YuMi Deadly Maths
Prep to Year 9

Review

VERSION 1, 15/07/2015

Prepared by the YuMi Deadly Centre
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ACKNOWLEDGEMENT

The YuMi Deadly Centre acknowledges the traditional owners and custodians of the lands in which the mathematics ideas for this resource were developed, refined and presented in professional development sessions.

DEVELOPMENT OF THIS BOOK

The YuMi Deadly Maths Review book is a final book added to the seven previous books (Overview, Number, Operations, Algebra, Geometry, Measurement, and Statistics and Probability) to close off the YuMi Deadly Maths training program. It looks back across these seven previous books, highlighting commonalities, and looks forward to developing YDM as a sustainable program within schools.

YUMI DEADLY CENTRE

The YuMi Deadly Centre is a research centre within the Faculty of Education at QUT that is dedicated to enhancing the learning of all students to improve their opportunities for further education, training and employment, and to equip them for lifelong learning.

“YuMi” is a Torres Strait Islander Creole word meaning “you and me” but is used here with permission from the Torres Strait Islanders’ Regional Education Council to mean working together as a community for the betterment of education for all. “Deadly” is an Aboriginal word used widely across Australia to mean smart in terms of being the best one can be in learning and life.

YuMi Deadly Centre’s motif was developed by Blacklines to depict learning, empowerment, and growth within country/community. The three key elements are the individual (represented by the inner seed), the community (represented by the leaf), and the journey/pathway of learning (represented by the curved line which winds around and up through the leaf). As such, the motif illustrates the YuMi Deadly Centre’s vision: Growing community through education.

The YuMi Deadly Centre can be contacted at ydc@qut.edu.au. Our website is research.qut.edu.au/ydc

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ABOUT YUMI DEADLY MATHS

From 2000–09, researchers who are now part of the YuMi Deadly Centre (YDC) collaborated with principals and teachers predominantly from Aboriginal and Torres Strait Islander schools and occasionally from low socio-economic status (SES) schools in a series of small projects to enhance student learning of mathematics. These projects tended to focus on a particular mathematics strand (e.g. whole-number numeration, operations, algebra, measurement) or on a particular part of schooling (e.g. middle school teachers, teacher aides, parents). They resulted in the development of specialist materials but not a complete mathematics program (these specialist materials can be accessed via the YDC website, research.qut.edu.au/ydc/).

In October 2009, YDC received funding from the Queensland Department of Education and Training through the Indigenous Schooling Support Unit, Central-Southern Queensland, to develop a train-the-trainer project, called the Teaching Indigenous Mathematics Education or TIME project. The aim of the project was to enhance the capacity of schools in Central and Southern Queensland Indigenous and low-SES communities to teach mathematics effectively to their students. The project focused on Years P to 3 in 2010, Years 4 to 7 in 2011 and Years 7 to 9 in 2012, covering all mathematics strands in the Australian Curriculum: Number and Algebra, Measurement and Geometry, and Statistics and Probability. The work of the TIME project across these three years enabled YDC to develop a cohesive mathematics pedagogical framework, YuMi Deadly Maths, that covers all strands of the Australian Curriculum: Mathematics and now underpins all YDC projects.

YuMi Deadly Maths (YDM) is designed to enhance mathematics learning outcomes, improve participation in higher mathematics subjects and tertiary courses, and improve employment and life chances. YDM is unique in its focus on creativity, structure and culture with regard to mathematics and on whole-of-school change with regard to implementation. It aims for the highest level of mathematics understanding and deep learning, through activity that engages students and involves teachers, parents and community. With a focus on big ideas, an emphasis on connecting mathematics topics, and a pedagogy that starts and finishes with students’ reality, it is effective for all students. It works successfully in different schools/communities as it is not a scripted program and encourages teachers to take account of the particular needs of their students. Being a train-the-trainer model, it can also offer long-term sustainability for schools.

YDC believes that changing mathematics pedagogy will not improve mathematics learning unless accompanied by a whole-of-school program to challenge attendance and behaviour, encourage pride and self-belief, instil high expectations, and build local leadership and community involvement. YDC has been strongly influenced by the philosophy of the Stronger Smarter Institute (C. Sarra, 2003) which states that any school has the potential to rise to the challenge of successfully teaching their students. YDM is applicable to all schools and has extensive application to classrooms with high numbers of at-risk students. This is because the mathematics teaching and learning, school change and leadership, and contextualisation and cultural empowerment ideas advocated by YDC represent the best practice for all students.

YDM is now available direct to schools face-to-face and online. Individual schools can fund YDM in their own classrooms (contact ydc@qut.edu.au or 07 3138 0035). This Review resource is part of the provision of YDM direct to schools and is the final in a series of eight that fully describe the YDM approach and pedagogical framework for Prep to Year 9. It overviews (a) mathematics and its teaching; (b) the proficiencies, affects, and language and literacy in mathematics; (c) assessment, diagnosis and remediation; (d) enrichment and extension, including problem solving and big ideas; and (e) sustaining YDM in schools. Because YDM is largely implemented within an action-research model, the resources undergo amendment and refinement as a result of school-based training and trialling. The ideas in this resource will be refined into the future.

YDM underlies three types of projects available to schools: (a) general training in the YDM pedagogy (through a variety of projects titled with YDM acronym); (b) Accelerated Inclusive Mathematics (AIM) training in remedial pedagogy to accelerate learning (through AIM projects, XLR8 projects and AIM Early Understandings projects); and (c) Mathematicians in Training Initiative (MITI) training in enrichment/extension pedagogy to build deep learning of powerful maths and increase participation in advanced Years 11–12 and tertiary maths subjects.
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List of Abbreviations

ACARA  Australian Curriculum, Assessment and Reporting Authority
AIM     Accelerated Inclusive Mathematics
CCSR1  Center for Comprehensive School Reform and Improvement
CCSSI  Common Core State Standards Initiative
EU     Early understandings
MITI   Mathematicians in Training Initiative
PTM    Pivotal teaching moment
QCAA   Queensland Curriculum and Assessment Authority
RAMR   Reality–Abstraction–Mathematics–Reflection
SES    Socio-economic status
XLR8   Accelerate
YDC    YuMi Deadly Centre
YDM    YuMi Deadly Maths
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1 Purpose and Overview

This is the final in the set of eight books describing the YuMi Deadly Maths (YDM) pedagogy. It is designed to:

(a) **review** the previous YDM training and associated resource books, looking for ideas that are common across the different topics and need to be considered in any school mathematics program; and

(b) **preview** the end of the training, looking at how to set up a school-oriented YDM program that will build sustainability into the future.

This chapter describes more fully the purpose of the book and overviews each chapter. The book is a first draft and will be modified in response to feedback from participants in YDC projects. As always, our focus is to produce resources that assist teachers to empower students to understand and use mathematics in their lives.

1.1 Purpose

The purpose of this book is to look back over the seven preceding resource books written to support YDM and to draw out generic ideas that apply to the books as a whole.

1.1.1 Background

The first seven YDM books are as follows:

1. **Overview** – describing the general components of the YDM pedagogy and discussing leadership, change and implementation;

2. **Number** – applying the pedagogy to the teaching of Number (Years F–9);

3. **Operations** – applying the pedagogy to the teaching of Operations (Years F–9);

4. **Algebra** – applying the pedagogy to the teaching of Algebra (Years F–9);

5. **Geometry** – applying the pedagogy to the teaching of Geometry (Years F–9);

6. **Measurement** – applying the pedagogy to the teaching of Measurement (Years F–9); and

7. **Statistics and Probability** – applying the pedagogy to the teaching of Statistics and Probability (Years F–9).

The last six books have a particular focus for their information; each of them deals with a strand (or part of a strand) of the Australian Curriculum: Mathematics. The first book, *Overview*, is more generic. It describes the components that make up the YDM pedagogy and apply to all topics:

- the structural focus on connections, big ideas and sequencing in determining teaching plans;
- the reality–abstraction–mathematics–reflection (RAMR) cycle in planning and teaching lessons; and
- the professional development, leadership, cultural and implementation ideas that apply to renewing a mathematics teaching and learning program.

This final *Review* book is more general and includes proficiencies such as understanding, fluency and problem solving; affects such as self-confidence; language and literacy in mathematics; assessment, diagnosis and remediation; and enrichment and extension. Like the *Overview* book, it deals with matters that apply across all the books, not just to certain mathematics topics (see subsection 1.1.2). The *Review* book also looks back at big ideas and relates YDM training to the other projects YDC offers, that is, Accelerated Inclusive Mathematics (AIM) and the Mathematicians in Training Initiative (MITI).
However, the role of this book is also to look beyond the YDM training and suggest how to build a sustainable YDM program for a school.

The YDM training process was as follows:

- four blocks of three days of PD (12 days in total) across two years, for four teachers per school, supported by books and online learning modules;
- in-school training of other teachers by the four trained teachers and in-school trialling of YDM ideas, supported online or in person by YDC staff; and
- development of a school program for implementing YDM during the PD, leading to a sustainable YDM program for each school that can continue after training finishes.

The Overview book introduced this process and the six topic-based books provided support for it; this Review book looks back at this process but also looks forward to what happens after the training program ends.

1.1.2 Generic ideas

The Australian Curriculum underpins YDM, providing generic pedagogical ideas and specific topic-based details.

1. Proficiencies. The Australian Mathematics Curriculum proposes four mathematical proficiencies and three content strands. The proficiencies are understanding, fluency, problem-solving and reasoning. They apply generally, seeking outcomes that go beyond a particular topic or strand. They deal with the quality of what is learnt from the content strands. For example, do learners understand number, are they fluent with number operations, can they solve problems in number situations, and can they reason numerically? YDM seeks to develop understanding (in abstraction), not rote learning, and supports fluency and problem solving (in the last two stages of the RAMR cycle: mathematics and reflection). However, is there sufficient focus on reasoning (other than at the end of Chapter 2 of the Number book)?

2. Affects. With the identification of proficiencies, the affective side of learning becomes relevant. We have focused on cognitive aspects in these books, but what about motivation and self-concept? We do look at motivation in the first step of the RAMR cycle but other affects are not included as much as they could be. There is also attribution (what students perceive are the causes for success and failure) and the attitudes useful in problem solving – perseverance and resilience. It seems obvious that affects should be part of any mathematics program openly involved in teaching, as the best teaching is of no use if it leaves students hating mathematics, believing mathematics is not for them, or unable to see unsuccessful attempts as learning opportunities.

3. Language. What about language and literacy, that is, listening and reading (that contribute to the development of the specialist mathematical vocabulary) and speaking and writing? There are differences of opinion as to its importance as a method of teaching and solving problems, but communication does cut across all areas of mathematics. Language is an important part of mathematics teaching.

4. Assessment and extension. It has become evident that the topics covered in two other YDM specialist pedagogies, which emerged from the general YDM pedagogy through YDC’s work with schools, were not complete within the existing YDM books. These two specialist pedagogies are (a) an acceleration pedagogy delivered through the AIM project that involves assessment, diagnosis and remediation; and (b) an enrichment and extension pedagogy delivered through the MITI project for deep learning of powerful mathematics ideas.

5. Mathematics itself. Given our basic belief that mathematics and its teaching is about having many perspectives, not a correct answer, a chapter on different visions of mathematics and its teaching is included in this book.
These ideas are included in this Review book because they are important in terms of sustainability. Looking towards the end of YDM training and considering the particular students in a school, the mathematics pedagogy developed from that training would include the following:

(a) a vision for the forms of mathematics to be taught to best meet the needs of the students;
(b) an emphasis on identifying and developing proficiencies, positive affects and appropriate language skills;
(c) a focus on diagnosing what students know and remediating any weaknesses by accelerating learning (catering for students who fall behind or need extra material to meet their talents); and
(d) a focus on enriching and extending students’ knowledge so they become the best they can be (providing opportunities to go beyond the curriculum to develop and cater for gifts and talents).

Any school program that uses YDM as a basis to cater for the uniqueness and diversity of the school’s students – using YDM ideas in each school’s own special ways – benefits from consideration of proficiencies, affects, language, remediation and extension.

1.2 Overview

This Review book looks at a variety of topics common across the previous seven books. The book is not designed to cover everything but rather to draw together the previous books and support sustainability of YDM in schools following the training. Three supplementary YDM books – Big Ideas of Mathematics, Problem Solving, and Literacy in Mathematics – provide more detailed information about some of the topics covered in this book.

1.2.1 Chapter overview

After this Chapter 1 giving purposes and overview, there are five further chapters.

Chapter 2: Visions of mathematics and its teaching

Mathematics can be thought of in a variety of ways or from a variety of perspectives. In this chapter we outline some of these and relate them to the previous books. Some of these perspectives are central to YDM and some are not, but all provide insight. The task for you as the reader is to choose the visions from which your school would benefit.

Chapter 3: Proficiencies, affects and language/literacy

The topics within each mathematics content strand all have situations where understanding, fluency with procedures, problem solving, and reasoning are important. In the first major section of this chapter, we look at these four proficiencies and how they can be integrated with YDM lessons to build a stronger program. Then they are reconsidered in the light of a mathematics program that would be sustainable.

The ideas in the preceding books impinge on learners cognitively in terms of understanding. However, they also impinge affectively in terms of interests, beliefs and values. In the second major section of this chapter we look at four clusters of affects: those built around motivation, self-concept, attribution, and resilience. We then ask you to relate these affects to your students and discuss an affects sub-program that would be effective long term.

The topics in the preceding books need to be talked about, explained, discussed and written about. This requires the use of language, and skill and understanding in terms of literacy with respect to mathematics – to communicate with others and to argue, critique and defend ideas and solutions. In the third major section of the chapter we provide ideas for doing this. We then ask the question of how best to integrate language and literacy into mathematics learning so as to meet the language needs of students.
Chapter 4: Assessment, diagnosis and remediation

Central to the YDM RAMR cycle should be assessment of understanding, diagnosis of strengths and weaknesses, and instruction to remediate/teach what is necessary. Once the level of knowledge is known, remedial instruction can be planned to meet the students’ needs. In this chapter we look at assessment, diagnosis and remediation. We also briefly relate the material in the YDM books to the materials in other programs, such as AIM, developed by YDM for the purpose of enabling underachieving students to access Year 10 mathematics.

Chapter 5: Enrichment and extension

Enrichment and extension represents the opposite direction to diagnosis and remediation. It focuses on developing and supporting more able and motivated students to complete the higher-level mathematics subjects needed in Years 11 and 12 for entry to university mathematics-based courses. In this chapter we focus on pedagogies to extend YDM to deep learning of powerful mathematics ideas. We also briefly relate the materials in the YDM books to the tasks and the ideas in MITI that have been developed by YDC to increase participation in high-level mathematics subjects.

Chapter 6: Towards sustainability

This final chapter is the bookend to Chapter 1. In this chapter we look forward at how to set up processes and options for what to do when the YDM two-year training ends.

1.2.2 Chapter framework

In the preceding books, the beginning of each chapter almost always includes a plan for how the material in the chapter will be sequenced. In this Review book, the plan is included at the end of each chapter as processes and frameworks for what to do next. The last section of each chapter takes the ideas in the chapter and places them in a framework to assist you to consider the information in the entire chapter. Thus, each chapter broadly operates as follows:

![Figure 1.1 Framework of chapters in the Review book](image-url)
2 Visions of Mathematics and its Teaching

The nature of mathematics lies at the foundation of the YDM pedagogy. The pedagogy is based on understanding mathematics and how mathematics topics are connected and sequenced. In this chapter we look at different ways mathematics can be perceived, that is, different “visions” and what this brings to teaching mathematics. We reinforce the power of mathematics as a variety of perspectives rather than correct answers.

This chapter is divided into three sections:

(a) **Nature of mathematics** (2.1) – definitions of mathematics (one based on logic and the other on patterns); components of mathematics (objects, relationships and transformations); functions of mathematics (structure, language and tool for problem solving); and features of mathematics (semantic and syntactic knowledge).

(b) **Mathematics teaching and learning** (2.2) – perspectives (instrumental, relational and organic); elements (knowledge of mathematics content, mathematics pedagogy and lesson planning); components (concepts, principles and strategies); and structures (connections, big ideas and sequencing).

(c) **Visions for the future** (2.3) – integration of all the above into a framework for consideration as part of sustainability.

It is best to see mathematics from multiple points of view, so it is useful to consider the options in this chapter. Each of them could be a helpful way for you to see mathematics and its teaching. At the end of the chapter, we frame the information in the chapter to help answer the question of what should be in your school’s vision.

### 2.1 Nature of mathematics

Mathematics is a paradox. Invented by humankind as a process for problem solving, it is often trivialised to a product of facts, rules and procedures. Culturally bound and idiosyncratically formed, it is often reified to an abstraction that has been interpreted as a reflection of the structure of the human intellect. Driven by the problems of the world, growing through intuition and refutation, it is praised for the elegance of its deductive systems.

This review of the nature of mathematics is based on the belief that this paradox can be answered in the synthesis of its two sides and in the commonalities that emerge therein. Mathematics is both a process and a product: a process that encompasses logical and visual thinking, deduction and executive processes; and a product that encompasses a coherent, succinct and powerful language and a structure of concepts and principles. Mathematics is both private and public knowledge: the particular workings of an individual mind, and the refined thinking developed by, and transmitted across, generations. However, in all these things, it is based on system, pattern and structure; and it is recursive in that it can be turned back on itself.

#### 2.1.1 Definitions of mathematics

There is a strong belief inherent in many definitions that mathematics is an objective “science” which is independent of planet, country, culture, class and discourse. For example, the esteemed mathematician David Hilbert is quoted as saying that “mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country” (Eves, 1972). Mathematics has been called variously “the science of space and quantity”, “the science of pattern and deductive structure”, and “the science of the infinite”. It has been proposed as the key to unlock the functioning of the universe.
**Logical underpinning**

Logical reasoning is at the foundation of all mathematical thinking. Problems or situations that involve logical thinking call for structure, a systematic approach, relationships between facts, and chains of reasoning that make sense. The basis of all logical thinking is sequential thought. This process involves taking the important information in a problem and arranging it in a chain-like progression that takes on a meaning in and of itself. To think logically is to think in steps. Reasoning that is logical is often described as being **valid**. The ability to think logically is central to success in mathematics.

Learning mathematics is a highly sequential process. Later work builds on earlier understandings. If a certain concept, fact, or procedure is not understood, a student will not understand other concepts, facts or procedures built on that knowledge. For example, fractions require an understanding of division, simple algebraic equations require an understanding of fractions, and solving some word problems depends on knowing how to set up and manipulate equations.

Logical reasoning is consistently used in mathematics to reach conclusions. A historical analysis of mathematics shows that, although mathematics is written and organised deductively, it was created and developed inductively. **Induction** is the process of discovery through observing patterns, while **deduction** commences with basic rules that apply to a situation (called axioms) and applies logic to those rules to prove that another, more complex, fact is true (see Figure 2.1). Most mathematics was generated by induction and then reorganised and rewritten to be deductive.

<table>
<thead>
<tr>
<th><strong>INDUCTION</strong></th>
<th><strong>DEDUCTION</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Gather information (experiment)</td>
<td>Develop axioms/propositions</td>
</tr>
<tr>
<td>Organise/observe patterns</td>
<td>Apply rules of logic</td>
</tr>
<tr>
<td>Form conclusion/generalisation</td>
<td>Prove result (“theorem”)</td>
</tr>
</tbody>
</table>

*Figure 2.1* The processes of induction and deduction

Training in logical thinking encourages students to think for themselves (both in mathematics and elsewhere), to question hypotheses, to develop alternative hypotheses, and to test those hypotheses against known facts. In fact, mathematical logic has been proposed as the only certainty in our problematic world.

These ideas have been questioned by the constructivist perspective that mathematics is an invention of the human mind. The objectivity of mathematics is also now disputed, with writers arguing that mathematics is culturally based (Wilder, 1981), represents the views of a particular class and background (Walkerdine, 1992) and is a consequence of humans arguing over proofs (Lakatos, 1976). This leads to arguments that mathematics teaching is best seen as **enculturation** (Bishop, 1988). Each of the preceding seven YDM resource books has attempted to provide a cultural perspective.

However, many people appear to continue to believe that mathematicians are special in terms of their thinking and their activity. To the general public, mathematics appears to be a collection of complicated rules and procedures which only a few “egghead” mathematicians can understand. Furthermore, this view is confirmed by an inspection of mathematics books that present mathematics in a refined, abstract, symbolic form. Mathematics is shown in its deductive formal state (as symbols, definitions, variables, axioms, propositions, theorems, and so on).

**Patterns**

The view of mathematics as a study of patterns has proven to be very powerful. Patterns can be represented visually, symbolically, in words and in tables, thus linking all forms of mathematical representations. The search for order and pattern is one of the driving forces in the teaching of mathematics: helping students see those patterns is an important pedagogical tool. The study of similar patterns occurring in different situations (called **isomorphisms**) assists students to see the connections between different areas of mathematics. For example,
patterns of square numbers can be found when arranging counters in a square pattern, when adding on odd numbers \((1 + 3 + 5 + \ldots)\), when squaring a whole number \((1^2, 2^2, 3^2, \ldots)\), when measuring the area of a square with whole-number side lengths, or in the graph of the parabola \(y = x^2\).

We have already noted that patterns are the basis of inductive thinking that has led to most mathematical developments. Identifying a pattern is also an important problem-solving strategy. Patterns are important in all aspects of mathematical content; for example:

- the decimal numeration system depends on repeating additive and multiplicative patterns;
- most methods of calculation rely on patterns, either in number (for example, adding on and multiples) or algorithms;
- in geometry, patterns are evident in symmetry, iteration and transformations;
- patterns occur in the measurement of time and angle, and in the metric system of measurement; and
- repeating patterns lead to generalisation, algebra and the study of functions.

### 2.1.2 Components of mathematics

**Objects, relationships and transformations**

As we have seen in the preceding books (in particular *Algebra* and *Geometry*), one of the really big ideas of mathematics is its structure. In this global big idea, mathematics is categorised as having only three foci: things; relationships between things; and operators, transformations or changes between those things. Things are the objects on which mathematics operates. These things or objects may be of our world or imagination (e.g. people, triangles, Martians) or the products of relations and transformations on these things (e.g. whole numbers, fractions, variables, functions, surds). Within this paradigm of mathematics, importance lies in the relationships and transformations, not in the things themselves.

This is not a trivial conclusion. If we think of variables and whole numbers as being just different contexts for mathematics to operate on, we start to see the beginnings of connections that make mathematics a lot simpler than it is commonly presented in texts (see Figure 2.2).

<table>
<thead>
<tr>
<th>24</th>
<th>(2a + 4b)</th>
<th>32</th>
<th>(3a + 2b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>35</td>
<td>(3a + 5b)</td>
<td>4</td>
<td>(c)</td>
</tr>
<tr>
<td>59</td>
<td>(5a + 9b)</td>
<td>128</td>
<td></td>
</tr>
</tbody>
</table>

**Figure 2.2** Connections between addition and multiplication for whole numbers and variables

Relationships are patterns, rules or procedures that relate the things. For instance, \(2 + 3 = 5\) describes the relationship where 2 and 3 are related to 5 via addition. Similarly, \((a + b)(a - b) = a^2 - b^2\) describes a relationship that is known as the factorisation of the difference of two squares. Operations, changes or transformations are ways in which the things can be changed from one to another. For instance, whole numbers can be changed by adding 2 to them (e.g. 3 \(\rightarrow\) 5, 8 \(\rightarrow\) 10, 567 \(\rightarrow\) 569), and length and width of a rectangle can be changed to area by multiplication. This form of mathematics is not as widely practised in schools as it could be.

The crucial point is that mathematics is often reduced to just two activities which are two perspectives of the same thing. That is, everything that is a relationship can be rethought as a change and vice versa (see Figure 2.3).

<table>
<thead>
<tr>
<th>RELATIONSHIP</th>
<th>TRANSFORMATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A = L \times W)</td>
<td>can be rethought as (L \rightarrow A)</td>
</tr>
<tr>
<td>(2 + 3 = 5)</td>
<td>can be rethought as (2 \rightarrow 5)</td>
</tr>
</tbody>
</table>

**Figure 2.3** Two perspectives of mathematics: relationship and transformation
2.1.3 Functions of mathematics

Mathematics appears to have three functions: structure, language and tool. These can be considered as three ways to think about mathematics.

**Structure**

The first function of mathematics is structure; that is, considering mathematics as a conceptual structure. As Skemp (1976) stated, mathematics is best understood as a structure of concepts and principles rather than a list of rules and definitions. As a structure, it is sequenced and connected, and has recurring themes that provide a framework, or macrostructure, across the topics. For example, addition and multiplication follow the same principles across number and algebra and the part-whole notion recurs across whole numbers, fractions and ratios (see Figure 2.4).

<table>
<thead>
<tr>
<th>Whole numbers</th>
<th>Decimal fractions</th>
<th>Fractions</th>
<th>Algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>42</td>
<td>4.2</td>
<td>$\frac{42}{5}$</td>
<td></td>
</tr>
<tr>
<td>+ 53</td>
<td>+ 5.3</td>
<td>$\frac{53}{9}$</td>
<td></td>
</tr>
<tr>
<td>95</td>
<td>9.5</td>
<td>5 a + 2 b</td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>3.2</td>
<td>3 a + 2 b</td>
<td></td>
</tr>
<tr>
<td>× 4</td>
<td>× 4</td>
<td>3 a + 2 b</td>
<td></td>
</tr>
<tr>
<td>128</td>
<td>12.8</td>
<td>$3a + 2bc$</td>
<td></td>
</tr>
</tbody>
</table>

In this view of mathematics, it is a few integrated macrostructures or superstructures, not a large collection of disparate rules and procedures. This changes the way mathematics is taught so that the focus is on how knowledge connects to other knowledge. For instance, in this perspective, the area of a triangle would be introduced as half the area of a rectangle, not as a formula of its own (see Figure 2.5). It would also result in common notation.

**Figure 2.4** Mathematics as a connected structure of concepts and principles

Area of rectangle = $L \times W$

Area of triangle = Area of rectangle + 2

**Area of triangle = (L × W) ÷ 2**

**Figure 2.5** Introducing the area of a triangle as half the area of a rectangle
Language

The second function of mathematics is language; that is, mathematics is considered as a powerful succinct language for describing relationships and change. Thinking of mathematics as a language alters the way mathematics is taught. Time should be spent on developing meanings for words and symbols, and in translating from real-world situations to mathematical language and vice versa. These ideas are developed further in section 3.3.

Tool (problem solving)

The third function of mathematics is a tool for problem solving; historically, mathematics grew from the need to solve problems. In this approach, mathematical expertise is seen as the ability to solve problems. Two points should be noted:

(a) problems need the integration of three levels of knowledge – domain-specific knowledge (content), strategic knowledge (cognitions), and metastrategic knowledge (metacognition); and

(b) problems need skills in discovery (pattern and problem finding), solution, communication (reporting), and validation (proof).

