YuMiDeadly
Growing community through education

YuMi Deadly Maths
Mathematicians in Training Initiative

MITI
OVERVIEW BOOK

VERSION THREE

MARCH 2018

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ACKNOWLEDGEMENT

The YuMi Deadly Centre acknowledges the traditional owners and custodians of the lands in which the mathematics ideas for this resource were developed, refined and presented in professional development sessions.

YUMI DEADLY CENTRE

The YuMi Deadly Centre (YDC) is a research centre within the Faculty of Education at the Queensland University of Technology which is dedicated to enhancing the learning of all students to improve their opportunities for further education, training and employment, and to equip them for lifelong learning.

"YuMi" is a Torres Strait Islander Creole word meaning "you and me" but is used here with permission from the Torres Strait Islanders' Regional Education Council to mean working together as a community for the betterment of education for all. "Deadly" is an Aboriginal word used widely across Australia to mean smart in terms of being the best one can be in learning and life.

YDC's motif was developed by Blacklines to depict learning, empowerment, and growth within country/ community. The three key elements are the individual (represented by the inner seed), the community (represented by the leaf), and the journey/pathway of learning (represented by the curved line which winds around and up through the leaf). As such, the motif illustrates YDC's vision: *Growing community through education*.

YDC can be contacted at ydc@qut.edu.au. Its website is https://research.qut.edu.au/ydc.

ABOUT THE MITI PROJECT

Mathematicians in Training Initiative (MITI) is a YDC mathematics project designed to improve Years 7–12 students' mathematics learning and thereby employment and life chances. The focus of MITI is to enrich and extend mathematics teaching to build confidence and interest in, and deep understanding of, powerful mathematics ideas in order to improve participation in high-level senior secondary mathematics subjects, university entrance and mathematically based careers, particularly for low-SES schools. It is a two-stage project: Stage 1 – investigations, problems, and seamless sequencing of powerful mathematics; and Stage 2 – deep applications in futures contexts. The overarching aim of MITI is to develop the capacity of high schools to build strong, high-level mathematics classes and university opportunities for their students.

To join a project, contact ydc@qut.edu.au.

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Abbreviations

AIM	Accelerated Inclusive Mathematics
AIM EU	Accelerated Inclusive Mathematics – Early Understandings
ICT	information and communication technology
MITI	Mathematicians in Training Initiative
PD	professional development
RAMR	Reality–Abstraction–Mathematics–Reflection
REAL	Review–Explore–Analyse–Link
SES	socio-economic status
STEM	science, technology, engineering, mathematics
XLR8	Accelerating Mathematics Learning
YDC	YuMi Deadly Centre
YDM	YuMi Deadly Maths

1 Background, Stages and Beliefs

The YuMi Deadly Centre (YDC) is a research and service centre in QUT's Faculty of Education dedicated to improving mathematics teaching and learning, and thus employment and life chances, of all students. To achieve its purpose, YDC has developed three types of teacher-training projects based on the mathematics pedagogy developed by YDC staff called **YuMi Deadly Maths** (YDM), as follows: (a) YDM general pedagogy projects; (b) three YDM accelerated learning projects: Accelerated Inclusive Mathematics (AIM), Accelerating Mathematics Learning (XLR8), and AIM Early Understandings (AIM EU); and (c) a YDM enrichment and extension project, the Mathematicians in Training Initiative (MITI).

The three types of projects have different purposes: YDM general pedagogy projects provide training in the basic YDM pedagogy for teachers of mathematics in Foundation to Year 9; YDM accelerated learning projects provide training in an acceleration pedagogy for teachers of students whose mathematics learning falls behind their year level; and MITI provides training in an enrichment and extension pedagogy for teachers preparing students for success in high-level mathematics subjects

This Overview book describes MITI in relation to the YDM pedagogy. After this first chapter, the remaining chapters of the book:

(a) summarise the structural basis of the YDM pedagogy and its relation to MITI;

(b) summarise the YDM Reality–Abstraction–Mathematics-Reflection (RAMR) teaching cycle and its relation to MITI;

(c) describe the proposed two stages in MITI projects as they are currently conceived; and

(d) discuss how MITI could be implemented in schools.

This chapter introduces MITI by providing background about its purpose, objectives and outcomes, summarising the two stages of MITI, and discussing the beliefs behind MITI.

1.1 Background of MITI

The focus of MITI is threefold: (a) to improve mathematics performance in schools; (b) to increase participation in Years 11 and 12 high-level mathematics subjects for all students; and (c) to achieve equity between Indigenous or low-SES students and other students in high-level mathematics subjects. The aim is to increase the number of students meeting the mathematics entry requirements for university courses leading to science, technology, engineering and mathematics (STEM) professions; particularly for those from schools where this has been historically low, such as schools with high enrolments of Indigenous and low-SES students. Attracting more students to the study of high-level mathematics and entry into STEM disciplines will enhance Australia's economic future.

1.1.1 Purpose

At present, urban secondary schools with significant numbers of low-SES students face real difficulty in preparing academically high-performing students for university entry into STEM-related courses. These schools are confronted by the prospect that their ethos and culture is becoming increasingly anti-academic; a status reinforced by home, community and teacher low expectations (Sarra, 2003) and peer pressure to reject academic work as a viable option upon which to develop a future (Carroll et al., 2009). Diminished academic performance may be attributed to the following factors: past enrolment at an underperforming primary school; attendance and behaviour difficulties (Cooper, Baturo, Warren, & Grant, 2006); living in poverty; violence and substance abuse in dysfunctional homes; cultural and language backgrounds that conflict with the culture of schooling (Bellert, 2009); and a lack of academic role models.

However, YDC believes these low-SES schools do have students with the potential to be academically high performing, since low performance is often a function and consequence of inadequate prior experiences and low cultural capital (Bourdieu, 1993), not intelligence. Unfortunately, these high-potential students are often neglected because schools struggle to develop and sustain a high-performing academic stream that caters to the needs of these students. Despite this trend, there are examples of school change where new mathematics teaching and learning practices have resulted in mathematically high-performing students.

The MITI project represents a convergence of educational, social and economic benefits. Additionally, although it focuses on mathematics, the project facilitates capacity building of academic studies in low-SES secondary schools, so that all students have opportunities to meet their academic potential in all school subjects and gain entry into university courses that lead to high-value employment, through a general uplift in abilities. It represents a win-win solution for government, community and individuals in relation to “closing the gap” (Department of Families, Housing, Community Services and Indigenous Affairs, 2009; Steering Committee for the Review of Government Service Provision, 2009).

1.1.2 Focus

The design of MITI has been influenced by an analysis of previously successful YDC mathematics projects in terms of ontology, pedagogy and methodology. The project reflects a convergence of cognitive, affective and cultural research built around a framework for building academic studies through use of information and communication technologies (ICTs) and professional development (PD) for teachers and a focus on deep learning for students (see Figure 1). The design has three components:

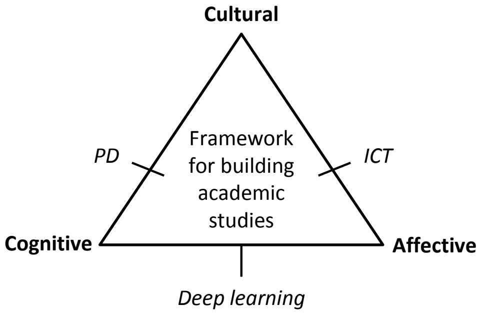


Figure 1 Design of MITI project

1. **Cognitive**. The project focuses on structural learning of mathematics through abstraction or generalisation (Cooper & Warren, 2008), aiming to reveal the big ideas of mathematics (e.g. the abstract schema of Ohlsson, 1993), and developing a “mathematical eye” through which to view and interpret the world. The pedagogy is based on social constructivism (English & Halford, 1995), but built around a cycle of instruction, YDM’s Reality–Abstraction–Mathematics–Reflection or RAMR teaching cycle, which was influenced by Wilson’s Activity Type cycle (Ashlock, Johnson, Wilson, & Jones, 1983), and the instructional levels of Baturo, Cooper, Doyle, and Grant (2007).
2. **Affective**. The project is based on integrating mathematics instruction with strategies for whole-school change similar to those advocated by the Stronger Smarter Institute (Sarra, 2003), namely: to challenge behaviour, build pride, make identity positive with respect to learning, ensure high expectations, enable local leadership, and integrate school and community. The project also focuses on making students’ and teachers’ affective responses more positive: building interest, motivation, resilience, confidence and positive attribution with respect to mathematics and with respect to liking and succeeding in mathematics.
3. **Culture**. The project is based on contextualising mathematics to students’ culture and language. The research of YDC staff in communities where language, activity and culture are different from that of mainstream schools has shown that: (a) local language is a major part of mathematics instruction (Schäfer, 2010; Young, van der Vlugt, & Qanya, 2005); (b) local knowledge must be made legitimate within the classroom; and (c) local culture must be celebrated and not negated (Baker, Street, & Tomlin, 2006; Ewing, Cooper, Baturo, Matthews, & Sun, 2010). In this way, students’ mathematics learning begins with what is already known.

The framework for building academic studies is based on PD to improve teacher and school capacity, ICT use to relate mathematics learning to 21st-century skills and enable authentic investigations, and a sequencing of instruction that enables growth in deep understanding of mathematics. In particular:

1. **Professional development** (PD) takes account of theories of effective teacher change evident in Baturo, Warren, and Cooper (2004) and Lamb, Cooper, and Warren (2007), particularly around interaction between researcher input and teacher need and the role of success when trialling new ideas. It builds communities of practice in which teachers act together with the student population (Wenger, 1998).

2. **Information and communication technology** (ICT) uses leverage to amplify the effect available from ICT (Jonassen, 1992) when used to model mathematics. Growth in deep understanding is based on theories from ARC Linkage Project LP0348820 (Cooper & Warren, 2011), which argues such understanding is based on generalisation and abstraction and this occurs across models and representations when these follow a nested structured sequence.

1.1.3 Objectives and outcomes

To bring about change with regard to participation in mathematics, MITI aims to focus on mathematics attitudes, beliefs and learning with the objective of developing teacher capacity to build students' interest, confidence and knowledge in mathematics; that is, to develop **deep understanding of powerful mathematics**. In turn, this leads to success in high-level mathematics.

The particular **objectives** are as follows:

(a) *develop pedagogy and resources* enabling teachers to facilitate deep learning of powerful mathematics ideas in order to renew the learning profiles of schools with regard to mathematics, particularly for Indigenous and low-SES students;

(b) *trial and document the pedagogy and resources* in relation to outcomes with regard to PD without and within the schools, classroom teaching practices and student learning outcomes;

(c) *identify actions and behaviours* linked to effective and ineffective student learning trajectories, classroom and school practices, and teacher PD and community involvement activities that facilitate or inhibit the growth of high-level mathematics capability;

(d) *compare and refine* outcomes between and across schools and classes, and pedagogy and resources, and reconstruct a theory of enrichment and extension behind pedagogy and resources; and

(e) *draw implications* for building capacity in the study of high-level mathematics (and academic studies in general) in urban secondary schools with high enrolments of Indigenous and low-SES students, to increase students' opportunities to enter high-value professions (improving employment and life chances).

The particular **outcomes** are as follows:

(a) *student learning* – motivation, confidence and understanding to progress and to succeed at post-compulsory mathematics at the highest level;

(b) *resources* – culturally and contextually appropriate resources to enable teachers to facilitate deep learning of powerful mathematics;

(c) *services* – PD workshops to train teachers in MITI pedagogy and in using the resources, and an online support framework covering email communication, discussion forum and training modules; and

(d) *research* – involving all teachers in action research on their practices, analysing data provided by teachers on researcher actions ←→ teacher practices ←→ student learning, and drawing implications for theory, pedagogy, resources and PD.

1.2 Summary of MITI stages

The MITI pedagogy is developed through two stages as follows. These stages prepare for teaching in Years 7–12 and in advanced mathematics subjects (Mathematical Methods and Specialist Mathematics).

1.2.1 Stage 1: Investigations, problems and seamless sequencing of powerful mathematics

The objectives of this stage are to enrich and extend the YDM pedagogy to enable mathematics teachers to:

(a) identify and understand the structure of high-level, powerful mathematics with respect to sequences, connections and big ideas;

(b) build these structures and develop their strength with respect to recalling mathematical ideas, solving problems and laying the foundation for later learning;

(c) identify, modify, construct and effectively teach investigations and problems appropriate for their students in a manner that helps build powerful mathematics ideas;

(d) build student motivation, confidence and understanding as students move from lower years to and through high-level mathematics in Years 10 to 12; and

(e) use digital technologies to assist with the above.

The stage focuses on the training of teachers to improve teaching and learning with respect to the following:

(a) the development of mathematics as a vertical and horizontal schema of interconnected and sequenced ideas built around key big ideas (see Chapter 2), a rich schematic structure of big ideas that fully includes definitions, connections, applications and experiences;

(b) the RAMR teaching cycle of the YDM pedagogy (see Chapter 3), bringing in the Renzulli (1976) approach to teaching mathematics to able students (see Chapter 4), changing instructional approaches from textbook pages of exercises to problems and investigations, and developing classroom techniques to get the most from these; and

(c) the creation of engaging instructional sequences that teach lower level ideas in a way that seamlessly extends to higher level forms of the same idea, particularly using digital technologies to assist in developing these sequences, following the Review–Explore–Analyse–Link (REAL) approach to their use (see Chapter 4).

This stage is supported by an online community and exemplar materials which include: (a) this MITI Overview book; (b) a collection of 45 investigations designed to cover all topics across Years 7 to 9; (c) an initial collection of ideas for teaching with technology and seamlessly sequencing topics for the transition from Years 7–9 to Years 10–12; and (d) supplementary YDM resource books on big ideas, problem solving and literacy in mathematics.

1.2.2 Stage 2: Deep applications in futures contexts

The objectives of this stage are to further enrich and extend the YDM pedagogy to enable mathematics teachers to:

(a) identify and understand how mathematics ideas develop through abstract symbols from lower level ideas that relate easily to real-world situations to higher level ideas that can exist only in the mathematician's mind;

(b) understand the structures (patterns and relationships) that enable mathematics to exist outside of normal reality, yet provide the underpinning to that reality;

(c) build mathematics ideas across year levels and topics, and gain generic understandings of how new mathematics can be added to existing knowledge with least difficulty and how this can be placed within a global understanding;

(d) maintain students' interest, motivation, confidence and understanding as they complete high-level mathematics subjects in order to increase participation in STEM subjects in tertiary institutions; and

(e) build the above through strong use of digital technologies and applications.

The stage focuses on the training of teachers to improve teaching and learning with respect to the following:

(a) the building of big ideas through model-based structured sequences of instruction across year levels and the placement of mathematics ideas into more global structures that we call **superstructures** to allow for more seamless learning, holistic recall and relationship to effective schematic/organic learning;

(b) the extension of the YDM RAMR pedagogy to a double RAMR model as a pedagogical structure to model the development of the mathematics that is built on symbol structures that, in turn, come from abstraction of number and arithmetic;

(c) the relationship between applications and understanding underlying mathematics, providing real and authentic contexts to powerful mathematics in order to balance mathematics understanding as worthwhile in its own right with mathematics understanding as a valuable tool in the world of applications and employment opportunities; and

(d) the identification of the mathematics on which 21st-century activities and applications are based, particularly those related to technology and mathematics content covered in the highest level mathematics subjects, and to make this mathematics visible to students in a manner that enables deep learning of the mathematics and strong capacity to understand applications; and to connect this mathematics to learning in the wider STEM field (science, technology, engineering and mathematics).

This stage is also supported by an online community and exemplar materials which include: (a) the resource books from Stage 1; (b) a collection of teaching ideas for pre-emptive transition from Years 7–9 to Years 10–12 with use of technology where appropriate; and (c) a collection of ideas on futures-oriented industry applications of mathematics and how to use them to teach mathematics.

Overall the two stages and their resources can be summarised as shown in Figure 2.

