YuMi Deadly Maths

Year 9 Teacher Resource:

MG – Terrific triangles

Prepared by the YuMi Deadly Centre
Faculty of Education, QUT

YuMi Deadly Maths Year 9 Teacher Resource: **MG – Terrific triangles**

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Growing community through education

ACKNOWLEDGEMENT

We acknowledge the traditional owners and custodians of the lands in which the mathematics ideas for this resource were developed, refined and presented in professional development sessions.

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Year 9 Measurement and Geometry

Terrific triangles

Learning goal

Students will:

- identify and name sides as hypotenuse, adjacent and opposite in a right-angled triangle
- calculate trigonometric ratios using known angle or side length values.

Content description

Measurement and Geometry – Pythagoras and trigonometry

- Investigate Pythagoras' theorem and its application to solving simple problems involving right-angled triangles (ACMMG222)
- Use similarity to investigate the constancy of the sine, cosine and tangent ratios for a given angle in right-angled triangles (ACMMG223)
- Apply trigonometry to solve right-angled triangle problems (ACMMG224)

Big idea

Geometry – Change vs relationship, Pythagoras' theorem, trigonometric relationships

Resources

Maths Mat and elastics; cards with the following words/symbols: angle, hypotenuse, opposite, adjacent, sin θ, cos θ, tan θ

Reality

Local knowledge

Discuss that this lesson is going to be about the branch of mathematics called **trigonometry**. The word trigonometry comes from the Greek words for triangle (*trigōnon*) and measure (*metron*). Trigonometry studies the ratios and relationships between the sides and angles of triangles, particularly right triangles.

Discuss where students see right triangles in the local environment, e.g. buildings, construction, ladders against walls, ship masts with sails.

Prior experience

Provide examples to check that students are able to:

(a) identify the hypotenuse as the side opposite the right angle;

(b) understand that the hypotenuse is the longest of the three sides;

(c) find the third angle in a right triangle if one angle is given;

(d) find the length of the third side given the lengths of the other two sides using the Pythagorean rule that the square of the hypotenuse is equal to the sum of the squares of the other two sides, or $c^2 = a^2 + b^2$; and

(e) identify similar right triangles as having the same angles and sides in the same ratio (see Year 9 MG lesson on similarity: *Look alikes*).

Kinaesthetic

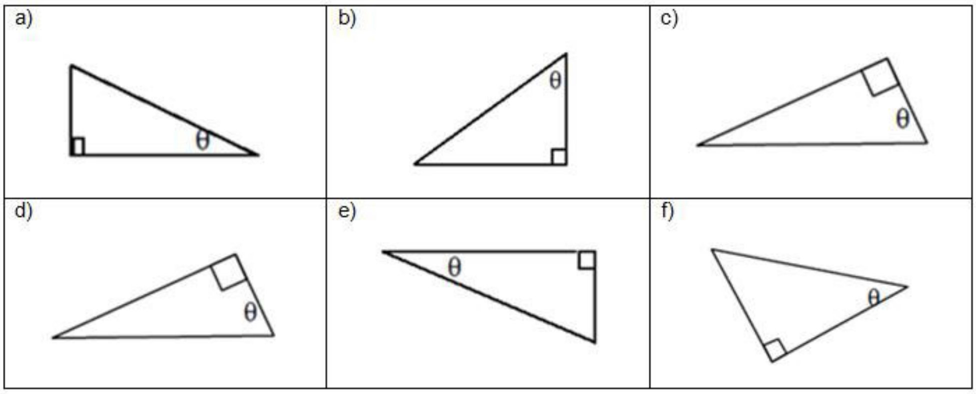
Maths Mat and elastics

Line all the students up on one horizontal side of the mat and get them to walk to the opposite side. Line them up on one vertical side and get them to walk to the opposite side:

- What are you doing when you go to the opposite side?
- Do the opposite sides actually touch each other?
- *What does the word adjacent mean?* [next to, joining, having a common boundary or point].

Ask a student to take an elastic and place it on any single line segment of one square in the middle of the Maths Mat. *How many line segments are adjacent to this?* [4] Mark them out with elastics of different colours. From the original line segment, have two students step away in both directions to lines that are opposite to the original. Reinforce that adjacent lines join but opposite lines never touch.