Understanding of mathematics concepts, principles, strategies and processes, rather than computational procedures such as algorithms, is crucial in solving problems. We have stressed this in the Operations book – in the two sides of operations (meanings and computation), meanings provide the basis of problem solving.

Wilson’s Activity Type Cycle (Ashlock, Johnson, Wilson, & Jones, 1983) on which RAMR is partially based (see Overview book) argues that mathematics began as a way of thinking that could help to manage day-to-day situations. On the way, certain behaviours or techniques (e.g. the addition algorithm) were found to be useful in a variety of situations. These techniques, along with the ways of thinking, became part of what was considered important in mathematics. This was efficient, because those techniques were a shortcut to solving many problems. The use of routine facts and algorithms reduces the cognitive load, allowing a focus on the non-routine aspects of the problem. However, the consequence has been that, for many people, the shortcuts (i.e. the algorithms) have come to define what mathematics is.

Once again, this perspective changes instruction in mathematics. A teaching program emphasising problem solving is very different from one emphasising algorithms. As detailed in the YDM Problem Solving supplementary resource, it requires understanding, strategies and metacognition along with procedures, whereas learning algorithms becomes a process of remembering facts and procedures.

2.1.4 Features of mathematics

In terms of understanding, mathematical knowledge can be classified as semantic (conceptual) or syntactic (procedural).

Semantic knowledge

Semantic knowledge includes the meaning behind the symbols, that is, mathematical concepts and underlying principles. It is a deeper level of understanding that goes beyond surface features. For example, semantic understanding of the decimal number 3.42 would include knowledge of place value, fraction, multiplicative structure, and renaming (see Figure 2.6).

<table>
<thead>
<tr>
<th>Place value</th>
<th>3.42</th>
<th>3.42</th>
<th>3.42</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 ones</td>
<td>×10</td>
<td>O</td>
<td>Multiplicative structure</td>
</tr>
<tr>
<td>4 tenths</td>
<td>t</td>
<td>t ×10</td>
<td>h</td>
</tr>
<tr>
<td>2 hundredths</td>
<td>÷10</td>
<td>h ÷10</td>
<td></td>
</tr>
<tr>
<td>Fraction</td>
<td>3 wholes</td>
<td>4 parts in 10</td>
<td>2 parts in 100</td>
</tr>
<tr>
<td>4 parts in 10 of the whole</td>
<td>42 hundredths</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 parts in 100 of the whole</td>
<td>34 tenths</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 ones</td>
<td>—</td>
<td>—</td>
<td>2 hundredths</td>
</tr>
<tr>
<td>—</td>
<td>—</td>
<td>—</td>
<td>342 hundredths</td>
</tr>
<tr>
<td>2 ones</td>
<td>12 tenths</td>
<td>22 hundredths</td>
<td></td>
</tr>
</tbody>
</table>

Figure 2.6 Semantic understanding of 3.42
Similarly, semantic understanding of $3a$ would include knowledge of variable and, in this example, that the variable is multiplied by 3 and represents any number that is multiplied by 3 or added three times. Semantic understanding would also assist in accepting a statement such as $4 < 7$, but rejecting $7 < 4$.

**Syntactic knowledge**

Syntactic knowledge consists of symbol manipulation and rules which are often unconnected to meaning and are often influenced by surface features. For example, syntactic understanding of 3.42 would be restricted to the digits and the number of digits ("length" – particularly of the decimal fraction). Such knowledge could result in the following misconceptions:

- 4.55 is greater than 4.7 because 4.55 is "longer" than 4.7; or
- 4.03 is the same as 4.3 because zeros have no value.

The semantic–syntactic differentiation is very important for computers and mathematics learning and teaching because the medium for computer–student interaction is most often the written word. Here, syntactic understanding can lead to error. For instance, a student could read $\frac{3}{5}$ as "three over five" with no understanding that it represents, for example, three parts out of a whole that has been divided into five equal parts.

Although YDM is presented in terms of understanding concepts, principles and strategies, the semantic–syntactic classification has not been included. It is particularly important in algebra because in the transition from arithmetic to algebra, there are changes in syntax that can make the transition difficult. For example:

- algebraic multiplication syntax ($3a$) looks more like place value (e.g. 37) than multiplication ($3 \times a$); and
- algebraic addition and multiplication syntax is horizontal, not vertical like arithmetic syntax.

### 2.2 Mathematics teaching and learning

Mathematics teaching and learning also has some classifications which we have not highlighted in the preceding books. Four are considered here: perspectives, elements, components and structures.

#### 2.2.1 Perspectives

There are three perspectives to teaching and learning mathematics (described by Thompson, 1992) with respect to how learners view mathematics and mathematicians: instrumental, relational, and organic.

1. **Instrumental**: The instrumental approach holds that mathematics is a disparate set of rules, facts and skills accumulated over time. It considers mathematicians to be trained artisans.

2. **Relational**: The relational approach (also called Platonist) holds that mathematics is a unified, integrated, connected body of knowledge. However, it also holds that this knowledge is static and discovered (not created or constructed) and is based on logic. It considers mathematicians to be discoverers.

3. **Organic**: The organic approach (also called problem solving) holds that mathematics is a human construction and that it is continually growing and dynamic. It also holds that, as a human invention, mathematics is a subjective social construction which can be reviewed and changed. It considers mathematicians to be dreamers, constructors and imaginers.

This organic view of mathematics is important. If mathematics is subjective like this, then it is not an absolute that holds true across time and culture, but rather a construct that serves social needs. This has led to the ethnomathematical and socially critical views of mathematics and to the focus in mathematics teaching on using mathematics to empower and emancipate students.
2.2.2 Elements

There has been extensive research on the knowledge teachers need in order to teach. The most common model sees teaching mathematics as built around three knowledge types, first described by Shulman (1986). We have referred to these knowledge types throughout the YDM series of books; they are listed below, with the terms that Shulman used in brackets after the names.

1. **Subject matter knowledge** (mathematics content): what teachers need to know about the subject if they are to help students learn the subject matter in ways that allow them to see connections within and between the various mathematical domains and between the mathematical subject matter and their lives in the real world.

2. **Pedagogical content knowledge** (mathematics pedagogy): knowledge of how particular topics, principles, strategies, and so on in specific subject areas are typically understood or misunderstood by students, and learned or likely to be forgotten.

3. **Lesson structure knowledge** (lesson planning): the skills needed to plan and implement a lesson, to create a smooth transition from one lesson to another, and to explain material clearly.

All three knowledges are important but pedagogical content knowledge or pedagogy is crucial (see figure on right). It links content to teaching.

Interestingly, mathematics subject matter knowledge or content is itself considered to have four types of knowledge:

1. **Substantive knowledge.** Not only should teachers be able to “do” mathematics to generate “correct” answers, they should also have a sense for the mathematical meanings underlying the concepts and processes. Furthermore, their subject matter knowledge should not be merely a collection of disparate facts and procedures; instead it should be a collection of interconnected concepts and procedures (Ball, 1988).

2. **Knowledge about the nature and discourse of mathematics.** Ball (1988) believed that teachers need to know: (a) what counts as an “answer” and that justification is as much a part of the answer as the answer itself; (b) that how the truth or reasonableness of an answer is established is an important issue in mathematics; (c) that “doing” mathematics is not “figuring columns of sums or performing long division” but instead consists of creative activities such as examining patterns, formulating and testing generalisations, and constructing proofs; and (d) what can be derived logically versus what could be defined as mathematical conventions.

3. **Knowledge about mathematics in culture and society.** Mathematics plays a crucial role in the development of modern technological societies (Mayer, 1992; Mathematical Sciences Education Board et al., 1989) because it not only provides the utilitarian skills required for normal participation in society and the problem-solving skills required for employment, but also produces the abstract, as well as the creative, knowledge required for technological innovation (Baturo & Cooper, 1993). These different outcomes of a mathematics education reflect an increasing level of mathematical difficulty in the transition from the real world to abstraction. If teachers are to play a role in making the learning of mathematics relevant to their students, they need to understand these various roles played by mathematics in our society. Furthermore, teachers need some awareness of how the mathematical ideas used in our society evolved.

4. **Dispositions towards mathematics.** As well as developing understandings of the substance and nature of mathematics, students also develop dispositions towards the subject (Ball & McDiarmid, 1989). To a large extent, their dispositions towards the subject will be influenced by their teachers’ liking for particular topics and activities and their teachers’ propensities to pursue certain questions and kinds of mathematical investigations.
2.2.3 Components

Mathematical structure is hierarchical, integrated, and consists of a linked system of concepts, principles and strategies. Therefore teaching it needs to have these latter three components, which are also highlighted in some of the seven preceding YDM resource books and the YDM supplementary resource Big Ideas of Mathematics.

1. **Concepts.** A concept is a large scheme that contains knowledge on a mathematical idea (e.g. the concept of number, the concept of fraction) in an integrated and connected structure. It has the following attributes: (a) it defines the idea; (b) it contains links to other related ideas; (c) it contains information on where the idea can be applied; and (d) it provides information on the learners’ experiences with the idea. A concept should be seen as the structural extension of a definition.

2. **Principles.** A principle is a relationship that holds for different contexts. It is the structural extension of a law (e.g. commutative principle). Principles are similar to abstract schemas (Ohlsson, 1993). Abstract schemas are schema frameworks (without things or objects) into which a variety of numbers, variables, objects, and so on can fit. They are a structure of relationships that hold for different objects. (The term *recurrent theme* has also been applied to some types of principles, for example, the part-whole relationship in fractions, percent, and so on).

3. **Strategies.** A strategy is a rule of thumb that may lead to a solution of a problem or procedure (e.g. draw a diagram). Strategies exist for problems (e.g. guess and check) and for processes (e.g. $5 + 7$ can be calculated by doubling and counting on 2). Strategies are mental processes with which to handle data. Methods to control strategies are called metastrategies or metacognitive processes. Metastrategies include planning, monitoring, regulating (decision-making) and evaluating.

It should be noted that there is ambiguity in these definitions. Ambiguity can be defended as one of the powers of mathematics. To some extent, good mathematicians have a composite view of these components, moving easily between them. Moreover, ideas in mathematics can often be considered as more than one of these components, depending on the perspective taken at the time.

2.2.4 Structures

Taking a structural view of teaching is useful because it assists learning. This has been widely encouraged in the preceding books. Structural knowledge is similar to what Skemp (1976, 1989) calls relational; procedural knowledge, the opposite to structural, is similar to what Skemp calls instrumental. Skemp’s terms are widely used.

**Connections**

The structure of concepts, principles and strategies that makes up mathematics is hierarchical and integrated (i.e. interconnected). The hierarchy gives the components of a particular piece of knowledge, and describes how they are related to each other and how they are included in each other. It enables larger constructs to be built from smaller and enables the new to be related to the old. This provides insight into how teaching may be sequenced. Knowledge of sequencing can allow a teacher to build on the past knowledge of students and to teach new knowledge in a form appropriate for the future. This appropriateness should be in terms of the sequence (i.e. preparing the students for the next step in the sequence), and in terms of availability of the new knowledge for a diversity of future areas (i.e. making the knowledge wide enough so it can be used in other topics). It is important that the knowledge being built is rich enough to cater for the future.

The integrations, or interconnections, provide information on how the various components of different pieces of knowledge are related. They can be used to allow the teacher to take advantage of the power of similarities and differences in enabling the human mind to retain and retrieve information. That is, if the teacher can identify how a new piece of knowledge is related to other pieces of knowledge, he/she can direct the students’ attention to these connections; and this may enable the students to remember the new piece of knowledge by its relation to the other
pieces of knowledge that the students already know. This is a superior form of retention to just learning the new piece of knowledge by rote as an isolated piece of knowledge.

**Big ideas**

The structures of mathematics recur continuously throughout its content in different forms. It is easier to remember one structure and all the times this structure appears than it is to remember the instances of that structure as disconnected pieces of information. As well, because mathematics has to be presented to students across the developmental level, the meanings often taught in the early years are compromises that do not always give the complete mathematical understanding. As students develop, these incomplete notions need to be expanded and, in some cases, replaced with more complex and mathematically accurate knowledge (this is called *schema change*).

The structure of mathematics enables the _big ideas_ that recur throughout the year levels to be identified and shown to students. It also helps identify inadequate notions, and gives an indication of when they need to be replaced. The big ideas that underlie connections can be used for sequencing and detecting when a schema change is necessary. Most importantly, mathematical structures facilitate abstraction of the salient mathematics, and abstraction reduces cognitive load.

**Sequencing**

Knowledge of structure also provides illumination as to what should be the focus of the teaching of a particular piece of knowledge by supplying information on the sequence of which this knowledge is part (identifying earlier and later knowledge). It can also enable necessary instructional activities to be identified. Further, it prevents wrong information from being stressed (such as “you cannot subtract 5 from 2”).

For example, the multiplication algorithm procedure is structurally important because it is an example of the distributive principle. This can be seen by comparing $32 + 4$ with $32 \times 4$ as below:

$$
\begin{array}{c}
\text{THESE ARE CORRECT} \\
32 + 4 \\
\hline
36 \\
\end{array}
\quad
\begin{array}{c}
\text{WHY AREN’T THESE?} \\
32 \times 4 \\
\hline
128 \\
\end{array}
$$

It is crucial for students to see that $32$ multiplied by $4$ is a combination of $2$ multiplied by $4$ and $30$ multiplied by $4$. When this is seen, the algorithmic procedure can be completed from basic facts ($2 \times 4 = 8$ and $3 \times 4$ tens $= 12$ tens). So it is crucial to show that the $4$ must multiply both the $2$ and the $30$, and for students to be able to identify that something different happens to $32 + 4$. This can then be extended to $(a + b) \times c$ where the $c$ multiplies both $a$ and $b$ to give $ab + ac$.

To build this understanding, it is important to use materials to assist teaching. For example, MAB can be placed on a Place Value Chart as follows (| stands for a ten, • for a unit) for the example $4 \times 32$. The multiplication is conceived as $4$ lots of $32$ and MAB is used to show this.

<table>
<thead>
<tr>
<th>TENS</th>
<th>ONES</th>
</tr>
</thead>
<tbody>
<tr>
<td>••</td>
<td>••</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>TENS</th>
<th>ONES</th>
</tr>
</thead>
<tbody>
<tr>
<td>••</td>
<td>••</td>
</tr>
</tbody>
</table>

<table>
<thead>
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<th>TENS</th>
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*Note:* This is a general point regarding the use of materials – materials should show processes, strategies and principles, not answers. The structure indicates how the materials should be used.
2.3 Visions for the future

Sections 2.1 and 2.2 contained different visions of mathematics and its teaching. Putting them all together, we have the concept map in Figure 2.7.

In the map, we have placed mathematics and mathematics teaching at the centre, with the mathematics components and options above and the mathematics teaching components and options below.

In many cases, the options in each box or circle are not alternatives because good mathematics teaching needs all these components. This is particularly the case for the mathematics teaching components concepts, principles and strategies, which are all important. Similarly, the three different knowledges in the Venn diagram are all important. What is open for discussion is the emphasis on the options/components. This emphasis can come from within the options. For example, syntactic is less important than semantic although both are necessary. The decision-making about emphasis should predominantly come from the needs of the students and the capacities of the teachers. For example, many teachers (and students) have little knowledge of the change/transformation approach to teaching mathematics; therefore, this should be emphasised for balance with the relationships approach. From the YDM perspective, we tend to emphasise inductive, semantic, and structure for mathematics and organic for mathematics teaching; the other components can be adjusted depending on school, teachers and students.

Your task from this chapter is to develop a plan of what your school should emphasise with respect to mathematics for a sustainable future.

Prepare a map or list. Use Figure 2.7 as a template for a framework for doing this if you wish.
In this chapter we look at three components in terms of their relation to the preceding seven YDM books, their effect on YDM programs, and their role in meeting the needs of your school into the future. The purpose of the chapter is to start discussion of how your school’s mathematics program should take account of these components.

The chapter is divided into four sections:

(a) Proficiencies (3.1) – based on the four proficiencies in the Australian Curriculum: understanding, fluency, problem-solving and reasoning.

(b) Affects (3.2) – four clusters of non-cognitive aspects that impinge on mathematics learning: attitude, motivation and engagement; self-confidence and self-efficacy; attribution; and perseverance and resilience.

(c) Language and literacy in mathematics (3.3) – listening and speaking, vocabulary, symbols, visual images, reading (decoding), writing (encoding) and teaching ideas.

(d) Implications for future programs (3.4) – a summary that can become the basis of your school’s approach to the above three components of a school mathematics program.

### 3.1 Proficiencies

The Australian Curriculum: Mathematics specifies three content strands: Number and Algebra, Measurement and Geometry, and Statistics and Probability; and four proficiencies: understanding, fluency, problem-solving and reasoning. In this regard, the Australian Curriculum follows the approach of the US Common Core State Standards for Mathematics (Common Core State Standards Initiative [CCSSI], n.d.), which also specify proficiencies (standards for mathematical practice: http://www.corestandards.org/Math/Practice/). In both situations, the proficiencies tend to be differentiated from the mathematical content by looking upon them in the following ways:

(a) as actions students can engage in when learning and using content;

(b) as ways the content is explored and/or developed; and

(c) as the thinking and doing of content, the ways of working mathematically.

Proficiencies are specified so teachers can check that they are being developed across year levels. Their development should be constructed through direct experience and reflection (Schliemann & Carraher, 2002). In the following subsections, we look at each proficiency in terms of the YDM resource books. While some definitions are provided, we recommend reading the descriptions and examples provided in the Australian Curriculum for understanding the proficiencies.

#### 3.1.1 Understanding

Mathematical understanding is a widely and divergently used word to describe mathematics learning. Everyone appears to agree that understanding should be the aim of instruction, but what do they mean?

**Understanding and connections**

YDM tends to follow the early work of Skemp (1976) who argued that it is not enough for students to understand how to perform various mathematical tasks (instrumental understanding); for full conceptual understanding students must also appreciate why each of the ideas and relationships work the way they do (relational understanding). As Sullivan stated, referring to Skemp’s work, “well-constructed knowledge is interconnected,
so that when one part of a network of ideas is recalled for use at some future time, the other parts are also recalled” (Sullivan, 2011a, p. 1). Figure 3.1 depicts this concept of interconnected knowledge.

![Diagram of Instrumental and Relational Understanding](image)

*Figure 3.1 Interconnected knowledge for mathematical understanding*

Understanding as the ability to move between representations is a commonly accepted meaning in mathematics education. The different forms include symbols, formal mathematics language, models (real-world, physical, virtual, pictorial and patterning), graphs and real-world situations. YDM tends to call this structural knowledge, in contrast to procedural knowledge. It draws on the work of Piaget (1977) and others, where the structural knowledge of mathematics may also be referred to as rich schema or big ideas.

As we have said before, highly schematic or interconnected knowledge is efficient for recall because the mind remembers connected ideas far better than a collection of unconnected ideas. It is also useful for problem solving because it provides access to peripheral knowledge through connections to what is being focused on. Problem solving is required when the focus of thinking is not enough for a solution; following connections can lead to finding knowledge that may initially appear to be peripheral to the task, but is found to be central and useful in solving the problem.

Understanding is also organic (that is, characterised by the systematic and/or organised arrangement of parts) and provides frameworks for later learning. When symbols, words, relationships and contexts are part of a web of ideas, different representations can be connected and any of them can be used to build new ideas (that are subsequently connected). The YDM MITI project, which focuses on deep learning of powerful ideas, focuses on how to build organic structures (or what MITI calls superstructures).

**Understanding in YDM**

All the YDM books stress connections, big ideas and sequencing (schematic knowledge) as the most powerful forms of understanding. YDM sees mathematics in terms of big ideas and breaks these into conceptual, principle, strategy and global big ideas in order that every mathematics idea can be understood (see Big Ideas of Mathematics supplementary resource book). This means that ideas like addition should be understood in a connected way, covering conceptual, principle, strategic and global understanding as follows:

(a) **Conceptual** understanding of addition means it is understood as each of the following: joining, the reverse of subtraction, comparison, and part-part-total, and that it applies to any form of numbers, variables, expressions and functions.

(b) **Principle** understanding of addition means it is understood as closed, having an identity, having inverses and following the laws of commutativity, associativity and distributivity (with multiplication).

(c) **Strategic** understanding means it is understood that addition computation is based on separation, sequencing and compensation and that these strategies apply to both numbers and variables.

(d) **Global** understanding means addition is understood as a fundamental operation that is common to both arithmetic and algebra. This global understanding contains the principles in (b) above, as well as the principles of equals (reflexivity, symmetry and transitivity), and the principles of the two approaches
where addition is seen as both a relationship (equivalence or balance rule, e.g. \(2 + 3 = 5\)) and a transformation (arrowmath and backtracking, e.g. \(2 \xrightarrow{+3} 5\)).

Like other pedagogies, YDM pedagogy advocates that understanding should be the objective of all teaching. Overall, structural or relational understanding involves the why as well as the how. It enables transfer of knowledge across contexts, multiple representations, identification of similarities and differences, and interpretation, construction and discussion about different aspects of mathematics content.

**Forms of understanding**

What does mathematical understanding look like? In the preceding books, we have looked at understanding as connecting between representations in different situations. In Number, we focused on relationships between models, language and symbols using counters, bundling sticks, MAB, place value charts/mats, and number lines. As we have progressed through the year levels, the mathematics has become more complex and we have connected different symbol forms and in turn connected all these to different models, often for the same number (e.g. \(43 = 30 + 13 = \text{three bundles of ten and 13 sticks}\)). We have also needed to become more flexible (e.g. \(0.345 = 3.45 \times 10^{-1}\)). Finally, we have looked at equivalence – for fractions, ratios (proportion), and then into algebraic expressions.

Thus, understanding mathematics as meaning that can move between different representations is not as simple as it was in earlier years. With algebra comes the connection of number laws and properties to algebraic terms and expressions, and relationships between numbers (i.e. equations) being represented geometrically in graphs.

Some educators believe it is the use of “why” that identifies understanding. They argue that there is a large difference in understanding between students who can rote-follow a procedure to multiply \(35 \times 6\) and students who can explain why the algorithm works. The “why” allows the second set of students to move on to \(35 \times 64\), \((a + b)(c + d)\), and on to factorisation. These educators look at the ability of students to justify their answers to enable them to determine understanding. This leads to teaching approaches where students have to explain what they are doing and why they are doing it to others, and to the flexibility to generate new procedures and answers.

The Reflection component of the YDM RAMR teaching framework includes Extension where students are asked to be flexible, reverse, generalise and change parameters. These generic pedagogies are included in the framework because they are important in developing deeper understanding.

### 3.1.2 Fluency

Fluency is usually associated with speed and accuracy in symbolic procedures, and has been over-emphasised in traditional mathematics teaching. As Sullivan (2011b, p. 8) stated,

> fluency is disproportionately the focus of most externally set assessments, and therefore is emphasised by teachers especially in those years with external assessments, often to the detriment of the other mathematical actions.

YDM was built to move the emphasis away from fluency. It does provide the stage called Mathematics in the RAMR framework to cover fluency, because it is useful to have skill and accuracy in procedures. However, the Mathematics stage of RAMR follows Reality and Abstraction and includes connections as a component. This recognises the need for contextual building of understanding and structure as the important facet of all mathematics learning. Emphasis on the fluency proficiency within YDM depends on how fluency is defined and what it involves, described further below.

**The nature of fluency**

The Australian Curriculum: Mathematics (ACARA, 2016) defines fluency in mathematics as follows:
Students develop skills in choosing appropriate procedures; carrying out procedures flexibly, accurately, efficiently and appropriately; and recalling factual knowledge and concepts readily. Students are fluent when they calculate answers efficiently, when they recognise robust ways of answering questions, when they choose appropriate methods and approximations, when they recall definitions and regularly use facts, and when they can manipulate expressions and equations to find solutions. (http://www.australiancurriculum.edu.au/mathematics/key-ideas)

In the USA, the Common Core State Standards for Mathematics (CCSSI, n.d.) use the term procedural fluency for the same thing and define it similarly as “knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently” (National Research Council, 2001, p. 121). The standards argue for a focus on procedural fluency across all grades with practice and support, and note that procedural fluency and conceptual understanding are interwoven. This supports arguments for seeing fluency as more than narrowly based on procedures or computation.

The National Council of Teachers of Mathematics (n.d.) argues that procedural fluency relates to applying procedures, transferring procedures to different problems and contexts, building and modifying procedures, recognising appropriateness of strategies, integrating concepts and procedures, and creating own informal strategies and procedures (http://www.nctm.org/Standards-and-Positions/Position-Statements/Procedural-Fluency-in-Mathematics/). They use words such as justify choices with regard to what fluency is – words which we have seen as important to understanding. They see procedural fluency as an extension of what they call computational fluency – more than fast and accurate use of computation. They argue that analysing situations is fluency in some cases. They emphasise connection of procedures with underlying concepts.

The YDM approach also sees fluency as more than computation. We agree with McClure (2014) that fluency “demands more of students than memorising a single procedure – they need to understand why they are doing what they are doing and know when it is appropriate to use different methods” (“What is fluency?”, para. 7). Proficiency includes: (a) efficiency – students have efficient strategies they can carry out, keep track of and use to solve problems (and sub-problems); (b) accuracy – students are accurate through using knowledge and recording and checking strategies; and (c) flexibility – students are flexible in that they know more than one approach, can choose the best approach and can use a different checking procedure to the solution procedure (Russell, 2000, as cited in McClure, 2014).

Although we acknowledge the value of practice with ideas before they are used to construct new ideas, that is, the importance of fluency for knowledge growth, we also agree with Rohrer (2009) that teachers need to be warned of the dangers of overdoing practice. It is important also to practice in a manner that makes links between situations in which procedures are used. This is done by having memorisation/practice occur with connections (as it does in the RAMR framework) because then the procedures are based in a schema/web of connected ideas and so are hard to forget.

Reasons for fluency and actions to improve it

The most cogent reason for spending time on fluency is to reduce cognitive load. Fluency is a way to reduce load on working memory (which is of limited capacity). Fluency allows more capacity for other mathematical actions and removes concern that students’ memories will be filled with trying to work out, for instance, what terms such as “average” mean. If students already understand mathematical terminology and have ready recall of, for example, basic number facts, this frees up working memory for more complex tasks. As Sullivan (2011a), paraphrasing Pegg (2010), stated:

If students do not know what is meant by terms ..., then the instruction using those terms will be confusing and ineffective since so much of their working memory will be utilized trying to seek clues for the meaning of the relevant terminology.... if students can readily recall key number facts, these facts can facilitate problem solving and other actions. (p. 2)

Furthermore, lack of fluency may affect conceptual understanding. As Ginsburg (2012, para. 1) stated in his blog:
Like all of us, students have finite energy. The more energy they use for procedures, the less energy they have for problem solving. And the less energy they have for problem solving, the less likely they are to gain conceptual understanding. A lack of procedural fluency can therefore contribute to a lack of conceptual understanding.