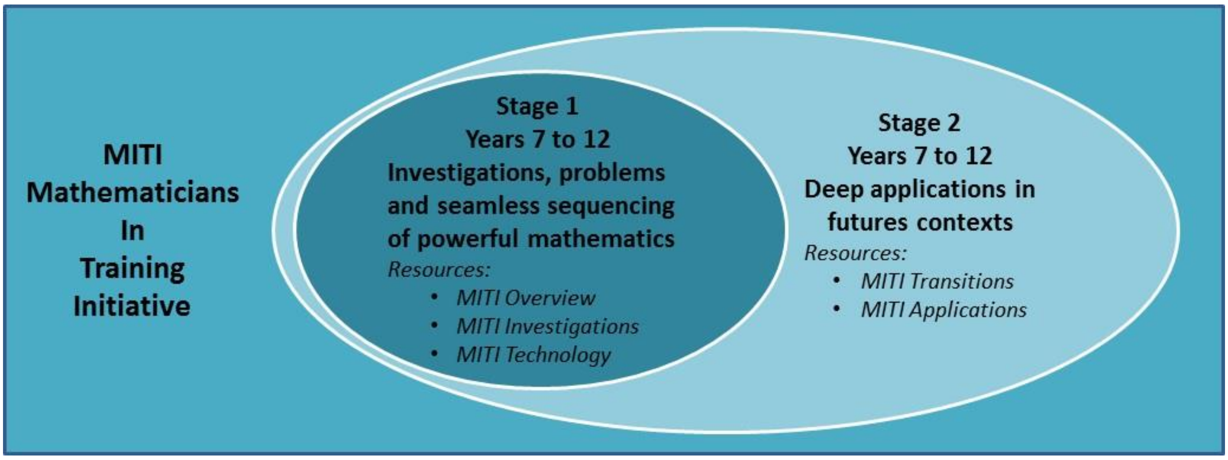


Figure 2 The two stages of MITI including resources

1.3 Imperatives and beliefs

Along with other YDM projects, MITI represents a convergence of social, economic and educational benefits, because it is designed to:

(a) give students the opportunity to change their future to include tertiary education that will provide quality employment;

(b) increase the pool of people with the mathematical understanding needed to succeed in STEM and other high-level professions, the lack of which is putting at risk Australia's economic development; and

(c) illuminate learning theory with regard to deep learning of powerful mathematics.

YDC staff realise that, like the other projects, MITI is a challenging project. Many teachers of mathematics are not secondary mathematics trained; they are teaching out of field. They have to learn mathematics and mathematics education, and to implement these ideas in challenging classroom situations.

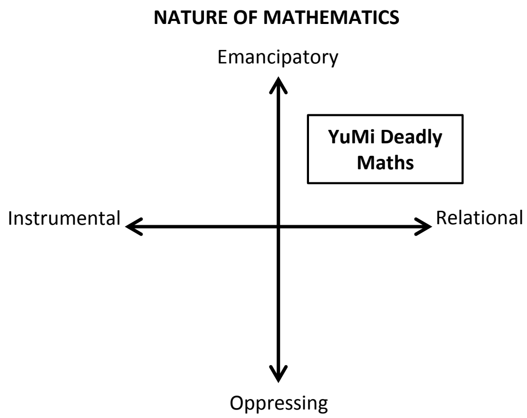
1.3.1 Imperatives

All YDC projects follow the imperatives below.

1. All people deserve to learn the deepest mathematics that empowers them to understand their world and solve their problems, and this is possible if mathematics is taught as a conceptual structure, life-describing language, and problem-solving tool.
2. All people can excel in mathematics and remain strong and proud in their culture and heritage if taught actively, contextually, with high expectations, and in a culturally safe manner.
3. All teachers can be empowered to teach mathematics with the outcomes above if they have the support of their school and system and the knowledge, resources and expectations to deliver effective pedagogy.
4. All communities can benefit from strong, empowering mathematics programs that profoundly and positively affect students' future employment and life chances if school and community are connected through high expectations in an education program of which mathematics is a part.

YDC positions its mathematics projects based on these imperatives in four areas, namely, mathematics, mathematics learning, mathematics teaching, and school–community relationships. These positions are shown in relation to dichotomies that reflect the four areas, and where the horizontal axis is cognitive and the vertical is social, giving rise to four quadrants

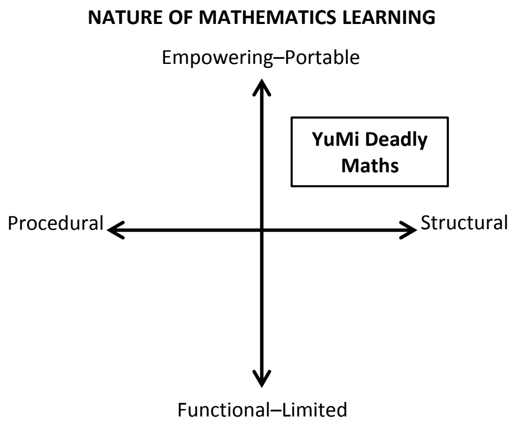
1.3.2 Mathematics positioning



For mathematics, the two dichotomies are **instrumental–relational** and **oppressing–emancipatory**. From a cognitive perspective, the dichotomy first described by Skemp (1989), instrumental versus relational, reflects two ends of perceptions of the nature of mathematics. Mathematics is seen from an instrumental perspective as a collection of definitions, rules and procedures that find answers in particular situations; from a relational perspective it is seen as a structure of concepts, strategies and principles that provide meaning and underpin applications and problem solving. From a social perspective, mathematics can be seen as emancipatory or oppressing. Emancipatory mathematics contains the ideas that enable students to understand their position in the world and to analyse, and take control of, the factors that determine this role; oppressing mathematics contains only the ideas that enable students to fit in.

As the diagram on the right shows, YDM is in the **emancipatory/relational quadrant**. Its aim is to reveal mathematics as a connected structure that provides students with the knowledge to take control of their lives and become what they wish. YDC projects include activities that effectively build this form of mathematics.

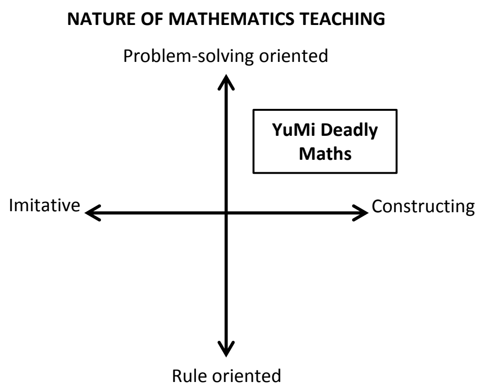
1.3.3 Mathematics learning positioning



For mathematics learning, the two dichotomies are **procedural–structural** and **functional–empowering**. From a cognitive perspective, procedural learning is focusing on the rote learning of instrumental mathematics content where the student learns disparate facts, rules and procedures as pieces of knowledge to be recalled, while structural learning is focusing on acquiring relational mathematics content as rich schema, where mathematically equivalent understandings are connected and form integrated networks of information. From a social perspective, learning can focus on functional outcomes, that is, mathematics content **limited** to the immediate need for it, or it can focus on empowering outcomes, that is, mathematics content that is **portable** and can be translated or transferred to a wide variety of situations. The power of mathematics lies in its portability, and portability depends on structural understanding (see section 2.3). Rich schema enables knowledge to be applied in all the components that are connected. It facilitates recall because knowledge is stored as whole structures not individual components, and it enhances problem solving because the connections enable other knowledge to be considered as well as the knowledge that is the focus of the problem.

As the diagram on the right shows, YDM is in the **empowering–portable/structural** quadrant. Its aim is to reveal mathematics as a connected structure that provides students with portable knowledge that is theirs and not reliant on memory of a rule. YDC projects include many activities that effectively enable this type of mathematics learning.

1.3.4 Mathematics teaching positioning



For mathematics teaching, the two dichotomies are **imitative–constructing** and **rule–problem**. From a cognitive perspective, imitative teaching is the traditional textbook exposition teaching where a worksheet or a textbook page of exercises is provided, the teacher shows how the first example is completed, then the teacher works through one or two more with the students, and finally the rest are given to the students to be done by **imitating** the teacher's process (the simplistic "I do, we do, you do" exposition approach) while the teacher wanders, checks and helps. On the opposite end, constructing teaching focuses on providing experiences from which the students can construct their own knowledge in a context where discussion with teachers and peers leads to development of language and symbols (that is, social constructivism). Imitative teaching leads to only being able to reproduce procedures when specific examples are provided; students are often confused by small changes in the form of presentation of examples. Constructing requires new knowledge to be accommodated, and partly generated, within students' existing knowledge and leads to ownership, flexibility and meaning.

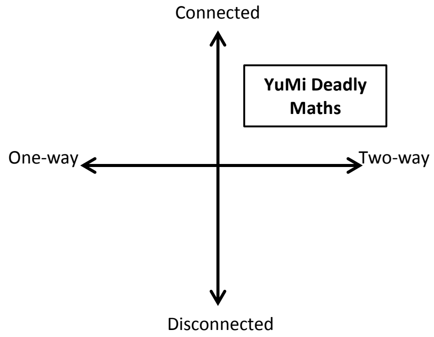
From a social perspective, rule-oriented teaching emphasises recall of ideas as definitions, and learning of procedures as rules; it provides collections of rules and procedures to be learnt by repetition. Against this, problem-oriented teaching starts from real problems (from the perspective of the student), acts out and models these problems looking at a variety of models and strategies, develops the mathematical language and symbolic activities that will solve the problem, connects this new mathematics knowledge to appropriate existing knowledge, finally translating the solution back into the problem situation from whence the teaching process started.

As the diagram on the right shows, YDM is in the **problem-solving oriented/constructing** quadrant. Its aim is to teach students to solve problems through revealing the structure of mathematics. YDC projects include many activities that effectively enable this type of mathematics learning.

1.3.5 School–community connections positioning

For school–community connections, the dichotomies are **one-way–two-way** and **disconnected–connected**. From a cognitive perspective, schools and their mathematics programs can relate to the community as if they are the experts, always operating the knowledge flow from school to community; or they can see themselves in a collaboration or co-construction where there is mutual learning and a two-way flow of knowledge. From a social perspective, schools can be disconnected from their community, operating in their own world and their own way within the school fence and seeing themselves as independent from the community (i.e. a world within a world); or they can be connected, with strong relationships between school, students, parents/carers and community members, and seeing themselves as part of the community, allowing the community access and some control over the schools' facilities.

NATURE OF SCHOOL–COMMUNITY CONNECTIONS



This involvement with, or connection to, community should also be part of school education with respect to mathematics. Examples of possible activities include looking at community methods for measurement, bringing in a tradesperson to show how they do mathematics in their trade, or looking at local community building activities (e.g. fixing the road, building a park) or events (e.g. catering for a celebration). Indigenous communities are particularly strong in mathematics in terms of patterns and relationships.

As the diagram on the right shows, YDM is in the **connected/two-way** quadrant. Its aim is to be part of a connected school which values community knowledge and welcomes community members into the school to share their knowledge. YDC projects provide examples of how this can be one of the most effective ways for Indigenous and low-SES students to learn, and this includes mathematics.

2 Connections, Sequences and Big Ideas

MITI is based on the YDM pedagogy, which has structural and teaching-cycle components. This chapter summarises the YDM pedagogy with regard to structure, namely connections, sequences and big ideas. We describe these structural components as they are presented in YDM and then discuss their role in MITI.

Connections and sequences provide the basis of the structure of mathematics, characterised as a network or schema. They also provide the basis of understanding mathematics, characterised by being able to move easily between representations of mathematics ideas, namely, symbols and language, pictures and images, materials and models, and real-world situations. Knowledge of connections and sequences assists mathematics teaching by providing information on effective sequencing of ideas and connections between ideas.

Big ideas are schemas that cover more than one topic over more than one year. Knowledge of a big idea means being able to understand more than one mathematical topic from one idea. Big ideas are also organic, in that they facilitate new learning. Their small number means they represent a powerful way to teach mathematics.

The chapter concludes by discussing the extension of the role of structure from YDM to MITI.

2.1 Connections and sequences

Connections and sequences form mathematical ideas into structure, a network or **schema**. One of the bases of YDM pedagogy is that mathematics should be understood and taught as a structure or schema – as ideas collected and formed into structure through sequences and connections.

2.1.1 Implications and outcomes

We argue that learning and teaching based on schema has important implications, namely, that learning mathematical ideas as a schema leads to the following.

1. **Deep understanding**. The idea can be understood not just on its own but in relation to the things to which it is connected and to the sequences they form. That is, the whole schema increases the depth of understanding of its parts. This is the basis of what it means to understand. For example, the mathematical idea of addition is connected to subtraction (inverse) and multiplication (repeated addition) and obeys the same properties (e.g. identity, inverse, associativity, commutativity) across a variety of topics (e.g. whole numbers, fractions, algebra, functions, and so on). Knowing all this deepens understanding of addition.
2. **Defining, connecting, applying and memorising**. The idea comes with knowledge to cover four aspects: (a) it fully defines the idea; (b) it includes all connections to and from the idea; (c) it covers all applications of the idea; and (d) it keeps, in memory, experiences with the idea. Thus, a schematic understanding of, for example, addition would mean knowing all the meanings of addition, knowing all things to which addition is connected, knowing all the applications of addition, and remembering experiences with addition.

However, to ensure an idea is learnt as schema, we argue that the following needs to occur.

1. **Student construction of knowledge**. Students have to fit the idea into their existing mathematics and non-mathematics knowledge structure. Thus, important mathematics cannot be told to students; rather, the students must construct the mathematics themselves from experiences. The role of the teacher is to provide the experiences and the questions to scaffold the construction.
2. **Teacher knowledge of structures**. Teachers need to know the structures of the ideas they are teaching so they can use and highlight the structures in their teaching. Connected and sequenced mathematics ideas are

best taught by using the mathematics connections and sequences in teaching, repeating the earlier models and remembering how later ideas are the same as and different from earlier ideas.

We therefore advocate that knowledge of structure, connections and sequences is a major component of effective teaching because:

(a) it enables teaching, learning and problem solving through these connections and sequences (schema);

(b) it enables teaching to build on what precedes and prepare for what follows the subject matter being taught (pre-empting); and

(c) it allows mathematics to be considered as a whole and teaching to move from whole to part.

2.1.2 Teaching, learning and problem solving

We argue that knowing mathematics as rich schema facilitates:

(a) *understanding* – the network structure of the rich schema relates new knowledge to existing knowledge and enables easy movement between representations;

(b) *recall* – it is easier to remember a structure than a collection of individual pieces of information; and

(c) *problem solving* – the content needed to solve problems is usually peripheral to central or focal thinking and peripheral ideas are better found if there is a structure of connections that can be followed from the focal thinking.

As a consequence, we contend that knowledge of the sequenced and connected structure of mathematics can assist teachers to be effective and efficient in teaching mathematics. It enables teachers to do the following.

1. **Determine what mathematics is important to teach**. Mathematical ideas with many connections and/or which form a part of sequences are more important to teach than mathematical ideas with few connections or little use beyond the present. Secondly, the connections and sequences to which mathematical ideas belong are as important as the ideas themselves and should be included in instruction.
2. **Link new mathematics to existing known mathematics**. Linking new mathematical ideas to existing ideas places the new learning within a schema of connections and sequences. This makes the new ideas easier to understand, recall and use in problem solving. Knowledge of the structure of mathematics enables teachers to ensure all connections and sequences are included when teaching new ideas.
3. **Choose effective instructional materials, models and strategies**. Mathematical ideas that are connected to, or in sequence with, other mathematical ideas can be taught with similar materials, models and strategies used in teaching the other ideas. Knowing structure means that appropriate materials, models and strategies are known, and using these materials, models and strategies also reinforces the structure.
4. **Teach mathematics in a manner that makes it easier for later teachers to teach more advanced mathematics (pre-empting)**. Knowledge of mathematics structure means knowing what mathematics follows present teaching. This enables teachers to teach in a form that lays a foundation for (pre-empts) the ideas the later teacher will use and makes teaching easier for the later teacher.

2.1.3 Seamless sequencing and pre-empting

The YDM pedagogy is based on sequencing between connected ideas being **seamless**; that is, the movement from one idea to the next should not be inhibited by concepts being taught for the first idea that do not transfer to, or worse still do not work for, the second idea. Such seamless sequencing is a major feature of acceleration because the previous learning assists with the next learning. The following two examples provide insight into the need for this type of sequencing.