Now have students mark out the following right triangles on the Maths Mat, placing cards with θ symbol to mark the angles as in the diagram below:



Have six students place the **hypotenuse** cards on the appropriate side of each right triangle (the side opposite the right angle). Ask another six students to stand in the angle marked θ in each of the triangles. Get them to walk from that angle to the opposite side of the triangle. Have six students place the card with **opposite** against those sides. Have the students return to stand in the θ angle and get them to tap with their foot the sides that are adjacent to the θ angle:

- What is one of those sides called? [hypotenuse]
- What is an appropriate name for the other? [adjacent]
- If we put the θ symbol on the other angle in the right triangle, what side would still have the same name [hypotenuse] and what sides would need to change? [adjacent and opposite]

Have students change the angle and then change the opposite and adjacent cards appropriately.

- Would it make any difference to the positions if the triangles were rotated or translated or reflected? [no]
- What do you know about the squares on the sides of right triangles? [Pythagoras]

Explain to students that, just as we can calculate the length of the sides in right triangles because of this Pythagorean relationship and so develop Pythagorean ratios (e.g. 3:4:5, 5:12:13, and so on), trigonometry also has constant ratios between the measure of the sides and an angle. This depends on being able to determine the hypotenuse and sides that are opposite and adjacent to a given angle. Special names are given to these relationships that always remain constant: **sine** (sin θ), **cosine** (cos θ) and **tangent** (tan θ).

Have another six students stand in each θ angle and then walk the relationships as they are named, repeating:

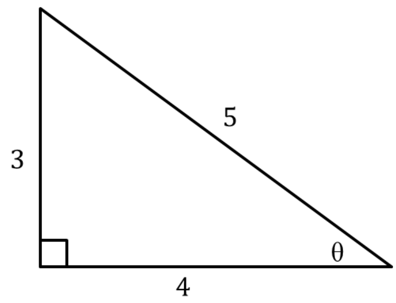
- the sine of an angle is the opposite side divided by the hypotenuse;
- the cosine of an angle is the adjacent side divided by the hypotenuse;
- the tangent of an angle is the opposite side divided by the adjacent side.

Write these as fractions:

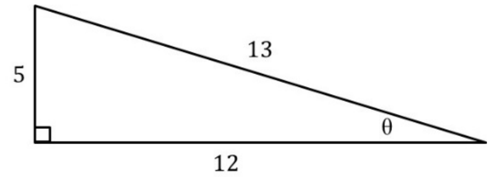
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Make each triangle shown below, one at a time, on the Maths Mat with elastics. Write the sin θ, cos θ and tan θ as fractions.

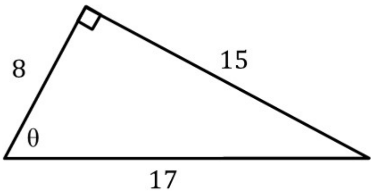
(a)



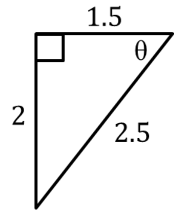
(b)



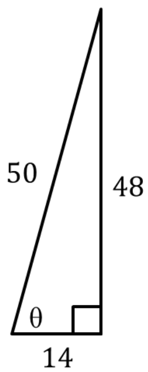
(c)



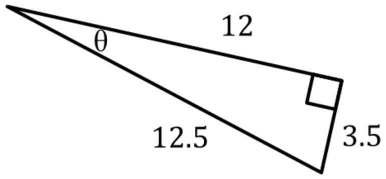
(d)



(e)



(f)



Then make similar triangles for each of the given triangles so that the same ratio of the sides is kept; e.g. 3:4:5 → 6:8:10 → 1.5:2:2.5 → and so on. Calculate the sin θ, cos θ and tan θ as fractions.

What conclusions can you draw from these examples? [If the **angles** in a right triangle are the **same**, the sides will be in the same ratio irrespective of size since they are all similar triangles.]

Because one angle is 90° in a right triangle, if the angle at one other corner is known, the third angle can be found. This means that all right triangles with the same first angle are similar. For example, a right triangle with an angle of 30° is similar to all other right triangles with an angle of 30° no matter what the size. (*Note*: We are using division although it is ratio – they are the same.)