Fluency also helps another aspect of mathematics instruction that is useful in learning – student talk. This is not just describing but also explaining and justifying – why something worked and how it is different to other methods. Discussing multiple methods requires not just terms and isolated rules but also the ability to make connections.

Classroom activities to build fluency with connections that also help understanding include:

(a) fostering individual and group communication, through discussion and reports (building student talk);
(b) going past “knowing facts” (knowing what) and being skilful in procedures (knowing how), to appropriately using procedures in the right situations (knowing when);
(c) modelling procedures for the students then allowing students to discover multiple solution strategies (multiple ways of using the procedures); and
(d) establishing connections between existing and new knowledge, that is, undertaking practice activities alongside activities to build connections (as in the Mathematics stage of RAMR).

Overall, then, we have a sequence for the development of fluency as shown in Figure 3.2.

![Figure 3.2 Development of fluency in mathematics](image)

### 3.1.3 Problem solving

The set of YDM resources includes a supplementary resource on problem solving. This YDM Problem Solving book defines problem solving as “What you do when you do not know what to do” and discusses problem solving with respect to its steps and components. Problem solving has three steps as shown in Figure 3.3:

![Figure 3.3 The three steps of problem solving](image)

It uses six components: mathematics content, affects, strategies, plans of attack, thinking skills and metacognition. The book also relates different problems to these components; routine problems that have a domain of knowledge (e.g. algebra problems) are best solved by content knowledge, while creative problems that do not have a domain of knowledge (e.g. matchstick puzzles) are best solved by thinking skills (Figure 3.4).

![Figure 3.4 Problem solving for routine problems versus creative problems](image)

The book concludes by proposing a teaching sequence from Years P to 9 to enable the development of all these components, and provides particular information for word problems.

These aspects of problem solving are presented in more detail in section 5.1 of this Review book. However, the full Problem Solving book contains further information. Rather than repeat what is already in the book, we
provide other insights below. It should also be noted that problem solving overlaps other proficiencies and that the difficulties in problems, “being in the eye of the solver”, depend on what the problem solver knows.

**Problem solving in curricula statements**

The Australian Curriculum: Mathematics (ACARA, 2016) defines the problem-solving proficiency strand in mathematics as follows:

> Students develop the ability to make choices, interpret, formulate, model and investigate problem situations, and communicate solutions effectively. Students formulate and solve problems when they use mathematics to represent unfamiliar or meaningful situations, when they design investigations and plan their approaches, when they apply their existing strategies to seek solutions, and when they verify that their answers are reasonable.

(http://www.australiancurriculum.edu.au/mathematics/key-ideas)

The Australian Curriculum stresses many aspects that are covered in YDM’s *Problem Solving* book. For example, the curriculum refers to using mathematics knowledge to rethink, restate and change the view of the problem; using affects such as perseverance; and encouraging the use of a wide variety of strategies such as looking for similar and simpler problems. The curriculum also recommends analysing the problem, making conjectures and planning a solution pathway (the first three steps of the SEE–PLAN–DO–CHECK plan of attack [Polya, 1957] in the *YDM Problem Solving* book), and continually checking progress, asking “does it make sense”, and monitoring and evaluating thinking (the basis of metacognition, the highest component of problem solving in the book). The curriculum also stresses formulating problems (problem posing) and justifying solutions which are the first and third of the three steps in the *Problem Solving* book.

Certain activities are highlighted:

1. **Mathematical modelling** – applying mathematics to solve problems in everyday life. This often means assumptions and approximations, analysing relationships and drawing conclusions, and using technology for displays and analysis.
2. **Using appropriate tools strategically** – including rulers, calculators, spreadsheets, and graphs. This requires sufficient familiarity with the tools to make sound decisions on what will be helpful and to be able to pose and solve problems.
3. **Making use of structure** – to discern pattern and identify components and use this to detect significance and identify profitable and non-profitable directions. This requires ability to step back and shift perspective.
4. **Looking for and expressing regularity** – being involved in activities where students can generalise to a formula. This requires maintaining oversight of process.

**Arguing, evaluating and justifying**

There appears to be strong support among educators for problem solving being related to arguing, evaluating and justifying. The US *Common Core State Standards for Mathematics* argue that constructing viable arguments and evaluating others’ reasoning are core mathematical practices in which students “justify their conclusions, communicate them to others, and respond to the arguments of others” (CCSSI, n.d., pp. 6–7). Skill in making and justifying claims is related to investigations; but discussing the nuances of mathematical text provides an additional context in which to develop reasoning and justification skills. This dual focus on speaking and writing is a subsection of the Mathematics stage of the RAMR framework; however, this discussion should begin in the Abstraction stage, because it is easier to discuss when students are active.

Section 3.3 of this chapter considers writing in more detail but, here, we look at arguing and writing with respect to improving problem solving. Five activities are briefly described (Edwards, Weinstein, Goetz, & Alexander, 2014):

1. **Ask whole-class questions** – assign a question to the whole class, get students to individually answer the question in writing and then explain the problem to the whole class. Students progress from symbols to language as they answer why they did what they did.
2. Give opportunities for writing and talking – continually stop lessons and ask why something is being done or why an algorithm is the way it is. It is not possible to achieve talking and writing in problem situations without giving time (note that the Problem Solving book has a template which students use to write and then discuss results of strategies).

3. Assign fewer problems but expect more from them – require students to solve problems in different ways and write down what they do. Ask questions – What does the problem mean? Are there different ways to look at it? Is there more than one solution? – then have group and class discussions and debates.

4. Ask open questions; for example, put out a calendar and ask what students can find or give a shape and say “investigate” (the shape on the right is excellent for this).

5. Give problem-solving tasks that focus on building writing and arguing skills by, for example, leading students in discussing alternatives, asking students to write a report, and so on.

6. Ask “philosophical” problems (e.g. Is telling the truth always right?) for which answers vary and discussion will always happen and there is no right or wrong answer, only justification of position. Learn and use techniques for discussion, such as that a student can only speak if they first paraphrase what the previous speaker said (e.g. “He/she said … but I say …”).

One of the most powerful ways to teach problem solving is simply to give students a problem even if you think it will be difficult – students cannot learn if they are not given problems to solve. Also the reverse of this process is important (reversing is a major part of the YDM pedagogy), so give students solutions to problems and ask the students to pose problems.

**YDM approach**

YDM has a particular approach to problem solving in mathematics domains that is not commonly in descriptions of this proficiency. Most discussions on problem solving focus on decoding the text of the problem, often into parts, in order to find the knowledge required to solve it. From this approach has arisen the focus on language and reading as important components of problem solving. In other words, the problem text is broken down into parts and reconstructed to identify the process that is required.

We believe moving from the detail to the “big picture” is not how people with expert mathematics knowledge solve problems, nor is it an approach used in Indigenous cultures. The YDM approach advocates that mathematics problem solving is best done and taught from knowledge to problem, not from problem to knowledge, as shown in Figure 3.5.

![Problem solving](image)

*Figure 3.5 Expert (YDM) vs non-expert approach to problem solving*

This is best explained with an example. Consider the problem:

*Tom had money for the shop, he was given $11 more, he then had $37; how much money did he have to start with?*

Decoding the text of the problem often leads students to erroneously add $11 and $37 because money had been given (joining other money) and there is the word “more” (and for many students, “more” means add). However, YDM students would have knowledge about addition/subtraction problems in a connected way as part of the teaching of addition/subtraction (i.e. in YDM’s knowledge structures, students are taught definition, connections,
and applications). They would know that addition/subtraction problem types can be (a) joining/taking away, (b) comparison, (c) the reverse of subtraction/addition, and (d) inaction, and that mathematical models for addition are part-part-total and the number line. With this knowledge, YDM students would start by looking for the relationships in the problem text, linking it to their knowledge of possible problem types and models. They would see that the relevant problem type is the reverse of addition and the model is part-part-total, where one of the parts is unknown. Both approaches indicate that this is a subtraction problem. At this stage, the values can be extracted from the problem text ($P_1 = ?, P_2 = $11 and $T = $37) and the solution can be found by subtracting. In this example, it is the knowledge of mathematical relationships, rather than reading and language, that determines the method of solution.

This approach can be used to find the problem solution (answer), given the problem and its mathematical relationships, or to determine a problem (problem posing), given the mathematical relationships and the answer.

3.1.4 Reasoning

Reasoning and problem solving are closely related. In fact, the communication and justification step of problem solving should be built around strong reasoning skills, as should the PLAN and DO sections of Polya’s (1957) four-stage plan of attack for solving problems. The consequence of this is that the arguing, evaluating and justifying section in 3.1.3 above could just as easily have been put in this section on reasoning.

The Australian Curriculum: Mathematics (ACARA, 2016) defines reasoning in mathematics as follows:

Students develop an increasingly sophisticated capacity for logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying and generalising. Students are reasoning mathematically when they explain their thinking, when they deduce and justify strategies used and conclusions reached, when they adopt the known to the unknown, when they transfer learning from one context to another, when they prove that something is true or false, and when they compare and contrast related ideas and explain their choices.

(http://www.australiancurriculum.edu.au/mathematics/key-ideas)

The US Common Core State Standards for Mathematics (CCSSI, n.d., p. 6) add to this by stating that students with proficiency in reasoning

... bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved.

This is an important reversing, saying that the reasoning has to go from language to symbols and from symbols to language (Figure 3.6):

![Figure 3.6 Reasoning as decontextualising (abstracting) and contextualising (reflecting)](image)

This is the basis of the RAMR framework, moving from reality to mathematics by abstraction and then from mathematics back to reality by reflection. Thus YDM is based on mathematical reasoning.
Role of logic

Reasoning requires logic. It is not just explaining – there must be a line of argument (a logical progression) from the first statement to the last. As the US Common Core State Standards for Mathematics state:

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They ... are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is.

(CCSSI, 2016 pp. 6–7)

Logical lines of argument have the following attributes:

1. They start from an agreed position (of neutrality or acceptability to both sides). Many arguments are flawed by adopting a starting position that is untenable – or a starting position that assumes the ending proposition (the circular or cyclic argument or tautology).

2. Ideas follow logically from previous to next, taking into account context as well as findings; for example, a substance might kill germs in laboratory-controlled tests, but this does not mean it will kill germs in a mouth in a diluted form.

3. They have quality and clarity of purpose (including arguments for worthwhileness where important), coherent relationships between components (including introductions and summaries), consistency in sub-arguments (building specific to general or general to specific – not mixed up), and discriminating (not repeating the same thing).

4. Everything is covered as the argument moves from one statement to the next (all possibilities are exhausted); for example, if it is claimed that a substance kills 99% of bacteria, whether you are safe depends on what is happening to the other 1%.

5. They correctly use logical connectives and their opposites, such as:
   - **AND, OR, NOT, IF ... THEN** (or \( P \Rightarrow Q \), i.e. \( P \) implies \( Q \))
   - NOT (A AND B) is (NOT A) OR (NOT B)
   - NOT (A OR B) is (NOT A) AND (NOT B)
   - \( P \Rightarrow Q \) does not mean (NOT P) \( \Rightarrow \) (NOT Q); it means (NOT Q) \( \Rightarrow \) (NOT P)
     For example, if all boys (P) are tall (Q) then this does not mean that all girls (i.e. not P) are short (i.e. not Q), it means that short people (not Q) are not boys (not P); some girls (not P) may be tall (Q). This is called the contrapositive.
   - Some arguments work in one direction only (\( P \Rightarrow Q \)) while others work in both directions (\( P \Leftrightarrow Q \)). For example, all counting numbers are integers does not mean that all integers are counting numbers (since integers can include zero and negative whole numbers). On the other hand all even numbers are divisible by two also means that if a number is divisible by two then it is even.

6. They do not use tricks such as:
   - stating **4 out of 5 dentists recommend** (which may not mean **80% of all dentists recommend**);
   - using a decimal value to improve the apparent accuracy of a finding based on loose assumptions and approximations;
   - stating a burger made out of one rabbit and one cow is a 50% rabbit burger;
   - inappropriately adding percentages, e.g. stating a 10% increase in materials, advertising and wages is a 30% increase overall;
• stating a 50% decrease on 2011 wages is undone by a 50% increase on 2012 wages.

7. They do not have “straw person arguments” where positions are presented over-positively so it can be argued they are better than other alternatives that are presented negatively.

8. There are no gaps and switches in argument, no extreme person arguments (e.g. it works for this person so it will work for everyone), and no unsubstantiated claims (e.g. education is important).

9. They can withstand the five questions posed by Huff (1954): Who says so? How do they know? What’s missing? Did somebody change the subject? Does it make sense?

10. They use proof by contradiction correctly when it is effective – assume the opposite of what you want to prove and then show this leads to a contradiction. The example of why the square root of 2 is not rational (a fraction) is a good starting point for older students. (Note, however, that this method can be used only when there are two possibilities, in this case that the square root of 2 is rational or it is not, and there are no other possibilities.)

**Teaching reasoning**

We have already discussed the importance of logical reasoning in mathematics. We will look at two ways to teach reasoning. The first is **directly** by teaching meaning of connectives and training students to detect anomalies in written arguments. Three examples of this first way are given below.

1. In the *YDM Number* book, Chapter 2, section 2.5 covers complex reasoning and inference. In this section there are descriptions of activities and games with logic attribute blocks covering identifying attributes, Venn diagrams (see right), and logical connectives *NOT, AND, OR, and IF ... THEN*, which finish with a section on engineering problems, inference and Boolean algebra. These activities and games will help build the formal logic associated with reasoning; they are also a lot of fun and build the ability to detect differences in attributes and to understand the logical connectives.

2. Collect examples of incorrect logical reasoning (there are many available on the internet). Give these to students and ask them to detect where the gaps and errors in the logical argument are. Students could also edit each other’s writing.

3. Invite students to write incorrect logical reasoning excerpts. This could be done one error at a time and/or in relation to a task that is being covered.

The other method is to be **indirect**. Some ideas for this are as follows:

1. Give students opportunities to write logically in a supportive environment. Having students read each other’s work or discuss each member’s work in groups can be effective.

2. Be prepared to spend time on this writing – anticipate difficulties – stay out of the way as much as you can and ask questions. Let students have time to think through what they are doing – combine writing with verbal explanations – get students discussing options.

3. Focus on perseverance (see section 3.2.4) through encouragement and showing that it can be as important to them as it is to you.

4. Encourage students to explain their own thinking – try to not end a session early by getting the best answer first.

5. Build reflection into your classroom by giving time for students to review lessons, in writing and discussion, and by encouraging students to exchange ideas. In these reviews, do more than restate the lesson – get students to describe their thinking.
3.2 Affects

This section looks at four types of affect: (a) attitude, motivation and engagement; (b) self-confidence and self-efficacy; (c) attribution; and (d) perseverance and resilience. Each of these is described and then teaching ideas are given to show how to make them positive to mathematics learning.

3.2.1 Attitude, motivation and engagement

Attitude to mathematics

Attitude refers to the students’ beliefs and feelings about mathematics and their relationship to mathematics (e.g. fear, dislike, hatred, happiness, love, and so on). Three groups of factors play a vital role in influencing student attitudes:

1. Student-based factors. These are related to the students’ experiences with mathematics. For example, attitudes can be affected by mathematical achievement, anxiety, self-concept, and motivation.

2. Classroom-based factors. These are related to the school, teacher, and teaching. Students’ attitudes can be affected by such factors as teaching style, teaching materials, classroom management, teacher knowledge, and teacher empathy.

3. Environmental factors. These are related to the students’ home environment and society. Attitudes can be affected by home background, support from parents, educational background, and parental expectations.

Attitudes, in particular, are often used by teachers to explain their students’ success or failure or as an excuse for not being able to help a student (Di Martino & Zan, 2010, 2011; Polo & Zan, 2006). The most common experience of young people in mathematics classrooms is a focus on acquisition of skills, solution of routine exercises, preparation for tests and examinations and a need for speed in calculations. These practices are known to increase anxiety. If students are anxious, information entering their brains is less likely to reach the conscious thinking and long-term memory parts of the prefrontal cortex, and learning will not take place.

Motivation towards mathematics

Motivation refers to the students’ interest in doing mathematics and affects the ways in which students choose to behave and their self-confidence, ability to overcome obstacles and challenges, and capacity to recover from setbacks. Student motivation determines whether or not students engage in a particular pursuit and is affected by their beliefs about what is important. Motivation can have a large effect on learning. As Sullivan (2011a, p. 55) stated: “... low-achieving students are particularly at risk in so far as their inappropriate motivation may inhibit their learning opportunities”.

Teachers are not always successful at determining their students’ motivational beliefs or predicting what engages them or does not engage them. This is often because students are very good at hiding their true motivational beliefs and because teachers misinterpret some cues (Bobis, 2012). Also, it is important to note that students can hold multiple goals simultaneously; it is possible for a student to be both mastery-approach-oriented and performance-approach-oriented. Such a student truly wants to learn and master the material but is also concerned with appearing more competent than others.

Engagement with mathematics

Engagement refers to the extent and willingness of students to undertake and try to complete mathematical activities. It is based on thoughts, behaviours and actions and incorporates behavioural, cognitive and affective traits as shown in Figure 3.7.
It is believed that engagement occurs in mathematics when students: enjoy and value learning it; see it as relevant in their own lives (now and in the future); and see connections between what they learn at school and the mathematics they use outside school. As Sullivan (2011b, p. 2) stated:

Students truly “get” math when they see it applied in real-life ways they care about—in other words, when they see math as a tool they need and want. Students may feel that they can’t control their learning in mathematics. If such feelings and thoughts are allowed to continue, the student may eventually disengage by giving up mathematics altogether.

Teaching ideas

The following are ideas that educators believe will improve student attitude, motivation and engagement. Improvement comes from making these suggested activities interesting and providing students with situations where they are supported and achieve success.

1. Find ways to encourage memories of positive school experiences, and use those memories to activate students’ motivation. Discuss some of these positive experiences with your students, talking about how and why their attitudes toward maths changed for the worse at some point before they started in your class.

2. Be aware of what stimulates students’ interest in mathematics so students maintain engagement, even when faced with difficult mathematical problems. Ensure students see setbacks as temporary and specific.

3. Encourage students to gain some control and manage their learning by having them set personal goals that are achievable, accompanied by some specific task management strategies, such as breaking down problematic tasks into achievable chunks.

4. Provide tasks at an appropriate level of challenge to prevent boredom (but not be outside of ability) and that are considered by students to be purposeful (so they hold their interest). Include active learning situations involving concrete materials and/or games and tasks that require students to take the mathematics out of the classroom and into the school playground. In the words of a student interviewed by Johnston-Wilder and Lee (2010, p. 7): “if we’re actually up and active then we’ve got the energy to do something, but it still sort of processes more”.

5. Use a variety of rich and challenging tasks that allow students time and opportunities to make decisions, have many possible solutions and engage students in productive exploration, and use a variety of forms of representation. Such tasks enhance motivation through increasing the students’ sense of control. Having some control over their learning is empowering for students.

6. Build on what students know, mathematically and experientially, including creating and connecting students with stories that both contextualise and establish a rationale for the learning. Question students to clarify
their thinking or consider their next step; model questioning skills to the whole class, with students subsequently practising the skills in pairs or groups. Allow students to talk to each other about their work, to work cooperatively to achieve success in mathematics, but also provide opportunities to work individually. As students interviewed by Johnston-Wilder and Lee (2010, p. 9) stated:

*I think just because like you’re normally just sat there doing like in a book you just write down and you don’t really say anything, when you’ve actually got to say [speak], you’ve got to think about what you’re saying. ... with teachers, people normally just say oh yeah I understand that, but with your friends you’ll just say if you don’t understand it, and with your friends, if you don’t understand it you can keep asking them ’til you definitely know you know how to do it ... you can like get on with your own work and figure it out for yourself. So you’re not bored.*

7. Provide a positive and caring learning environment in the classroom and at home. Encourage parents to act as “maths allies” and to find ways to integrate real-world maths into their child’s hobbies and interests. Explain that the first step to maths success is a positive attitude towards the subject matter, not just to the grades associated with it.

### 3.2.2 Self-confidence and self-efficacy

There are a variety of “self-something” affects. We have labelled this section with two of them, self-confidence and self-efficacy, but relate two others to them: self-esteem with self-confidence and self-actualisation with self-efficacy, as shown in Figure 3.8.

![Figure 3.8 “Self” affects needed in mathematics](image)

**Self confidence and self-esteem with respect to mathematics**

Self-confidence is the feeling of trust in one’s abilities, qualities, and judgement with respect to mathematics, a measure of belief in ability to do and learn mathematics. Self-esteem reflects overall subjective emotional evaluation of own worth. It is a judgement of oneself as well as an attitude toward the self.

Low self-confidence and esteem with respect to mathematics is often a trait of low-achieving students – they come to believe that they cannot learn mathematics. Low self-confidence and esteem can be the cause of low achievement as well as its consequence.

**Self-efficacy and self-actualisation with respect to mathematics**

Self-efficacy refers to belief in capabilities to learn or perform in mathematics. Self-efficacy influences academic motivation, learning, and achievement and is one of the key issues for teaching mathematics. It refers to belief in capacity to execute behaviours necessary to produce specific performance attainments (Bandura, 1997). Self-efficacy reflects confidence in ability to exert control over motivation, behaviour, and social environment. Self-actualisation is a concept from the theory created by Maslow (1999) and other psychologists. It represents fulfilment of one’s full potential and satisfying those physical, social and intellectual needs that give life purpose and meaning.
**Teaching ideas**

Positive self-confidence, esteem and efficacy (and even actualisation) are built around giving students success. Being able to successfully do and learn mathematics improves all the “selves”. However, this may not be easy because the success often requires effort as well as well-chosen activities. Students should know they have teachers’ respect but need to be encouraged to learn that challenge and effort enhance self-esteem and are not threats.

It is important to realise that teaching is not just the effective presentation of content, but a means of ensuring students have a positive self-concept that will allow them to persist long enough to overcome at least the immediate challenge they confront. Many students avoid risk-taking and do not persist through challenges confronting them in completing a task. Teachers can sometimes avoid the challenge of dealing with students who have given up, by reducing the demand of the task, but this can become evident to the students.

Teaching ideas to enhance self-confidence and self-efficacy include:

- activities that built positive attitude, motivation and engagement from subsection 3.2.1;
- carefully designed active instruction accompanied by high expectations; this communicates to students that teachers expect them to learn and can become a self-fulfilling prophecy;
- ensuring students do not repeatedly experience failure and organising them to set personal goals that are achievable; and
- organising lessons so students feel they have some control over their learning (this can be as little as choosing which exercises they will do) and using this to build self-belief.

As Sullivan (2011b) argued, classroom culture needs to build community, encourage effort and acceptance of errors, and not only tolerate, but celebrate, difference.

### 3.2.3 Attribution

**Positive and negative attribution**

This affect is particularly important. It covers what students perceive to be the causes for success and failure and assumes that their understanding of these causes influences their future actions. Psychologist Bernard Weiner developed an attribution theory that identifies four types of cause – ability, effort, task difficulty, and luck (Weiner, 1985). These can be classified along three causal dimensions: (a) locus of control – internal or external; (b) stability – change or do not change over time; and (c) controllability – within a person’s control, such as skills, or outside one’s control, such as luck and the actions of others.

<table>
<thead>
<tr>
<th>Internally perceived locus</th>
<th>Externally perceived locus</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Ability</strong></td>
<td><strong>Effort</strong></td>
</tr>
<tr>
<td>Positive attribution: cause for success</td>
<td>Positive attribution: cause for success (or failure if there was lack of effort)</td>
</tr>
<tr>
<td>Negative attribution: cause for failure</td>
<td></td>
</tr>
<tr>
<td><strong>Chance/Luck</strong></td>
<td><strong>Task difficulty</strong></td>
</tr>
<tr>
<td>Positive attribution: cause for failure</td>
<td>Positive attribution: cause for failure (if task considered “hard”)</td>
</tr>
<tr>
<td>Negative attribution: cause for success</td>
<td>Negative attribution: cause for success (if task considered “easy”)</td>
</tr>
</tbody>
</table>

*Figure 3.9 Student perceptions of causes of success and failure in relation to positive and negative attribution (adapted from Weiner, 1985)*

As shown in Figure 3.9, attribution is considered to be **positive** when students perceive their causes for success are internal or within their control (i.e. ability and effort), and causes for failure are external or outside their control (i.e. task difficulty or luck). Task difficulty is controllable but external to the student, because it is usually assigned by the teacher. Attribution is considered to be **negative** when students perceive their causes for success are external or within their control (i.e. luck or effort), and causes for failure are external or outside their control (i.e. task difficulty or luck).
are external (i.e. luck or task difficulty) and their failure is due to lack of ability, an internally perceived locus. Negative attribution is damaging. Attributing failure to an internal cause such as low ability leads to the expectation that failure will continue in the future, which, in turn, may cause an individual to stop pursuing a task. Because having high ability is often strongly valued, a perceived lack of ability can lead to negative emotions and decreased self-esteem.

Attribution in students can be difficult to change where students perceive a lack of control. Success alone may not work because if students have a negative attribution they will put success down to luck and/or the task being easy and still expect failure as the normal situation. Students need both success and convincing that this success is due to their ability and effort.

The dimension of stability is also directly related to students’ expectancy for success. If a student perceives a cause of their success or failure could change in the future (e.g. if the teacher explains that the student used inappropriate strategies), the student is likely to be motivated to improve. This means that failure can be changed through appropriate communication to students to develop positive attribution (see Teaching ideas below). Negative attribution can lead to three problems as shown on the right:

1. **Performance avoidance** where failure is so expected that students invest little effort in trying to do the mathematics and are then able to use that lack of effort as an excuse.

2. **Learned helplessness** where students feel that nothing can be done – this is the result of consistent negative attribution over an extended period of time. Students feel that mathematics is hopeless and they are helpless; this takes a long time to turn around.

3. **Mathaphobia** where learned helplessness reaches a point where students stop working in mathematics class and do nothing.

Teacher expectations are also important because they act as a self-fulfilling prophecy. If teachers think students can learn (independent of whether the students are learning or not), the students achieve well, and if teachers think that particular students will experience difficulty in learning, then those students do so. Some teachers respond to students experiencing difficulty by providing easier tasks for them, thereby reinforcing low achievement. If teacher treatment is consistent over time, and if students do not actively resist or change it, it can negatively affect motivation, self-concept and attribution.