1. **Whole number algorithms to algebra calculation**. If we teach that whole numbers are added by adding like place values (and renaming, if needed), this can translate to algebraic addition being adding like variables – this is even stronger if we use vertical addition for algebra as below:

$$\begin{array}{r} 4\,6\,2 \\ \underline{+\,2\,3\,5} \\ 6\,9\,7 \end{array} \quad \longrightarrow \quad \begin{array}{r} 3a + 7b \\ \underline{+\,5a + 2b} \\ 8a + 9b \end{array}$$

2. **Decimal numbers to percent**. If we teach flexibility of decimal notation (e.g. 24 = 2.4 tens = 0.24 hundreds), we can teach that percent being hundredths is place value with the decimal point after the hundredths instead of after the ones – this means that 0.075 ones = 7.5 hundredths = 7.5%.

Mathematics is replete with such examples, and this focus of the YDM pedagogy on seamless sequencing and pre-empting is one of its major strengths. The YDM pedagogy ensures prerequisite mathematics is taught so it is an easy transition from simpler mathematics in earlier years to more difficult mathematics in later years.

2.1.4 Whole-to-part teaching

One of the powerful ways to teach mathematics is to understand the major relationships in mathematics. For example, simplistically, European mathematics could be considered to grow out of two views of reality: number and shape. The basis of number was the unit, the one. Large numbers were formed by grouping these ones, and small numbers (e.g. fractions) by partitioning these ones into equal parts. The operations of addition and multiplication, and the inverse operations of subtraction and division, were actions on these ones which joined and separated sets of numbers.

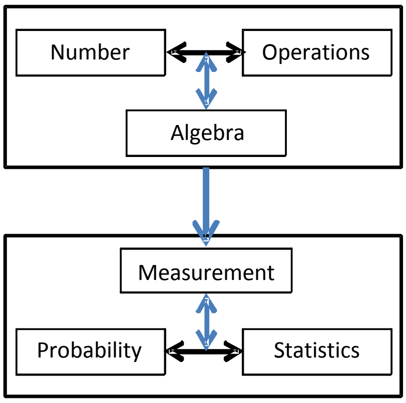


Figure 3 Major connections

Algebra was constructed by generalising number and arithmetic, and representing general results with letters. With input from geometry, this gave rise to applications within measurement, and statistics and probability. This is illustrated by the relationships in Figure 3.

These relationships give a framework for Foundation to Year 9 that enables mathematics as a whole to be considered. This leads to major sequences and connections; for example, the principles of arithmetic are the same as the principles of algebra, meaning there is commonality in relationships between arithmetic and algebra. It also provides an overview and sequence for the connections upon which teaching should be built; for example, number and geometry before measurement, and fractions before numerical probability. Similarly, all strands and topics of mathematics have internal structures (connections and sequences) that provide the same holistic views. The YDM resource books use diagrams to show these structures for the strands and build lesson ideas around them.

2.2 Big ideas

Big ideas are mathematics ideas that can be used in many year levels and across different topic areas. Knowing mathematics in terms of big ideas is a powerful way to learn mathematics and knowing big ideas represents **deep learning of powerful mathematics**.

An example of a big idea is that mathematics can always be seen from two perspectives – as a *relationship* (static) or as a *change* or transformation (dynamic). In the example in Figure 4, addition of 4 and 3 to make 7 can be seen as a relationship of balance: 4 joined with 3 is the same as 7; and as change: 4 changes to 7 by the action of +3.

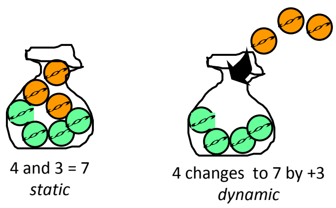


Figure 4 Relationship vs change perspective of mathematics

This section describes big ideas in YDM and should be read in conjunction with **Appendix A**. The supplementary YDM resource book, *Big Ideas of Mathematics*, provides more detail.

2.2.1 YDM pedagogical position on big ideas

YDM pedagogy emphasises big ideas in mathematics. Big ideas cover significant concepts in mathematics and have a wide effect, making them an efficient way to learn mathematics. YDM emphasises beginning a new topic by determining the big ideas important in that topic so they can be referred to where relevant to build understanding. YDM defines big ideas to have some or all of the following **properties**:

1. **They provide generic approaches to a wide range of ideas** – they encompass viewpoints that cross boundaries. For example, many mathematical actions can be considered as relationship (*static*) and transformation (*dynamic*), as in the addition example discussed above.
2. **They apply across topic areas** – they have some generic capabilities that are not restricted to a particular domain (e.g. the inverse relation in division between divisor and quotient also applies to measurement, fractions and probability).
3. **They apply across year levels** – they have the capacity to remain meaningful and useful as a learner moves up the grades (e.g. the concept of addition holds for early work in whole numbers, work in decimals, measures, common fractions, and algebraic variables).
4. **Their meaning is independent of context and content** – it is encapsulated in what they are and how they relate, not to the particular context in which they operate. For example, the commutative law says that *first number + second number = second number + first number* irrespective of content type (e.g. whole numbers, decimal and common fractions, algebra or functions).
5. **They are teaching approaches that apply across ideas** – they have the capacity to apply to many situations. For example, the teaching approach of reversing (reversing the order of activity in a lesson) applies everywhere, including going whole to part and part to whole, shape to symmetry and symmetry to shape, algorithm to answer and answer to algorithm.

Big ideas are very effective ways to learn mathematics at a deep level, for the following reasons:

1. **One big idea can apply to a lot of mathematics**. This makes big ideas powerful ways to teach and understand mathematics. For example, part-part-whole, multiplicative comparison (double number line) and start-change-end diagrams can solve most fraction, percent, rate, and ratio problems (i.e. they reduce cognitive load).
2. **One big idea can cover work that would need many procedures and rules to be rote learnt**. For example, the distributive law and area diagrams can be used to understand and solve 24 × 37, $2/5 \times 4/5$ and $(x-1)(x+2)$ problems.
3. **Big ideas are organic in that they assist later learning**. Big ideas build structural connectivity across domains of mathematics, thus developing rich schema that can easily accommodate new ideas. For example, building the notion of inverse as "undoing things" and teaching the inverse relationships between $+2$ and -2; $\times 5$ and $\div 5$; x^2 and $\sqrt{x}$; p^3 and p^{-3}; $(p)^n$ and $(p)^{1/n}$; $f(x) = 2x + 1$ and $f(x) = (x-1) \div 2$ can make it really easy to understand integration as the inverse of differentiation in calculus.

2.2.2 Big idea types

YDM identifies five types of big ideas:

1. **Global big ideas**. These relate to nearly all mathematical ideas and all year levels. For example, the *commutative principle* is not a global big idea because it only refers to addition and multiplication situations; on the other hand, *transformation and relationship* is global because it refers to all mathematics, saying that every idea can be considered both as a change and as a relationship.

2. **Concept big ideas**. These are the meanings of ideas that are common across mathematics. For example, the meanings behind *equals* and *multiplication* – such meanings have large impact and can help in many topic areas, from operations to algebra and measurement to statistics.
3. **Principle big ideas**. These are relationships where meaning is encoded in the relation of the parts, rather than in their content. The *commutative principle* (turnarounds – e.g. 1st + 2nd = 2nd + 1st) is an example of a principle big idea because it also holds for many contexts (e.g. whole numbers, decimals, fractions, variables, functions), while 2 + 3 = 5 is contentful because it only holds for 2, 3 and 5.
4. **Strategy/model big ideas**. These are ways of solving exercises and problems that apply to a range of mathematics across year levels. For example, the *part-part-total* (PPT) strategy and model underpins all operations and is a powerful strategy in solving word problems and fraction, percent and ratio problems.
5. **Pedagogy big ideas**. These are ideas for teaching that are generic in their application – they can apply to the teaching of many mathematics ideas. For example, the teaching approach of *reversing* where the teaching direction between teacher and student is reversed (e.g. from "what is 5 + 8?" to "what addition facts give answer 13?") can apply in many situations other than addition.

Appendix A at the end of this book lists some of the more important big ideas. A more complete list with descriptions is provided in the YDM Supplementary Resource 1 on Big Ideas available through the MITI online learning Blackboard site as a resource for schools involved in MITI projects.

2.3 Extending YDM structure to MITI

The three YDM pedagogy positions outlined in sections 2.1.2 to 2.1.4 regarding connections and sequences are important in MITI because MITI deals with deep understandings and powerful mathematics. Rich schema is the deepest and most powerful form of mathematics. MITI extends the YDM pedagogy to Years 10 to 12 mathematics where the growth in mathematical ideas requires schema.

2.3.1 Mathematics structure

A structural approach to teaching mathematics that involves connections and sequencing remains equally or even more important in MITI. In its coverage of mathematics across Foundation to Year 9, YDM has only started to build the structure that underlies the mathematics of Years 10 to 12 and tertiary years. As the mathematics knowledge to be understood grows across the years, it becomes even more important for mathematics to be seen as a connected and sequenced structure. Existing structure provides a framework into which new mathematics can fit to build a combined structure containing new and earlier ideas, which in turn can be used to accommodate even newer ideas. Thus students who have structural knowledge of mathematics find it easier:

(a) to retain Foundation to Year 9 mathematical ideas, which are still the basis of much that is in Years 10 to 12, while learning the new knowledge;

(b) to learn, recall and solve problems with the new knowledge (in particular, they do not have to laboriously rote learn new materials and processes); and

(c) to prepare for new ideas with confidence and engagement, to make sense of these ideas and know where they fit into the scheme of mathematics.

Retaining ideas in the form of rich schemas that define, connect, apply and remember is even more crucial for the mathematics that makes mathematicians. It is the learning and thinking in this schematic form that we call *deep learning of powerful mathematics*.

This focus on structure will be seen across both stages of MITI, although it will predominate in Stage 2, as discussed in the next section.

2.3.2 Pre-empting

Knowing the structure of mathematics means knowing how mathematics fits together, and particularly what comes before a mathematical idea and what comes after. Thus, teaching mathematics in earlier years in a way that prepares for the mathematics to be taught in later years remains a strong part of MITI. This is called pre-empting in both YDM and MITI and is part of the YDM pedagogy where we stress the need to teach mathematics in a way that makes the teaching in later years easier and builds on the teaching in earlier years (it is called seamless transitions from lower years to higher years in parts of this book). It is both a consequence of structure and an important part of building structure.

As MITI moves into Years 10 to 12, it is important to ensure that teaching in Foundation to Year 9 prepares students for what they do in later years in a way that makes the later teaching and learning easier. This may make teaching in the earlier years a little more difficult but once started pays off in so many ways and reduces the time needed later. Two examples should help here:

1. In Years 11 and 12, teaching calculus introduces the notion of limit in differentiation and integration. It would prevent a lot of difficulties if some simpler form of the limit idea was introduced in Years 7 to 9.
2. In Years 9 and 10, it is common to look at trigonometry in terms of similar right-angled triangles and ratios. This bases trigonometry on similarity and proportion, which is a good sequence. However, in Years 11 and 12, trigonometry is extended to circles. Therefore, while retaining the positive aspects of triangles, similarity and proportion in teaching trigonometry, it is important to also introduce it using circles so that Years 11 and 12 trigonometry is made easier to learn.

2.3.3 Holistic ideas (organic learning)

The idea of seeing mathematics as a whole structure, not a collection of rules and definitions, is also important in MITI as it is in other YDM-based projects (if not more so). This can be seen in a single example. The calculus of mathematics in Years 11 and 12 has two actions, differentiation and integration. These actions relate to each other like addition and subtraction, multiplication and division, square and square roots: they are inverses. Seeing calculus in terms of the big idea of inverse helps calculus at all levels, including enabling students to see that calculating an integral is doing the opposite to calculating a differential. The idea of seeing that differentiation and integration are inverses, in addition to the connections of calculus to other big ideas such as rates and areas, is powerful whole-to-part teaching.

This use of holistic structure has been called *organic* because it facilitates growth of knowledge. As will be seen in Chapter 4, it remains particularly important in MITI.

3 RAMR Teaching Cycle

The YDM pedagogy is based on the Reality–Abstraction–Mathematics–Reflection (RAMR) cycle (see Figure 5). This cycle is a pedagogic framework for planning, teaching and learning mathematics. It proposes:

(a) working from student reality and local culture (prior experience and everyday kinaesthetic activities);

(b) abstracting mathematics ideas from everyday instances to mathematical forms through an active pedagogy (kinaesthetic, physical, virtual, pictorial, language, and symbolic representations, i.e. body → hand → mind);

(c) consolidating the new ideas as mathematics through symbols and language, practice and connections; and

(d) reflecting these ideas back to reality through a focus on applications, problem solving, flexibility, reversing and generalising.

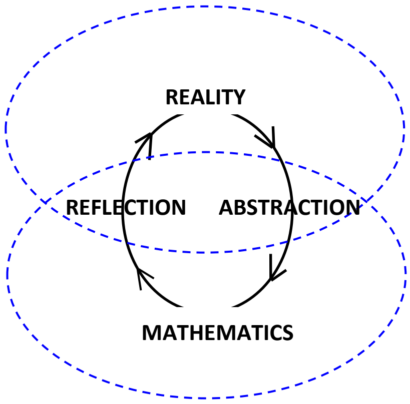


Figure 5 The RAMR cycle

The innovative aspect of RAMR is that the right half develops the mathematics idea while the left half reconnects it to the world and extends it. For example, whole-number place value built around the pattern of threes where hundreds-tens-ones applies to ones, thousands, millions, and so on, can easily be extended to metrics by considering the ones to be millimetres, the thousands to be metres and the millions to be kilometres.

3.1 Basis of RAMR cycle

The RAMR cycle is based on a philosophy of mathematics teaching and learning that has an Indigenous beginning. This has been extended to a pedagogical framework by adding the best of instructional strategies to the four sections of the cycle. It has been one of the successes of YDM.

3.1.1 Philosophical model

To have an approach to teaching mathematics that takes account of the cultural capital students bring to the classroom and negates the traditional Eurocentric nature of school mathematics, it is necessary to consider the nature of mathematics. Mathematics starts from observations in a perceived *reality*. An aspect of a real-life situation is selected and *abstracted* using a range of mathematical *symbols*. The resulting *mathematics* is used to explain reality and solve problems. It is validated and extended by being *critically reflected* back to reality. The cycle from reality to mathematics and back means that abstraction and reflection are *creative* acts; the invented mathematics as a structure, language and problem-solving tool is built around *symbols*; and the mathematics and how it is used in reality is framed by the *cultural bias* of the person creating the abstraction and reflection. The act of abstraction requires learners to move from reality to symbols, and the act of reflection requires learners to extend this knowledge by relating symbols back to reality. This cyclic process is encapsulated in Figure 6.

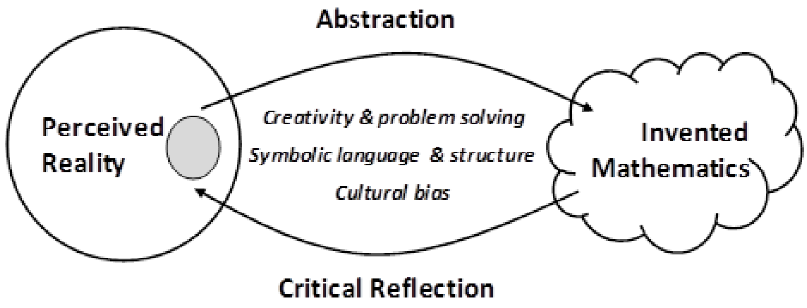


Figure 6 Relationship between perceived reality and invented mathematics (adapted from Matthews, 2009)

Creativity, symbols and cultural bias are features of the model in Figure 6. The first, *creativity*, is particularly evident in the abstraction and critical reflection cycle. It is important to note that this cycle is similar to other artistic pursuits such as dance, music, painting and language as different forms of abstractions. Therefore, mathematics can be considered as another art form and, in theory, relates to these other forms of abstractions. In essence, it is possible to develop empowering pedagogy that allows students to be creative and express themselves in the mathematics classroom. This allows students to learn mathematics from their current knowledge (i.e. from the students' social and cultural background), thereby providing agency through creativity and ownership over their learning.

As a product of the abstraction process, *symbols* and their meanings are important features of the model since they connect the abstract representation with reality. However, it is common for students not to make these connections easily and to view mathematics as just sums with no real meaning. This is further exacerbated for students when they first learn algebra, and letters are suddenly introduced into mathematics without any obvious reason except that we are now learning algebra. Focusing on creativity within mathematics provides an opportunity for students to generate their own symbols to represent their understanding of the mathematical process. These symbol systems can then be compared to and assist in understanding the meanings of current symbols, symbolic language and their connection to reality. This can also lead to the teaching and learning of the underlying structure of mathematics, providing students with a holistic view of mathematics.

