the ‘work backwards’ strategy”). Instead, select several problems that involve the selected strategy and allow students to identify the strategy that they have in common. After students have recognised the strategy for themselves, it is appropriate to give it a name.
5. Content – This component has a curriculum that states when ideas are to be taught. However, in general, basic ideas come first and are then built into more complex forms. It is also true that the early years are predominantly additive (e.g. counting and addition) and specific (e.g. arithmetic) while the later years are multiplicative (fractions and rate/ratio) and generalised (e.g. algebra).
6. Affect – This component may be the hardest of all to develop and maintain. This is particularly the case if students come to school without that eagerness and belief in their own ability to learn that can is the hallmark of many students at school entry. Positive self-image, resilience, motivation and engagement have to be built and maintained through and for problem solving.

It is good problem-solving practice to have an approach to teaching that allows content and problems encountered in day-to-day class work to direct instruction to problem solving. In other words, the problem-solving teacher should always be trying to bring problem solving into all content being covered and should seize any problem thrown up in class as an opportunity for problem solving.

Taking all the above into account, the following table summarises an effective problem-solving program. It distributes the six components across the clusters of year levels.

Program for teaching mathematics problem solving

COMPONENTS	EARLY PRIMARY YEARS P–2	MIDDLE PRIMARY YEARS 3–4	UPPER PRIMARY YEARS 5–6	JUNIOR SECONDARY YEARS 7–9
Stages	Focus on solving and begin characterising problem types	Continue solving (extend to all types) Begin finding or posing simple problems Develop communication and justification		Continue finding, posing and solving Begin to formalise communication
Metacognition	Introduce planning, monitoring and checking	Introduce overseeing, evaluating and making decisions	<u>Major focus</u> Teach awareness and conscious control of thinking	Focus on evaluation and conscious judgement Continue previous work
Thinking skills	<u>Major focus</u> Directly teach skills • *Logical thinking* • *Visual thinking* • *Patterning*	Consolidate thinking skills by applying them to problem-solving, investigations, and learning experiences. Directly teach new skills • *Creative and flexible thinking* • *Evaluative (critical) thinking and decision making*		
Plan	Develop procedures for comprehending, planning and checking	<u>Major focus</u> Develop Polya's four stages	Directly teach Polya's plan with poster Discuss and solve problems in terms of the stages	
Strategies	Develop core strategies • *Reread the question* • *Identify given and wanted* ** • *Act the problem out* • *Check your solution* • *Look for a pattern*	Continue to develop strategies • *Identify given, needed and wanted* ** • *Restate problem in own words* • *Make a model* • *Make a drawing, diagram, graph* • *Find another way to solve it* • *Construct a table* • *Identify sub-goal*	Continue introducing new strategies Begin relating to Polya • *Write a number sentence* • *Select appropriate notation* • *Study the solution process* • *Find another solution* • *Guess and check* • *Work backwards* • *Solve a simpler problem*	<u>Major focus</u> Systematically develop a repertoire of strategies related to Polya's 4 stages • *Account for all possibilities* • *Change your point of view* • *Check for hidden assumptions* • *Generalise*
Content	Build well-sequenced and clearly understood collections of basic concepts, principles and strategies Understanding to pre-empt later work		Systematically connect topics into a comprehensive scheme Move from additive and specific to multiplicative and general	
Affects	Interest and motivation Willingness to take risks	Attitude and perseverance	Self-concept, self-image Attribution	Continued engagement and maintenance of resilience

Note: ** strategy is given in two steps

7.3 Relating plan and strategies

It is recommended that the plan, Polya's four stages, be directly taught using a poster, with strategies added to the poster as they are learnt.

Below is one proposal for how strategies and plan can fit together. The strategies marked *** could also be in SEE.

Plan and strategies proposal

POLYA'S 4 STAGES	STRATEGIES	
SEE	• Reread the problem • Identify given, needed and wanted • Restate the problem	• Act the problem out • Make a model • Make a drawing, diagram or graph
PLAN	• Guess and check • Solve a simpler problem • Look for a pattern	• Work backwards • Identify a sub-goal • Change your point of view • Check for hidden assumptions
DO	• Select appropriate notation *** • Write a number sentence ***	• Construct a table • Account for all possibilities (systematically)
CHECK	• Check you solution • Find another way to solve it	• Find another solution • Study the solution process • Generalise

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Appendix A: Words Suggesting Arithmetic Operations

Addition	Subtraction	Multiplication	Division
accumulate	backwards	array	half, third, etc
add	decrease	by	distribute
altogether	debit	commission	divide
all told	debt	double, triple, etc	divisible
and	deduct	twice, thrice, etc	divisor
another	difference	square, cube, etc	factor
augment	diminish	two-fold, three-fold, etc	fraction
bigger (than)	discount	factor	groups
credit	down	groups of	left over
deposit	exceed	lots of	out of
extra	fall	magnify	parts
faster (than)	fewer (than)	multiple	per
forward	from	multiply	per cent
further (than)	gone	of	portion
gain	leave	product	rate
greater (than)	left (over)	repeated	reciprocal
grows	less (than)	taxation	remainder
heavier (than)	lighter (than)	times	quotient
higher (than)	lose	times more (than)	share
increase	lower (than)		split
longer (than)	off		
more (than)	narrower (than)		
older (than)	nearer (than)		
positive	net (eg: income)		
plus	minus		
rise	negative		
sum	reduce		
taller (than)	remaining		
thicker (than)	remove		
together	reverse		
total	thinner (than)		
up	shorter (than)		
wider (than)	slower (than)		
with	subtract		
	take (away)		
	withdraw		
	younger (than)		

Appendix B: Framework for Getting Started

FRAMEWORK FOR GETTING STARTED WITH WORD PROBLEMS

PROBLEM:

DRAWING:

GIVEN:

WANTED:

NEEDED:

NOT NEEDED:

STORY:

WORKING:

ANSWER:

Prep to Year 9: Supplementary Resource 2 – **Problem Solving**

YuMiDeadly
Growing community through education

QUT Faculty of Education
School of Teacher Education and Leadership
S Block, Room 404
Victoria Park Road
KELVIN GROVE QLD 4059

CRICOS No. 00213J

Phone: +61 7 3138 0035
Fax: +61 7 3138 3985
Email: ydc@qut.edu.au
Website: http://ydc.qut.edu.au