**Teaching ideas**

Some specific teaching ideas to develop positive attribution in students are as follows.

1. Have high expectations and give specific feedback, focusing on changing attribution beliefs from negative to positive. Explain clearly and in some detail how performance can be improved, as this helps build adaptive attributional beliefs.

2. Demonstrate interest in your students as individuals. To acknowledge that they may have had previous negative mathematics experiences, ask questions they can respond to in a mathematics autobiography, a class discussion, or a private conversation. Once doors reopen that were previously closed by negative feelings, mathematics is revealed to students as an accessible, valuable tool to help them understand, describe, and have more control over the world they live in.

3. Reassure all students that if they want to achieve high grades, they will have opportunities that will allow them to regain some sense of control, such as retests. Retests provide opportunities to re-evaluate answers and make corrections, as necessary. Teachers’ primary goal is to have students learn the appropriate material so they can move forward with an adequate background for success. Incorporating accountability into retesting allows students to build skills related to self-reliance, goal planning, and independent learning. If the original test and retest scores are averaged together, students understand that they remain...
accountable for that first test grade. Retesting takes time but it shows respect to students for their capacity to be responsible and successful learners.

4. Maintain a carefully chosen program that enables success but continually links the students’ success to positive things that are in their locus of control.

5. Point out that simply copying the question will help students build their maths brains, and after they review their homework, they will have more success when they return to problems they copied down instead of facing a blank page.

6. Allow and/or require students to provide evidence of corrective action, such as participating in tutoring, doing skill reviews, or finding textual examples that correctly demonstrate how the type of problem is solved.

7. Capture students’ imaginations. Show the extended values of maths in ways students find inspiring, that is, show students the ways they benefit from mathematics and how it is applicable to their areas of interest.

8. Report cards and grades are often high-stress experiences that remain as strong negative memories. Clearly explain your policies concerning credit for partial work (if a serious attempt was made to solve the problem) and for homework corrections.

3.2.4 Perseverance and resilience

Mathematical perseverance

Perseverance in mathematics is having the confidence and sheer doggedness to keep trying to solve a problem; it is keeping going and not giving up. Sometimes perseverance can enable a lower ability student to do better than a higher ability student.

Mathematical resilience

Resilience in mathematics is the strength of character to keep trying to solve new problems when not succeeding in earlier problems. Mathematically resilient learners approach mathematics with confidence, persistence and a willingness to discuss, reflect and research. Students need mathematical resilience to engage with mathematics when not having success, struggle through problems when solutions are hard to find, deal with barriers and misunderstandings and keep working on mathematical ideas.

It is important not to prevent students from gaining resilience by providing too much help. Students who see themselves as being good at mathematics when at school may not develop mathematical resilience if every time they get stuck on a problem, their teacher helps them to work it out (Wigley, 1992).

Mathematically resilient students:

(a) have the skills they need to decide what a question is asking of them and to function mathematically in the world beyond school, and the willingness to continue their mathematical development as and when needed;

(b) know that, if they think hard, talk to others, read about mathematical ideas and reflect on the information gained, they will be able to make headway with seemingly difficult ideas and problems;

(c) are less dependent on external rewards and persist even when facing challenges and are not vulnerable when extrinsic motivational rewards stop and the mathematics becomes challenging to the point where they may experience less success;

(d) have a reflective and thoughtful stance towards learning mathematics and persevere when faced with difficulties because they know that the more they work at mathematics the more successful they will be; and

(e) adapt positively to the difficulties presented by mathematics, enabling them to be in a position to consider continuing to develop their mathematics beyond compulsory age.
Teaching ideas

The focus of teaching for perseverance and resilience is to enable a positive adaptive stance to mathematics, which will allow learning to continue despite barriers and difficulties, as in the following ideas.

1. Allow students to reformulate problems presented to them. Students generally pose problems at the level where they are most comfortable. The teacher’s challenge is how to move them into a less comfortable stage by proposing more complex tasks, possibly with other students.

2. Refrain from telling students how to solve a problem. Give students time to struggle through a problem. Students who solve difficult problems on their own, without the help of other students or teachers, often gain a better understanding of the maths concepts than their classmates.

3. If students need assistance to overcome a barrier, ask questions to diagnose the particular cause of the barrier (for example, reading, comprehension, the context of the problem, mathematics skills, interpretation of the outcome – see page 40) and limit the assistance to that issue. Then allow the student to resume work on the solution by themselves.

4. Give problem-posing activities. They require students to ask themselves questions, which assists students in exploring the potential of many tasks. This capacity to pose questions is one of the goals of mathematics teaching, and has important elements of strategic competence.

5. Encourage collaborative working where learners support one another in learning. This can help change mindsets and overcome current negative attitudes to mathematics. Support students to know when and where to put in their learning effort.

6. Encourage students to see that learning takes effort but that this effort will result in improvement. Discuss resilience and what it can offer.

7. Seek to engage the whole community of the school in the quest to improve the mathematical attainment of the school. Use non-specialists, as they have much to offer the process of developing confidence in mathematics.

8. Shift in focus from “ability” to “learning strategies” for working at mathematics, taking into account affective aspects of learning such as what mathematics looks like in the adult world of home and work, and what strategies adults have developed to cope with situations involving mathematics.

3.3 Language and literacy in mathematics

We have already seen that mathematics is a powerful succinct language for describing relationships and change. Literacy is a General Capability of the Australian Curriculum. According to the curriculum:

students become literate as they develop the knowledge, skills and dispositions to interpret and use language confidently for learning and communicating in and out of school and for participating effectively in society. Literacy involves students listening to, reading, viewing, speaking, writing and creating oral, print, visual and digital texts, and using and modifying language for different purposes in a range of contexts. (ACARA, 2016, Home / F-10 Curriculum / General capabilities / Literacy / Introduction)

In particular, in the context of mathematics, the curriculum states that:

students learn the vocabulary associated with number, space, measurement and mathematical concepts and processes. This vocabulary includes synonyms, technical terminology, passive voice and common words with specific meanings in a mathematical context. Students develop the ability
to create and interpret a range of texts typical of mathematics ranging from calendars and maps to complex data displays. Students use literacy to understand and interpret word problems and instructions that contain the particular language features of mathematics. They use literacy to pose and answer questions, engage in mathematical problem-solving, and to discuss, produce and explain solutions. (ACARA, 2016, Home / F-10 Curriculum / General capabilities / Learning area specific advice / Mathematics)

Because literacy in mathematics differs from literacy in other learning areas, it must be taught in the context of mathematics. While mathematics teachers draw on the foundations developed in English lessons, it cannot be assumed that literacy skills taught in other contexts are automatically transferred to mathematics. According to the Queensland Curriculum and Assessment Authority (QCAA), “the responsibility for developing and monitoring students’ abilities to use effectively the forms of language demanded [in mathematics] rests with the teachers of mathematics” (QCAA, 2014, p. 37).

This section explains the unique features of literacy in mathematics and provides some strategies for developing the necessary skills. While the term mathematical literacy is used by some as a synonym for numeracy (particularly in the USA), here mathematics literacy is used to describe the different ways of communicating in the subject of mathematics, as shown in Figure 3.10.

This section is based on the supplementary YDM resource book, Literacy in Mathematics, which can be obtained from the YuMi Deadly Maths Professional Learning Online Blackboard site.

An important point to note is that mathematical symbols can be considered as a language that tells stories. However, in this section, we focus on the language alongside the symbols that refers to particular concepts, principles, strategies and procedures.

3.3.1 Listening and speaking

Listening and speaking in the mathematics classroom is critical to the development of mathematical language and an important part of learning foundational mathematics concepts. Language is a social concept developed through interactions with others for communication purposes (Vygotsky, 1986). Language acquisition involves not only a child’s exposure to words but also an interdependent process of growth between thought and language. Children learn through social experiences in their communities. Through these interactions, teachers impart skills, values and knowledge to their students. Vygotsky viewed language as humans’ greatest tool. When directed to others, it is a means for communicating with the outside world and, when used internally as private speech, it is a medium for thinking.

Vygotsky hypothesised that from around three years of age speech and thought are interdependent, with thought becoming verbal and speech becoming representational. He argued that “children solve practical problems with the help of their speech, as well as with their eyes and hands” (Vygotsky, 1978, p 26). It follows that conversation (listening and speaking) or discourse in the mathematics classroom is vital and encourages the dual purposes of language: as a medium for students to develop their mathematical thinking; and as a strategy for teachers to engage students in mathematics as they assess and plan. Teachers need to use conversation to engage students in mathematics and to continue to enable them to converse throughout the stages of the mathematics lesson. Establishing a culture of shared ideas and guiding students as they talk about their mathematical understanding enriches mathematical thinking. Conversation needs to be modelled, taught and revisited many times within the mathematics context/classroom.
Children first learn to communicate mathematically by listening to speech. They then learn to communicate mathematical ideas through speech, for example when counting and identifying fundamental concepts such as more and less. It is the means of developing the lexicon that is the foundation of mathematical communication. The development of the mathematical lexicon in the early years focuses on everyday English words and simple mathematical concepts such as number and shape, so it can be integrated with general vocabulary development. Discussing problems from the students’ perspective provides opportunities to develop real mathematical conversations. Although written forms of mathematical communication begin to be developed as students progress in their schooling, speech remains the most important form of mathematical communication.

Talking gets students involved. The mathematics classroom should be an exciting place when we establish a culture of shared ideas and guide students as they discuss their mathematical understanding. Using the language of mathematics in a meaningful way is an everyday necessity. Mathematics classroom conversations should allow students to converse with the teacher and each other. As with any other language, speaking and listening should be practised regularly.

3.3.2 Vocabulary

The nature of mathematical vocabulary

In the mathematical vocabulary, many words are defined with a greater precision than normal English usage. The mathematical lexicon (vocabulary) consists of three classes of words: those used in everyday English; words with different ordinary English and mathematical meanings (called sub-technical words); and words that have meaning only in the context of mathematics (called technical words; Pierce & Fontaine, 2009). Some types of everyday English words appear frequently in mathematical texts and are crucial to their understanding. They include:

- prepositions such as to, from, under, over, along;
- adjectives such as plus, minus, approximately;
- adjectives with comparative and superlative forms such as long/longer/longest, far/further/furthest, less/lesser/least;
- quantifiers (another form of adjective) including numbers, both cardinal (one, two, three) and ordinal (first, second, third) and other words that show quantity such as dozen;
- logical operators such as and, or, not, if ... then (see also section 3.1.4);
- words derived from other parts of speech; for example, the nouns division and divisible are derived from the verb divide; the adjectives different, differing, differentiable and indifferent, the verb differentiate, the adverb differently, nouns difference, differentiation and differential are all derived from the noun differ (Nippold & Sun, 2008); and
- complex and sometimes contradictory strings of words and phrases, such as lowest common denominator, mutually exclusive, increasing at a decreasing rate, if and only if (Gough, 2007; Halliday, 1990; Spanos, Rhodes, Dale, & Crandall, 1988).

Words with different meanings in ordinary English and mathematics include root, linear, variable, square, power, rational, function. In some cases the mathematical meaning of these words is related to, but more precise than, the ordinary English meaning, for instance, equality, similar. A word such as rectangle has to be strictly interpreted in mathematics, while in English usage it can apply to objects that approximate the mathematical definition. Examples of technical words that have meaning only in a mathematics context include coefficient, percent, median, hectares, binomial, denominator, and vinculum.

Mathematics has many words with irregular plural forms, such as die/dice, or no plural form, such as dozen. Often they originate from Latin or Greek: axis/axes, radius/radii.
These examples show that the task of developing a comprehensive mathematical vocabulary in students is both extensive and complex. The strategies used for teaching mathematical vocabulary are influenced by the complexity of the meaning of the word. Words that have relatively straightforward meanings, for example triangle, do not require a lot of teaching effort. However, some words have complex, varied and/or conceptual meanings that must be developed carefully, for example, infinity, limit (where a sequence of numbers gets closer and closer to another number without ever reaching it), point (having no dimension), independent (both variables and events), and proof.

Pedagogical strategies that can be used to develop or deepen an understanding of a word are discussed in more detail in the supplementary YDM resource Literacy in Mathematics. However, it is important to note that strategies such as writing a word on the board, explaining its meaning, and/or providing a glossary, are insufficient for developing a deep understanding of the mathematical vocabulary.

**Spelling**

Spelling is an important part of knowing a word and is usually introduced after the word is used in speech. Minor variations in spelling can change the meaning of the word (for example, minutes/minus, complement/compliment, principle/principal). Homophones such as two/to/too, sum/some, pi/pie and sign/sine can be a source of spelling confusion for some students (Durkin & Shire, 1991). In the upper primary years, mathematical words should be included as part of normal spelling activities. In secondary classrooms, mathematics teachers should periodically test students’ spelling of mathematical words (possibly as a lesson starter, using NAPLAN-style questions). Incorrectly spelled words (whether mathematical or not), wherever they occur, must be corrected by the teacher. Pedagogical strategies for teaching spelling are discussed in more detail in the supplementary YDM resource Literacy in Mathematics.

**The teacher’s dilemma**

Mathematics teachers often confront a dilemma between, on the one hand, using vocabulary that students can understand easily and, on the other hand, using the appropriate mathematical terms. The QCAA advises the following.

> When writing, reading, questioning, listening and talking about mathematics, teachers and students should use the specialised vocabulary related to the subject. Students should be involved in learning experiences that require them to comprehend and transform data in a variety of forms and, in so doing, use the appropriate language conventions. (QCAA, 2014, p. 37)

Abedi (2009) found that it is important for students to be continually exposed to the value and richness of the mathematical lexicon. However, there is a fine line between vocabulary that is an essential part of the mathematical content and vocabulary that adds unnecessary complexity. If students are unlikely to understand the essential mathematical vocabulary, teachers have the option of (a) pre-teaching the relevant vocabulary; or (b) providing explanations when the unfamiliar word(s) arise. Regardless of the strategies used to teach vocabulary, time invested in developing students’ understanding of a rich mathematical lexicon will yield dividends when the literacy focus is extended to reading and writing.

### 3.3.3 Symbols

Mathematics uses symbols as a concise and coherent way to describe the world around us. For example, symbols such as $2 + 3 = 5$ describe an operation ($+$) which acts on two numbers to give a third, which can be used to represent many real-life situations, such as two ducks on a lake being joined by three more to give five ducks.

There are many types of symbols commonly used in mathematics:

- numerals (that can be combined in different ways to represent numbers of various types);
- pro-numerals (variables, unknowns and parameters);
• those that represent operations;
• grouping symbols;
• those used to compare values (equality and inequalities); and
• abbreviations (mathematical, units of measurement and geometric).

The range of mathematical symbols is discussed in more detail in the supplementary YDM resource *Literacy in Mathematics*.

The symbolic language of mathematics can describe complexities that are difficult to describe in everyday language. Algebra is an important part of the symbolic mathematical language. For those familiar with it, complex things become simple. For instance, \(2x + 5 = 13\) can be interpreted as:

\[
\begin{align*}
x & \quad \text{(unknown)} \\
\times 2 & \quad \text{(multiply by 2)} \\
+ 5 & \quad \text{(add 5)} \\
= 13 & \quad \text{(end up at 13)}
\end{align*}
\]

Understood in this way, \(2x + 5 = 13\) can be solved by quite young students: *I am a number. Multiply me by 2. Add 5. I am now at 13. What number was I?* The solution is reached by reversing or *backtracking* the activity as shown below:

<table>
<thead>
<tr>
<th>START</th>
<th>FINISH</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n)</td>
<td>(2n)</td>
</tr>
<tr>
<td>Forward</td>
<td>(\times 2)</td>
</tr>
</tbody>
</table>
| Reverse | \(\leftarrow 2\) | \(\rightarrow 5\) | \(13\)

Algebraic symbols, when understood, can simplify the presentation of many problems, plus help us to see what is wanted and provide the solution, as the following three examples show.

**Example 1 (language to symbols):** A problem is translated to symbols by using \(x\) to represent the starting number:

“I was a number. I was multiplied by 7, 12 was subtracted from me and then I was divided by 4. I am now at 25. What number was I?”

\[(7x - 12) \div 4 = 25 \quad \text{or} \quad \frac{7x - 12}{4} = 25\]

**Example 2 (language to symbols to solution):**

Two-thirds of 15

\[
\begin{align*}
\text{Take the group (15),} \\
\text{divide it into 3 equal parts (each 5), and consider two of the parts (10)}.
\end{align*}
\]

\[
\frac{2}{3} \text{ of } 15 = 10
\]

**Example 3 (language to solution via symbols):** One-quarter to the power of a negative half is 2

\[
\begin{align*}
\text{4 is the square of 2; the negative (-) power means} \\
\text{the } \frac{1}{4} \text{ becomes 4; the power, } \frac{1}{2} \text{ means square root. The answer is therefore 2.}
\end{align*}
\]

\[
\frac{1}{\left(\frac{1}{2}\right)^2} = 2
\]
3.3.4 Visual images

Mathematics is not only composed of words and symbols; it is also a pictorial language that uses visual models to communicate. From a mathematics perspective, a visual image is a print or electronic (digital) picture or representation of something or someone.

Visual images are an important form of mathematical communication, in school and extending into post-school education, training and employment. The expectation that students of mathematics should be able to encode and decode visuals that provide both quantitative and qualitative information requires a broad approach to developing strategies for using these visuals. Creating visual images in mathematics requires students to think visually, that is, to visualise. They need to explore what visual model a word conjures up for them and teachers should provide frequent opportunities to visualise and construct.

Students are likely to encounter a wide variety of visual images in mathematics. The extent of this variety makes it difficult for students to learn about and practise every possibility. It follows that it is neither practical nor likely to be successful for teachers to try to teach every variation or type of visual image. The traditional approach to teaching visuals is by purpose, with little transfer of knowledge between contexts. For example, number lines, linear measuring scales, measuring gauges, and timelines all rely on the interpretation of a scale in a single dimension, but they are often taught separately in mathematics and also in different learning areas, implying that they differ from each other. However, it is the characteristics (properties) of a visual (e.g. scale, direction, shape, colour) that primarily determine how it is decoded. A teaching approach that focuses on the properties of visual images is more likely to assist students in making meaning and in transferring knowledge between visual images with similar properties (Hipwell, Carter, & Barton, 2018). This is explained further in the supplementary YDM resource Literacy in Mathematics.

The nature of a visual image determines how it is used and how we write about it. Mathematical visual images can be classified into three broad groups (Hipwell at al., 2018):

- **tables** that provide information in a compact, summarised form;
- **informative** visual images that show factual information, including graphs of all types, number lines and Cartesian planes, timelines, all types of maps, plans and blueprints, scale drawings, geometric diagrams, Venn diagrams, flowcharts, tree diagrams, hierarchies and networks; and
- **narrative** visual images that are used in many situations to add interest and tell a story, for example, illustrations, photographs and sketches.

The purpose and properties of a visual image determine how they are used.

3.3.5 Reading (decoding)

**The nature of mathematical texts**

In this section a text is anything that can be read or analysed. It is not a random accumulation of information, but has been presented in a considered way to achieve a particular purpose. Mathematical texts are harder to decode than most other texts. First, the reader must be able to interpret the words. This requires an understanding of the mathematical lexicon, but also the ability to negotiate the complex syntactical (grammatical) structures often used in mathematical texts, including abstract and impersonal presentations (Abedi, 2009; Abedi & Lord, 2001). Second, the reader must be familiar with any symbols used, the conventions of presenting information in tables, and the wide variety of visual images used to present quantitative information (Newman, 1983).

The concise nature of mathematical texts adds to the problem of decoding. A lot of information is often packed into few words (called lexical density). Unlike other texts, there is little textual redundancy. Textual redundancy occurs when the same information is presented in several ways so that the meaning of an unfamiliar word can often be deduced from the surrounding text. In many mathematical texts, the inability to understand a key word can affect the understanding of the entire text. As Orton (2004) stated, “mathematics texts generally cannot be read quickly, for every word and every symbol is essential to the extraction of meaning” (p. 161).
School students generally encounter written mathematical texts in four situations: (a) teacher presentations, usually accompanied by verbal explanation; (b) written notes, often in a textbook or handout; (c) written instructions; and (d) worded problems. The latter two provide the greatest challenge for students because they are often required to interpret them without assistance.

**Interpreting written instructions**

Written instructions generally use task words, such as *calculate, simplify, verify, define, evaluate*, to tell students what to do (Carter & Hipwell, 2013). Some task words are synonyms, for example, *show and demonstrate*. Others may appear to be similar, but have important differences, for example, *approximate and estimate*. Some task words are used frequently in mathematics, for example, *calculate and simplify*, and some have particular mathematical meanings, for example, *evaluate*. It is critical that students know and can use the task words used regularly in mathematics. This is a process that starts in the earliest years of primary school with words such as *add and write* and continues to the end of secondary school with the higher order skills such as *analyse, assess and generalise*.

**Interpreting worded problems**

Decoding the text of worded problems is challenging. Information is presented in words, but there may also be symbols, tables and visual images. Studies have found that up to 64% of errors in mathematical problem solving can be attributed to errors in the interpretation of the problem text (Cummins, Kintsch, Reusser, & Weimer, 1988; De Corte & Verschaffel, 1985). The importance of teaching students how to interpret a problem text cannot be overstated.

Mathematical problem texts should be read three times. When first encountering a problem text, it should be **scanned** (read quickly) to gain an overall feel for the problem. Scanning is followed by **close reading** to gather the required details from the text. Finally, once a solution has been found, the text should be **reread** to ensure that the proposed solution fully answers the question.

During the close reading stage, students must locate all relevant information. There are four ways of extracting information from a problem text (Raphael, 1982, 1986) as shown in Figure 3.11.

![Figure 3.11 Four ways of extracting information from a problem text (Raphael, 1982, 1986)](image)

Questions students should ask of themselves as they read a worded problem text include:

- What facts am I given in the text? [often easily identified as numbers]
- What am I asked to do? [look for the question mark]
- What additional facts do I know that are relevant? [‘author and you’ approach]
- Can I summarise the information in a table or diagram?
- What processes can I use?
If the problem involves multiple steps, it can be difficult to process all the information at once. Students may have to read the text more slowly, or several times, highlighting or underlining important sections. It may help to break the text up into meaningful chunks and phrases. The burden on short-term memory may result in earlier information being forgotten by the time the student reaches the end of the problem (Barton & Heidema, 2002). In such cases a graphic organiser may assist students in sorting through the information in a problem text.

Some conventions about mathematics problem solving are summarised in the four “rules” (or assumptions) of worded mathematical problems (Kintsch & Greeno, 1985) below:

- Only the relationship between, and changes to, mathematical ideas is important. Other information such as names and context are mathematically irrelevant.
- All values are assumed to be exact, unless modified by words such as approximately, at least, more than.
- All relevant information is assumed to be in the text of problem, so the reader does not have to consider whether extra information is needed.
- The problem has to be solved within the bounds of the question, no matter how unrealistic.

The first of these rules says that contextual information is irrelevant. That is only partially true. Contextual information is used in a mathematical problem to give it authenticity. The context also provides clues as to the mathematical processes required. Additionally, while names and contexts may be unimportant during the calculation stage, any answer has to be interpreted and presented in the context of the original problem.

In the artificial environment of the mathematics classroom, the data in a problem is usually limited to the information relevant to the solution. This can provide valuable clues as to how to solve the problem (if you have not used all of the information given, then you must have made a mistake). Students should occasionally be exposed to problem texts that do include irrelevant information, including illustrations.

Rules 3 and 4 represent a cynical view of problem solving in mathematics classes. However, they apply more often than we may care to admit, especially in examinations. We might expect students to introduce some memorised factual knowledge (for example, the number of days in a week), but they are not generally expected to collect additional data to solve a problem (nor are they able to do so in a closed examination). More importantly, in the case of rule 4, we do not expect students to introduce additional information, even if it is realistic. For example, when calculating the number of trips required to take 33 people upstairs in an elevator if the maximum load is eight people, students are not permitted to make the reasonable assumption that at least one person would choose to use the stairs rather than wait for the fifth trip of the elevator (B. Cooper & Dunne, 2000). These assumptions should be discussed with students in the context of interpreting mathematical problem texts.

**Diagnosing student decoding problems**

Where students are unable to correctly respond to a worded mathematics problem, Newman (1983) proposed five questions that could be used to diagnose the nature of the error.

<table>
<thead>
<tr>
<th>QUESTION</th>
<th>DIAGNOSIS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Read the question aloud to me.</td>
<td>Difficulty with basic reading.</td>
</tr>
<tr>
<td>What does the question ask you to do?</td>
<td>Unable to understand what has been read.</td>
</tr>
<tr>
<td>How are you going to find the answer?</td>
<td>Unable to determine the process(es) needed to solve the problem.</td>
</tr>
<tr>
<td>Show me how to get the answer. As you do it, talk aloud about what you are doing.</td>
<td>Unable to apply the process(es) needed to solve the problem.</td>
</tr>
<tr>
<td>What does your answer mean?</td>
<td>Unable to interpret the answer in the context of the problem.</td>
</tr>
</tbody>
</table>
The first two questions diagnose decoding issues. Asking these five questions every time a student needs assistance ensures the assistance is targeted appropriately. Students should be helped to overcome only the diagnosed weakness and then be allowed to continue with the rest of the solution independently.

3.3.6 Writing (encoding)

Writing is the process of composing a text. In the case of mathematics, the texts are informative. There are two important aspects of writing in mathematics: the mathematics must be accurate and the writing must be grammatically correct (Carter & Hipwell, 2013).

It is easy for experienced writers to underestimate the challenges for beginning writers. The process of writing is complex and requires control of eight processes:

- identifying rhetorical positions, such as purpose, audience, context, and stance;
- marshalling the required ideas and knowledge;
- planning what and how the ideas and knowledge will be presented;
- using the codes and conventions of written language;
- balancing processes strategically;
- monitoring the writing process;
- reading to review the text; and
- editing the text by correcting and revising.

Students can become overwhelmed if they attempt all eight processes simultaneously, especially in extended writing tasks. Teachers can assist developing writers by separating the processes into steps that can be dealt with sequentially, as shown in Figure 3.12. More detail of the strategies for teaching writing is provided in the supplementary YDM resource *Literacy in Mathematics*.

![Figure 3.12 Steps involved in mathematical writing](image)

3.3.7 Teaching ideas

**Pedagogical strategies**

Writing, as we have said, is a complex process. Effective writers take years to develop. Scaffolding is required in the early years, gradually withdrawn as the students gain experience. Writing tasks should initially be structured and brief, becoming longer and more open-ended over time, for example:

- Fill in the missing information in the sentence(s) (cloze)
- Complete the sentence
- Rephrase this sentence to make the answer
- Write a sentence about …
- Write a paragraph about …
- Write several paragraphs about …

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Teachers should require students to present ideas in sentences and paragraphs regularly and use these opportunities to explicitly teach the skills of mathematical writing.