The third feature, *cultural bias*, exists in all aspects of the abstraction and critical reflection cycle. The observer expresses their cultural bias in the way they perceive reality and decide on which aspect of reality they wish to focus. In the abstraction process, the form a symbol takes and the meanings attached to this symbol or group of symbols are biased by a cultural perspective. Finally, the critical reflection processes are underpinned by the cultural bias within the abstraction process and the observer's perception of reality. If we have an understanding and appreciation of the cultural bias within mathematics, new innovative pedagogy can be developed that moves beyond some cultural biases so students can relate to mathematics but also gain a deep understanding for the current form of mathematics and how mathematics is used.

3.1.2 Components of RAMR

The philosophical relationship of Figure 6 can be deconstructed into four components: reality, abstraction, mathematics, and reflection. The nature of each of these components is as follows.

Reality

The reality component of the cycle is where students: (a) access knowledge of their environment and culture; (b) use existing mathematics knowledge prerequisite to the new mathematical idea; and (c) experience real-world activities that act out the idea. The focus in this component is to connect the new idea to existing ideas and everyday experiences. Among the kinaesthetic, physical and visualisation activities that predominate in this component, it is vital for students to be provided with opportunities to generate their own experiences and verbalise their own actions. This generation and verbalisation provides the students with ownership over their understanding of the mathematical idea.

Abstraction

The abstraction process is where students experience a variety of representations, actions and language that enable meaning to be developed that carries mathematical ideas from reality to abstraction. Representations, actions and language will predominantly be as in Figure 7 below; however, students should also be provided with opportunities to create their own representations, including language and symbols, of the mathematical idea initially experienced through physical activity. This allows students to have a creative experience that will, firstly, develop meaning and, secondly, attach it to language and symbols. The sharing of other students' representations provides students with alternative views of the same idea attached to varied symbolic representations. Discussions on the use of different symbols enables students to: (a) critically reflect on their journey (enabling them to justify and "prove" their ideas); (b) understand the role of symbols in mathematics (enabling them to understand the relation between symbol, meaning and reality); and (c) be ready to appropriate (Ernest, 2005) the commonly accepted symbols of Eurocentric mathematics.

Representations	**Actions**	**Language**
REALITY		
Real world (e.g. biscuits) and replicas (e.g. toys)	Whole body	Students'
Manipulatives (e.g. counters, MAB)		
Virtual (computer replications)	Hands	Model
Pictures (e.g. drawings, diagrams, lines)		
Symbols patterns (e.g. calculators, spreadsheets)	Mind (image)	Mathematics/symbol
ABSTRACT MATHEMATICS		

Figure 7 Abstraction sequence from reality to mathematics

The act of abstraction requires the learner to generalise a mathematical idea from examples in the world to symbols in the artificial world of mathematics. It means the learner has to move from reality to symbols; for example, connecting the real-life situation of three children joining two children to make five children with the symbols 2 + 3 = 5. The recommended way to do this is to move through a sequence of representations of the mathematical idea from reality to abstract (as in Figure 7). The representations can be external (real-world activities, materials, images, pictures, language and symbols) or internal (mental images of external representations), with learning occurring when structural connections are made between the two (Halford, 1993). The external representations facilitate the internal representations while accompanying language and actions become increasingly abstract (as in Figure 7).

Mathematics

The mathematics component of the cycle is where students: (a) appropriate the formal language and symbols of Eurocentric mathematics; (b) reinforce the knowledge they have gained during the abstraction phase; and (c) build connections with other related mathematical ideas. The focus is to assist students to construct their own set of tools (filling their "mathematical toolbox") that will enable them to recognise and recall mathematical ideas from the language and symbols associated with the ideas, thus adding to their bank of accessible knowledge. The connections between new and existing ideas enable better recall of mathematical ideas and improve problem solving. It is easier to remember ideas in terms of how they are related to each other (structural understanding) than as many disconnected pieces of information. The ideas that help in problem solving are often connected peripherally to the central idea to which the problem refers.

Reflection

The critical reflection process is where the new mathematical ideas are: (a) considered in relation to reality in order to validate/justify understandings; (b) applied back to reality in order to solve everyday life problems; and (c) extended to new and deeper mathematical ideas through the use of reflective strategies, namely, flexibility, generalising, reversing and changing parameters. As well as reflecting on the mathematics they have learnt in relation to the world they live in, this process involves students' consideration of the journey they took from reality to mathematics via abstraction in developing the mathematical ideas. It requires reflection on what they learnt, how they learnt it, and why they learnt it. It also requires them to justify their outcome.

Reflection is more powerful than it seems at first glance. It requires the learner to validate their mathematics learning against their everyday life, thus generating ownership of the knowledge. However, it is also a method of extending learning as the reflection acts on the abstracted mathematics in relation to reality. For example, students can reflect on 3 + 4 = 7 and see that if one addend, say the 3, was reduced by 2, then the sum, 7, has to be reduced by 2 to keep the equation equal, the beginning of the balance rule. The extension of knowledge through critical reflection can be assisted by the use of the four strategies: flexibility, generalising, reversing and changing parameters.

Along with abstraction, reflection forms an important cycle (thesis-antithesis-synthesis) with perceived reality and mathematics. Through this cycle mathematics knowledge is *created, developed* and *refined*. Mathematical knowledge is created (the thesis) by abstraction from perceived reality. This knowledge is trialled within itself for consistency (proof) and against reality for effectiveness (application). Problems that emerge in proof or application (the antithesis) are used to amend the mathematics (the synthesis) and the cycle continues.

3.1.3 Planning with the RAMR cycle

Planning the teaching of mathematics can be based around the RAMR cycle, deconstructed into components that are applied to a mathematical idea. By breaking instruction down into the four parts and taking account of the pedagogical approaches described earlier, the cycle can lead to a structured instructional sequence for teaching the idea. Figure 8 briefly outlines how this can be done. Prerequisite mathematical ideas are considered in the reality and mathematics components of the cycle, while extensions and follow-up ideas are considered in the reflection component.

REALITY • **Local knowledge**: Identify local student cultural-environmental knowledge and interests that can be used to introduce the idea. • **Prior experience**: Ensure existing knowledge prerequisite to the idea is known. • **Kinaesthetic**: Construct kinaesthetic activities, based on local context, that introduce the idea.	**ABSTRACTION** • **Representation**: Develop a sequence of representational activities (physical to virtual to pictorial materials to language to symbols) that develop meaning for the mathematical idea. • **Body-hand-mind**: Develop two-way connections between reality, representational activities, and mental models through body → hand → mind activities. • **Creativity**: Allow opportunities to create own representations, including language and symbols.
REFLECTION • **Validation**: Facilitate discussion of the idea in terms of reality to enable students to validate and justify their new knowledge. • **Applications/problems**: Set problems that apply the idea back to reality. • **Extension**: Organise activities so that students can extend the idea (use reflective strategies – *flexibility*, *reversing*, *generalising*, and *changing parameters*).	**MATHEMATICS** • **Language/symbols**: Enable students to appropriate and understand the formal language and symbols for the mathematical idea. • **Practice**: Facilitate students' practice to become familiar with all aspects of the idea. • **Connections**: Construct activities to connect the idea to other mathematical ideas.

Figure 8 Outline of the RAMR cycle

3.2 Pedagogies on which RAMR is based

The RAMR cycle is related to and has been constructed from mathematics pedagogies that have been commonly and successfully used in mathematics for many years. Seven of these are briefly described under three headings as follows.

3.2.1 Cycles, triangles and representations

Wilson's Activity Type cycle

Wilson's Activity Type cycle (see Wilson, 1976 and Ashlock et al., 1983) is a pedagogy used by YDC staff for many years with success. Wilson's cycle specifies five steps (see Figure 9): (a) *Initiating*: teach the idea informally in real-world situations, representations and informal language; (b) *Abstracting*: introduce the formal mathematics language and symbols; (c) *Schematising*: undertake activities specifically to connect the new knowledge to existing knowledge; (d) *Consolidating*: practice (games, practice activities and worksheets); (e) *Transferring*: apply knowledge to solve problems, and see if students can undertake activities that can extend knowledge to new knowledge without having to go through all five steps. The cycle also advocates continuous checking and *diagnosis* of students' understandings to ensure no errors become habituated. The RAMR cycle was based on ideas from Wilson's Activity Type cycle, particularly in abstraction, mathematics and reflection. However, RAMR has the added perspective of emphasising building from reality back to reality.

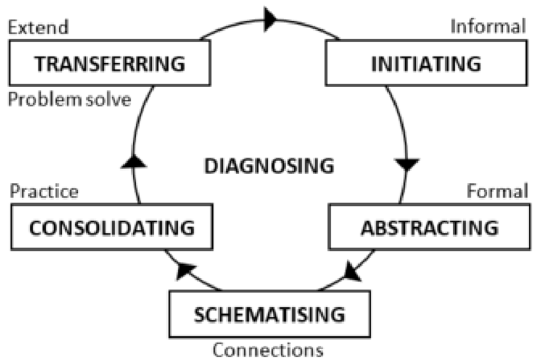


Figure 9 Wilson's Activity Type cycle

Payne and Rathmell triangle

As a framework, Payne and Rathmell's (1977) triangle model relates real-world situations, representations, language and symbols (see Figure 10). Real-world situations are identified and modelled with body, hand and mind. The physical, pictorial and virtual materials, and accompanying mental-visual models, are connected to language and symbols, studied and reinforced as two-way connections. The abstraction process in RAMR is designed to cover the ideas in this triangle.

The triangle model advocates an order in teaching that must follow: story → model → language → symbol, then work back relating all the parts in both directions.

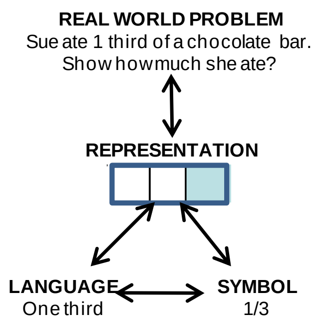


Figure 10 Payne and Rathmell triangle framework for teaching fraction one-third

Multi-representational instruction

This approach involves using many representations in teaching episodes (e.g. MAB, place-value charts, calculators and pen/paper) to relate model with language and symbol. It involves continuously linking representations. As argued by Duval (1999), mathematics comprehension results from the coordination of at least two representational forms or registers; the multi-functional registers of natural language and figures/diagrams, and the mono-functional registers of notation systems (symbols) and graphs. Learning is deepest when students can integrate registers. Multi-representations work well with the Payne and Rathmell triangle and are the basis of the abstraction component of the RAMR cycle.

3.2.2 Types, levels and learner-centred principles

Baturo/Leinhardt knowledge types

The four components of the RAMR cycle were also a product of Baturo's (1998) modification of Leinhardt's (1990) four knowledge types: (a) entry (knowledge of mathematical ideas before instruction – from experience); (b) representational (knowledge of physical materials, virtual materials and pictures used to develop the ideas); (c) procedural (knowledge of definitions, rules and algorithms); and (d) structural (knowledge of relationships and concepts). All four types have to be developed in a unit of work with the final goal being structural.

It is easy to see that these four knowledge types are similar to the RAMR components – entry is reality, representational is abstraction, procedural is in the first part of mathematics, and structural is in the last part of mathematics joined with reflection.

Levels of instruction

Baturo et al. (2007) identified three levels of instruction that need to be taken into account in lesson planning:

(a) *technical* – becoming familiar and proficient with the use of materials;

(b) *domain* – knowing what materials and what activities will provide experiences effective for learning the topic being taught; and

(c) *generic* – knowing instructional strategies that hold for all topics. These three levels of instruction should be taken into account for all activities within the RAMR cycle.

Four of the most important generic strategies have particular application to the reflection component of RAMR: (a) *flexibility* (experiencing the mathematical idea many ways); (b) *reversing* (teaching in the opposite direction, e.g. whole to a fraction and then fraction to a whole); (c) *generalising* (developing the idea into a generality); and (d) *changing parameters* (considering what would happen if something changed).

As well, most other components of RAMR are also important; for example, starting from what the students know and are interested in, and making connections. In fact, the RAMR model is itself a generic strategy as it can be used with all topics. However, these are not the only such strategies. For example, continuously diagnosing students' knowledge, and asking students to reflect on what they have learnt are also important generic strategies.

Learner-centred principles

Overall, effective mathematics teaching means taking account of good principles of teaching and learning in mathematics. A good set of principles has been developed from a review of the literature by Alexander and Murphy (1998) as follows. These should always be taken into account when using the RAMR model.

1. **Knowledge base**. One's existing knowledge serves as the foundation of all future learning by guiding organisation and representations, by serving as a basis of association with new information, and by colouring and filtering all new experiences.
2. **Situation/context**. Learning is as much a socially shared knowledge as it is an individually constructed enterprise.
3. **Development and individual differences**. Learning, while ultimately a unique adventure for all, progresses through various common stages of development influenced by both inherited and experiential/ environmental factors.
4. **Strategic processing**. The ability to reflect upon and regulate one's thoughts and behaviour is essential to learning and development.
5. **Motivation and affect**. Motivational or affective factors, such as intrinsic motivation, attributions for learning, and personal goals, along with motivational characteristics of learning tasks, play a significant role in the learning process.

3.2.3 Available and accessible knowledge

There are two ways to make errors: (a) not having the knowledge available (i.e. not having learnt the knowledge); and (b) having the knowledge available but not accessing it (i.e. having learnt the knowledge but not realising that it can be applied in the task situation). Thus, mathematics teaching needs to build strong available **and** accessible mathematics knowledge. In particular, accessible knowledge is very productive as it is a form of deep learning and enables performance to continue into higher mathematics.

Teaching for accessible knowledge requires more than teaching for available knowledge. Accessing mathematics knowledge means knowing it on its own merits and knowing when, where and how it can be applied to task situations. Thus, accessible knowledge is based on mathematical ideas being held in *rich schemas*. As we have seen, such schemas have four characteristics: (a) they **define** – completely describe all the ways the mathematical idea can be thought of; (b) they **connect** – store knowledge so that all relationships are evident and all related knowledge is connected; (c) they **apply** – contain all the different ways the mathematical idea can be used in reality; and (d) they **remember** – organise and store all experiences students have had with the mathematical idea.

The RAMR cycle has been designed to be effective in developing rich schema and, therefore, accessible knowledge. It involves abstraction and reflection and begins and ends with reality, thus ensuring mathematical ideas are well defined, connected, applied and experienced. It identifies with techniques particularly useful for accessibility such as explicitly relating the mathematical ideas to as wide a collection of out-of-school experiences as possible and teaching that mathematics and its symbols form a highly connected structure, a language to describe the world and a set of tools for solving the world's problems.

Overall, the secret to accessible knowledge is to do **both sides** of the RAMR cycle: *abstract* the mathematical idea and *practise* the idea but not to stop there; to draw a breath and move on to *connections* and *reflections*. In particular, the **reflective strategies** are useful: *flexibility* gives the students a rich set of applications (particularly non-prototypic activities); *generalising* integrates knowledge which means there are fewer things to apply; *reversing* often means that real-world instances are constructed (the best way to teach interpreting the world); and *changing parameters* builds connections.

As well as improving applications and problem solving, improving accessible knowledge has the potential to improve results in testing, which is more than just repeating known procedures.

3.3 Extending RAMR cycle to MITI

In this section we look at RAMR's role in expanding MITI through to Year 12, looking at structure first, process of inquiry second and problem solving at the end.

3.3.1 Structure

As argued across Chapters 2 and 3, the RAMR cycle is a great basis for MITI. RAMR enables teaching and learning that takes account of the structure in mathematics and assists students to have structural understanding of mathematics. In so doing, the cycle enables deep learning of powerful mathematics. In particular, the following components of RAMR are particularly important for MITI.

1. **Reality/prior experiences**. Starting from the interests of the students is important because it grounds the new learning with the everyday structure of the students' existing knowledge. It does not start learning from an artificial position where it sits alone and unconnected as classroom knowledge. This building of cognitive structure is an important part of Stage 2 of MITI in Chapter 4.
2. **Abstraction/body**. The initial use of the body to act out the new mathematics idea builds visual imagery; developing a picture in the mind of how a mathematical idea works is the basis of deep understanding. We will look again at this in Stage 1 of MITI in Chapter 4.