Therefore, if we know the angle of a right triangle, we know the three ratios or proportions that have been designated with special names as below:

- Longest side is called hypotenuse, side next to angle is called adjacent, and side opposite to angle is called opposite; and
- $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$ $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$ $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$

Abstraction

Body

On lawn or cement, mark out a series of large right triangles with the same angle but different sides in the same ratio. Using some paper, cut the shape and size of one angle, other than the right angle, in the first right triangle. Compare it with the same angle in all the other right triangles to demonstrate that the triangles all have the same angles. Have students pace the sides of each triangle and record the measures. Have students check the ratios of the sides, which should be the same for each right triangle. Find the sine, cosine and tangent using the same angle in all triangles. *What conclusion can be drawn?*

Hand

Scientific calculator

Make sure the mode of your calculator is set to **degrees**, the **D** or **deg** button. The trigonometric ratios are not dependent on the exact side lengths, but on the **angles**. There is one fixed value for every angle, from 0° to 90°. Your scientific calculator knows the values of the sine, cosine and tangent of all of these angles. Depending on your calculator, you should have [SIN], [COS] and [TAN] buttons. Use these to find the sine, cosine and tangent of any acute angle. For example, find the trigonometric value of the examples below, using your calculator to enter the following and rounding to three decimal places:

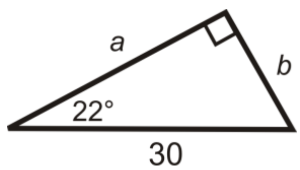
(a)	sin 78°	sin	78	=	Answer 0.978				
(b)	cos 60°	cos	60	=	Answer 0.5				
(c)	tan 15°	tan	15	=	Answer 0.268				
(d)	cos 6.35°	cos	6	.	3	5	=	Answer 0.994	

Draw right triangles and mark the degrees of one angle other than the right angle. Mark the sides as opposite, adjacent and hypotenuse and using a calculator find the sine, cosine and tangent of the given angle. Now calculate the sine, cosine and tangent of the other angle.

Example A

One application of the trigonometric ratios is to use them to find the missing sides of a right triangle. All you need is one angle, other than the right angle, and one side.

1. Find the value of each variable, a and b, on the triangle below. Round your answer to the nearest tenth.



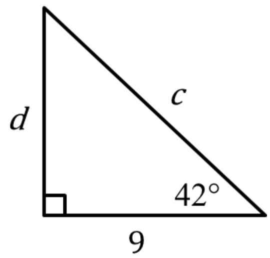
We are given the hypotenuse. Use sine ($\frac{\text{opposite}}{\text{hypotenuse}}$) to find b and cosine ($\frac{\text{adjacent}}{\text{hypotenuse}}$) to find a. Use your calculator to evaluate the sine and cosine of the angles.

$$\sin 22 = \frac{b}{30}$$
$$30 \times \sin 22 = b$$
$$b \approx 11.2$$

$$\cos 22 = \frac{a}{30}$$
$$30 \times \cos 22 = a$$
$$a \approx 27.8$$

Note: Some calculators use the dot button instead of the multiplication sign.

2. Find the value of each variable, d and c, on the triangle below. Round your answer to the nearest tenth.



We are given the adjacent length to the angle of 42°. Use cosine ($\frac{\text{adjacent}}{\text{hypotenuse}}$) to find c and tangent ($\frac{\text{opposite}}{\text{adjacent}}$) to find d.

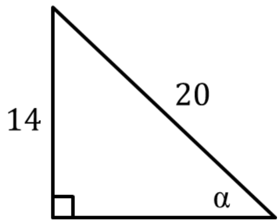
$$\begin{aligned} \cos 42 &= \frac{9}{c} \\ c \times \cos 42 &= 9 \\ c &= \frac{9}{\cos 42} \\ c &\approx 12.1 \end{aligned}$$

$$\begin{aligned} \tan 42 &= \frac{d}{9} \\ 9 \times \tan 42 &= d \\ d &\approx 8.1 \end{aligned}$$

Example B

If the lengths of any two sides of a right triangle are known, then the sine, cosine and tangent can be used to find the angles of the triangle.

1. Find the angle marked α in this triangle correct to one decimal point.



$$\begin{aligned} \sin \alpha &= \frac{14}{20} \\ &= 0.7 \end{aligned}$$

$$\alpha = \sin^{-1} 0.7 = 44.4°$$

A similar process is used to find the cosine and tangent of angles given relevant sides.