Students need regular feedback on their writing and teachers need to resist the temptation to start by getting out the red pen and correcting the spelling errors. Instead, use the same process to provide feedback as the student followed in writing, starting with the big picture and then gradually zooming in to the detail:

- Audience and purpose
- Ideas and planning
- Structure and sequence
- Conformity to conventions of mathematical writing
- Language use (spelling, grammar, punctuation, etc.)

**Mathematics writing tips**

Mathematics teachers can begin to incorporate writing into their classrooms following these tips.

1. **Start small.** If you currently do no writing in your maths classes, start with just one sentence. When you ask a question, rather than just having students raise hands and you picking one volunteer, have every student in the class write a sentence. This way, students can be active and can get the feel of writing about maths.

2. **Read in maths class.** Have students read articles from *Math Horizons*, published by the Mathematical Association of America (http://www.maa.org/press/periodicals/math-horizons), or Martin Gardner’s *Scientific American* columns (https://www.scientificamerican.com/author/martin-gardner/). When students get a feel for what writing about maths can look like, they will be able to do it better themselves.

3. **Have students keep a maths journal.** In their maths journals, students can write about things they are having trouble with or things that they’ve figured out. Putting these thoughts into words can help students get a more concrete handle on the logic of their ideas.

4. **Do some of your own writing.** To understand the writing process, teachers have to write also. Start by writing about your family or yourself. Get “uncomfortable” as you try to make your writing efficient and interesting. Just as students are uncomfortable doing unfamiliar maths, teachers should experience that same feeling of being out of their comfort zone, and writing is a good way to do that.

5. **Just jump in and start.** Encourage students to put the pen to hand or the fingers on the keyboard and just start writing. They could write about the maths that’s confusing to them or about that “Eureka!” moment that led to a solution.

6. **Learn from other specialities.** Ask teachers who specialise in language arts for ideas (and remember, mathematics may have some writing ideas to share as well).

7. **Write about big ideas.** Get students to write about the big picture or a mathematical big idea. Give some examples that represent a big idea and ask students to describe what big idea they think the examples demonstrate. Then reverse this and give students a big idea and ask them to figure out and write about some examples of it.

8. **Do simple obvious writing.** For example, ask students to write about the properties of a rectangle. Writing their solutions to creative problems (see section 3.1.3) gives students the opportunity to write mathematically without being concerned with calculations.

9. **Use writing as a learning tool.** For example, confusion about perimeter and area can be assisted by getting students to write about the two and discuss similarities and differences.

10. **Use writing tasks as assessment.** For example, ask students to write all they know about fractions.

We cannot assume that students will learn to write about mathematics in other subjects. If students are required to read or write mathematical texts in assessment, then these skills must be developed in mathematics lessons before the assessment occurs.
3.4 Implications for future programs

If you continue to follow YDM, your school’s future program will be based on the pedagogy and the teaching approaches of the previous seven YDM books, and will be built around effective teaching of Number and Algebra, Measurement and Geometry, and Statistics and Probability. However, it will also be important to cover proficiencies, affects, and language and literacy. Figure 3.13 summarises the ideas covered in this chapter on these three areas in preparation for analysis of what is important.

**Figure 3.13 Proficiencies, affects, language and literacy needed for a school mathematics program**

| PROFICIENCIES | • Understanding  
|               | • Fluency  
|               | • Problem solving  
|               | • Reasoning  

| AFFECTS | • Attitude, motivation and engagement  
|        | • Self-confidence and self-efficacy  
|        | • Attribution  
|        | • Perseverance and resilience  

| LANGUAGE AND LITERACY | • Listening and speaking  
|                       | • Vocabulary  
|                       | • Symbols  
|                       | • Visual images  
|                       | • Reading (decoding)  
|                       | • Writing (encoding)  

**Your task from this chapter is to develop a plan for taking account of proficiencies, affects and language/literacy.**

Look at the summary in Figure 3.13 and determine what your emphases would be in your school if you could implement what you want. Following the steps below may help.

1. Consider your school and prepare a table giving your perception of your school’s current situation with regard to proficiencies, affects and language.

2. List proficiencies, affects and language that you would like to be the focus of your activity – make three groups – *major* (needs attention at once); *important* (will need attention after major); and *secondary* (can be focused on when major and important are under control).

3. Beside each item in your list, write the actions you would implement to make these listed areas positive for your school (use the ideas in this chapter).

4. Develop a program of activities for the year levels in your school for each of the proficiencies, affects and language ideas, using a table as shown in Table 3.1 on the next page.
<table>
<thead>
<tr>
<th></th>
<th>YEAR P–2 ACTIVITIES</th>
<th>YEAR 3–4 ACTIVITIES</th>
<th>YEAR 5–6 ACTIVITIES</th>
</tr>
</thead>
<tbody>
<tr>
<td>PROFICIENCIES</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Major</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Important</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Secondary</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AFFECTS</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Major</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Important</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Secondary</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LANGUAGE</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Major</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Important</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Secondary</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Introduce this table or program with a general discussion of your school and complete it by describing why you have set up the program that you have. Describe your school in terms of proficiencies, affects and language, showing awareness of the needs of your school, and justify what is included and not included in your program.
4 Assessment, Diagnosis and Remediation

An important part of teaching is being able to tell whether students are learning or not learning what is being taught. Teachers are required to continuously diagnose what students know in order to make decisions about future instruction. For example, do students know enough to move on to the next topic? Do we need to reteach any parts of the old topic? Is the cause of the difficulties something that should have been learnt some years before? Do we need to revisit past topics and rebuild them? Student difficulties need to be diagnosed and remediated through assessment and teaching.

YDC’s previous research and collaboration with schools revealed that many students entered secondary school underperforming in mathematics. These students were unable to progress in mathematics and tended to disengage from school. For this reason, YDC developed an Accelerated Indigenous Mathematics project for underperforming Indigenous schools and this became Accelerated Inclusive Mathematics training for other schools (both titled AIM). A similar accelerated learning project called XLR8 was developed for Brisbane secondary schools as part of an Australian Research Council Linkage project. The purpose of these projects is to accelerate the learning of these students across junior secondary so they can rejoin their class level with knowledge sufficient to handle mathematics subjects in Years 10 to 12. In these contexts, YDC uses the term accelerated learning to refer to the process of learning at a faster pace so students can catch up to the curriculum expectations for their age, often referred to as remediation. This differs from the use of the word accelerated by others to refer to moving talented students beyond the curriculum expectations for their age.

Students can underperform in all year levels, so it is important to have processes that accelerate the learning of those students so they can reach curriculum expectations. YDC has now developed AIM-style materials for the early years, covering Foundation to Year 2 content and the learnings that usually occur in the years before starting school. This set of materials is called AIM Early Understandings (EU). The materials use the same module structure and process as the junior secondary AIM materials, but with different content.

An important component of all YDC’s AIM project materials is diagnostic testing through pre- and post-tests to determine if the acceleration is working. The AIM tests are based on earlier work with diagnostic testing called Cognitive Diagnostic Assessment Tasks (CDAT). These tasks were developed by YDC staff to help teachers determine the knowledge held by their students so the teachers could make teaching decisions.1

This chapter is divided into four sections:

(a) Assessment (4.1) – forms of assessment (formative and summative), how classroom assessments improve teaching, and how classroom assessments improve learning.

(b) Diagnostic testing (4.2) – the theory behind cognitive diagnosis and the techniques for administering assessment.

(c) Remediation through acceleration (4.3) – acceleration structure and curriculum implications of acceleration.

(d) Implications for future teaching (4.4) – how to set up sustainable ways for assessment, diagnosis and remediation to operate in your school.

1 Information on AIM and AIM EU can be obtained from the YDC website under YDM accelerated learning projects. The CDAT book, Developing Mathematics Understanding Through Cognitive Diagnostic Assessment Tasks, can be downloaded from the Student learning resources page of the website.
4.1 Assessment

Both formative and summative assessment can be used to inform and improve teaching and learning. The following subsections define these two forms of assessment and provide ideas for using assessments in the classroom.

4.1.1 Forms of assessment

**Formative and summative assessment**

Assessments are used for a variety of purposes (a 2008 UK House of Commons Children, Schools and Families Committee report lists 22(!) purposes of assessment), come in a variety of forms (e.g. supervised tests, open-ended group tasks, portfolios, performances) and are held at different times (e.g. the end of the school year, when the teaching of a topic has been completed, or some other date determined by external education authorities). The purpose, form and timing all affect the nature of the assessment. Assessments designed for ranking are often held at the end of the year (just before students move on to another a new teacher) and contain a broad range of questions from simple to very challenging so that every student can be distinguished from the others and placed on a continuum. They are generally not good instruments for diagnosing student learning for teachers to improve their instruction or modify their approach to individual students. Assessments can be a vital component in improving education, but if they are used only as a means to rank schools and students, their most powerful benefits as indication of needed teaching will be missed (Guskey, 2003).

Diagnostic assessments require questions that focus on the knowledge and skills that have just been taught or that are prerequisites for what is about to be taught. They also need to be scheduled while students are learning new content so that the teacher has the opportunity to provide additional instruction, if needed. Stiggins (2004) argued for a more balanced approach to assessment, using assessment of learning and assessment for learning. This approach uses assessment both to actively and continuously measure a learner’s progress and to obtain useful data to inform a teacher’s instructional practice (Stiggins, 2004). Thus, school assessment is commonly used to obtain evidence of students’ learning for two purposes (Popham, 2013):

- **Formative assessment** – evidence is used by teachers to adjust their ongoing instructional procedures or by students to adjust their current learning approaches; and
- **Summative assessment** – evidence is used to make success/failure or ranking decisions following completion of instructional activities.

According to Marzano (2007), most researchers consider that formative assessment might be one of the most powerful weapons in a teacher’s arsenal. He also argues that the frequency of formative assessment is related to student academic achievement.

The US Center for Comprehensive School Reform and Improvement (CCSRI, 2008, p. 1) argued that assessment is vital to effective teaching:

> Assessment can be one of the most difficult aspects of teaching. The educational, emotional, and formative ramifications of judging a young person’s work can weigh heavily on the mind of a teacher. In spite of the anxiety it poses, knowing how to assess students in order to improve instruction is a core principle of effective teaching.

CCSRI (2008, p. 1) contended that “collecting data on student understanding is an essential step in moving students toward full understanding of important concepts and standards”. However, often students see assessment “as a means of competing with classmates for the highest grade instead of as a mile marker on the journey to increased knowledge and understanding” (CCSRI, 2008, p. 1). YDC is supportive of a predominantly...
formative role for testing, particularly for low-achieving students, because regular gathering of information on student progress empowers teachers and students to improve achievement.

Changing teaching practice

Within the YDM approach, an important role of both formative and summative assessment of students is to bring about change in teaching practice. Student achievement should be used to guide continual improvement of teaching using an action research approach. This is the focus of YDC and the basis of the implementation of YDM into schools. Therefore, assessment should be built in to teaching to maintain sustainable effective mathematics teaching and learning.

Interpreting assessment information, understanding the implications for practice, and learning how to teach in different ways in response to that information is complex and requires:

- knowledge and skills in the use of assessment data;
- interpreting what assessment data indicates about the changes needed to teaching practice;
- knowledge and skills to check the impact of any changes to pedagogy, creating a cycle of inquiry; and
- knowledge of the curriculum and how to teach it effectively.

School leaders must know how to lead the kinds of change in thinking and practice required and provide opportunities for teachers to acquire the new knowledge and skills in the curriculum, pedagogy and assessment.

4.1.2 How classroom assessments improve teaching

Data for instruction

The best classroom assessments serve as meaningful sources of information for teachers, helping them identify the effectiveness of instruction – what worked, what did not, and what to do next. This can begin very simply – just with records of how students handled the various test items (questions). The following marking rubric could be used (Marzano, 2007):

<table>
<thead>
<tr>
<th>Description of Learning Outcome</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>The student has gone beyond the content and skills taught by the teacher</td>
<td>4</td>
</tr>
<tr>
<td>The student has achieved the learning goals, unassisted</td>
<td>3</td>
</tr>
<tr>
<td>The student understands the simpler content and/or can do the simpler aspects of the task</td>
<td>2</td>
</tr>
<tr>
<td>The student cannot demonstrate understanding of the learning goals unassisted</td>
<td>1</td>
</tr>
<tr>
<td>The student cannot demonstrate understanding of the learning goals, even with help</td>
<td>0</td>
</tr>
</tbody>
</table>

This data can be recorded in a table or spreadsheet to cross-reference students by test items. The scores for each student give an indication of individual student performance across the test and the scores for each test item give an indication of the whole-class performance in each item. This enables the teacher to determine which students are doing well and who needs further help, and the items that a significant number of students are failing. This latter information may indicate that extra instruction is needed and/or that instruction needs to be improved (see Figure 4.1). Generally, if fewer than half of the students in the class achieve a level 3 or better (that is, correctly answer a question), then the teacher’s method of instruction needs to change/improve or the question needs redesigning.

![Figure 4.1: Assessment to determine the focus of teaching](image-url)
Assessments should include some items that give students the opportunity to demonstrate the highest level of understanding (level 4) and should reflect the concepts and skills that the teacher emphasised in class. There should be no “gotcha” questions about isolated or obscure concepts. Some criticise this as “teaching to the test”. However, if what is taught aligns, in both content and emphasis, with curriculum standards, teachers are testing the curriculum. These assessments are fair measures of important learning goals (CCSRI, 2008).

**Activities and analytic points**

Overall, teachers and students share responsibility for learning; teaching effectiveness is not defined on the basis of what teachers do but rather on what their students are able show that they know and can do. However, we can use student assessment results to gain insight into teacher capacity. To ensure we get the best data and the best outcomes, the following have been suggested by educators (Guskey, 1998, 2000, 2003; Timperley, 2011):

1. Focus on the links between particular teaching activities, how different groups of students respond to those activities, and what the students actually learn. Use a variety of teaching techniques to cover all student responses.

2. Provide students who have few or no learning errors to correct with enrichment activities to help broaden and expand their learning. At the same time, provide the students who need additional time with follow-up instruction. Pairing high- and low-achieving students for a cooperative activity can benefit both learners as well. In addition to the obvious benefits for those receiving assistance, those providing the assistance also benefit through structuring their own thinking, internalising learning intentions and reflecting on success criteria in the context of another’s work.

3. Follow up with instructional alternatives that present the concepts in new ways and engage students in different and more appropriate learning experiences.

4. Use a range of methods to assess students informally and formally. Judging impact requires the use of assessment information on a daily, term-by-term and annual basis (see also section 4.1.3). Remember that the frequency of formative assessment leads to higher student achievement.

School leaders should offer structured professional learning and collaboration opportunities to help teachers share strategies and collaborate on teaching techniques.

It is important not to allow minor errors to become major learning problems. In not doing this, teachers better prepare students for subsequent learning tasks, eventually needing less time for corrective work and proceeding at a more rapid pace in later learning units (Guskey, 2003; Whiting, Van Burgh, & Render, 1995). Most important is to actively seek evidence of students’ failure to learn, use this evidence to improve and change teaching, then further seek evidence of the effect of the change (Hattie, 2012), as shown in Figure 4.2.

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**Questions**

Teacher reflection on testing and test responses is important. Use the test and the responses to ask the following questions (Bovell, 2014, Assessment/Literacy/School page):

- Which students have not achieved the learning goals?
- How is my teaching preventing students from making progress?
- What do I need to do next with each student to improve their learning?
- How should I change what I am doing to ensure the students achieve the learning goals?
- With which colleagues can I discuss these results?
• Which colleagues can advise on instruction methods?
• Do I fully understand the subject matter I am teaching?
• Do I need coaching in the subject/topic?

4.1.3 How classroom assessments improve learning

Incorporating formative assessment into pedagogy

Popham (2013) described five pedagogical practices associated with formative assessment that can be used to promote student learning, based on the work of UK assessment expert Dylan Wiliam.

1. Share learning goals: By sharing with students the learning goals and criteria for success they will learn more effectively than if they had no idea what they were supposed to learn. This should be expressed in the language of students, not the curriculum documents, and should cover the full range of tasks that students might be expected to accomplish.

2. Use effective classroom discussions, activities, and tasks to obtain evidence of learning: There is a wide array of formal and informal techniques from which evidence about students’ learning can be gathered. While tests are one method of collecting evidence, others include posing questions during classroom discussions and giving students opportunities to demonstrate their skills during carefully planned activities.

3. Provide feedback that promotes learning: Providing feedback to students can have positive or negative outcomes. Feedback should lead to further student reflection about what to do next rather than dwelling on past failures. It should engender a cognitive reaction rather than an emotional reaction.

4. Use students as instructional resources for each other: As mentioned above, peer assessment and peer tutoring harness the power of collaborative and cooperative learning.

5. Activate students as owners of their own learning: Although teachers determine the learning environment, it is only learners who create learning. The better students are better able to manage their own learning through a process of reflection, the more likely it is that they will learn well.

Complicity

We have looked at how students’ responses in assessments reflect on teachers and provide insights into teaching. Now we look at how the same responses can reflect on the students and provide the students with insights on how to improve their learning. This is making the students complicit in their own learning, and in their own improvement of learning. It can be very powerful.

The basis of this is to involve students in the testing and in the analysis. For example, discuss the test data with students, provide feedback on the students’ learning progress, and help the students to identify learning problems (Stiggins, 2004). Formative assessment allows students to observe the increase in their learning over time and to see a direct relationship between how hard they work and how much they learn (Marzano, 2007). There are several ways to involve learners in their assessment:

(a) provide students with rubrics or checklists that clearly explain the standard against which their work will be evaluated;
(b) show students work that is excellent and work that needs improvement along with help to analyse the differences between them;
(c) allow students to mark and analyse their own responses;
(d) have students chart their own progress on the learning goals; and
(e) teach students to use test items to analyse their own thinking.

However, if this approach is to be taken, the feedback should be given as soon as possible after the assessment occurs so it can influence the next steps in the learning process. High-quality instructional feedback is timely, useful and appropriate (Sternberg, 1994). As well, the test responses can be the basis of questioning; this can
give insights into student thinking that can guide teachers’ refinement of future lessons and also helps students reflect on their own thought processes, which builds metacognition (Burns, 2005). In one case a teacher taught her students how to analyse their thinking and to keep a record of their thinking during all classroom activities. In this way, the students developed their thinking and learning themselves.

**Corrective instruction**

High-quality corrective instruction is not the same as reteaching. Teachers must use approaches that accommodate differences in students’ learning styles and intelligences (Sternberg, 1994). As students become accustomed to this corrective process and realise the personal benefits it offers, the teacher can drastically reduce the amount of class time allocated to such work and accomplish much of it through homework assignments or in special study sessions before or after school. Some points on corrective instruction are provided below.

1. Developing ideas for corrective instruction and enrichment activities requires support. In this, teacher meetings devoted to assessment results and instructional strategies can be highly effective.

2. Corrective instruction gives students a second chance to demonstrate their new level of competence and understanding. This second chance helps determine the effectiveness of the corrective instruction and offers students another opportunity to experience success in learning.

3. Students who do well in a second assessment after corrective instruction have also learnt well. They deserve the same grades for the same test performance as those who passed the first time (as in, for example, a driving test).

4. Corrective instruction develops learning-to-learn skills in students (e.g. learning from mistakes). It often results in the best learning as it gives direction on how to improve (Wiggins, 1998), and can show the student how they best learn.

5. Corrective instruction must also cater for those who do not need the corrective instruction (e.g. provide enrichment activities).

6. Teachers who develop useful assessments, provide corrective instruction, and give students second chances to demonstrate success can improve their instruction and help students learn (Guskey 2003), as illustrated in Figure 4.3.

![Corrective instruction flowchart]

**Figure 4.3** Assessment and instruction practices leading to improved teaching and learning

4.2 **Diagnostic testing**

As we have seen, assessment is undertaken for a variety of reasons, but none so important as when it is undertaken to inform the teaching and learning process by determining the extent of individual student knowledge and the effectiveness of teaching. Such assessment is even more powerful if it can ascertain what students know about important ideas and provide insight into where corrective instruction or remediation should begin.

Over the years that we have been constructing, trialling and evaluating teaching ideas, YDC staff have used pre-tests and post-tests to ascertain whether the students have learnt what the teaching ideas were aiming to teach.
The trials have had the aim of both (a) understanding and improving learning (and developing learning theory); and (b) refining teaching and instructional material (and developing instructional theory).

The assessment tasks have been based on the learning and instructional theories so the students’ responses would illuminate both learning and instruction (resulting in a cyclic process designed to continuously refine theory). As we have always been interested in structural approaches to mathematics instruction, we have used a cognitive approach to diagnosis. Thus, the tests are labelled cognitive diagnostic assessment tasks (CDAT).

YDC staff have developed two books following this CDAT approach:

(a) a CDAT book, Developing Mathematics Understanding Through Cognitive Diagnostic Assessment Tasks, covering whole number, decimal number, common fractions, and probability – available to download from the YDC website (Student learning resources); and

(b) a second CDAT book covering operations (concepts, principles and strategies) – draft version available to download from the Operations section of the YDM Professional Learning Online Blackboard site.

### 4.2.1 Theory behind cognitive diagnosis

**Pedagogically based assessment**

The tasks in the CDAT books are designed to elicit students’ understanding of the big ideas in the topics covered. The tasks are sequenced across year levels and placed in year-level tests. The important theoretical frameworks for the development of these tests are given in this subsection.

The tasks in the CDAT books are based on the pedagogy of the YuMi Deadly Maths framework as described in the YDM Overview book. This includes the following:

(a) a structural understanding of mathematical development;

(b) a cycle of learning (RAMR) that moves from reality (world of student) through abstraction (body → hand → mind) to formal mathematics to reflection back into the world of the student; and

(c) a focus on generic pedagogies such as flexibility, reversing and generalising, and specific pedagogies for particular mathematics topics.

The idea is for the tasks to monitor what mathematical concepts and processes students understand before, during, and at the conclusion of teaching. This means that the CDAT books have three roles:

(a) giving insights into how students learn the core ideas;

(b) providing teachers with a springboard for intervention (corrective teaching) or prevention (good first-time teaching); and

(c) deepening teachers’ understanding of core ideas in elementary mathematics, enabling them to modify or extend their instruction (a good test is as much a teaching tool as an assessment tool).

Figure 4.4 illustrates this development of the CDAT books and their outcomes.
**Stages of diagnostic assessment**

There are four main stages of assessment in the diagnostic cycle (see Figure 4.5), namely:

1. **Sweep** – test instrument includes items from all mathematics domains (e.g. whole number, fractions, measurement). This is used when a student has several mathematics learning difficulties but the source is not known. The test is scored and analysed in terms of the mathematics domains being assessed.

2. **Scan** – test instrument includes all concepts and processes within a specific domain (e.g. whole number). The results are scored and analysed in terms of the concepts and processes related to the domain.

3. **Probe** – test includes specific concepts and related processes (to determine the order of remediation). Ideally the test is a teacher-made diagnostic instrument which is administered in individual interviews. The student’s responses are analysed for misconceptions and error patterns.

4. **Intervene** – teach the idea and see how students react – good for specific concepts and related processes. Ideally, these should be teacher-made tasks designed to remediate the particular problems (sometimes the best way to find out what students know is to teach them, watching how they react to instruction).

CDAT tests are **scans** because they include all concepts and processes related to a particular mathematics domain. Teachers can then pinpoint at which stage in the learning cycle the student has experienced problems in understanding. The assessment provides the basis of remediation because the tasks can be replicated by the teacher to use as intervention/instructional tasks.

**Pedagogical underpinnings of CDAT**

One of the most important underpinnings to the CDAT tests is Baturo’s (1998) modification of Leinhardt’s (1990) mathematical knowledge types as follows (relation to the RAMR cycle is shown in the description of each knowledge type).

1. **Entry knowledge** – knowledge constructed before formal instruction of a new concept or process (related to the **Reality** stage of the RAMR cycle). Entry knowledge needs to be examined for appropriate or inappropriate mathematical thinking before teaching a new concept or process.

2. **Representational knowledge** – knowledge constructed during instruction (related to the **Abstraction** stage of the RAMR cycle). This type of knowledge provides the student with mental pictures of a concept or process.

3. **Procedural knowledge** – knowledge constructed during instruction with respect to “working out” how to solve a given problem or task (related to the **Mathematics** stage of the RAMR cycle).

4. **Structural knowledge** – knowledge abstracted after instruction in the topic is complete (related to the **Reflection** stage of the RAMR cycle). Structural knowledge is the **macro-knowledge** constructed by experts.
and should be the goal of all mathematics learning. Its construction requires teacher and learner to share an active role, both cognitively and kinaesthetically.

Type 2 and 4 knowledge types were the most important for developing the CDAT tests. The tests are divided into two types, representational and structural. The representational tests assess students’ knowledge of how to use the materials common in instruction, while the structural tests are symbolic.

The CDAT tests are designed in reverse of the teaching sequence. Hence the structural tests are designed to precede the representational tests if both tests are given together. This is because:

(a) the structural tests determine if students know the mathematics and at what level they know it (the quality of the knowledge is almost more important than whether it exists); and

(b) the representational tests determine what students know about the common ways to teach the knowledge so that the starting point and starting materials of corrective instruction can be determined.

4.2.2 Administering assessment

Let’s say you have a unit of work you wish to assess for learning and teaching. You have some tasks and perhaps even a test from a book or testing system. How do you administer this to your class of students with their special abilities and needs? We now look at this in general and then for pre- and post-tests.

Administering diagnostic assessments

The first general rule of assessment for diagnosis is that it is done in the opposite direction to the teaching sequence. When teaching, we move, in general, from easy to hard and prerequisite to final idea. But for diagnosis, we move in the opposite direction. We start at the endpoint and work backwards until we find something that students know (as in Figure 4.6). An example of this is in the CDAT tests where, as we said at the end of 4.2.1, we administer the structural tests before the representational tests.

The reason for the opposite direction is that our aim is to find out what the students know (not what they don’t know). This gives us a bedrock for teaching – a starting point where there is knowledge. The second reason is if we start with earlier ideas, we may not be starting where the students are, and risk having to repeat the process. The third reason is that the students may pass later tests by the knowledge they gain from experiences in the earlier tests and we cannot be certain they actually know the work.

The reverse direction or “peel-back” for assessment is particularly useful if using an interview instrument rather than a written test (which may be a necessity in early years, before students’ reading skills are sufficiently developed). Here, one can probe for nuances of understanding. For instance, in an interview into two-digit seriation, one could ask: What is 10 more than 42? If the student cannot answer, the peel-backs begin: can the student answer if the question is:
(a) put into a verbal context – You have $42 and I give you $10 more, how much money do you have now?