3. **Mathematics/connections**. A powerful component of RAMR is the focus on connections in the Mathematics phase. This is based on Wilson's Activity Type cycle (Ashlock et al., 1983) and is underpinned by the argument that ideas are better consolidated by being placed in a structure (that is, connected to other knowledge) than through practice. For deep learning, knowledge is constructed in the form of schema; for powerful mathematics, the structure of the mathematics has to require schematic understanding. A specific focus on instruction that connects is a crucial part of the power of RAMR for MITI.

4. **Reflection/validation**. At the end of the cycle, the knowledge developed in mathematics is reflected back to the everyday life of the students, the position it came from at the start. This closing the circle, along with the applications and problem solving that are part of the Reflection component of RAMR, ensures all new knowledge learnt is brought back together and then connected to the structure of the students' everyday knowledge. This prevents separation of school knowledge from everyday knowledge.

5. **Reflection/extension**. This is the main component of RAMR for MITI. In this component, the teacher seeks to extend and deepen the knowledge of the students. This is done by focusing on the generic strategies or pedagogies of *flexibility*, *reversing*, *generalising* and *changing parameters*:

   (a) Flexibility seeks to attach the newly learnt idea to as wide a set of topics as possible. For example, $^3/_4$ can represent 45 minutes or 270 degrees of turn as well as 750 mm and 75%.

   (b) Reversing works to ensure that connections are in both directions. For example, $^3/_4$ of 12 is 9 (whole → part); and $^3/_4$ is 12 means whole is 16 (part → whole).

   (c) Generalising requires understanding to be generic, for any situation. For example, a whole divided into 4 equal parts is $^1/_4$; a whole divided into Fred equal parts is $^1/_{\text{Fred}}$.

   (d) Changing parameters gives an opportunity for gestalt leaps of understanding built on big ideas. For example, in generalised terms, adding two 2-digit numbers involves adding the ones, adding the tens and doing any necessary renaming. It is possible to extend this generalisation by changing parameters to adding metres and centimetres, hours and minutes, or algebraic addition of two variables.

3.3.2 Inquiry

The process by which students construct their knowledge is important. The inquiry approach builds a community of learning in the classroom by organising students to cooperate in investigating what has to be learnt. It involves **co-opting** the students to be **complicit in their own learning**, to be **co-constructors** of their knowledge and to discuss and debate without the necessity of common conclusions. It assumes this will be done in a way that is culturally safe with high-expectation relationships. It includes *social constructivism* where students construct their own knowledge in discussion with peers and teachers. It makes social sharing a component of individual learning.

The inquiry approach involves planning lessons that continuously link different representations (e.g. symbols, language, models, materials and graphs) and the ideas they represent. Mathematics comprehension is enhanced if instruction, and learning, can coordinate and integrate at least two representations/ideas. This is the **multi-representational** teaching mentioned in section 3.2.1.

In order to co-opt students as co-constructors of knowledge, instruction needs to enable students to understand themselves as learners, and to involve the students in their learning as **co-researchers**. This includes such activities as teaching students to understand learning, social negotiation and research, thus allowing the students to monitor and measure their own learning and to regulate their own behaviour and learning actions.

3.3.3 Problem solving

As we move into higher level mathematics dealing with problems and applications, we have to be more specific with involving problems and problem solving. YDC has available a supplementary resource book, *Problem Solving*, which looks at this area in detail (available through the MITI Blackboard site). This subsection looks at what is required to solve one-step and many-step problems, covering plans of attack, general problem-solving strategies, and strategies particular to many-step problems.

1. **Plan of attack**. The basis of powerful problem solving is strong structural knowledge of mathematics, resilient affective traits, metacognition, and powerful thinking skills. Metacognition is the ability to control one's own thinking; this means being able to monitor, coordinate, check, evaluate and modify thinking and make decisions. Metacognition can be codified into a plan of attack. The best plan of attack is Polya's four stages:
   - *See* – work out what you have to do
   - *Plan* – make a plan to do it
   - *Do* – do the plan
   - *Check* – check your answer and see what you can learn from the problem.

2. **Problem-solving strategies**. Metacognition is supported by strong thinking skills, namely, spatial or visual thinking, flexible thinking, creative thinking, logical reasoning, and patterning skills. Thinking skills can also be enacted into strategies. An effective list of strategies is as follows.
   - *Verbal-logical* – reread the problem; identify given, needed and wanted; restate the problem; write a number sentence.
   - *Visual-spatial* – act out the problem; make a model; make a drawing, diagram or graph; select appropriate notation.
   - *Organising* – look for a pattern; construct a table; account for all possibilities systematically.
   - *Checking* – generalise; check the solution; find another way to solve it; find another solution; study the solution process.
   - *Restructuring* – guess and check/trial and error; work backwards; identify a sub-goal/break the problem into parts.

3. **Particular strategies for multi-step problems**. In multi-step problems there are two components: determining the steps, and determining what to do in each step. There is also an emphasis on doing each step in turn, finishing each step before moving on to the next, and making sure you take everything into account. For simpler two- and three-step problems, there are three major strategies:
   - *Make a drawing* – draw something useful that will help solve the problem.
   - *Given, needed and wanted* – determine what is given, what is needed to get you to where you want to go, and what is wanted (where you want to go).
   - *Restate the problem* – rethink the problem in your mind so it becomes easier.

   For more complex multi-step problems, three other strategies become important:
   - *Break the problem into parts* – check that you have not missed any parts.
   - *Make a table or chart* – this helps ensure nothing is missed and keeps things systematic.
   - *Exhaust all possibilities* – go through and do everything on a list.

3.3.4 Conclusions

Looking back across section 3.3, we can see that:

(a) RAMR is flexible and powerful enough to be the basis for driving mathematics in structural terms at MITI level;

(b) RAMR will remain a powerful framework into which inquiry fits and is supported, particularly in the Body → Hand → Mind and in Reflection; and

(c) RAMR enables problem solving which is specifically stated as a step in the Reflection phase and also the starting point for the Abstraction phase (the real-world situation).

One of the bases of YDM is to see mathematics in three ways, and to teach to achieve these ways:

(a) *as a structure* – with teaching built around schema (connections and sequencing) and big ideas emphasising similarities and differences between topics and their representations, and ensuring seamless/smooth sequencing and pre-empting;

(b) *as a life-describing language* – with teaching built around ensuring that the formal notation and words are seen as a concise life-describing language; and

(c) *as a tool for problem solving* – with teaching built around developing mathematics thinking so that ideas assist problem solving both within a domain of mathematics and outside domains of mathematics.

RAMR was designed to achieve these three outcomes. More will be discussed in the next chapter.

4 Pedagogies, Proposals and Stages

Having provided background and beliefs and described the RAMR planning/teaching cycle, in this chapter we look at MITI itself and the two stages for its implementation. For each of these stages we provide information on the pedagogy which, along with YDM, drives that stage.

4.1 Stage 1: Investigations, problems and seamless sequencing of powerful mathematics

As well as introducing the YDM pedagogy, this first stage of MITI focuses on the training of teachers to improve teaching and learning with respect to the following:

(a) the development of mathematics as a vertical and horizontal schema of interconnected and sequenced ideas built around key big ideas (see Chapter 2), a rich schematic structure of big ideas that fully includes definitions, connections, applications and experiences;

(b) the RAMR teaching cycle of the YDM pedagogy (see Chapter 3), bringing in the Renzulli (1976) approach to teaching mathematics to able students (see this chapter), changing instructional approaches from textbook pages of exercises to problems and investigations, and developing classroom techniques to get the most from these; and

(c) the creation of engaging instructional sequences that teach lower level ideas in a way that seamlessly extends to higher level forms of the same idea, particularly using digital technologies to assist in developing these sequences, following the REAL approach to their use (see this chapter).

To do this, Stage 1 looks at both students and teachers. For students, it needs to introduce teaching methods that will be positive in terms of student affect. For teachers, Stage 1 looks at ensuring the maths ideas are taught in a way that enables Years 7–9 teaching to connect seamlessly to Years 10–12 teaching, using inquiry methods of teaching mathematics (see section 3.3.2). This initially focuses on Years 7–9 but extends over Years 7–12.

This section, therefore, overviews: (a) a special pedagogy for investigation (Renzulli triad) and its relation to RAMR; (b) technology use (and the REAL pedagogy); (c) teaching approaches regarding affects and investigations (and their implementation); and (d) implications and the plan for MITI Stage 1 PD activity.

4.1.1 Renzulli triad pedagogy

In the development of investigations (rich tasks), YDC has used the pedagogy of Renzulli (1976) – that mathematics ideas should be developed through the following three stages for able students.

1. **Motivation** – base instruction on activities that will interest the students and will assist them to engage with mathematics.
2. **Prerequisite skills** – use diagnosis to determine and then teach all necessary mathematics ideas needed to undertake the motivating activity.
3. **Open-ended investigation** – end the teaching sequence by setting students an open-ended investigation based on the activity identified in Stage 1. This is to allow the students freedom to explore the activity as far as their interest and ability will go.

Note: This triad was developed by Renzulli (1976) as the most appropriate way to teach mathematically gifted and talented students.

The Renzulli pedagogy combined with YDM can provide insight into how to enrich and extend students in mathematics. We now look at this in terms of the three components of the Renzulli triad, namely, motivation, skills and rich tasks or investigations.

Motivation

Motivation can be considered in relation to the first step in the RAMR cycle, Reality. Reality looks at prerequisites, local knowledge and kinaesthetic activity, with the aim of building interest and engagement by starting from the interests of the students and being active in instructional tasks. Therefore, the implication is that the motivation sought for able students must be based on a real interest of the students not of the teacher, and that this interest will most likely lie within experiences from the social situation of the students and their culture, and be reinforced with the active way the ideas are first experienced.

Skills

This component of Renzulli interacts strongly with the YDM pedagogy and relates to mathematics structure. If able students are to prosper in advanced mathematics subjects, their mathematics should be: (a) structural (or relational, in terms of Skemp, 1989), that is, highly connected and deep; (b) based on big ideas, with real language understandings that relate to models; and (c) able to reflect learning that covers all the steps in the RAMR cycle.

1. **Structural mathematics** means rich schema, which requires students' knowledge to have four components: (a) completely defines the mathematical idea; (b) provides all applications of the knowledge; (c) has all connections identified between the knowledge and other mathematics knowledge; and (d) contains critiqued and recoverable experiences of that knowledge.
2. **Big ideas** means that students' knowledge: (a) covers all the different concepts making up a mathematics big idea; (b) encompasses a strong understanding of mathematics language, particularly as a shorthand concise symbolic structure that tells stories and describes life; and (c) has a rich repertoire and understanding of models, representations, and strategies.
3. **RAMR** means that, other than starting from the students' interests, students can: (a) connect their knowledge across body, hand and mind activities; (b) be creative in constructing their knowledge; (c) be familiar with their knowledge as a result of effective practice; and (d) understand their knowledge in terms of the four generic Reflection strategies (i.e. flexibility, reversing, generalising and changing parameters).

Investigations

This component of Renzulli is very open but the students doing investigations should expect to experience the mathematics ideas in the investigations with all the power discussed under skills. For example, the investigation should provide experiences that: (a) reveal/provide structure, (b) involve the four generic Reflection strategies, and (c) require use of problem solving through a plan of attack and strategies.

4.1.2 Technology use

To develop and maintain interest and motivation, and meet the needs of the 21st century, MITI will make extensive use of technology to enable students to undertake 21st-century-relevant authentic tasks. Graphical calculators and other forms of technology can be put to use in many different ways to enhance a student's learning and understanding of mathematical concepts. These are discussed in more detail in the supplementary MITI resource book on Technology.

REAL learning with technology

One often-quoted example is the use of technology to enable students to explore real-life related mathematical applications. This requires the student to use the technology to overcome the barriers that arise when the student's current level of mathematical knowledge is not sufficient to cope with the application, however interesting and engaging it might be. This is a valuable method of providing extension but often not suited to all students in a class. Therefore, MITI proposes a new pedagogy called **REAL learning with technology** (Lowe, 2016).

REAL learning with technology describes a process to link the students' current mathematical studies with current issues and events in the real world. It is based on REAL as an acronym and emphasises four steps:

Review → Explore → Analyse → Link

Using the technology to adapt information obtained from real-life events and developing a learning activity directly linked to the current program of mathematics study is a much more effective method of bringing the real world into the classroom.

Four steps of REAL

1. **Review**. Read widely and keep up-to-date with topical issues in both the print and electronic media to find topics that students might have some awareness of and find interesting. The range of media needs to be more comprehensive than the usual commercial sources. Mathematics teachers seeking to engage and encourage students to continue their mathematical studies at the highest levels need to demonstrate and share their own interest in mathematics by sharing insights gained from such media with students. Review the suggestions for wider media sources available in the MITI Technology book with an eye and an ear open for topics suitable for mathematical treatment by students. These may be major sporting events, isolated but significant events (e.g. sinking of *Costa Concordia* a few years ago), long-term trends such as global warming or local topical issues (e.g. local grand final, new stadium or road tunnel).

2. **Explore**. Seek further information and possible data sources, images and interactive tools, and so on, related to the topic that will enable the "mathematising" of the topic. This could be in the form of tabulated data, diagrams to help explain the topic or interactive simulations that students could use. Web searches can be progressively refined as information is obtained until you find what is required. For example, during the swine flu epidemic of 2009 a general search for "swine flu cases" produced a number of starting points. Using the information produced it was possible to further interrogate websites until WHO data for the number of notified cases in Australia, tabulated according to the date, could be obtained as the basis for a student investigation.

3. **Analyse**. From the available material decide which parts would be useful in the classroom. Decide if the data could be used in the current format or requires modification. For example, can dates included in the data be used in the original format or do they need to be modified to Day 1, Day 2, and so on.

4. **Link**. Identify connections between current curriculum topics and the topic being investigated. Look at the current program to determine if the available data/picture/tool could be used in a way to develop students' mathematical understanding. A dataset may be capable of being used in a variety of ways. Look beyond the obvious.

4.1.3 Teaching approaches

The success of MITI projects depends on how they are implemented and this implementation involves changing both teachers' and students' affects. As well, the more open procedures in the classroom require different teaching techniques from those used with textbooks, including feedback and pivotal teaching moments.

1. **Implementation**. The success of MITI projects is based on teachers implementing a mathematics program which may require major changes in the teachers' knowledge, attitudes and beliefs about mathematics and mathematics teaching. This change is best achieved when external input leads to success in the classroom. This means that the PD and teacher-change principles and procedures documented in Baturo et al. (2004) and Darling-Hammond and Bransford (2005) should be followed. Also, the processes used with teachers have to overcome the contradictions between research and teacher needs documented in Lamb et al. (2007). (This is discussed in more detail in Chapter 5).

2. **Affects***.* MITI projects will only be successful in terms of cognitive growth if mathematics instruction is linked with positive changes in student and teacher affects, teacher and school practices and community involvement. If students do not believe they can learn mathematics and do not have the motivation to continue even when they are successful at solving mathematics problems, they are unlikely to persevere to high-level mathematics in Years 11 and 12 (English et al., 2008). A student's success in undertaking high-level mathematics is critically dependent upon teacher expectations; teachers must believe that low-SES students can be successful in this endeavour.

3. **Support and feedback**. Having groups or individuals working on investigations is not an opportunity for the teacher to wait until a hand goes up. It is important to move and observe and ask questions. Then it is important to have questions that could support those not making progress. Finally, when the time comes for feedback it is important to choose an order in which groups report that maximises learning. The pedagogy YDM follows is:

   Anticipate, Monitor, Select, Sequence, Connect (Smith & Stein, 2011)

   Anticipation is before the lesson where the teacher prepares for all the questions and errors that will require support. *Monitor* and *select* are where decisions are made about feedback. *Sequence* is the order of feedback – students who make less progress before those who are successful. Finally, *connect* is to remind teachers to relate the investigation back to their normal teaching. All of these are discussed in the MITI Stage 1 PD days – see section 4.1.4.

4. **Pivotal teaching moments**. These are points in a lesson where there is an opportunity to use a students' comments/questions to draw an important point from students' activities. YDM focuses on three forms:

   Anticipated, Pedagogy-based, Unanticipated

   As expected, *unanticipated* is the most difficult to handle, *anticipated* is when your experience and knowledge expects an opportunity, and *pedagogy-based* is when you have designed the lesson to maximise chances of a useful comment/question at certain times. This will be discussed in section 4.1.4.

4.1.4 Implications and Stage 1 PD activity

This section discusses the implications of the above and summarises the proposed PD activity.

Implications

The discussion above with respect to Renzulli and teaching approaches is a major part of the PD days for MITI Stage 1. However, three interactions important for enrichment and extension are worth highlighting.