Mind

It is helpful to be able to recall the trigonometric ratios from memory. Ask students to see the sine, cosine and tangent ratios in their minds. *Take the initials of the words in each ratio and see them in your mind: Sine = Opposite divided by Hypotenuse – SOH; Cosine = Adjacent divided by Hypotenuse – CAH; Tan = Opposite divided by Adjacent – TOA; this gives SOH-CAH-TOA. Now write a sentence with words beginning with those letters, e.g. Songs of happiness can always help to offer amusement.*

Creativity

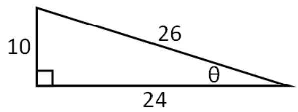
Make your own mnemonic from the SOH-CAH-TOA to help you remember the formulas. You may like to set it to music; choose a song you really like and that will help you remember the trigonometric ratios.

Mathematics

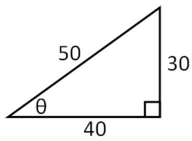
Language/ symbols Pythagoras, right triangle, hypotenuse, base, height, scientific calculator, trigonometric ratios, opposite, adjacent, sine, cosine, tangent

Practice

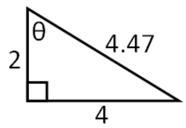
1. For each triangle, calculate the value of the given trigonometric ratio:



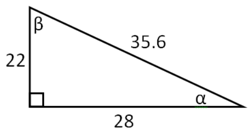
(a) tan θ



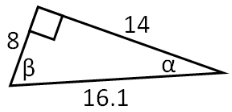
(b) sin θ



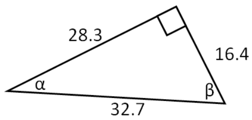
(c) cos θ



(d) tan α



(e) sin α



(f) cos β

2. Explain why the following are true statements, using diagrams and words to justify your reasoning:

   (a) $\tan 45° = 1$

   (b) $\sin 25° = \cos 65°$

   (c) sin θ and cos θ are never > 1

   (d) tan θ can be > 1

3. Use a calculator to complete the table below. If any calculations produce an error message on your calculator, also enter that message into the table.

Given angle (θ)	sin θ	cos θ	tan θ
0°			
10°			
20°			
30°			
40°			
50°			
60°			
70°			
80°			
90°			

   (a) Identify and describe the patterns occurring for each of the trigonometric ratios. What happens to the ratio as the angles increase?

   (b) Complete the following statements: (The first one is done for you.)

      i) Tan θ values get [larger] as the given angle approaches 90°. This is because the opposite side length becomes very large in relation to the adjacent side length.

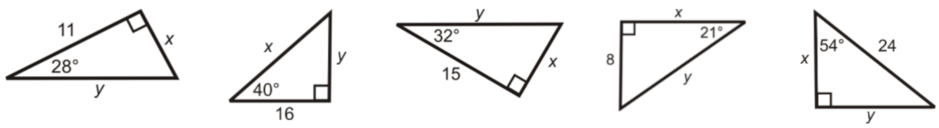
ii) Sin θ values approach ☐ as the given angle approaches 90°. This is because:

iii) Cos θ values approach ☐ as the given angle approaches 90°. This is because:

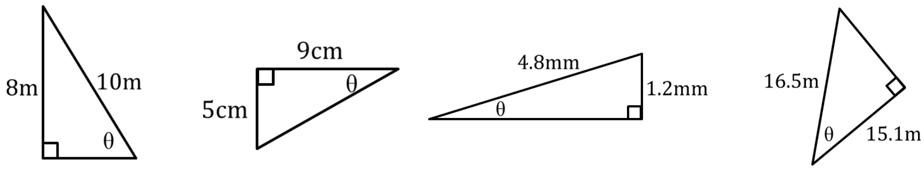
iv) Where does sin θ = cos θ?

v) Generalise all instances where sin θ = cos θ.

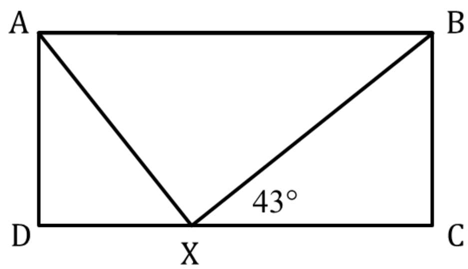
4. Find the length of the missing sides and round your answers to the nearest tenth:



5. Find the angle θ:



6. In the diagram below, ABCD is a rectangle (not drawn to scale). The point X on DC is such that AX = XC = 12 cm and CXB = 43°



Calculate correct to one decimal place:

(a) the length of CB

(b) the length of DX

(c) the size of the angle DXA

Connections Pythagoras, sine, cosine rules in any triangle; ratios in similar right triangles.