(b) put in a context familiar to the student – You are playing netball, your team scores 42 points but the opposition beats you by 10, how many do they score?

(c) given with materials – Here are four $10 play-money notes and a $2 play-money coin [or 4 MAB longs and 2 MAB units], now I give you one more $10 in play money [or here is one more MAB long], how much do you have now?

(d) put in a real situation – use real money, go to a shop and act out needing $10 more to buy the present.

This leads to the second general rule of assessment for diagnosis – use more than one approach and have the examples going from sweep → scan → probe (based on Figure 4.5). For example, the whole class could receive a scan test (such as in CDAT). Those who cannot do the items associated with the topic are given a probe interview, based on asking them why they answered the items as they did. Then work out the corrective instruction needed and intervene for a deeper understanding of what the students know and do not know.

Finally we hit the third general rule – do not expose students to too much failure early. This one has to be taken in conjunction with the other two. Peel-back can result in a lot of failure before reaching the students’ level. If this will adversely affect your students (see Affects in section 3.2), it means that you may need to use your knowledge of the students to start at a lower point or you may need to convince them that, while you want them to try, you only want to know what they know and there are no worries about not answering a task and no implications for test scores. Peel-back difficulties can also be ameliorated by mixing the ways of assessing as suggested above in the second general rule.

**Administering pre-post assessments**

Many assessment situations involve diagnosis with achievement testing. Before teaching a unit of work, a pre-test will allow you to diagnose students’ starting knowledge and therefore the starting point for teaching; after finishing a unit of work, a post-test will enable you to assess students’ achievement (did the unit do its job?) and diagnose reasons for success and lack of success along with directions for corrective instruction. As an example, CDAT tests are most beneficial when administered before new concepts and/or processes are introduced (entry knowledge) and then re-administered when teaching of the concepts/processes is complete (exit knowledge).

Because each school’s teachers and students are different, the relationships between pre- and post-tests vary and the following parameters need to be followed.

1. If students are not strong in the mathematics topic being assessed, the pre-tests should normally include only lower level items, otherwise the pre-test would result in continuous failure because of the low performance of the students; teachers need to choose the level at which to end the pre-test, noting point 3 below.

2. The post-tests would need to include all the higher level items for the mathematics topic, but might not consist of all the early items in the pre-test, otherwise assessment will take too long; once more, the teacher can make the appropriate selection, noting point 3 below.

3. It is important that the two tests (pre and post) have items in common so they can be used for comparison. However, the higher items in the post-test that were not in the pre-test have to be such that they can be assumed as zero for the students for the pre-test, and similarly the lower items from the pre-test that are not put into the post-test have to be assumed as correct for the post-test.

4. If the above assumptions cannot be made or are uncertain, all items, or a reasonable number of items, need to be included in both the pre-test and the post-test, and the post-test must show the highest level of learning.
Overall, teachers need to do the following with regard to administering pre-post assessment (see Figure 4.7):

1. Use their knowledge of their students to ensure that pre-tests are based on item types that students might reasonably know but do not become a continuous stream of “don’t knows” – this means (a) not putting items in the pre-tests that it is known students cannot do; and (b) ensuring students know that it is all right in pre-tests to not answer items because the pre-test is given before instruction (it is to find out if they know anything before it is taught).

2. Reduce the number of item types to make the pre-tests as short as possible without removing important questions – it is not necessary to give students all possibilities when they have shown they cannot answer any possibility.

3. Ensure that there are sufficient common assessment item types in both the pre- and post-tests to allow for pre-post comparisons, and ensure the post-test covers all possibilities.

4. Encourage students to try all items in post-tests and do as well as they can, because these final tests show how far they have progressed and learnt.

While pre- and post-tests are effective in demonstrating that a student has learned, the question remains as to whether they have learned enough? To answer this question, it is recommended that pre-tests and post-tests be compared by effect size (Hattie, 2012). To do this, calculate the difference between the pre-test and post-test means and divide this by the average of the standard deviations of the pre- and post-tests. If the resulting calculation is higher than 0.4 (the higher the better), this indicates that teaching the unit was more effective than could be expected from regular teaching. However, if the students do very well in a pre-test (either because the students are gifted, or the pre-test was poorly designed), then it is difficult for them to show gains in the post test. This is called the ceiling effect.

### 4.3 Remediation through acceleration

The form of remediation used in YDM is to accelerate the learning of students from what they know to what they should know by using a vertical pedagogy. To this end, YDC has developed the AIM, XLR8 and AIM EU pedagogies and materials for accelerating underperforming students, discussed at the beginning of this chapter. They enable up to six years of mathematics to be learnt in three years. This section describes the theory behind this remediation pedagogy.

#### 4.3.1 Acceleration structure

**Imperatives**

In the YDM approach, acceleration of learning (i.e. learning at a faster pace so students can catch up) such as that used in AIM, XLR8 or AIM EU is based on four imperatives.

1. **Imperative 1: Chunking.** Mathematics topics are taught in large chunks, not as small pieces spread over years. The focus is on building a holistic understanding around whole numbers as a complete multiplicative system – the approach to teaching that YDC has found to be effective with Indigenous students.
2. **Imperative 2: Structure.** Mathematics topics are taught as a structure (i.e. a connected set of ideas), both within domains – such as place value of number, and across domains – such as place value and metric measures. Students are led to recognise the big ideas of mathematics (see section 5.3).

3. **Imperative 3: Active pedagogy.** Mathematics topics are taught actively, with actions and material leading to ideas in the mind (i.e. mental models); teaching follows the sequence body → hand → mind. Mathematics topics relate to real-world problems (from everyday experiences) and relate back to real-world problems, integrating real-world situations, physical, virtual and pictorial models, language, and symbols.

4. **Imperative 4: Culture, community and school focus.** The cultural implications of Western mathematics are made visible, frameworks for learning relate to the local community, and mathematics is contextualised into culture and related to home language. The local community is encouraged to actively participate in the school and local knowledge to be made legitimate in the classroom. Schools adopt change processes similar to those advocated by the Stronger Smarter Institute (http://strongersmarter.com.au/), so that school processes build pride in heritage, change identity to where there is a belief that students can learn, confront poor behaviour, ensure cultural safety and high expectations, and are based on local leadership.

The framework for acceleration used in AIM/XLR8 is based on a theory for building big mathematical ideas (see Chapter 5 of this book and the YDM supplementary book *Big Ideas of Mathematics* for further information on big ideas). Knowledge growth for big ideas is through structured sequences across models/representations not within a model/representation.

**Structured sequence theory**

YDC’s theory of structured sequencing that leads to the development of big ideas and allows acceleration argues that structured sequences have the following properties.

1. **Isomorphism.** Effective models and representations have strong isomorphism (similar in objects and actions of operations) to desired internal mental models, few distracters, and many options for extension. In other words, these models/representations grow with the idea. For example, $3 \times 4$ can be modelled by a 3 by 4 array. Arrays can be extended into the area model (although this is not as simple as it seems) and the area model can be used as the basis of multiplication of larger numbers, such as $7 \times 23$, fractions, such as $\frac{4}{5} \times \frac{2}{3}$, and algebra, such as $x(x + 2)$.

2. **Sequence.** Sequences of models/representations develop so there is increased flexibility, decreased overt structure, increased coverage and continuous connectedness to reality. For example, the balance model for equations moves from a physical balance to a pictorial balance to an abstract balance that can handle division and negatives.

3. **Nestedness.** Ideas behind consecutive steps are nested wherever possible. That is, later thinking is a subset of earlier. For example, the first understanding of equals should be “same value as”, and the second understanding should be that it is the result of a calculation, which is a particular form of “same value as”. That is, the meaning of equals in $4 + 2 = 6$ is nested within $4 + 2 = 5 + 1$, so $4 + 2 = 5 + 1$ comes first.

4. **Integration.** More complex and advanced mathematical ideas can be facilitated by integrating models. For example, solving algebraic equations is a combination of number-line understanding of inverse operation and balance-beam understanding of the balance rule.

**Note:** Some complex and advanced ideas may require the development of superstructures if complexity leads to compound difficulties. For example, the compensation principle for addition is to do the inverse to the first action (e.g. $8 + 5 = 10 + 3$ by adding 2 to the 8 and subtracting 2 from the 5), while the compensation principle for subtraction is, simplistically, the opposite (e.g. $14 - 6 = 18 - 10$ by adding 4 to both numbers). This opposite difference between them can cause confusion and, thus, is called a compound difficulty. However, if a superstructure of understanding is built around subtraction as the inverse of addition and division as the inverse of multiplication, it is almost self-evident that there has to be an opposite process for subtraction in relation to addition.
5. **Comparison.** Abstraction is facilitated by comparison of models/representations to show commonalities. For example, $2 + 3 = 5$ makes more sense when seen in joining counters (set model) and steps along a number track (number-line model.)

The implication is that acceleration is enhanced if instruction is divided into vertical units of work (called **modules** in AIM/XLR8) that structurally sequence mathematical ideas from early childhood to junior secondary. In this way, the modules can build big ideas as well as concepts, strategies and processes. The first stages of a module (covering the early years of the sequence) have to be completed slowly and carefully to build the connections that frame out the big mathematical ideas. The later stages (covering the later years) can then be covered quickly in gestalt-like leaps of understanding, that is, leaps to holistic ideas or superstructures which enable deeper understandings of the original parts.

This implies that mathematics learning of underperforming students can be accelerated if: (a) instruction, models, representations and language follow a nested line of abstraction from ability level to age level; and (b) instruction has its basis in carefully built, contextually and culturally appropriate foundational ideas enabling rapid and gestalt advances across higher level and more generalised topics.

### 4.3.2 Curriculum implications

**Vertical curriculum**

Accelerating mathematics learning by modules that develop a big idea from early childhood to junior secondary has implications for curriculum if the project is to cover, as it must, all the mathematics in a given time (for AIM this means covering Years 4–9 mathematics in Years 7–9).

It means that only part of the mathematics curriculum can be covered in each year. This is because modules change the way the mathematics is accelerated. Instead of building all topics in three stages as in the “normal” growth diagram on the left in Figure 4.8, one-third of the topics are completed in each year, with the Years B and C filling in the gaps left from Year A, as in the “vertical” growth diagram on the right in Figure 4.8. In this way, the mathematics curriculum moves **from a horizontal to a vertical structure**.

![Figure 4.8 Normal/horizontal and unit-based/vertical mathematics growth](image)

Both structures reach the same outcome by the end of the third year but in different ways.

**Internal module structure**

The AIM modules have the following components:

1. An **introduction** section that describes the focus of the module, the connections and big ideas that underlie the module, and the vertical sequence that the units will move through.

2. The **vertical sequence of units** that take the big ideas from Year 4 to Year 9, but modified to ensure that (a) a proper foundation for the sequence is first built (i.e. sometimes the first unit includes work from Years F–3); and (b) the last units can be covered from the earlier units (i.e. sometimes the sequence ends early and later work is left to the end of other modules whose knowledge is needed for the later work to be understood).
3. **Pre-post test items** for the teachers to be able to (a) select the appropriate starting unit for their students, and (b) check that students understand all the units when the module is finished.

Where possible, modules are built around a central mathematics big idea, or collection of ideas, or a theme or topic in mathematics. In many cases, this allows for the modules to have a single sequence of units. These units may contain more than one idea but each idea is integrated into a single sequence with the other ideas. However, there are differences, as shown in Figure 4.9.

![Figure 4.9 Different AIM module types](image)

The left-hand-side section of Figure 4.9 shows a module as a strict vertical sequence of coherent connected ideas; the middle section shows a module where there are three vertical sub-sequences running side-by-side; and the right-hand-side section shows a hybrid, with a strong single vertical sequence to the module but one of the steps has more than one option.

Another factor that affects module structure is the need for strong foundations. Often there is a preponderance of activities in the early units with later units covering more years of work in fewer activities. This is because, when foundations are set, **acceleration occurs through the later units.**

The module structure has been found to have real advantages:

(a) it gives teachers the freedom and confidence to drop down and teach at ability level because they see that such instruction accelerates the students’ knowledge up to age level;

(b) it gives students a sense of progress and confidence that it is possible to reach age-appropriate standards;

(c) it enables teachers to realise that it is appropriate to spend time on one topic area as long as learning is in connected and integrated chunks to enable students to make progress;

(d) it enables students to experience big ideas across year levels and facilitates strong generalisations of these big ideas;

(e) it enables teachers to see growth of students’ ideas as the students move through the units and to make on-the-run, in-class assessments of progress; and

(f) it allows students to have the most powerful form of acceleration, where strong foundations enable gestalt-type leaps in later knowledge to superstructures.

**Testing framework**

Each module grows knowledge in particular mathematics ideas. When all modules are combined, they grow knowledge across all ideas. The growth within modules can be shown by the difference in achievement between the pre- and post-test results. To maximise implementation, the pre-tests and the post-tests have to be both diagnostic (formative) and achievement-related (summative). This happens within and across modules as follows.

1. **Within modules.** Each module includes diagnostic achievement tests called **subtests** for each of the vertical units (see Figure 4.10). The highest level subtest is normally associated with the highest level unit. This enables the subtests to be used by teachers to:

   (a) determine where their students should start the module,

   (b) check that their students complete the module, and

   (c) gain information on how to teach so their students progress to the end of the module.
2. **Across modules.** Knowing where students start and finish a module in terms of vertical units can be used to show overall growth of knowledge. The units have been designed so that students’ responses to year levels can be translated in terms of Year 4 to Year 9 knowledge with respect to the Australian Mathematics Curriculum. For such an overall testing regime, the relation of subtests to vertical units as shown in Figure 4.10 could be aggregated to enable an overall grade in terms of year level of mathematics understanding. Such a grade would be:

(a) based on rapid changes in some mathematics ideas (as modules covering these ideas are completed);

(b) based on no change in other ideas (where modules covering these other ideas have not yet been taught); and

(c) best recorded in terms of levels within the six topic areas (Number, Operations, Algebra, Geometry, Measurement, and Statistics and Probability) as well as an overall level.

![Figure 4.10 Sequenced structure of a module](image)

### 4.4 Implications for future teaching

Any program adopted by schools needs a diagnostic assessment framework. This framework needs to take into account assessment, diagnosis and remediation. The concept map in Figure 4.11 summarises the important points in this chapter that deal with these three issues.

One of the questions to face as your school leaves the training phase of YDM and moves on to setting up a sustainable way for YDM to operate in your school is how to ensure mathematics learning is taking place so it meets the needs of your students. In this chapter we looked at diagnosis and remediation procedures for YDM, and their basis, which is assessment. The concept map below provides a summary framework of some of what we have covered. How would your school like to take these into account?

In doing this, you may like to look at YDC’s AIM, XLR8 or AIM EU projects and the modules that are available. The first two projects focus on remediation in Years 7–9 but can also be used in upper primary years. The AIM EU modules focus on Years F–2. However, the main focus for you with regard to this chapter is to develop an assessment framework for your school.
Your task from this chapter is to develop a plan for the assessment framework at your school.

Look at the summary in Figure 4.11 and determine what your emphases would be in your school if you could implement what you want. The questions below may help.

1. What will be the assessment framework for your school?
2. How will you use this for diagnosing what your students know (and do not know) and the quality of that knowing?
3. How will you organise remediation (acceleration/corrective reteaching)?
5 Enrichment and Extension

The other side of remediation of underperforming students is enrichment and extension of able students. Therefore, this chapter looks at teaching to develop the deep learning of powerful mathematics ideas together with the motivation and engagement that will enable more students to participate, and want to participate, in the highest level mathematics subjects, and go on to mathematics-based courses in tertiary education.

Like the AIM program, this area of mathematics needed a special extension of the YDM pedagogy. It led to the Mathematicians in Training Initiative (MITI), which was developed around problem solving and investigations, analysing the understandings of mathematics that best prepare students for high-level subjects, and knowing the pedagogies to teach these skills and understandings. This means that there are now three types of programs based on YDM:

- YDM general pedagogy projects for all students (Years F to 9), in which this Review book is the eighth resource;
- accelerated learning projects (AIM, XLR8, AIM EU) for remedial students (Years F to 2 and 7 to 9); and
- enrichment and extension projects (MITI) for advanced study (Years 7 to 12).

This chapter focuses on enrichment and extension and is divided into four sections:

(a) Problem-solving components and their teaching (5.1) – major skills required to be effective in problem solving and investigations (metacognition and thinking skills; plans of attack and strategies; content knowledge and affective traits), and general tips for teaching problem solving.

(b) Mathematics structure (5.2) – components and teaching of deep and powerful mathematics.

(c) Mathematics big ideas (5.3) – nature and justification of big ideas and types of big ideas (structures of mathematics) that best constitute deep learning of powerful mathematics ideas.

(d) Implications for future programs (5.4) – taking into account each of the three components above to cater for deep learning of powerful mathematics in your school.

5.1 Problem-solving components and their teaching

Problem solving is made up of three stages as shown in Figure 5.1. The first stage involves finding and defining the problem and determining whether it is a problem, the solution of which is worthwhile finding. The second stage, solving, can be a messy, idiosyncratic, doodling, acting out, and making lists type of activity. The third stage, communicating and justifying, is where the solver rewrites the solution so it can be followed by others, following the protocols laid down by that topic (usually under the euphemism “show all working”). This means that problem solving, and communicating and justifying, follow each other and must work together.

![Figure 5.1 The three stages of problem solving](image-url)

Information on these projects can be obtained from the YDC website.

\(^2\) Information on these projects can be obtained from the [YDC website](#).
This section summarises the components of problem-solving expertise: metacognition and thinking skills, plans of attack and strategies, and content knowledge and affective traits. These components make up “the expert problem solver” – see right.

Teaching these components of problem solving is also important, so we will refer to this as we go along, and provide some general hints at the end of this section. The supplementary resource book *YDM Problem Solving* expands on the information summarised here (see YDM Blackboard).

5.1.1 Metacognition and thinking skills and their teaching

This subsection looks at metacognition and thinking skills, describing the ideas within them and giving examples of their use.

**Metacognition**

Metacognition is a term from the study of psychology that means the awareness and understanding of one’s own thought processes. So, in the context of problem solving, understanding when to make decisions, and which decisions to make, is a result of metacognition. Metacognitive processes oversee, monitor and evaluate the problem-solving process. They coordinate planning and checking. In particular, problem-solving performance is enhanced when the problem solver is able to develop the ability to be aware of, and consciously control, these overseeing and monitoring functions. This is because metacognitive processes coordinate the application of specific knowledge as well as oversee other problem-solving processes.

We identify the following processes as important:

1. **Overseeing and monitoring** – the ability to continually keep tabs on what is going on in any attempts to solve problems, that is, to have part of the thinking tracking what is being done and has already been done.

2. **Checking and evaluating** – the ability to continually check how thinking is going, and to evaluate if progress is satisfactory.

3. **Planning and predicting** – the ability to continually look ahead, to make plans and to predict what is possible.

4. **Decision-making** – the ability to make decisions with regard to new directions. This is a particularly important trait in problem solving as you have to make decisions on what directions, strategies, and so on to trial. It is also problematic as most students are given little chance to make their own decisions in classrooms.

5. **Being aware of and controlling** – being conscious of what you are thinking and being able to report on and discuss thinking. It is also, then, consciously controlling the direction of this thinking.

Unconscious metacognitive processes are thought to exist from early childhood. Awareness of these processes, and conscious control over them, is believed to emerge in middle childhood. It is believed that metacognition can be enhanced by interaction with others with more expertise. This is particularly so for conscious control, where the learner moves to conscious self-regulation.

**Teaching metacognition**

There are group techniques to enhance metacognition. The best of these are the following:

1. **Reporting** – requiring students to report on their problem solving by stating what and how they thought, as well as the answer (using the third stage of problem solving, communicating and justifying, to improve metacognition). An extension of this is to have groups solve the problems but have a reporter, who has to explain thinking and justify the solution, chosen by lot. Then the group has to ensure all members have sufficient information and knowledge to be the reporter if chosen.
2. **Reciprocal teaching** – giving students opportunities to act as teachers in the classroom, leading others’ learning (summarising, questioning, clarifying, predicting and checking). One way to do this is to lead a student through a problem and then, later or the next day, to let them work with a small group leading this group to solve the problem.

3. **Think–pair–share** – organising the class to attempt to solve a problem on their own, to discuss their solution with a partner (form pairs) and then to come to one solution within a group of four (two pairs) through discussion and negotiation. This makes students explain and justify their answers, describing their thinking.

4. **Social-interaction roles** – the students solve problems in groups where members are given roles. An effective and simple way is to form groups of three: one is designated leader (makes decision if there are differences of opinion in the group), one is the checker (continuously checks that direction is the best to go), and the third is recorder/reporter (the only one with pen/paper to record findings, and explains the group’s solutions). Roles are changed for each problem.

Finally, as decision making is so important, it is useful to prepare students for problem solving by **allowing them to make decisions in other mathematics**; for example: (a) discussing examples with students to encourage them to determine what information will be needed for the example; (b) giving students examples and letting them choose the ones they answer (e.g. *Here are 20 questions, do any 10 you like*); and (c) giving examples with more than one answer or more than one solution method and asking students for all the answers and methods.

**Thinking skills**

These are abilities to think in different ways. The following seem to be most important in problem solving:

1. **Logical thinking** – being able to interpret facts and relationships and organise them in a logical sequence. Logical thinking is a valuable skill not only in problem solving but in most other mathematical tasks as well. It is related to verbal logical thinking and thus has a relationship with language in problem solving. Unfortunately, it is often taken for granted that students are natural logical thinkers; rarely do mathematics programs for the early years incorporate specific activities to help develop logical thinking. The topic “sets and logic”, part of many mathematics curricula, has usually been a mask for a variety of set symbols. The logic component has rarely surfaced. Consequently, many students are ill-equipped for problem solving in the middle and upper year levels, irrespective of the heuristics and strategies they might be taught. Students in early years need to be particularly exposed to logical thinking activities.

2. **Visual thinking** – being able to distinguish one shape from another, recognise specific shapes when they are in different positions, and transform shapes; being able to picture situations and mentally manipulate images. It includes abilities found in flipping–sliding–turning, symmetry, tessellations, and dissections. For example, wooden cubes can be effectively used for a variety of spatial activities such as constructing shapes from drawings. Activities where students have to select a shape to fit a gap are also excellent for visual imagery. So are tangrams and other shape puzzles.

3. **Evaluative thinking** – being able to monitor whether activity is productive in terms of the problem and conclusions are accurate and make sense in terms of the problem. It covers checking but also learning from checking and looking for other options (solutions and solution methods).

4. **Patterning thinking** – being able to recognise patterns and determining relationships. It is at the core of thinking mathematically and solving problems. Patterning skills can be developed through activities with beads on string, attribute blocks, coloured cubes and a variety of other concrete and pictorial aids. Patterns can be followed and invented.

5. **Creative thinking** – having a creative approach to solving a problem; being able to generate a number of different solutions to a problem. Creative thinking is not in the exclusive domain of Language Arts. It plays an important role in mathematical problem solving and should be encouraged through activities which direct students to think of alternative ways of doing things.
6. **Flexible thinking** – being able to change one’s point of view to see a problem from more than one perspective. Students who experience considerable “traditional” mathematics teaching often have difficulty in making decisions regarding what to do next, so activities should be provided which enable students to make decisions. Such activities will help them become more flexible in their approach to problem solving. This thinking skill works well in combination with creative thinking.

### 5.1.2 Plans of attack and strategies

**Plans of attack**

Metacognition can be operationalised through a good plan of attack (or plan as we will call it). A powerful heuristic plan from which to attack the problem can assist when, in a problem situation, a learner’s content knowledge appears, initially, to be inadequate. This is particularly so when the plan is linked with a strong repertoire of strategies.

The best plans are based on Polya’s (1957) **four stages**, which are often represented as SEE, PLAN, DO and CHECK. This heuristic plan is most widely advocated as facilitating problem solving. It has four stages as shown in Table 5.1.

<table>
<thead>
<tr>
<th>Table 5.1 Polya’s four-stage plan of attack</th>
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<tbody>
<tr>
<td>1. <strong>SEE</strong></td>
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<td>2. <strong>PLAN</strong></td>
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<td>3. <strong>DO</strong></td>
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<td>4. <strong>CHECK</strong></td>
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Different problem types have differing requirements with regard to plans; for example, the main difficulty with word problems lies in the SEE section. Plans are best taught directly. Two methods are very useful. First, put Polya’s four stages up as a **poster** and then teach to it following the four stages. Give a problem, then:

(a) have the class discuss the problem in terms of SEE;

(b) when this is finished, have the class discuss a PLAN to solve the problem;

(c) let the class DO this plan; and

(d) bring the students back together to discuss CHECK.

Second, use Polya’s four stages in problems as a basis for **hints for students**, particularly for those having difficulty with a particular problem. For example, problems could be written on index cards, then hints could be written on the reverse of the cards.
Strategies
A major part of any problem-solving program is teaching a rich repertoire of strategies. These are general “rules of thumb” that direct to an answer. We recommend the strategies shown in Table 5.2, which we have related to the thinking skills. These strategies are defined in greater detail, with examples, in the YDM Problem Solving supplementary resource book (available from the YDM Professional Learning Online Blackboard site).