1. **Focus of activities and investigations**. The focus of the mathematics within the Renzulli components of skills and investigations has to be, as far as possible, based on a genuine interest of the students and developed in a way that maintains interest. This means the Renzulli first component, motivation, is particularly important.
2. **Depth of knowledge from normal mathematics teaching**. Effective MITI teaching should build deep knowledge within which new ideas can be easily structured. This means the following:
   (a) strong diagnosis to detect weaknesses and rebuilding of knowledge from first principles where there is little understanding;
   (b) smooth seamless progressions of ideas, ensuring all meanings are included;
   (c) strong repertoires of models and strategies, and not accepting correct answers from rote-learnt processes without understanding of underlying knowledge;
   (d) implementation of student-centred practices with an acceptance that students have to construct their own knowledge by looking for similarities and differences in activities; and
   (e) high expectations for quality knowledge and the understanding that enrichment and extension are more than adding an investigation to existing rote teaching.
3. **Complexity in investigations**. It is important to construct and use investigations that are:
   (a) multi-representational in terms of integrating models and representation, but also multi-topic in terms of bringing many ideas together to reach solution ("built in" not "bolt on");
   (b) challenging to the point of failure for many students but enabling progress for all students; and
   (c) bridging affect, skills and reflection, particularly in terms of technical, domain and generic knowledges.

Proposed Stage 1 PD activity

The **goals** for the PD in MITI Stage 1 are to have teachers who can use YDM pedagogy and Renzulli to:

(a) identify, modify, construct and effectively teach investigations and problems appropriate for their students in a manner that helps build powerful mathematics ideas;

(b) build student motivation, confidence and understanding as students move from lower years to and through high-level mathematics in Years 10 to 12, in particular by using digital technologies;

(c) identify and understand the structures of high-level, powerful mathematics with respect to sequences, connections and big ideas, and examine Years 7–9 teaching to ensure sequencing to Years 10–12 is as seamless as possible; and

(d) build these structures and develop their strength with respect to recalling mathematical ideas, solving problems and laying the foundation for later learning.

The **resources** available in Stage 1 are: (a) this MITI Overview book; (b) a collection of 45 investigations designed to cover all topics across Years 7 to 9; (c) an initial collection of ideas for teaching with technology and seamlessly sequencing topics for the transition from Years 7–9 to Years 10–12; (d) supplementary YDM resource books on big ideas, problem solving and literacy in mathematics; and (e) a QUT Community Blackboard site.

The **program** for the Stage 1 PD workshop days will be developed for each cohort and will include identifying good tasks, Renzulli's ideas, modifying and constructing tasks, and developing sequences for teaching maths seamlessly from Years 7–9 to 10–12.

4.2 Stage 2: Deep applications in futures contexts

As well as continuing with the YDM pedagogy, this second stage of MITI focuses on the training of teachers to improve teaching and learning with respect to the following:

(a) the building of big ideas through model-based structured sequences of instruction across year levels and the placement of mathematics ideas into more global structures (which we call **superstructures**) to allow for more seamless learning, holistic recall and relationship to effective schematic/organic learning;

(b) the extension of the YDM RAMR pedagogy to a double RAMR model as a pedagogical structure to model the development of the mathematics that is built on symbol structures that, in turn, come from abstraction of number and arithmetic;

(c) the relationship between applications and understanding underlying mathematics, providing real and authentic contexts to powerful mathematics in order to balance mathematics understanding as worthwhile in its own right with mathematics understanding as a valuable tool in the world of applications and employment opportunities; and

(d) the identification of the mathematics on which 21st-century activities and applications are based, particularly those related to technology and mathematics content covered in the highest level mathematics subjects, and to make this mathematics visible to students in a manner that enables deep learning of the mathematics and strong capacity to understand applications; and to connect this mathematics to learning in the wider STEM field (science, technology, engineering and mathematics).

Thus, this second stage focuses on applications to provide motivation and to give insight into the deep and powerful mathematics knowledge needed to be successful in Years 11 and 12 advanced mathematics subjects, and to continue this success on to university.

This section, therefore, looks at deep learning and the plan for PD in MITI Stage 2. It covers (a) structured sequence theory and its implications; (b) an extension of RAMR to double RAMR; (c) pre-empting and superstructures (extension of big ideas); and (d) the plan for MITI Stage 2 PD activity.

4.2.1 Structured sequence theory

Building big mathematics ideas

The framework for learning used in YDM projects is based on a theory (Warren & Cooper, 2009) for building big mathematical ideas. This theory is built around understanding of mathematical ideas being the ability to move between four representations of these ideas: (a) symbols; (b) language; (c) physical, virtual and pictorial; and (d) graphical representations (if they are appropriate). It also involves being able to move between various models of these representations such as set, number line, array, and so on.

Warren and Cooper's theory argued that mathematics knowledge growth for big ideas is through structured sequences **across** models/representations not within a model/representation. It also argued that, if the foundations are well learnt, there is acceleration in knowledge growth with reduced time having to be spent on learning, which can be used for remediation in AIM or deeper learning in MITI.

Thus, Stage 2 of MITI, with its focus on big ideas, requires us to do the following: (a) **look *wider*** in terms of pedagogy, to vertical curricula (structured sequences) and to the use of technology; (b) **look *deeper*** in terms of how mathematics topics grow from Years 7 to 12; and (c) **be *cleverer*** in developing Years 7–9 mathematics teaching that enables connections and sequences to be identified that make it easier to teach Years 10–12 mathematics **with concomitant savings in time**.

Components

The theory argues that these structured sequences have the following properties.

1. **Isomorphism**. Effective models and representations have strong isomorphism to desired internal mental models, few distracters, and many options for extension. In other words, these models/representations grow with the idea – for example, 3 × 4 is an array which can grow into the area model which is the basis of multiplication of larger numbers (7 × 23), fractions ($\frac{4}{5} \times \frac{2}{3}$), and algebra (x (x + 2)).
2. **Sequence**. Sequences of models/representations develop so there is increased flexibility, decreased overt structure, increased coverage and continuous connectedness to reality. For example, the balance model for equations moves from a physical balance to a pictorial balance to an abstract balance that can handle division and negatives.
3. **Nestedness**. Ideas behind consecutive steps are nested wherever possible. That is, later thinking is a subset of earlier. For example, the first understanding of equals should be "same value as", and the second understanding should be the result of the calculation which is a particular form of "same value as". That is, the meaning of equals in 4 + 2 = 6 is nested within 4 + 2 = 5 + 1, so 4 + 2 = 5 + 1 comes first.
4. **Integration**. More complex and advanced mathematical ideas can be facilitated by integrating models. For example, solving algebraic equations is a combination of number-line understanding of inverse operation and balance-beam understanding of the balance rule. However, some complex and advanced ideas may require the development of *superstructures* if complexity leads to *compound* difficulties. For example, the compensation principle for addition is to do the inverse to the other number (e.g. 8 + 5 = 10 + 3 by adding and subtracting 2), while the compensation principle for subtraction is, simplistically, the opposite (e.g. 14 − 6 = 18 − 10 by adding 4 to both numbers). This opposite difference between them can cause confusion and, thus, is called a compound difficulty. However, if a superstructure of understanding is built around subtraction as the inverse of addition, and division as the inverse of multiplication, it is almost self-evident that there has to be an opposite process for subtraction in relation to addition.
5. **Comparison**. Abstraction is facilitated by comparison of models/representations to show commonalities that represent the kernel of desired internal mental model. In simple terms, 2 + 3 = 5 makes more sense when seen in joining counters (set model) **and** steps along a number track (number-line model.)

Implications

The implication of this theory is that learning is enhanced and accelerated if instruction takes account of the vertical sequences in mathematics topics. Early trials of sequences have shown that the first stages of a sequence (covering the early years of the sequence) have to be completed slowly and carefully to build the connections that frame out the big mathematical ideas. Then the later stages (covering the later years) can be accelerated to cover the idea quickly, often in gestalt-like leaps of understanding. This leads to the development of holistic ideas or superstructures which enable deeper understandings of the original parts.

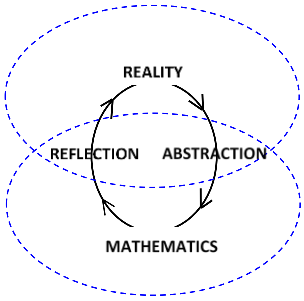
Thus it is important that Years 7–9 and 10–12 are considered together. This ensures that the ideas in Years 10–12 are pre-empted by the ideas in Years 7–9. A slow and careful start in Years 7–9 can mean a quick finish in Years 10–12, and a saving in time that can be spent on developing deeper understandings. It is the intention of MITI to use Stage 2 to set up Years 7 to 12 sequences that will make time savings in Years 10–12.

4.2.2 Deep learning and double RAMR

Summary of RAMR model

The RAMR model was developed from a philosophical position with regard to the creation of mathematics; that mathematics knowledge was created through the ontological relationship between reality and mathematics, and that mathematics is a symbolic, culturally biased system creatively abstracted from, and then reflected back to, reality (Matthews, 2009) – see discussion in section 3.1.1.

From this philosophical relationship, the YDM Reality–Abstraction–Mathematics–Reflection (RAMR) cycle pedagogy was developed (see Figure 5 at the beginning of Chapter 3 – repeated below right). This cycle has four stages and two sides or halves:



(a) Reality – working from reality and prerequisites;

(b) Abstraction – moving from everyday instances to mathematical ideas and forms through an active pedagogy and creativity (kinaesthetic, physical, virtual, pictorial, language, and symbolic representations or body → hand → mind);

(c) Mathematics – consolidating the new ideas through introducing formal language and symbols, practice and connections; and

(d) Reflection – validating and extending these ideas through a focus on applications, problem solving, flexibility, reversing, generalising and changing parameters.

Innovatively, the right half of the RAMR model develops the mathematics idea while the left half reconnects it to the world and extends it.

Deepening the RAMR model

The RAMR model was bolstered by the integration with other pedagogies. Learners who completed all RAMR stages acquired strong schematic and, therefore, deep understanding of the ideas being taught. However, deep and powerful mathematics is based on big ideas and second-level abstractions. Many big ideas are based on principles (the abstract schema of Ohlsson, 1993) where meaning lies in the relationships between components not the components themselves (e.g. the commutative principle). Because of this, big ideas and principles are often second-level abstractions (and reflections) of the previously abstracted ideas. For example, arithmetic is a result of abstracting objects to operations and numerals, while algebra is the result of a second-level abstraction from arithmetic (operations with numbers/numerals) to algebra (letters and variables).

To meet and understand the nature of deep mathematics and how it is learnt, YDC proposes the creation of an extension of the RAMR cycle to the double RAMR cycle as illustrated in Figure 11. The double cycle shows, for example, how reality (objects and actions) is abstracted to the mathematics (in this case, arithmetic) which in turn is abstracted again to the deeper mathematics (algebra), with reflection moving backwards from deeper mathematics (algebra) to mathematics (arithmetic) and from mathematics (arithmetic) to reality (objects and actions). Some very advanced mathematics could be a result of three (or more) levels of abstraction.

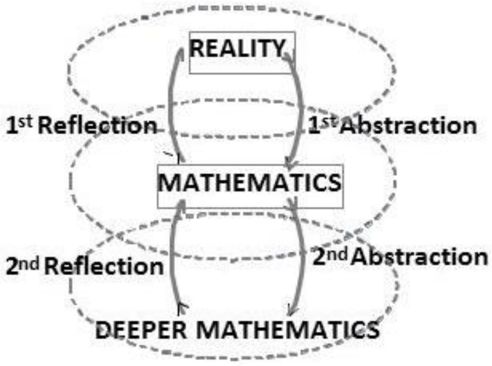


Figure 11 Double RAMR cycle

The double cycle illustrates the difficulties of high-level or deep and powerful mathematics: it takes two (or more) abstractions; it is distant from reality; and it relies on success at the first level (previous levels) of abstraction. This must be taken into account when teaching mathematics in Years 7 to 12. Some ideas can be developed in one level of the RAMR cycle, but others may need two levels. In the latter case, the questions are: *Do the students know the first level? Has this been properly abstracted? Is their knowledge of the first level strong enough to bear a second level?*

4.2.3 Pre-empting and superstructures

Pre-empting

Pre-empting is part of YDM's vision for sequencing. Mathematics is best understood as a structure of connections and sequences. For example, fractions are connected to division and division has a sequence that covers: meaning of division → basic facts → algorithms/multi-digit division → decimals/fraction division → algebra, and so on. The role of division with regard to fractions helps teach fraction ideas. For example, (a) $\frac{1}{7}$ is less than $\frac{1}{3}$ because dividing by 7 gives a smaller amount than dividing by 3; and (b) division of fractions has to be taught by the most powerful division idea, multiplication by inverse or reciprocal.

This has two implications: (a) it is important to ensure that sequences are seamless and move as easily as possible from lower to higher level knowledge; and (b) most mathematics ideas will be in a sequence and it is efficient to ensure that the sequence helps with learning. There are two aspects to sequencing that depend on position in the sequence. First, lower steps prepare for later steps (**pre-empting**) and, second, later steps consciously build on the earlier steps (**building schematic structure**). This is why we advocate, for example, never to restrict multiplication by 10 for whole numbers to the rule "add a zero" because it does not apply to decimals, the next step in the sequence.

Pre-empting is therefore a **very powerful pedagogical point**. We advocate that all teaching in lower levels pre-empts the teaching in higher levels. This often means not taking shortcuts at the lower levels. With respect to MITI, it means looking at the mathematics needs of Years 10–12 when deciding how to teach Years 7–9. This means looking not just at content but also at the models and contexts used. The obvious example is trigonometry being taught in relation to the circle as well as in relation to similar right-angled triangles. This is because similar right-angled triangles has been taught earlier and so should lay the foundation for sine, cosine and tangent, and in Years 10–12 use of trigonometry is related to circles. By doing both in Years 7–9, a teacher builds on the past and prepares for the future.

Superstructures

Superstructures are an important mathematics structure in terms of student learning. They follow on from pre-empting and big ideas and are part of enabling schematic learning. They are also known as **encompassing big ideas** and relate to what Skemp (1976) called the **organic** nature of what he called relational (and what we call structural) mathematics. So we argue that there is the following sequence in terms of structures in mathematics:

Connections and sequences	➡	**Schemas**: Maths ideas in a structure that defines, connects, applies and remembers	➡	**Big ideas**: Schemas that apply to many year levels and topics	➡	**Superstructures**: Global schemas covering ideas so new ideas fit in

Overall, superstructures are webs of mathematics knowledge in which new mathematics ideas can fit. They are organic in that they assist learning by having the mind prepared for the next ideas. A superstructure web contains big ideas but can have other less complete structures. A superstructure containing the algebraic ideas from arithmetic in a manner that the main structures are covered is an excellent idea of what a superstructure is and what it can do.

An example of a superstructure is described below for the concept of percent. It is designed so that if the superstructure is known, it is easy for students to understand and calculate with respect to the three types of percent problems: (a) *What is 23% of $90?* (b) *If 23% is $90, what is the total cost?* and (c) *What percent of $90 is $17?* To do the above, the percent superstructure would have to include the following.

1. **Direct lines of development** such as (a) percent as a fraction (whole ←→ part where there are 100 parts); (b) percent as a decimal number (percent is hundredths so whole numbers have to relate to percent by × and ÷ 100); and (c) percent as a rate (interest per each one hundred dollars).

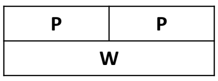
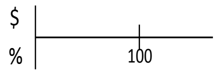
2. **Difficulties with the direct lines** such as from first problem type (23% of $90) and second problem type (23% is $90) for students to automatically write 23% as 23/100.

3. **Earlier and later activities in relation to the direct lines** such as sharing division, concepts of fractions, decimal measures, understandings that multiplication is normally rate by number, and role of formula *percentage = percent × amount*.

4. **Associated ideas** such as (a) integrating fraction and ratio and extending to percent ($^2/_5$ fraction part, $^3/_5$ other part, and 2:3 ratio; and (b) multiplicative structure of the decimal number system that means ones for whole numbers relate to 100 as percent.

5. **Things that are different** or don't fit in such as (a) percent as a fraction being inflexible (denominator is always 100 which makes equivalence difficult); and (b) inverse relation by denominator size not applying.

6. **Models** as well as definitions and actions, such as (a) the P-P-W model, (b) the double number line model, and (c) the change arrowmath model as below.