Reflection

Validation Both Pythagoras and trigonometry relate to right triangles. Use a Venn diagram to compare and contrast the elements of each. Find examples of both in the real world.

Application/ problems Provide applications and problems for students to apply to different real-world contexts independently; e.g. *A ship is 50 km south and 70 km west of its destination port. What bearing should it sail on to reach the port?*

Using your knowledge of trigonometric ratios and similarity, find the height of a given tree by comparing the length of its shadow with that of a metre ruler. (Hint: Calculate the angle in the right triangle formed by the metre ruler and its shadow.)

Extension **Flexibility**. Students become adept in using the appropriate formula that applies to given criteria and interchanging formulas given relevant measures. They are able to discern when to select Pythagoras and when trigonometric ratios are required to solve problems.

Reversing. Using trigonometric ratios in right triangles, students are able to move between a given angle and a side to calculate the other side and find the angle when given two sides.

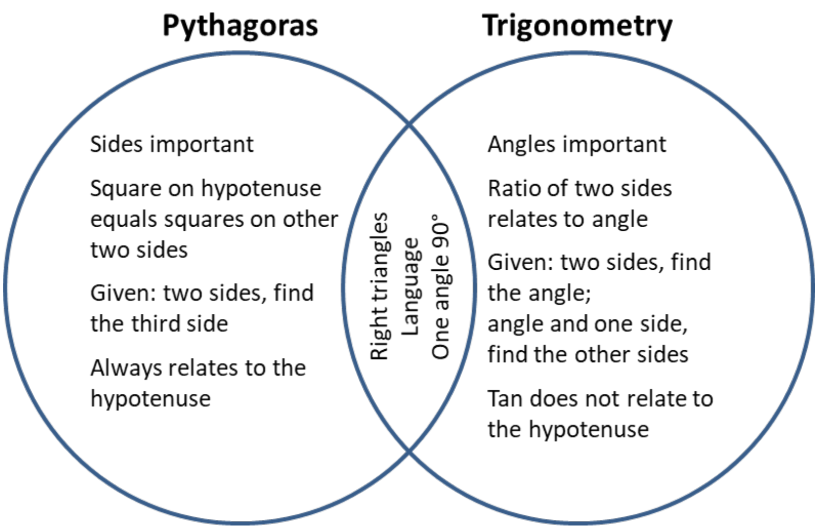
Generalising. Adjacent and opposite sides are dependent on the given angle. In a given right triangle, excluding the right angle, the sine of one angle equals the cosine of the other angle. Trigonometric ratios remain constant and depend on the size of the angles. The formulas for sine, cosine and tangent in right triangles are constant.

Changing parameters. Explore sine, cosine and tangent in similar right triangles. What conclusions can be drawn? Calculate angles and sides in any triangle using the formula:

$$\frac{a}{\sin \mathrm{A}} = \frac{b}{\sin \mathrm{B}} = \frac{c}{\sin \mathrm{C}}$$

Teacher's notes

- The Venn diagram in Validation could be drawn as below:



- Students need to be taught the skill of visualising: closing their eyes and seeing pictures in their minds, making mental images; e.g. show a pattern block of a right triangle, students look at it, remove the pattern block, students then close their eyes and see the right triangle in their mind; then make a mental picture of a different type of right triangle.
- Suggestions in Local Knowledge are only a guide. It is very important that examples in Reality are taken from the local environment that have significance to the local culture and come from the students' experience of their local environment.
- Useful websites for Aboriginal and Torres Strait Islander perspectives and resources: www.rrr.edu.au; https://www.qcaa.qld.edu.au/3035.html
- Explicit teaching that aligns with students' understanding is part of every section of the RAMR cycle and has particular emphasis in the Mathematics section. The RAMR cycle is not always linear but may necessitate revisiting the previous stage/s at any given point.
- Reflection on the concept may happen at any stage of the RAMR cycle to reinforce the concept being taught. Validation, Application, and the last two parts of Extension should not be undertaken until students have mastered the mathematical concept as students need the foundation in order to be able to validate, apply, generalise and change parameters.