Table 5.2 Strategies for problem solving

<table>
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<tr>
<th>Thinking skill</th>
<th>Strategies</th>
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| Logical thinking        | 1. *Reread the question (looking for crucial words).* Read carefully what the problem is saying. It is so easy to misread a word and obtain the wrong impression.  
2. *Identify given, wanted and needed information.* Defined as follows: (a) “given”, the information we work on and start from; (b) “wanted”, the required answer(s); (c) “needed”, anything we have to work out on the way to the answer; and (d) “not needed”, any unnecessary information (“red herrings”).  
3. *Restate the problem in your own words.* Internalise in your own words (and understandings) what it is that the problem wants you to do. A major source of failure in problem solving is spending too little time in understanding and defining the problem. A good problem solver reworks the problem in their own words so that it is more straightforward.  
4. *Write a number sentence.* This is an extension of 2 and 3 above. It requires the problem to be converted to mathematical equations/formulae. It is a very powerful method. |
| Visual thinking         | 1. *Act the problem out.* Often you have to visualise what is happening in a problem. To make picturing easier, you can gather actual materials and act out what has to be done.  
2. *Make a model of what happens in the problem.* If acting out the problem with actual materials is impossible, make a model to help visualise the problem. Easily accessible objects can represent harder-to-obtain or awkward-to-use real objects in the problem.  
3. *Make a table, drawing, diagram or graph.* Many problems have so much information it is difficult to organise it to see relationships. Tables, drawings, diagrams or graphs can make relationships apparent. This strategy is one of the most powerful in anyone’s repertoire.  
4. *Select appropriate notation to picture the problem.* The power of mathematics often lies in the wide descriptive powers of symbols. Putting a word problem into symbols might reveal a rule you can apply or a way of using the symbols to reach a solution. Appropriate symbols are not necessarily numbers but could be lines, dots, boxes, and so on. |
| Evaluative thinking     | 1. *Check your solution.* This is so simple but nearly all problem solvers make careless errors when they don’t do it.  
2. *Find another way to solve it.* In real life the same task is seldom resolved the same way. Different and better solutions are developed. The same can be done to good problems.  
3. *Study the solution process.* Often when solving problems (particularly with the step-at-a-time strategy) you can’t see the forest for the trees. Overall, what was your strategy? Once you’ve solved the problem you are free to consider this and you always should. This is how you learn strategies. So, go through your problem solutions – explain them to yourself and, if you can, to others – keep in your mind successful strategies so you can use them again.  
4. *Find another solution.* Mathematics teaching often directs students to expect only one correct solution. In real life this is often not the case and students must learn to find all solutions.  
5. *Generalise.* There are two possibilities here: (a) you may need to generalise from simpler cases to solve problems, but when you have solved them, generalising the solution is a fruitful practice of problem solving; and (b) you have a solution but can you make up a more general problem and generalise your answer? |
Thinking skill | Strategies
--- | ---
Patterning thinking | 1. *Look for a pattern.* This strategy involves active search as well as passive observation. You should try to identify and continue the pattern. You may also have to construct or identify the original data that gave the problem.
2. *Construct a table.* When you are looking for a pattern or relationship, it is much easier to discover it when you put the information in tabular form.
3. *Account for all possibilities (systematically).* Another strategy that goes hand in hand with constructing a table is to account for all possibilities. This strategy does not mean you have to examine all possibilities, but that you account for them in a systematic way.

Creative and flexible thinking | 1. *Identify a sub-goal (or break the problem into parts).* This is a way of making the problem smaller and getting started. Some problems require a series of steps to be performed before they can be solved. So identify a sub-goal (the first step) and pursue it, then move on to the next sub-goal. In other words, break the problem into parts to follow in sequence.
2. *Guess and check (trial and error).* This can be a very effective strategy (as long as the guesses are “educated”). The key element to its success is the “check”. This strategy can always enable you to do something when you are really blocked.
3. *Work backwards.* This strategy has very wide implications for process problems where mastering it allows for easy solution. The secret is to think of the opposite (e.g. losers instead of winners), or to work backwards through the problem from end to start (particularly important when we have been given the end).
4. *Solve a simpler problem (or use simpler numbers).* Information from simple cases (or similar cases) can give insight into how to solve more complex problems. This reference to simpler problems will not be contained within the statement of the problem. So, you need to always think of looking at simpler cases as a powerful way of helping to solve a complex problem.
5. *Change your point of view and check for hidden assumptions.* Your perception of a problem can make it difficult to solve—sometimes you can take an incorrect and non-productive perspective. So when you are stuck, you should consciously redefine or restate the problem, trying to take a new view and changing your focus. It could also be that you have made a fundamental assumption that makes it impossible to solve the problem. You have to find this assumption and remove it. These strategies require a major shifting of focus on the problem components, a major look at what the problem says and does not say, and your assumptions about this.

The advocated way to teach a problem-solving strategy is called the *match–mismatch* teaching approach:

- first, teach the strategy directly with an example;
- second, give a few examples, all of which will use that strategy; and
- third, provide a mixture of problems, only some of which will use that strategy.

### 5.1.3 Content knowledge and affective traits

In this section we look briefly at the final two components of the expert problem solver, namely content knowledge and affective traits. The first of these will be discussed more fully in section 5.2, and the second was discussed in section 3.2.

#### Content knowledge

Studies of experts have shown that effective content knowledge for problem solving in a particular domain (e.g. mathematics) has two aspects, namely:

(a) knowledge of the domain is structured into rich schemas; and

(b) basic knowledge is automated (i.e. learnt until its recall requires little to no working memory).
As will be described in section 5.2, expert problem solvers have knowledge that defines, connects, applies, remembers, and has its basics automated.

**Affective traits**

Affective traits were discussed in section 3.2, where we identified four clusters of affects as follows:

1. **Attitude, motivation and engagement.** The extent of interest of the learner partially determines the extent of thinking about the problem the learner is willing to do. A good problem solver has a “give it a go” attitude and the motivation to work hard and to spend time on a problem.

2. **Self-confidence and self-efficacy.** Poor self-concept/efficacy makes for an unwilling problem solver. Problems do not necessarily lead to success, even after a major effort. One needs to be able to feel that effort, not right answers, makes success, and that building a repertoire of methods rather than a record of correctness is what makes mathematics worthwhile.

3. **Attribution.** Too much failure can lead to poor attribution – attributing success to luck and task difficulty and failure to ability and effort. Such attribution cannot be changed with success as it reflects beliefs about oneself. It can lead to task avoidance, learned helplessness and mathaphobia.

4. **Perseverance and resilience.** Problem-solving skill is affected by a learner’s ability to handle stress and anxiety, take risks and make decisions, and hold off from premature closure. In particular, as problems take time and finding a solution is not assured, the student must have the perseverance to keep going when success is not immediate and the resilience to try again after failure.

### 5.1.4 Teaching problem solving in general

The following are some general ideas for teaching expertise in problem solving.

1. Take all the students through the three stages of the problem-solving process (defining/posing, solving, and communicating solution) as separate skills before requiring students to do all three in problem-solving situations.

2. Ensure that all components, namely, metacognition, thinking skills, plans, strategies, content and affects, are developed.

3. Ensure a positive classroom atmosphere to build resilient affects and ensure deep learning of mathematics (schema).

4. Do not get in the way of students’ self-learnt problem-solving ability – most students come to school as excellent problem solvers which schooling seems to diminish not enhance.

5. Facilitate a questioning, encouraging, enquiring approach to teaching that enables student participation in problem solving more than teacher-led traditional instruction.

6. Focus on how the solution was found rather than the answer itself. Prevent evaluation procedures harming the problem-solving program. Adopt the role of facilitator.

7. Cater for all problem types – those best solved by content and those best solved by thinking skills.

8. Most importantly – avoid partitioning problem solving from the remainder of mathematics instruction. Instead integrate problem solving with everyday teaching: do problem solving in every lesson, include problem solving and the teaching of problem solving in all subject areas, and be a problem solver yourself.

**Finally,** problem solving cannot be divorced from the concepts, skills and attitudes on which it is based or from the methods to teach those concepts, skills and attitudes. Teachers may have to **rethink their whole mathematics program** right down to their fundamental beliefs about what mathematics is and how students learn.
5.2 Mathematics schematic structure

One of the important findings from years of research on thinking and problem solving is that routine problems, that is problems that are based on particular content (called the domain), are best solved by powerful knowledge of the domain. (Note that although these problems are called routine problems, they are not “routine” in the everyday English definition of the word.) Consequently, enrichment and extension in mathematics must include the development of deep knowledge of powerful mathematics.

However, mathematics should also prepare students for problems that do not depend on particular content knowledge (called creative problems) and these require strong thinking skills. This means that all the components of problem solving remain an important part of mathematics development for enrichment and extension, with their impact as shown in Figure 5.2. The figure shows that routine problems are best solved by content and creative problems by thinking skills. Metacognition, plans, strategies and affects assist all types of problems. Thinking skills help content form in a way that assists problem solving. (See the YDM Problem Solving supplementary resource book for more on this, and also for some examples of routine and creative problems.)

![Figure 5.2 Components of problem solving and their impact on solving routine and creative problems](image)

This section looks at what needs to be developed mathematically for enrichment and extension in mathematics.

5.2.1 Components of deep and powerful mathematics

Deep learning of powerful mathematics requires that students connect new mathematical ideas to existing knowledge structures (or networks) so that they can store and remember mathematical knowledge in a schematic form. It is also important that students develop mathematical fluency so that crucial basic mathematical processes are automated (instantly recalled). This frees up cognitive capacity for students to focus on the structural aspects of the mathematics problem.

Thus, the mathematics required for enrichment and extension is as shown in Figure 5.3. Each of these knowledge components is discussed below.

**Definition knowledge**

Expert problem solvers in a domain have that domain completely defined. This means they can recognise an idea, say addition, anywhere. For example, they can recognise all the following as addition:

- Joining – Three cars are parked and four join them, how many cars are parked?
- Comparison – Three cars are parked in area 1, four more cars are parked in area 2 than in area 1, how many cars are parked in area 2?
• Inverse (of subtraction) – Four cars were taken out of the car park, three cars were left, how many cars were there at the start?

• Inaction – There were 3 Holdens and 4 Fords, how many cars?

Students should be given definitions that cover all the ways an idea can be represented. This also includes all models, which for addition are set and number line:

• Set – There were 4 logs in the trailer, 3 more were added, how many logs in the trailer?

• Number line – The 3 m log was placed end to end with the 4 m log, how long was the distance covered?

The consequence of this is that, for routine problems, the expert problem solver solves problems by starting with content knowledge and moving towards the information in the problem, not from the problem to the content knowledge. The end of subsection 3.1.3 described this.

**Connected knowledge**

Problems occur when the task appears not to be solvable by the knowledge that is being focused on (called focal processing). This lack of success results in the focus changing to the periphery (called peripheral processing). When knowledge is highly connected, there are many chances for this peripheral processing to find knowledge that will lead to a solution – the processing follows the connections. If knowledge is unconnected, there is little peripheral processing available and little chance of a solution. Expert problem solvers have been found to have highly connected knowledge. In pictorial terms, their knowledge resembles a collection of nodes joined by many connections.

For the example of addition, rich schema would result in connections to number, counting, all the methods of addition, subtraction (as the inverse – what undoes the adding), multiplication (through repeated addition), and all situations where adding is used other than in number (predominantly measures but also statistics and probability and some geometry).

**Application knowledge**

Rich schema includes a knowledge of the different applications for the idea/concept under consideration. For addition, this would involve all the definitions and models in “definition knowledge” above plus ways to use the idea in applications, as follows.

• **Part-part-total** – understanding that addition is two parts forming a total (P-P-T) and being able to use this to solve problems. For example, in the inverse problem above, the 4 taken out is a part, the 3 left is a part, while the number at the start is a total, so regardless of the language used it is a problem where you know the parts and want the total, so addition is the required operation.

• **Transformation** – seeing an example like 3 + 4 = 7 as a change like on right. For example, in the inverse problem this means that the change is subtraction, as in the bottom arrowmath example, but the answer requires the change to be reversed, meaning the solution requires addition.

• **Difference between addition and multiplication** – both addition and multiplication require sets of objects to be joined but for multiplication the sets have to be the same size as each other (not necessary for addition) and one of the numbers refers to the number of sets while the other number refers to the number of objects in each set. In addition, all numbers refer to the objects in the sets.

**Experiential (remembered) knowledge**

Rich schema contains the experience of the solver so that he or she can use situations in the past to help with present problems. A useful question is “Does this look like anything I’ve seen before?” Students often have little memory of their past mathematics experiences, preferring to forget them. If one uses the evaluative thinking strategies (see Table 5.2 in section 5.1.2) to analyse past experiences, and remembers this as part of the experience, it is even stronger.
**Automated knowledge**

Knowledge that has been over-learnt until it can be immediately recalled without hesitation (e.g. rote-learnt basic facts) is called *automated*. Automated knowledge has little to no cognitive load. This means that the knowledge is available to the problem solver without reducing thinking power.

It is important that students see rote learning (automated knowledge) as a shortcut to a concept that they understand well (for example, using a rule for the area of a triangle is quicker than conceptualising the triangle as half a rectangle). However, rote learning is not a substitute for understanding. Automated knowledge has limited value in problems but it can enable a solver to: (a) use the “solve a simpler problem” strategy (try for solution quickly with smaller numbers); (b) trial quickly a series of estimates to get a feel for the answer; and (c) have a feel for the reasonableness of numbers.

It is important to note that automated knowledge is best taught from strategies. The strategies enable the learner to get answers (albeit slowly) and the practice speeds up the strategy application until it is automated. Drill without strategies may not be as effective.

### 5.2.2 Teaching deep and powerful mathematics

The following strategies should be used in any program for teaching deep learning of powerful mathematics. They include two methods of turning worksheets of exercises into problems, ideas for constructing many problems from a few, how to implement investigations, and how to use students’ statements and actions to build a rich teaching program through pivotal teaching moments (see Figure 5.4).

**Turning worksheet exercises into problems**

It is important to reduce emphasis on worksheets and make the major focus for all lessons problem solving. One way to do this is to change worksheet lessons to problem-solving lessons based on the worksheets. Two main methods of changing exercises to problems are provided below with several examples in each.

**Method One: RAMR cycle**

The first method is based on using RAMR to change exercises to problems (with examples) – called “RAMRising” the exercises. Two exercises will be used as examples:

1. **47 + 26**
2. Calculate surface area of a cylinder and cone when \( r = 2 \) m, \( h = 3 \) m.

**Stage 1: Reality.** Find things in reality for the exercises including translating from real to symbol; translate exercises to word-type problems and vice versa:

(a) Write a story for the exercise. For example: *I had $47 and my dad gave me $26, how much do I have now?* Try to keep the values realistic for the context (e.g. you would not use *The car cost $47 and the car dealer added a $26 profit, what did he sell the car for?*). You can ask students for different contexts and types of problems, for example: *Write a 47 + 26 story about fishing with the word/action “take-away”*. 

(b) Set up a story that determines if students know what surface area is. For example: *Jill cut two circles and a rectangle from a sheet of metal which she then welded into a tank of diameter 4 m and height 6 m. Her boss was angry because the sheet of metal would have to be replaced. The boss asked Jill to work out the smallest size of the replacement sheet but Jill could not lay it flat to measure. How could she find the size?”*
2. **Stage 2: Abstraction.** Find a way to do Body \( \rightarrow \) Hand \( \rightarrow \) Mind for the worksheet exercises, particularly finding kinaesthetic body activities for the exercises but not neglecting hand activities:

(a) Use materials for tens and ones to represent \( 47 + 26 \).

(b) Use a flat sheet of paper to make a cone by finding the midpoint of a long side and folding around to make a cone shape. Hold the cone at the point so the cone is vertical and make a horizontal cut around the edges away from the point to make a proper cone with a flat base. Cut off excess paper. Measure the height \((h)\), the radius of the base \((r)\), and note that the slant length \((l)\) is found by Pythagoras \(l = \sqrt{r^2 + h^2}\). Open out the cone so it is flat and find area using a grid \((SA = \text{surface area})\). Repeat for different-sized pieces of paper and draw a graph of \(h\) to \(SA\) and \(r\) to \(SA\).

3. **Stage 3: Mathematics – connections.** Get students to find other topics with the same types of problems:

(a) Translate \( 47 + 26 \) to problems in decimals, metric measures, non-metric measures such as time, and algebra.

(b) Find a way to relate surface area of a cone (no base) to a circle. How do \( r \) and \( h \) relate? For a cone, how do volume and surface area relate?

4. **Stage 4: Reflection**

*Applications.* Turn exercises into applications and do this for a variety of contexts and meanings:

(a) An addition exercise could be finding perimeter.

(b) Find the surface area of composite shapes, for example, shapes made of cylinders, cones and hemispheres, a thick water pipe where you have inside, outside and ends to coat with sealant, and so on.

*Flexibility.* Find the same idea in a variety of contexts:

(a) Adding time (where no longer base 10).

(b) Find the surface area of an 8-sided dice (octahedron), a truncated cone, a loudspeaker with two truncated cones facing in opposite directions, a shuttlecock in badminton, and so on.

*Reversing.* Go from “exercise to answer” to “answer to exercise”, reversing order in question or in equation:

(a) The exercise \( 47 + 26 \rightarrow 73 \) is reversed by saying: \( 73 \) is the answer, what is the addition question? Also, *What is the easiest question? What is the hardest?*

(b) For cones, have students find \( h \) when given \( SA \) and \( r \); find \( l \) given \( SA \) and \( r \); and find \( r \) when given \( SA \) and \( h \) or \( l \). For example, Greg needed 20 m\(^3\) of topsoil to put on his garden. He paid for the soil to be delivered to his house and asked the truck driver to tip the load of topsoil in his driveway. His driveway is 8 m wide. Will the load of topsoil fit in the driveway? (Note that this is a good open-ended problem as students will have to make an assumption about the height of the cone formed by the pile of topsoil.)

*Generalising.* Ask students to see the general way to extend the exercise:

(a) For exercise \( 47 + 26 \), you can ask students to provide, in words, the procedure they would follow to add \( AB + CD \), or to add \( 475 + 268 \).

(b) Repeat the Abstraction cone-making activity more than once. Have students find the relation between \( SA \) and \( r \) and \( SA \) and \( h \). Construct a formula for \( SA \) in terms of \( r \) and \( h \).
Method Two: Using problem knowledge

The second method is based on problem solving itself. There are two techniques here and we use, initially, the same examples as before.

1. **Changing.** Change problems by making the answer part of the changed problem (a form of reversing):
   
   (a) For exercise $47 + 26$, put in the answer (e.g. $47 + 26 = 73$), then rewrite with 73 in the problem (e.g. **John had 26 more books than Jack, John had 73 books, how many books did Jack have?**).
   
   (b) **What height, slant length and radius would make the best conical tent (no base) if the tent was made using 622.86 m$^2$ of material? Argue why your solution is best.**

2. **Removing bits.** Write down the exercise, putting the answer in, then remove some of the question parts:
   
   (a) For exercise $47 + 26$, write in the answer, $47 + 26 = 73$, remove bits, $\Delta 7 + 2\square = 73$, then ask for the bits: **What numbers are $\Delta$ and $\square$?**
   
   (b) **A cone tent (no base) of height 3 m used 23 m$^2$ of material; what $r$ and $l$ are possible? A cone tent (no base) has $h = 4 m$; what $r$ and $l$ gives the greatest SA?**

3. **Extra steps.** Change the exercises to two or more step problems:
   
   (a) For exercise $47 + 26$, add in an extra step such as: **I bought jeans for $47 and a hat for $26, how much change will I get from $100?**
   
   (b) Construct a problem of a series of steps with SA having to be calculated in one of the steps. For example: **A worker is painting a giant cone, paint costs ?? per m$^2$, labour is ?? per hour, how much does the painting cost if the worker paints ?? m$^2$ per hour?**

4. **Justifying.** Have students give an argument for why the solution is as it is, why the procedure is as it is, or why the formula is as it is:
   
   (a) **Why when we add 47 + 26 is the 4 added only to the 2 but when we multiply 47 × 26 the 4 multiplies the 6 as well as the 2?**
   
   (b) **Provide a logical explanation for the surface area of a cone being what it is.**

5. **Relating to other topics.** Relate to some maths ideas and teacher activities that enable problems:
   
   (a) For $47 + 26$, we can relate to subtraction and ask for a story where 47 joining 26 is subtraction.
   
   (b) For surface area, we can relate to volume, which leads to investigation of why babies cool down and heat up more quickly than adults or why fast food stores mainly have shoestring chips. We can also use other things, for instance, show a tank pictured beside a person, give the height of the person and then ask for volume and surface area of the tank where students have to use proportion for determining or estimating height and radius.

YDC has learnt over the years that teachers are great at thinking up ideas – so use worksheets as starting points to lessons and **get together with other teachers and brainstorm ideas** to give the lessons a problem-solving focus. Also, any movement of students from passive to active is worthwhile – so even letting the students choose which exercises they will do is a good starting point. Another point to note is that students can do the changing themselves – give exercises and ask the students to change them. The process of constructing problems is a powerful way for students to learn problem solving.
Constructing problems

You will need many problems. We have shown how to create some by converting worksheet exercises into problems; now we look at how to use problems to generate/construct many more problems. The methods to do this are as follows. The problem being used as an example is the handshake problem: Six couples meet for the first time, how many handshakes?

1. **Introduce proof.** Change the problem from finding a solution to proving that the solution found is correct (e.g. in the handshake problem above, now you ask students to prove to you that there are 30 handshakes; you can also extend the problem by asking students to provide a proof that their strategy to find the solution is also correct, or ask for a different proof from that given).

2. **Reverse the problem.** Turn the problem around and construct a problem that goes in the reverse direction (e.g. If the number of handshakes is 42, how many couples?).

3. **Identify a special case.** Look for something special in the problem, a special case, that can be capitalised on (e.g. cultural implications of handshakes – What if the women cannot shake hands with the men? What if half the people touch noses and do not handshake – how many handshakes then?).

4. **Generalise the problem.** Look for the rule that relates number of couples to handshakes, that is, ask how many handshakes for \( n \) couples. Do this also for the reverse – going from the number of handshakes back to number of couples. What about singles handshaking and triples and groups of four, and so on?

5. **Extend the problem.** Place the handshakes within a two or more step problem (e.g. Jenny and Mark handshake with fresh gloves for each person; they meet 4 other couples, then 8 other couples, finally 5 singles – how many new gloves?) – let your imagination run riot.

Implementing investigations

Investigations are multi-step tasks that require application of problem solving to work out what to do and get a solution. For example: Fold the edges of a square of paper to make a box. What is the biggest box that can be made by folding (a) a 20 cm \( \times \) 20 cm square of paper; and (b) any square of paper?

Use investigations as well as exercises for practice and assessment. Step-by-step tasks can be modified to be investigations. YDC’s MITI project that is designed to enrich/extend YDM provides many investigations (if interested, contact YDC).

The investigations used in MITI are based on Renzulli (1976) who argues that building powerful maths requires:

- **Motivation** – finding a topic or an activity that engages students and set up the investigation in this area.

- **Skill development** – providing a classroom environment where knowledge can be acquired and ensuring students have the skills needed to do the investigation.

- **Open investigation** – giving investigations based on the engaging topic/activity that are open and students can go as far as they are able.

Like problems, it is important to do more with investigations than simply handing them out and then passively fielding questions. To get the most out of an investigation, we recommend the following process (see Figure 5.5):

1. **Anticipate** – look at the investigation and your students’ abilities, and anticipate both where the difficulties will lie so you can prepare hints and also where the opportunities for enriching/extending the students exist so you can apply them.
2. **Monitor** – always be moving around the room and looking at groups and individuals so that as the teacher you are aware of where each group/individual is, how they are tackling the problem and what progress they are making.

3. **Select** – as your monitoring gives you information on each group/individual, select students/groups to report on what they did and why, so as to cover a wide range of strategies and have examples to build enrichment and extension.

4. **Sequence** – when you are ready to have your students/groups report on their progress, use (2) and (3) above to determine the best sequence by which to give reports – the recommendation is to show simpler first and the best last.

5. **Connect** – use (2), (3) and (4) to run a period where you connect different strategies, showing similarities and differences between strategies and connecting ideas that are related or part of the same big idea.

6. **Reflect** – always end discussions/lessons with reflections in which you ask students what they have learnt. A good way to do this is to give the students the start of a sentence, such as: *I learnt today that …; The new strategy I learnt today was ….*

**Pivotal teaching moments**

One of the most powerful ways to enrich and extend mathematics knowledge is for the teaching in the lesson to emerge from something the students bring up or something they have just done. For example, a student wonders why $\frac{1}{7}$ is smaller than $\frac{1}{3}$ – this gives an opportunity to connect division with fractions. These opportunities are called **pivotal teaching moments** (PTMs; Stockero & Van Zoest, 2013).

Three types of PTMs have been identified by YDC staff, as shown in Figure 5.6. These are described below.

**Anticipated PTMs**

These are PTMs that can be prepared for by analysing what is being taught and trying to identify where a PTM might occur before beginning teaching. This analysis depends on understanding what precedes the material to be taught in the lesson and what follows. Knowing what follows the lesson is particularly important because PTMs provide a good opportunity to introduce the next step in teaching.

The idea is to pre-plan for PTMs by (a) looking for sequences and connections with and to the topic being covered; (b) preparing a list of possible extensions/connections that could be brought up if the time was right; and (c) looking for and visualising the type of student activity that might generate a PTM. However, it is important not to let your anticipation blind you to other unanticipated opportunities for PTMs, and not to force the PTM on the class when it did not come up as you anticipated. PTMs are an opportunity not a necessity.

**Pedagogy-based PTMs**

Try to use a pedagogy that offers much in terms of setting up PTMs. The RAMR pedagogy has built-in opportunities: (a) body and hand activities; (b) connection activities; and (c) extension activities such as reversing or flexibility, and particularly generalising. In abstraction, we also advocate doing something creative such as
letting students construct their own symbols for the operations as a starting point for symbolisation. This creative component of abstraction is good for PTMs.

Knowledge of big ideas also allows for PTMs because, as a big idea teacher, you should always be looking for ways to widen teaching to the big ideas that relate to the topic you are teaching.

Renzulli-type investigations, the ones advocated by MITI (motivation, skills and open investigations), are also useful in terms of PTMs. The open nature of the students’ activities means there are many different discussions and presentations that are just made for PTMs.

**Unanticipated PTMs**

Because they are unanticipated, these PTMs are difficult to identify and are often missed by teachers. Also, even when identified, there is little time to determine whether the opportunity should be pursued. However, the following approaches can facilitate creating and identifying unanticipated PTMs:

(a) **extending** – always look for ways to introduce extensions for topics being taught;
(b) **sense-making** – look for situations where students are confused and a PTM on sense-making would benefit;
(c) **incorrect mathematics** – always look for students’ errors as a way to enrich and extend (but do not embarrass);
(d) **mathematics contradiction** – look for where two things are said in contradiction to each other and have a PTM debate on options; and
(e) **mathematical confusion** – look for confusion and try to use it to bring in a clarification PTM.

Finally, the important teaching style is to be active both with and in responses.

### 5.3 Mathematics big ideas

The most effective way to learn and remember powerful mathematics is through big ideas. If most of a topic can be reduced to a few big ideas, this is a very effective way to understand mathematics and build knowledge that can handle high-level mathematics subjects. This section looks at the mathematics big ideas that we believe are important. It is a summary of the YDM supplementary resource book, *Big Ideas of Mathematics*.

#### 5.3.1 Nature and justification of big ideas

Big ideas are mathematics ideas that can be used in many year levels and across different topic areas. Knowing mathematics in terms of big ideas is a powerful way to accelerate mathematics learning. A big idea covers a lot of mathematics and so there are not as many to remember as if trying to memorise separate mathematics rules and procedures. Therefore, they represent an excellent basis for teaching mathematics (T. Cooper, Carter, & Lowe, 2016).

**Nature**

The nature of big ideas is that they have a wide effect. For YDM, big ideas have some or all of the following properties:

1. They provide generic approaches to a wide range of ideas – they encompass viewpoints that cross boundaries. For example, mathematical actions can be considered as both a relationship (static) and a change (dynamic), as in the addition example on right.