		Whole Amount $\xrightarrow{percent}$ Part percentage

7. **Big ideas that apply to the superstructure** such as (a) the part-part-whole big idea (the big idea that covers all part-whole situations); (b) the relationship vs change big idea (percent can be seen as a relationship and a change); and (c) the triad big idea (since there are three parts to percent problems, P = % × A, there are three problem types).

Another example of a superstructure is the number system with PV positions in mathematical notation. This superstructure would include:

1. **A focus on one** – as the basis of groups, singles and parts (and depending on point of view).

2. The **place value structure for whole numbers** – grouping to left, one is right-most digit, multiply and divide by 10 by moving place values left and right, role of zero, and pattern for larger place values.

3. The **place value structure for decimal numbers** – role of decimal point, determining place values (symmetry around one), similarities and differences to whole numbers, errors that can emerge from simplistic teaching of whole numbers, and prevention of errors.
4. **Placing mathematical notation into this superstructure** and maintaining role/position of one, flexibility (anything can be a one), multiplicative structure (e.g. 10^0 as 1 and 10^1, 10^2, 10^3 and so on as whole number place-value positions; and 10^{-1}, 10^{-2} and 10^{-3} and so on as decimal place-value positions).
5. Other topics that integrate/connect to number structure.

4.2.4 Stage 2 PD activity

Hidden applications

It has been said that a mobile phone is $10 worth of materials and a set of mathematics algorithms and that it is the mathematics that makes the phone more expensive. In a similar way, the internet and social media are all backed by mathematics and are the places that mathematicians now work. For example, the successful Angry Birds app is based on understanding the mathematics of parabolas as a way of explaining how the birds move through the air. The app would not be possible without understanding this mathematics.

Thus, much of what is interesting in the world is based on mathematics, but a mathematics that stays hidden. Making these applications visible is an excellent way of teaching mathematics and making this teaching motivating. This will be a major part of the PD in Stage 2.

Proposed Stage 2 PD activity and resources

The **goals** for the PD in MITI Stage 2 are to have teachers who can:

(a) identify and understand how mathematics ideas develop through abstract symbols from lower level ideas that relate easily to real-world situations to higher level ideas that can exist only in the mathematician's mind;

(b) understand the structures (patterns and relationships) that enable mathematics to exist outside of normal reality, yet provide the underpinning to that reality;

(c) build mathematics ideas across year levels and topics, and gain generic understandings of how new mathematics can be added to existing knowledge with least difficulty and how this can be placed within a global understanding;

(d) maintain students' interest, motivation, confidence and understanding as they complete high-level mathematics subjects in order to increase participation in STEM subjects in tertiary institutions; and

(e) build the above through strong use of digital technologies.

To help achieve these goals, the structure of MITI was designed so that teachers can: (a) use the time gained by the pre-empting of topics in Stage 2 during Stage 1 to support the increased depth of Years 10–12 mathematics topics; (b) present topics in a meaningful context and allow students to investigate applications of curriculum topics in a meaningful way; and (c) connect applications to the mathematics that is hidden in the applications.

The **resources** available, and being developed, include: (a) the resource books from Stage 1; (b) a collection of teaching ideas for pre-emptive transition from Years 7–9 to Years 10–12 with use of technology where appropriate; (c) a collection of ideas on futures-oriented industry applications of mathematics and how to use them to teach mathematics; and (d) a QUT Blackboard Community site.

As for Stage 1, the **program** for the Stage 2 PD workshop days will be developed for each cohort and will focus on constructing ideas to meet the needs of the schools involved.

5 Implementation

Like other YDC projects, the resources for MITI are written as exemplar ideas to support the pedagogy and the PD. MITI is conceived as a project focusing on training teachers in enrichment and extension, producing effective senior school mathematics teachers. Therefore, the implementation of MITI with teachers is central to the success of MITI. The resources exist as a basis for teacher activity and support for planning and development. The teacher is the central point of learning; the resources exist as a starting point for teacher activity that is developed for the particular needs of the teachers' classes.

This chapter looks at factors with regard to implementation: school change and leadership, teacher knowledge, training provisions and curriculum implications.

5.1 School change and leadership

Changing the teaching in schools requires a change at teacher and school level. In particular, MITI could require teachers to change from textbook "explain and practice" to inquiry/constructive teaching approaches. This requires effective and pertinent PD and leadership for change.

5.1.1 PD and teachers' practices

YDC sees PD and changing teachers' classroom practices as being a cycle of affective readiness for change, pertinent external input, effective classroom trials, positive student outcomes, and supportive reflective sharing that leads to further readiness (see Figure 12).

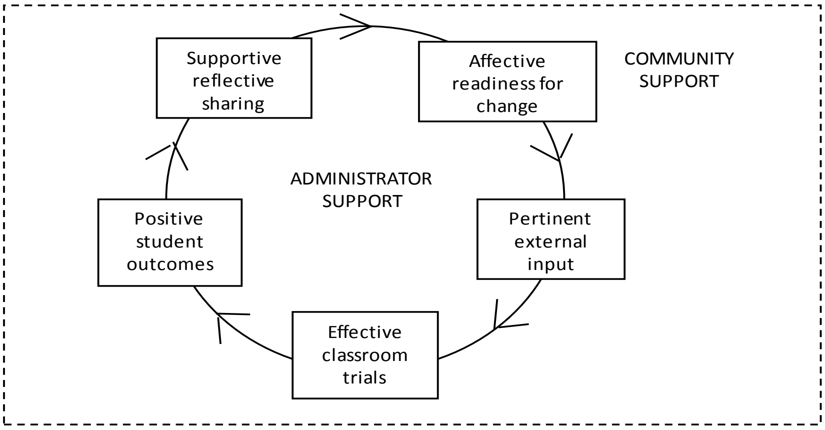


Figure 12 The YDC effective PD cycle (adapted from Clarke & Peter, 1993)

We acknowledge the importance of the **interaction** between researcher input and teacher need and readiness for this input, and the **role of success** (in terms of student outcomes) when trialling new ideas. We recognise that positive student responses along with initial readiness are crucial to successful change and that these are facilitated by: (a) pertinent, relevant and innovative ideas and resources at input (the MITI resources and PD workshops); (b) just-in-time support before and during classroom trials (assistance with planning, visits to model teaching approaches and provide feedback on teaching); (c) data gathering in an action-research process during classroom trials that can be seen as positive in terms of student outcomes; and (d) opportunities for teachers to share their successes and to attribute them to their ability. We also recognise the important role of the principal (and other administrative staff) and the local community. Without the support of the principal and other senior administrators, few if any interventions succeed in changing school practices. The same is true without the support of the local community.

We recognise that input through resources and PD workshops is a long way from student outcomes. Thus, for the project to be transformational within schools and improve student outcomes in mathematics, it requires six tiers of interaction, namely: (a) the community; (b) external researchers (YDC staff); (c) systems; (d) school administrators (principals and senior administrators such as heads of department and/or heads of curriculum – HODs and HOCs); (e) classroom teachers; and (f) students. YDC staff have control over resources and PD and, through visits, some involvement in school policies and classroom teaching. However, much of the teaching that affects student outcomes will be undertaken by teachers without involvement of YDC staff. It is this teaching that determines the central outcome of student success that, in turn, determines effectiveness of intervention. This distance between YDC staff and student outcomes gives importance to other people, meaning we need to consider: (a) key people in delivery (particularly principals, administrators and teachers); (b) parents, carers and other community members; and (c) students.

5.1.2 School change and leadership cycle

The provision of new mathematics teaching ideas is often insufficient for sustainable improvement in underachieving students' learning of mathematics. The new ideas have struggled to have positive effects when low attendance and negative behaviour are endemic across a school, when school practices and learning spaces disengage students, when positive partnerships are not formed between teachers and their teacher aides, when classrooms do not involve community leaders or acknowledge local knowledge, and when teachers do not believe the students are capable of the work. On the other hand, the ideas have been successful when they have been integrated into whole-school changes that challenge attendance and behaviour, integrate and legitimise local community knowledge, build in practices to support the culture of the students, and change teacher attitudes towards and relationships with the students. Thus, the MITI project is much more than a set of new teaching ideas; it integrates: (a) a particular teaching–learning approach to mathematics designed to empower students within their culture; and (b) an approach to PD and school change designed to facilitate change to support community involvement and student engagement.

The MITI approach to school change and leadership is based on the belief schools can only enhance mathematics learning with a program that focuses on mathematics and on school change together. In simple terms, we believe schools should see themselves as part of, not apart from, the communities in which their students live, and should see their role in terms of YDC's vision, as *growing community through education*. We believe school change can have a profound effect in creating emancipatory environments that actively seek to improve the educational outcomes and life chances for Aboriginal, Torres Strait Islander, low-SES, and indeed **all** students, and strong school leadership plays a critical role in acknowledging the existence of this student excellence.

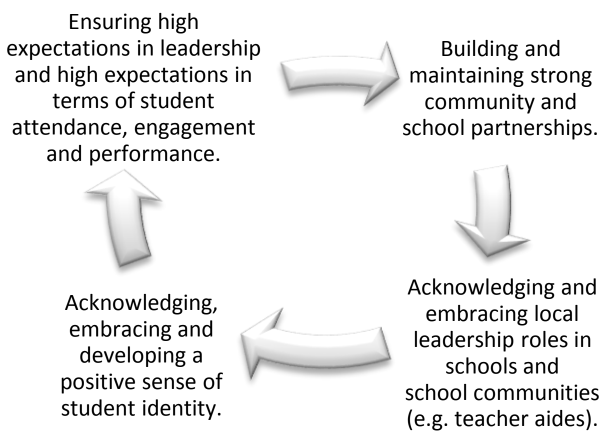


Figure 13 Cycle of school change and leadership

YDC has been influenced by the philosophy and success of the Stronger Smarter Institute, which argues that school change and leadership cycles through four requirements (see Figure 13): community–school partnerships; local leadership; positive student identity; and high expectations. It aims to develop not just new capabilities but also shifts in thinking individually and collectively. Maintaining the cycle of the four requirements ensures sustained growth towards enhanced learning. It creates and sustains emancipatory environments that enhance the opportunities of the students who attend, and challenges mechanisms and processes that continue to produce disengagement among many students within the schooling system.

5.2 Teacher knowledge

Low-SES students and, particularly, Indigenous students, tend to be holistic in learning style, moving from whole to parts, and not aligned with traditional procedural/algorithmic teaching which moves part to whole. To take advantage of this, the approach to mathematics teaching advocated in MITI (and in YDM that underlies it) focuses on big ideas (concepts, strategies and principles), vertical sequencing (from junior to senior secondary for MITI), and the RAMR cycle. However, this is a teaching approach that requires a lot from teachers – understanding of mathematics structure, use of technology, investigations and applications, active pedagogy and classroom control of behaviour. The MITI pedagogy also relies on teachers making decisions as to instruction themselves, based on their understanding of mathematics and their knowledge of the individual students. MITI provides teaching ideas and activities but not in the form of scripts or "recipes".

5.2.1 Shulman's knowledge types

MITI pedagogy therefore requires teachers to know all three of Shulman's knowledge types for effective mathematics teaching: *subject matter* (knowledge of mathematics content in terms of how it is structured, sequenced and connected), *pedagogic content* (knowledge of how to teach mathematics) and *lesson planning* (general knowledge of how to organise and run a lesson – includes behaviour management). Figure 14 illustrates these three knowledge types. To facilitate this, the activities of MITI are designed to build the capacity of teachers. However, in the past, it has taken time for teachers to come to terms with what is required to effectively implement a MITI-type mathematics program.

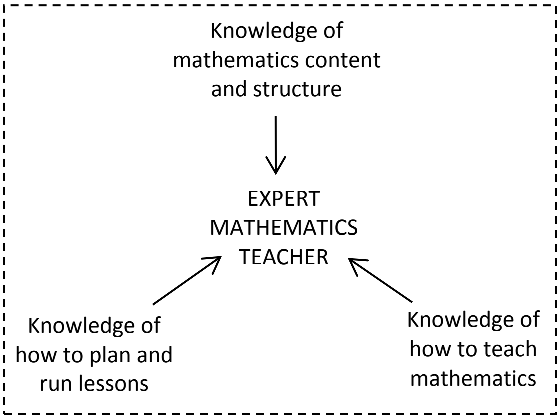


Figure 14 Shulman's knowledge types

Similar to teachers, teacher aides need subject-matter, pedagogic-content and lesson-planning knowledge. We believe in providing teacher aides with the same PD as the teachers and giving them the same resources. This does not mean simplification, because the highest quality knowledge is required for one-to-one tutoring as well as one-to-many teaching.

5.2.2 Building capacity

To build the capacity of MITI project schools to teach mathematics effectively, the plan consists of four steps (as in Figure 15):

(a) development of *mathematics resources* (with appropriate tests),

(b) provision of *PD and school support*,

(c) *classroom trials* of resources and *teacher feedback* on effectiveness, and

(d) *student testing at the start and end* of the trials.

The teacher feedback and any test results are used by YDC staff to modify materials. Thus in all MITI projects, YDC staff always plan to: (a) write or refine mathematics resources, (b) provide PD, and (c) visit each school to meet teachers and provide school-specific PD if this is part of the requested program.

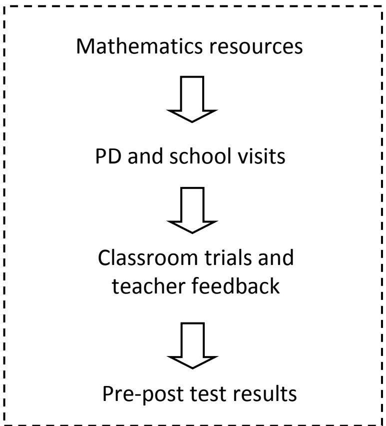


Figure 15 Student evaluation process

5.2.3 Teacher feedback

For quality control, to make the resources more effective and to enhance theory, YDC encourages all teachers trialling the MITI material to keep an annotated plan (or reflective journal) of what they do and their students' responses. Both the plan and the record of the students' responses will help the teachers and the researchers.

Teachers who trial new ideas and record how they go will get much more out of the trials. They will have systematic information on whether the new ideas are worthwhile, and information on how their teaching is going and how their practices could be improved. It takes time to become expert with a new approach and an action-research view of what is happening really helps. It is effective in improving teacher knowledge.

It is also important for such data gathering and reflection to be done as a group; YDC recommends schools set up communities of practice, groups where teachers can get support for their trials and can reflect on what happened together. For example, when physical materials were introduced in primary classes in the 1970s and 80s, there were often difficulties because students were not used to working other than in rows with worksheets or textbooks; they did not know how to behave, became excited with the material and out of control. However, after being given a chance to play with the material and having been taught how to act in groups with material, the new approach was found to work.

5.3 Training provisions

The focus of the MITI project is to evaluate and refine the enrichment and extension pedagogy, the tasks, and the PD workshops by studying teachers' and students' reactions to them in terms of achieving depth of mathematics understanding. This requires trials in classrooms by teachers who know the tasks and the pedagogy on which they are based. The model of PD and teacher change followed by YDC (see Figure 12 in section 5.1, and also Baturo et al., 2004; Lamb et al., 2007) is that a new approach to teaching is successful if:

(a) teachers believe there is a need to change their pedagogy;

(b) motivating PD is provided showing a new approach;

(c) teachers trial something of this new approach and find it successful; and

(d) teachers believe, on reflection, that this success is due to the effectiveness of the new pedagogy.

Thus, we believe implementing MITI successfully in schools requires teachers to be interested in changing their present pedagogy to MITI; to receive enthusing PD on MITI that convinces them they need to, and can, change; and to be supported in their first trials of the MITI pedagogy.

5.3.1 Implementation structure

In view of the above, to set up the implementation of MITI in schools, YDC does the following:

1. **Resources** – provides all schools with this Overview book, the extension tasks and the other resources for Stages 1 and 2 as outlined in Chapter 4.
2. **PD support** – provides four teachers (called trainers), and principals in certain sessions, with sufficient PD to be able to use the resources. These teachers then train the other teachers in their school.
3. **Online support** – provides all schools with online support in the form of a website with extra resources, a discussion forum, and a helpdesk to answer questions sent by teachers.
4. **Action research** – provides PD on how to use an action-research approach to trialling MITI resources so teachers can learn from the trials and provide feedback to YDC on quality of resources, PD and online support.

This would normally be six or more days of PD per year (and up to nine days if MITI is the only training being provided). The PD for each stage of MITI follows the plans outlined in Tables 1 and 2 in Chapter 4.

5.3.2 Additional services

There is more certainty of success with MITI if there are also in-school sharing experiences and in-school support, particularly to support the trainers. MITI training focuses on mathematics, mathematics education, and lesson planning. This requires a lot from the teachers and is difficult for trainers to provide in a rushed school situation.

Therefore, YDC can provide, at extra cost, the following additional services.

1. **Additional trainers**. Schools can negotiate for more than four teachers to become trainers.
2. **Expert practitioners**. Schools can negotiate for expert practitioners to visit their school, support the trainers, and work with the teachers in classrooms (modelling teaching, planning, observing, and providing after-school special PD).
3. **Teacher aides**. Schools can negotiate for special training for their teacher aides so that they can assist in the instruction.
4. **Other projects**. Schools can negotiate for teachers to be trained in YDC's basic training (YDM teacher development training) and YDC's remedial training (AIM or XLR8) as well as MITI. There can be cost reductions in integrating the basic, remedial and extension materials and pedagogies.

5.4 Cultural implications

An important component of MITI is to take account of the cultural differences between students' culture and the middle-class Western culture of the school and to ensure there is a focus on school change and community involvement in relation to the PD and other support sessions. This section focuses on cultural implications. The MITI project aims to improve mathematics performance of all students. This includes students who may not reflect the dominant culture of both school and Australian society, in particular, students who are Indigenous and low SES.

5.4.1 Indigenous students

The underachievement of Aboriginal and Torres Strait Islander students is, in part, a consequence of being part of a dispossessed people who have been considered by the dominant culture as primitive with no value for a modern society. This has implications for the way mathematics is taught to Aboriginal and Torres Strait Islander students. The devaluing of Indigenous cultures still continues today; the notion perpetuated by the education system that humankind evolved from hunter-gatherers to technologically advanced societies does not provide a sense of pride for Indigenous students about their culture. It ignores the reality that Aboriginal and Torres Strait Islander people have powerful and sophisticated forms of mathematical knowledge that enable the complexity evident in total ecosystems to be understood. It can lead to disengagement and non-attendance.

Aboriginal and Torres Strait Islander students predominantly come to school with a home language which is not standard English and with knowledge, skills, and patterns of interaction that are not appreciated by schools and do not match what facilitates success in school. This mismatch is particularly evident in the way mathematics is taught in schools. Aboriginal and Torres Strait Islander students tend to be active holistic learners, appreciating overviews of subjects and conscious linking of ideas (Grant, 1998). In fact, Indigenous people have been characterised as belonging to *high-context* culture groups using a holistic (top-down) approach to information processing in which meaning is extracted from the environment and the situation. In contrast, mainstream Australian culture is characterised as a *low-context* culture and uses a linear, sequential building block (bottom-up) approach to information processing in which meaning is constructed (Ezeife, 2002). School mathematics is traditionally presented in a compartmentalised form where the focus is on the details of the individual parts rather than the whole and relationships within the whole, a form of presentation that disadvantages Aboriginal and Torres Strait Islander students.

5.4.2 Low-SES students

Historically, educational institutions have favoured higher to middle-class backgrounds, beliefs and practices. This is due to a number of factors including the history of the purpose of schooling across its development and the socio-economic backgrounds of the majority of teachers and curriculum developers (Meadmore, 1999). As such, there are pre-existing patterns of communication and interactions (or discourses) endemic to education in Australia which are not favourable to lower SES students (Meadmore, 1999). Thus, the middle-class Eurocentric culture of Australian schools and implicitly understood patterns of communication and interactions serve to further marginalise students with low-SES backgrounds from school mathematics. The nature of discourses within school practices do not always successfully link to, nor validate, mathematical practices that may be part of low-SES students' out-of-school experiences (Baker et al., 2006), leading to insufficient links being made between students' existing mathematical knowledge and practices and school mathematics. In these cases, students may disassociate from school mathematics and feel they cannot succeed, particularly if their home skills and knowledge are not valued nor actively sought (Thomson, 2002).

Expectations may also pose difficulties for low-SES students as for Indigenous students. Low-SES parents may perceive mathematics as alienating and unnecessary or too difficult for their children to learn; this can lead to students not expecting to succeed in mathematics, having low expectations of themselves and their future roles in society, and not participating in mathematics classrooms. Teachers may also have low expectations of low-SES students and often believe that lower level or *life* numeracy is all that is needed for these students (Baker et al., 2006). The resulting emphasis of mathematics for these students becomes utilitarian, rote and procedural mathematics tasks that are not explicitly related to overarching mathematical structures.

5.4.3 Strengths and weaknesses of Eurocentric mathematics

Interestingly, the Eurocentric mathematics taught in Australian schools has weaknesses due to its cultural bias. Because of the way their culture was developing, European societies developed mathematics to help them explain their world and solve their problems, particularly to explain space, time and, eventually, number. When trading became a way of life, a need developed to be more precise in representing and quantifying value to have a shared agreement of how values could be compared ("is mine worth more than yours"), a more sophisticated process than quantification because it involves rate (e.g. 3 cows = 1 boat). Over time, the European mathematics' quantification and comparison system grew to encompass a variety of numbers (common and decimal fractions, percentages, rates and ratios), measures of time and shape (length, area, volume, mass and angle), and two operations (addition and multiplication) and their inverses (subtraction and division). The system was also generalised to findings that hold for all numbers and measures, and the resulting mathematics area of algebra has grown in importance as science and technology has expanded.

The weakness of European mathematics lies in its strength, the success of its quantifying and comparing systems in underpinning the growth of science and technology. This has resulted in longer and healthier lives and the devices that support work, home life and leisure. Western society now has the tools to change our environments to make living better; we can cool the hot, warm the cold, clear the land, bring in new plants and animals, and clothe, shelter, and feed large populations. However, this triumph has affected European culture and society. Success has come to mean increasing numbers and continued growth; smaller numbers and negative growth are to be avoided (and are given names that signify failure, such as "recession"). The culture appears to have little ability to understand harmony and act sustainably; it tries to dominate the land, the sea, the weather, and the animals, birds and fish, with little understanding of how things interact to allow human life, resulting in poverty, hunger, war, pestilence and climate change. Mathematics can be developed which would reinforce planetary equity and sustainability (see Figure 16); however, such mathematics requires less dominance by number, less need for growth, and an emphasis on living in harmony with land and sea; a non-Eurocentric form of mathematics.

Status depends on number and size → Success means larger numbers → Over production and climate change

The progress of Eurocentric mathematics

Less emphasis on number and size → Success is caring for country → Planetary sustainability

A better mathematical progress

Figure 16 Two perspectives of progress

5.4.4 Implications for teaching culturally diverse students

Traditional teaching of mathematics can confront Indigenous and low-SES students' cultures and perpetuate the belief that success in mathematics classrooms requires the rejection of culture (i.e. that one has to become "white" or middle class). To be effective, mathematics teaching needs to enhance mathematics outcomes but retain pride in culture and heritage. Approaches that seem to be effective are as follows.

1. **Discuss the role of mathematics**. Confront the Eurocentric nature of traditional school mathematics by making students aware of the cultural implications of mathematics teaching, and draw attention to which mathematics ideas change perceptions of reality. Discuss the role of mathematics in European culture and draw attention to its strengths and weaknesses. Make mathematics available to all students.
2. **Legitimise and contextualise the mathematics**. Legitimise local Indigenous or other cultural knowledge and integrate students' culture and mathematics instruction, to match the mathematics classroom to the cultural capital brought by the students (Bourdieu, 1973). Contextualise mathematics into the life and culture of students, using models and activities from the everyday lives of the students. Use a local cultural framework for learning. Take account of English not being the first or home language of students; develop language and be aware of different meanings for mathematics words.
3. **Modify teaching pedagogies**. Present mathematics as a holistic structure that can empower the learner by focusing on big ideas and using instructional strategies that relate acting, creating, modelling and imagining. Modify teaching pedagogies to include cultural perspectives.
4. **Integrate with whole-school changes**. Realise that approaches to improve mathematics learning cannot stand alone, and they need to be allied with whole-school activities that include the local community, challenge attendance and behaviour, have high expectations, develop local leadership and give a strong role to local teacher aides.

Appendices

Appendix A: Major big ideas by topic area and strand

Global big ideas	• *Symbols tell stories.* The symbols of mathematics enable the world to be described succinctly and in a generalised way (e.g. 2 + 3 = 5 means caught 2 fish and then caught another 3 fish, or bought a $2 chocolate and $3 drink, or joined a 2 m length of wood to a 3 m length, and so on). • *Change vs relationship.* Everything can be seen as a change (e.g. 2 goes to 5 by +3) or as a relationship (e.g. 2 and 3 relate to 5 by addition). • *Probabilistic vs absolutist.* Things are either determined by chance (e.g. will it rain?) or are exact (e.g. what is $2 + $5?). • *Accuracy vs exactness.* Problems can be solved accurately (e.g. find 5 275 + 3 873 to the nearest 100) or exactly (5 275 + 3 873 = 9 148). • *Continuous vs discrete.* Attributes can be continuous (smoothly changing and going on forever – e.g. a number line) or they can be broken into parts and be discrete (can be counted – e.g. a set of objects). Units break continuous length into discrete parts (e.g. metres) to be counted. • *Part-part-total/whole.* Two parts make a total or whole, and a total or whole can be separated to form two parts (e.g. fraction is part-whole, ratio is part to part; addition is knowing parts, wanting total).
Numeration big ideas	• *Part-whole/Notion of unit.* Anything can be a unit – a single object, a collection of objects, a section of a line, a collection of lines. Units can form groups and units can be partitioned into parts (e.g. if there are six counters, each counter can be a unit, making six units, or the set of six can be one unit.) • *Concept of place value.* Value is determined by position of digits in relation to ones place. • *Additive/Odometer.* All positions change forward from 0 to base, then restart at 0 with position on left increasing by 1, and the opposite for counting back (e.g. $2\frac{3}{5}$, $2\frac{4}{5}$, 3, $3\frac{1}{5}$, and so on). • *Multiplicative structure.* Adjacent positions are related by moving left (× base); moving right (÷ base). Base is normally 10 or a multiple of 10 in Hindu-Arabic system and metrics. • *Number line.* Quantity on a line, rank, order, rounding, and density.
Equals, operations and algebra big ideas	• *Concepts of the operations.* Meanings of addition, subtraction, multiplication and division. • *Equals and order.* Reflexivity/non-reflexivity: A = A but A is not > A; Symmetry/antisymmetry: A = B → B = A while A > B → B < A and A < B → B > A; Transitivity: A = B and B = C → A = C and A > B and B > C → A > C. • *Balance.* Whatever is done to one side of the equation is done to the other. • *Identity.* 0 and 1 do not change things (+/− and ×/÷ respectively). • *Inverse.* A change that undoes another change (e.g. +2/−2; ×3/÷3). • *Commutativity.* Order does not matter for +/× (e.g. 8 + 6 / 6 + 8; 4 × 3 / 3 × 4). • *Associativity*. What is done first does not matter for +/× (e.g. (8 + 4) + 2 = 8 + (4 + 2), but (8 ÷ 4) ÷ 2 ≠ 8 ÷ (4 ÷ 2)). • *Distributivity*. ×/÷ act on everything (e.g. 2 × (3 + 4) = 6 + 8; (6 + 8) ÷ 2 = 3 + 4). • *Compensation.* Ensuring that a change is compensated for so answer remains the same – related to inverse (e.g. 5 + 5 = 7 + 3; 48 + 25 = 50 + 23; 61 − 29 = 62 − 30). • *Equivalence*. Two expressions are equivalent if they relate by +0 and ×1 – also related to inverse, number, fractions, proportion and algebra (e.g. 48 + 25 = 48 + 2 + 25 − 2 = 73; 50 + 23 = 73; $\frac{2}{3} = \frac{2}{3} \times \frac{2}{2} = \frac{4}{6}$). • *Inverse relation for −, ÷ / direct relationship +, ×.* The higher the number the smaller the result (e.g. 12 ÷ 2 = 6 > 12 ÷ 3 = 4; $\frac{1}{2} > \frac{1}{3}$); the higher the number the higher the result (e.g. 4 + 3 < 4 + 7). • *Backtracking.* Using inverse to reverse and solve problems (e.g. 2*y* + 3 = 11 means *y* × 2 + 3, so answer is 11 – 3 ÷ 2 = 4). • *Basic fact strategies.* Counting, doubles, near 10, patterns, connections, think addition, think multiplication. • *Operation strategies.* Separation, sequencing and compensation. • *Estimation strategies*. Front end, rounding, straddling and getting closer.

Measurement big ideas	• *Concepts of measure*. Length, perimeter, area, volume, capacity, mass, temperature, time, money/value, angle. • *Notion of unit.* Understanding of the role of unit in turning continuous into discrete. • *Common units.* Must use same units when comparing and calculating (e.g. a 3 m by 20 cm rectangle does not have an area of 60). • *Inverse relation.* The bigger the unit, the smaller the number (e.g. 200 cm = 2 m). • *Accuracy vs exactness.* Same as Global principle (e.g. cutting a 20 cm strip usually does not give a length of exactly 20 cm). • *Attribute leads to instrumentation.* The meaning of an attribute leads to the form of measuring instrument (e.g. mass is heft or pushing down on hand, so measuring instrument is how long it stretches a spring). • *Formulae.* Perimeter, area, volume formulae. • *Using an intermediary.* Using string to compare length of a pencil with distance around a can.
Geometry big ideas	• *Concepts.* All types of angles, lines, 2D shapes and 3D shapes, flips-slides-turns, symmetries, tessellations, dissections, congruence, coordinates (Cartesian, polar), plotting graphs (slope, y-intercept, distance, midpoint), types of projections, similarity, trigonometry, topology, networks. • *Formulae.* Angle, length, diagonal and rigidity formulae and relationships – interior angle sums, Pythagoras, trigonometry (sine, cosine and tangent), number of diagonals, number of lines to make rigid. • *Reflection and rotational relationship.* Number of rotations equals number of reflections; rotation angle double reflection angle (holds for symmetry and Euclidean transformations). • *Euler's formula.* Nodes/corners plus regions/surfaces equals lines/edges plus 2 (holds for 3D shapes and maps). • *Transformational invariance.* Topological transformations change straightness and length, projective change length but not straightness, and Euclidean change neither. • *Visualising.* Mental rotation, choosing starting piece.
Statistics and probability big ideas	• *Tables and graphs.* Types of charts and tables, comparison graphs, trend graphs and distribution graphs. • *Concept of probability.* Chance (possible, impossible and certain), outcome, event, likelihood. • *Inference concepts.* Variation, error, uncertainty, distribution, sample, and inference itself. • *Experimental vs theoretical.* Knowing when something can be calculated or determined by trials. • *Equally likely outcome.* Outcomes as a fraction by number giving result ÷ total number. • *Formulae.* Mean, mode, median, range, deviation, standard deviation, quartiles. • *Integration of different knowledges.* For example, question *Do typical Year 7 students eat healthily?* requires some form of data gathering, determining typical, and determining healthy eating.
Pedagogy big ideas	• *Interpretation vs construction/Generation vs illustration.* Things can either be interpreted (e.g. what operation for this problem, what properties for this shape) or constructed (write a problem for 2 + 3 = 5; construct a shape of 4 sides with 2 sides parallel) – activities should generate students' knowledge not illustrate teachers'. • *Connections lead to instruction/Seamless sequencing.* Two connected ideas are taught similarly and progress from one to the other should not involve changing rules. • *Pre-empting and peel back/Compromise and reteaching.* Look forward and back – teach for tomorrow and rebuild from known – be aware what ends and what lasts forever and rebuild ideas not lasting. • *Gestalt leaps and superstructures.* Look out for ways of accelerating knowledge. • *Language as labels/Construction before explanation.* New ideas to be constructed not told. • *Unnumbered before numbered.* Big ideas are best started in situations without number. • *Creativity.* Let students create own language and symbols (particularly to support pattern). • *Triadic relationships.* When three things are related, there are three problem types (e.g. 2 + 3 = 5 can have a problem for: ? + 3 = 5, 2 + ? = 5, 2 + 3 = ?). • *Problem solving.* Metacognition, thinking skills, plans of attack, strategies, affects, and domain knowledge. • *RAMR cycle.* All components of RAMR cycle are big pedagogy ideas.

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