2. They apply across topic areas – they have some generic capabilities that are not restricted to a particular domain. For example, the inverse relation in division between divisor and quotient also applies to measurement, fractions and probability.
3. They apply across year levels – they have the capacity to remain meaningful and useful as a learner moves up the grades. For example, the concept of addition holds for early work in whole numbers, work in decimals, measures, common fractions, and algebraic variables.

4. Their meaning is independent of context and content – it is encapsulated in what they are and how they relate, not the particular context in which they operate. For example, the commutative law says that first number + second number = second number + first number irrespective of content type (e.g. whole numbers, decimal and common fractions, algebra or functions).

5. They are teaching approaches that apply across ideas – they have the capacity to apply to many situations. For example, the teaching approach of reversing (reversing the order of activity in a lesson) applies everywhere, including going whole to part and part to whole, shape to symmetry and symmetry to shape, algorithm to answer and answer to algorithm.

**Justification**

The justification for focusing on big ideas is that they are very effective ways to accelerate mathematics learning, for the following reasons:

1. One big idea can apply to a lot of mathematics – this makes them powerful ways to teach and understand mathematics. For example, part-part-whole, multiplicative comparison (double number line) and start-change-end diagrams can solve most fraction, percent, rate, and ratio problems (i.e. they reduce cognitive load).

2. One big idea can cover work that would need many procedures and rules to be rote learnt – for example, the distributive law and area diagrams can be used to understand and solve multiplication problems in whole numbers, fractions and algebra, such as 24 × 37, ⅓ × ⅗ and (x – 1)(x + 2).

3. Big ideas are organic in that later learning can fit within them – they build structural connectivity across domains of mathematics, developing rich schema that can easily accommodate new ideas. For example, building the notion of inverse as “undoing things” and teaching the inverse relationships between +2 and –2; x5 and ÷5; x² and √x; p³ and p⁻³; (p)ⁿ and (p)¹/n; f(x) = 2x + 1 and f(x) = (x – 1) ÷ 2 can make it easy to understand integration as the inverse of differentiation in calculus. This is a particularly strong reason for focusing on big ideas – the fact that they make later learning easier.

5.3.2 Types of big ideas

YDM identifies five types of big ideas (see Figure 5.7).

1. **Global big ideas** – these relate to nearly all mathematical ideas and all year levels. For example, the commutative principle is not a global big idea because it only refers to addition and multiplication situations; on the other hand, change vs relationship is global because it refers to all mathematics, saying that every idea can be considered both as a change and as a relationship.

2. **Concept big ideas** – these are the meanings of ideas that are common across mathematics, for example, the meanings behind equals and multiplication. Such meanings have large impact and can help in many topic areas, from operations to algebra and measurement to statistics.

3. **Principle big ideas** – these are relationships where meaning is encoded in the relation of the parts, rather than in their content. For example, the commutative principle (turnarounds – e.g. 1² + 2² = 2² + 1²) is an example of a principle big idea because it also holds for many contexts (e.g. whole numbers, decimals, fractions, variables, functions), while 2 + 3 = 5 is contentful (holds for 2, 3 and 5) and is not a big idea.

4. **Strategy big ideas** – these are ways of solving exercises and problems that apply to a range of mathematics across year levels. For example, the part-part-total (PPT) structure underpins all operations and is a powerful strategy in solving word problems and fraction, percent and ratio problems.
5. **Pedagogy big ideas** – these are ideas for teaching that are generic in their application – that can apply to the teaching of many mathematics ideas. For example, the teaching approach of *reversing* where the teaching direction between teacher and student is reversed (e.g. from *What is 5 + 8?* to *What addition facts give answer 13?*) can apply in many situations other than addition.

The table in **Appendix A** lists the major big ideas. A complete list is available in the supplementary YDM resource book *Big Ideas of Mathematics*. Each YDM teaching book also contains a list of the big ideas that are important for the topic being covered in the book.

### 5.4 Implications for future programs

Any program adopted by schools needs to provide for able students and should be aiming for the highest mathematics outcomes, particularly if the school has Aboriginal, Torres Strait Islander or low-SES students. The framework for this needs to cater for deep learning of powerful mathematics, which includes taking into account problem solving and investigations, mathematical schematic structure and big ideas. Using pivotal teaching moments is also important. The concept map in Figure 5.8 summarises the main points in this chapter that deal with these three components of a comprehensive mathematics program.

![Figure 5.8 Framework for enrichment and extension](image-url)
One of the questions to face as you leave the training phase of YDM and move on to setting up a sustainable way for YDM to operate in your school is how to ensure that able students are developed and supported. As can be seen in the concept map, this chapter looked at the components of problem solving, the nature and teaching of mathematics schematic structure, and the big ideas of mathematics. The concept map provides a summary framework showing some of what we have covered so that you can start thinking about and discussing how your school will take able students into account, and how it could develop more able students in the future.

Your task from this chapter is to develop a plan for how to ensure that able students are developed and supported in your school.

Look at the summary in Figure 5.8 and determine what your emphases would be in your school if you could implement what you want. The questions below may help.

1. How will your school cater for able students?
2. What framework will you put in place?
3. How will your school enrich and extend the mathematics of high-potential students who are not exhibiting ability?
4. How will you use assessment and change teaching programs to achieve this?

In doing this you may like to look at YDM MITI and the pedagogic focus and resources available in this project. In particular, the three supplementary YDM books *Big Ideas of Mathematics*, *Problem Solving*, and *Literacy in Mathematics* are worth attention. At present MITI is secondary oriented and aims to increase the number of students with both the skills and the motivation to participate in high-level mathematics subjects and tertiary courses based on mathematics.
6 Towards Sustainability

YDC would like the schools we work with to be able to take from YDM what works best for their students and maintain that into the future. We aim for sustainability of YDM in your school. To help achieve this goal, this final chapter draws together the ideas from all eight YDM books, this one plus the previous seven.

YDM is a mathematics pedagogical approach that focuses on teachers, working with you to develop better ways to teach students. While the YDM program includes resources, it does not rely on them in the way that a textbook series does. We expect that teachers will modify the resources to suit their students, using them as a model of good practice and a source of inspiration rather than a prescriptive process that must be followed in all circumstances.

Our experience shows that a school’s use of the YDM pedagogical approach stays strong under the following circumstances:

- trained teachers continue at the school and are active in passing on YDM ideas to new teachers;
- the school leadership team is supportive, actively promotes YDM and ensures that teachers have the opportunity to mentor each other and plan together; and
- the school has an ongoing training (induction) regime for teachers who are new to the school.

With these three pillars in place, YDM is sustainable within schools.

The question remains, as the YDC-led YDM training finishes, how do you implement this sustainability? This final chapter discusses and proposes ideas to address sustainability. It comprises the following sections:

(a) Summary of YDM (6.1) – provides a summary of all eight books, particularly the Overview and Review books, and draws implications for sustainability.

(b) Supported sustainability (6.2) – describes support available from YDC for sustainability, including access to online modules, practitioner visits and/or further YDM training, training in AIM, AIM EU or MITI.

(c) Independent or cluster sustainability (6.3) – discusses ways to build sustainability independent from YDC, using cluster coordinators, school coordinators, and school processes.

6.1 Summary of YDM

6.1.1 Summarising figure

The eight YDM books are summarised in Figure 6.1 on the next page. The structure of the diagram is as follows.

1. Overview book – This book is placed in the central red circle, with its summary of YDC’s position regarding beliefs and imperatives; philosophy, learning and culture; pedagogical framework; school change and leadership; and implementing YDM.

2. How-to-teach books – The application of the Overview ideas to the six “how-to-teach” books, covering Number, Operations, Algebra, Geometry, Measurement, and Statistics and Probability, are indicated by these books being spread around in the second circle (or the “donut”).

3. Integration – The ideas that emerge from the integration of the Overview book with the seven “how-to-teach books” are added to the centre – these consist of the focus on structure (connections, big ideas and sequencing) and the RAMR cycle.
4. **Review book** — Finally, the *Review* book is summarised around the outside of the circle in the four coloured boxes: visions of mathematics and its teaching; proficiencies, affects and language/literacy; assessment, diagnosis and remediation; and enrichment and extension.

![Diagram of YuMi Deadly Maths components](image)

**Figure 6.1 Summary of YuMi Deadly Maths components**

### 6.1.2 Implications for sustainability

Figure 6.1 gives an indication of what is involved in a sustainable YDM program for a school:

(a) It follows the YDM approach to structure (connections, big ideas and sequencing), pedagogy (reality–abstraction–mathematics–reflection or RAMR cycle) and implementation (whole school change, community involvement and professional learning cycle).

(b) It covers the ideas on how to teach in the resources for all the mathematics strands from Years P to 9, and renews these ideas as the school and community change.

(c) It determines an approach/plan for mathematics that takes account of the needs of students and community, using the ideas from the teaching books and this *Review* book.

(d) It has a process that trains new teachers in YDM.

To implement the YDM approach, YDC uses a train-the-trainer process which is based on training teachers to improve their capacity to teach mathematics in a way that can change students’ futures. Therefore, it is built
around trained teachers and processes that continually reflect on, act on, and provide feedback on the effectiveness of the approach for existing students. The central feature of schools that have sustained the approach after the two years of YDM teacher development training is that they have a process for training new teachers to maintain the total number of trained teachers in the school, and a plan for continual renewal of their mathematics program. The following two sections discuss ways to support this sustainability.

6.2 Supported sustainability

The first way to maintain sustainability of YDM in your school, called supported sustainability, is to continue an association with the YuMi Deadly Centre. This association can be in up to five ways, or any combination of the five, as shown in Figure 6.2. Schools wishing to avail themselves of any of these ways should contact YDC by phone (07 3138 0035) or email (ydc@qut.edu.au).

6.2.1 Online learning and practitioner visits

Schools that have completed YDM training, or one of the other projects based on YDM, can maintain interest and support new staff through continued access to the YDM Professional Learning Online Blackboard community site or continued visits by YDC staff, as described below.

**Online learning modules**

The YDM online Blackboard site contains:

1. learning modules to support YDM training for each of the YDM resource books;
2. copies of the latest editions of the YDM resource books and the three supplementary books, Big Ideas of Mathematics, Problem Solving, and Literacy in Mathematics, for downloading;
3. over 80 exemplar RAMR lessons, at least 8 for each year level from P to 9;
4. PowerPoint materials to support professional learning on the information in the books; and
5. a discussion board where teachers share ideas.

The Blackboard site can be used to support training of new teachers or simply used as a resource that new teachers can work through to give them knowledge of YDM. It contains talks and videos of activities along with the written information. It can be used to ask YDC practitioners, or other teachers, questions concerning implementation of YDM in the classroom, the answers to which are placed on the discussion board.

Access to this website, and YDM materials, can be negotiated by contacting YDC.

**YDC practitioner visits**

Schools that have completed YDM training but have limited opportunities to train new staff can negotiate with YDC for an expert practitioner to visit the school to work with the new staff. Ideally, this would be one day per term or four days per year but any time can be negotiated.

A YDM practitioner can visit the school and work with the new teachers as the school requires. After-school PD for all staff can also be negotiated. The visits require preparation and follow-up time and this is part of the negotiation. The school would let the practitioner know in advance what they want so that materials can be prepared and would then provide the practitioner with follow-up requirements that have emerged from the visit to be provided electronically.
This type of support could be provided electronically but the visits enable the practitioner to gain information on context that can allow better targeting of support.

6.2.2 Further YDM training

Schools that have completed YDM training, but feel that for better development of the program in their school they need more teachers with training, can arrange for additional teachers to attend a new training cluster. This option is usually only taken up by larger schools who find that four trained teachers is not enough to ensure that in-school trialling and training can be effective. However, some smaller schools have used this option after losing nearly all their YDM-trained teachers in transfers. Some schools have continued this option for ongoing training of new teachers. This is usually to ensure that a significant number of their teaching staff are trained through the workshops to meet in-school training needs.

Some schools have also used this option after some years when YDM has declined in their school. Their objective has been to renew teaching with YDM across all year levels – to start again. Other schools have simply chosen to have more than four trained teachers as part of their initial YDM training.

Funding for this additional YDM training option is negotiated with YDC, but can be at a lower rate per teacher than the original cost of the training.

AIM training

The YDM pedagogy has been extended to projects that focus on remediating underperforming students, accelerating their learning from their present ability level to their year level. The name given to these projects, and the resources supporting them, is Accelerated Inclusive Mathematics (AIM).

Schools that have many underperforming students can sustain YDM through being part of an AIM project. The same pedagogy is used but there is a focus on accelerating knowledge through vertical sequences. This pedagogy is described in section 4.3 of this Review book.

Once again, this option is negotiated with YDC. It can be an effective way of sustaining YDM as it builds YDM into the remediation processes of the school. Although AIM was originally developed for Years 7 to 9, the modules cover Year 4 mathematics to Year 9 mathematics, and the AIM Early Understandings (EU) project has developed number, operations and early algebra modules to cover Foundation to Year 2. This option is therefore available for primary as well as secondary schools.

MITI training

The YDM pedagogy has also been extended to projects that focus on enriching and extending students’ mathematics knowledge to develop deep learning of powerful ideas. The name given to these projects, and the resources that support them, is Mathematicians in Training Initiative (MITI). MITI projects, at this point, are limited to secondary as they focus on Years 7 to 12.

Secondary schools that have able students can sustain YDM through being part of a MITI project. Like AIM, MITI also uses the YDM pedagogy but there is a focus on enrichment, extension, deep understanding and powerful mathematics ideas. This pedagogy is referred to, and its main ideas covered, in Chapter 5 of this book.

Once again, this option is negotiated with YDC. It can be an effective way of sustaining YDM as it builds YDM into the enrichment and extension processes of the school.
6.3 Independent or cluster sustainability

The second way to maintain sustainability of YDM in your school, called independent or cluster sustainability, is independent of YDC. It involves developing processes for maintaining YDM training and school development within your school after having received training from YDC, or within a cluster that has received training. This approach is usually associated with a strong school plan to implement YDM, supported by the administration and driven by highly motivated staff. Some examples of how to do this are as follows.

6.3.1 Coordinators

Cluster coordinators

A group of schools in a close geographical or system cluster could make a group decision to maintain and sustain YDM in their schools. By pooling their funding, it could be possible for this cluster to employ a coordinator/trainer with respect to YDM. This person would take on a role similar to that of YDC staff by leading training of new staff and supporting existing staff to maintain YDM across the schools in the cluster. This approach would work best if the cluster coordinator/trainer could build a community of practice around YDM ideas across the schools and coordinate the sharing of teaching ideas. Negotiation with YDC staff could allow such a person to shadow YDC staff training of teachers in the second year of any project.

It is particularly productive for clusters of schools to share the induction training in YDM of teachers new to the school. Often there are too few teachers new to a school to justify the provision of a dedicated YDM training program for them. However, when resources are pooled across several schools, the new teachers in those schools become sufficient to warrant a dedicated program. Ideally, this induction training in YDM could occur at the beginning of the school year. It could be run by a YDC practitioner or one or more teachers from the schools in the cluster.

School coordinators

If a cluster is not available, it is possible to do the above within a school. A teacher could take the role of coordinator/trainer with respect to just the teachers in the school. With state and federal government funding supporting coaches, Heads of Curriculum, Heads of Department, and Master teachers, schools could appoint someone as the school YDM coordinator/trainer. Once again, negotiation with YDC staff could allow such a person to shadow YDC staff training of teachers in the second year of any project.

Schools that implement YDM most successfully usually have a staff member with some time allocation as school liaison or director of YDM.

State coordinators

A third option is for the mathematics specialists employed in state, territory or regional education authorities to be trained by YDC and accredited or licensed to deliver YDM training to schools in their jurisdiction. Accredited trainers would be able to access the intellectual property and resources developed in YDC and draw on the advice of YDC practitioners. This model would be difficult to implement. It is an idea in a very early stage of development. However, some education authorities have and are considering this approach, and support from principals and teachers may help in justifying the necessary allocation of resources.

6.3.2 Processes

Some schools develop processes within their school group structures to maintain YDM and inculcate new staff. The following three examples give an idea of what could be involved. These examples reflect composites of ideas from many schools. Figure 6.3 on the next page summarises these processes in diagrammatic form.

1. Regular PD, planning and sharing. One school set up a process of after-school PD sessions and school meetings where, once a month, staff prepared RAMR lessons by year levels. In the intervening three weeks, staff trialled these lesson with other teachers watching. An observation schedule was developed for staff to
critique and improve the lessons. Across a year, the staff came to understand YDM more deeply and became independent creators of YDM lessons, building a critical mass to maintain YDM in the school.

2. **Constructing own teaching materials.** There are two main ways to do this:

   (a) A second school set up a process with teachers to jointly plan and write a curriculum based on YDM-style lessons and units, built around the big ideas. As all teachers were involved, the preparation load was reduced. The resulting curriculum became the basis of teaching at the school and, thus, potentially built YDM into the teaching fabric of the school, a powerful form of sustainability.

   (b) A third school did similar to this second school by developing a variety of units across year levels and setting up a program to “fill in the spaces” as the years passed.

3. **Distributed leadership.** There are two ways of doing this as well:

   (a) A fourth school set up a process to build YDM around year levels. A teacher was identified in each year level to act as leader for that year and was given time to mentor other teachers in that year to trial YDM ideas and pedagogies.

   (b) A fifth school did the same as this but with respect to groups of year levels (e.g. F–2, 3–4 and 5–6). This distributed leadership form was effective at building a school ethos of YDM.

All these cases built sustainability around school practices and, as such, facilitated teachers coming to feel that they **owned YDM** in those schools. Many of these schools integrated what they were doing in their school with accessing YDM online learning modules, some practitioner visits, and targeted further YDM training.

The examples above show that it is possible for schools to build sustainability by developing a plan tailored to the needs of the school. YDC can support such sustainability and is available for negotiating blended programs.

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Your task from this chapter is to develop a plan for how to make YDC sustainable in your school. Use the ideas above and from Figures 6.1 to 6.3. Determine what way would be most successful for your school.
References


**Appendix A: Major Big Ideas**

Table A1  Major big ideas by topic area or strand

<table>
<thead>
<tr>
<th>Topic area</th>
<th>Big ideas</th>
</tr>
</thead>
</table>
| **Global big ideas**| • **Symbols tell stories.** The symbols of mathematics enable the world to be described succinctly and in a generalised way (e.g. 2 + 3 = 5 means caught 2 fish and then caught another 3 fish, or bought a $2 chocolate and $3 drink, or joined a 2 m length of wood to a 3 m length, and so on).  
• **Change vs relationship.** Everything can be seen as a change (e.g. 2 goes to 5 by +3) or as a relationship (e.g. 2 and 3 relate to 5 by addition).  
• **Probabilistic vs absolutist.** Things are either determined by chance (e.g. will it rain?) or are exact (e.g. what is $2 + $5$?).  
• **Accuracy vs exactness.** Problems can be solved accurately (e.g. find $5.275 + 3.873$ to the nearest 100) or exactly ($5.275 + 3.873 = 9.148$).  
• **Continuous vs discrete.** Attributes can be continuous (smoothly changing and going on forever – e.g. a number line) or they can be broken into parts and be discrete (can be counted – e.g. a set of objects). Units break continuous length into discrete parts (e.g. metres) to be counted.  
• **Part-part-total/whole.** Two parts make a total or whole, and a total or whole can be separated to form two parts (e.g. fraction is part-whole, ratio is part to part; addition is knowing parts, wanting total). |
| **Numeration big ideas** | • **Part-whole/Notion of unit.** Anything can be a unit – a single object, a collection of objects, a section of a line, a collection of lines. Units can form groups and units can be partitioned into parts (e.g. if there are six counters, each counter can be a unit, making six units, or the set of six can be one unit.)  
• **Concept of place value.** Value is determined by position of digits in relation to ones place.  
• **Additive/Odometer.** All positions change forward from 0 to base, then restart at 0 with position on left increasing by 1, and the opposite for counting back (e.g. $2\frac{2}{5}, 2\frac{1}{5}, 3, 3\frac{1}{5}$, and so on).  
• **Multiplicative structure.** Adjacent positions are related by moving left (× base); moving right (÷ base). Base is normally 10 or a multiple of 10 in Hindu-Arabic system and metrics.  
• **Number line.** Quantity on a line, rank, order, rounding, and density. |
| **Equals, operations and algebra big ideas** | • **Concepts of the operations.** Meanings of addition, subtraction, multiplication and division.  
• **Equals and order.** Reflexivity/non-reflexivity – $A = A$ but $A$ is not $> A$; Symmetry/antisymmetry – $A = B \rightarrow B = A$ while $A > B \rightarrow B < A$ and $A < B \rightarrow B > A$; Transitivity – $A = B$ and $B = C \rightarrow A = C$ and $A > B$ and $B > C \rightarrow A > C$.  
• **Balance.** Whatever is done to one side of the equation is done to the other.  
• **Identity.** 0 and 1 do not change things (+/− and ×/÷ respectively).  
• **Inverse.** A change that undoes another change (e.g. +2/−2; ×3/×3).  
• **Commutativity.** Order does not matter for +/× (e.g. $8 + 6 / 6 + 8; 4 \times 3 / 3 \times 4$).  
• **Associativity.** What is done first does not matter for +/× (e.g. $(8 + 4) + 2 = 8 + (4 + 2)$, but $(8 + 4) + 2 \neq 8 + (4 + 2)$).  
• **Distributivity.** ×/÷ act on everything (e.g. $2 \times (3 + 4) = 6 + 8; (6 + 8) \div 2 = 3 + 4$).  
• **Compensation.** Ensuring that a change is compensated for so answer remains the same – related to inverse (e.g. $5 + 5 = 7 + 3; 48 + 25 = 50 + 23; 61 - 29 = 62 - 30$).  
• **Equivalence.** Two expressions are equivalent if they relate by ×+1 – also related to inverse, number, fractions, proportion and algebra (e.g. $48 + 25 = 48 + 2 + 25 + 2 = 73; 50 + 23 = 73; 4 + 3 = 4 + 2 = 4 + 3\div 2 = 4 + 1\div 2 = 4 \div 2$).  
• **Inverse relation for −, ÷ direct relationship +, ×.** The higher the number the smaller the result (e.g. $12 \div 2 = 6 > 12 \div 3 = 4; 1\div 2 > 1\div 3$); the higher the number the higher the result (e.g. $4 \times 3 < 4 \times 7$).  
• **Backtracking.** Using inverse to reverse and solve problems (e.g. $2y + 3 = 11$ means $y \times 2 + 3$, so answer is $11 - 3 = 2 \times 4$).  
• **Basic fact strategies.** Counting, doubles, near 10, patterns, connections, think addition, think multiplication.  
• **Operation strategies.** Separation, sequencing and compensation.  
• **Estimation strategies.** Front end, rounding, straddling and getting closer. |
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| Measurement big ideas | - **Concepts of measure.** Length, perimeter, area, volume, capacity, mass, temperature, time, money/ value, angle.
- **Notion of unit.** Understanding of the role of unit in turning continuous into discrete.
- **Common units.** Must use same units when comparing and calculating (e.g. a 3 m by 20 cm rectangle does not have an area of 60).
- **Inverse relation.** The bigger the unit, the smaller the number (e.g. 200 cm = 2 m).
- **Accuracy vs exactness.** Same as Global principle (e.g. cutting a 20 cm strip usually does not give a length of exactly 20 cm).
- **Attribute leads to instrumentation.** The meaning of an attribute leads to the form of measuring instrument (e.g. mass is heft or pushing down on hand, so measuring instrument is how long it stretches a spring).
- **Formulae.** Perimeter, area, volume formulae.
- Using an intermediary. Using string to compare length of a pencil with distance around a can. |
| Geometry big ideas | - **Concepts.** All types of angles, lines, 2D shapes and 3D shapes, flips-slides-turns, symmetries, tessellations, dissections, congruence, coordinates (Cartesian, polar), plotting graphs (slope, y-intercept, distance, midpoint), types of projections, similarity, trigonometry, topology, networks.
- **Formulae.** Angle, length, diagonal and rigidity formulae and relationships – interior angle sums, Pythagoras, trigonometry (sine, cosine and tangent), number of diagonals, number of lines to make rigid.
- **Reflection and rotational relationship.** Number of rotations equals number of reflections; rotation angle double reflection angle (holds for symmetry and Euclidean transformations).
- **Euler’s formula.** Nodes/corners plus regions/surfaces equals lines/edges plus 2 (holds for 3D shapes and maps).
- **Transformational invariance.** Topological transformations change straightness and length, projective change length but not straightness, and Euclidean change neither.
- **Visualising.** Mental rotation, choosing starting piece. |
| Statistics and probability big ideas | - **Tables and graphs.** Types of charts and tables, comparison graphs, trend graphs and distribution graphs.
- **Concept of probability.** Chance (possible, impossible and certain), outcome, event, likelihood.
- **Inference concepts.** Variation, error, uncertainty, distribution, sample, and inference itself.
- **Experimental vs theoretical.** Knowing when something can be calculated or determined by trials.
- **Equally likely outcome.** Outcomes as a fraction by number giving result ÷ total number.
- **Formulae.** Mean, mode, median, range, deviation, standard deviation, quartiles.
- **Integration of different knowledges.** For example, question Do typical Year 7 students eat healthily? requires some form of data gathering, determining typical, and determining healthy eating. |
| Pedagogy big ideas | - **Interpretation vs construction/Generation vs illustration.** Things can either be interpreted (e.g. what operation for this problem, what properties for this shape) or constructed (write a problem for 2 + 3 = 5; construct a shape of 4 sides with 2 sides parallel) – activities should generate students’ knowledge not illustrate teachers’.
- **Connections lead to instruction/Seamless sequencing.** Two connected ideas are taught similarly and progress from one to the other should not involve changing rules.
- **Pre-empting and peel back/Compromise and reteaching.** Look forward and back – teach for tomorrow and rebuild from known – be aware what ends and what lasts forever and rebuild ideas not lasting.
- **Gestalt leaps and superstructures.** Look out for ways of accelerating knowledge.
- **Language as labels/Construction before explanation.** New ideas to be constructed not told.
- **Unnumbered before numbered.** Big ideas are best started in situations without number.
- **Creativity.** Let students create own language and symbols (particularly to support pattern).
- **Triadic relationships.** When three things are related, there are three problem types (e.g. 2 + 3 = 5 can have a problem for: ? + 3 = 5, 2 + ? = 5, 2 + 3 = ?).
- **Problem solving.** Metacognition, thinking skills, plans of attack, strategies, affects, and domain knowledge.
- **RAMR cycle.** All components of RAMR cycle are big pedagogy ideas. |