Accelerated Inclusive Mathematics Project

YuMiDeadly
Growing community through education

YuMi Deadly Maths

AIM Module A4

Year C, Term 4

Algebra:

Algebraic Computation

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ACKNOWLEDGEMENT

The YuMi Deadly Centre acknowledges the traditional owners and custodians of the lands in which the mathematics ideas for this resource were developed, refined and presented in professional development sessions.

YUMI DEADLY CENTRE

The YuMi Deadly Centre is a research centre within the Faculty of Education at the Queensland University of Technology which is dedicated to enhancing the learning of Indigenous and non-Indigenous children, young people and adults to improve their opportunities for further education, training and employment, and to equip them for lifelong learning.

"YuMi" is a Torres Strait Islander Creole word meaning "you and me" but is used here with permission from the Torres Strait Islanders' Regional Education Council to mean working together as a community for the betterment of education for all. "Deadly" is an Aboriginal word used widely across Australia to mean smart in terms of being the best one can be in learning and life.

YuMi Deadly Centre's motif was developed by Blacklines to depict learning, empowerment, and growth within country/community. The three key elements are the individual (represented by the inner seed), the community (represented by the leaf), and the journey/pathway of learning (represented by the curved line which winds around and up through the leaf). As such, the motif illustrates the YuMi Deadly Centre's vision: *Growing community through education*.

The YuMi Deadly Centre (YDC) can be contacted at ydc@qut.edu.au. Its website is http://ydc.qut.edu.au.

DEVELOPMENT OF THE AIM MODULES

The Accelerated Inclusive Mathematics (AIM) modules were originally developed from 2010 to 2013 as part of the Accelerated Indigenous Mathematics project funded under the Commonwealth Government's *Closing the Gap: Expansion of Intensive Literacy and Numeracy* program for Indigenous students. The project aimed to assist secondary schools with beginning junior-secondary Indigenous students who were at Year 2/3 level in mathematics by developing a junior-secondary mathematics program that accelerates the students' learning to enable access to mathematics subjects in the senior-secondary years and therefore enhance employment and life chances. The project developed three years of modules (Years A to C) that are vertical sequences of learning to take students from their ability level to their age level in mathematics. The YuMi Deadly Centre acknowledges the role of the Department of Education, Employment and Workplace Relations in the development of the AIM project and these modules.

Contents

Module Overview

This is the fourth algebra module, following A1, A2 and A3. It also follows on from Module O4 *Arithmetic and Algebra Principles*. It completes the algebra work in AIM. It focuses on: (a) substitution, simplification, expansion and factorisation; and (b) extensions into simultaneous equations and quadratics. For (a) we use information from Module O4 and, for (b), we use real-world situations for investigations.

This module is particularly connected to O4 *Arithmetic and Algebra Principles* where it uses the principles translated from arithmetic and algebra to undertake algebraic computation. The principles that are especially useful come from three structures that are common to arithmetic and algebra and are built around *equals* (the equivalence class principles – the reflexive, symmetric and transitive laws), *order* (the order principles – the nonreflexive, antisymmetric and transitive laws) and *operations* (the field principles for addition and multiplication – closure, identity, inverse and the commutative, associative and distributive laws). This module focuses on equals and operations.

Background information for teaching algebraic computation

Algebraic computation is an extension – in fact, a generalisation – of arithmetical computation. However, where arithmetic focuses on computational methods to achieve answers, algebra uses generalisations of arithmetic – the principles of arithmetic – to substitute, simplify, expand and factorise. This gives rise to two basics of algebra.

1. First, algebra provides **relationships that hold for any number**. If we have a problem such as *John's discount was $10 less than the marked price*, we can represent that algebraically by $p - 10$, where p can be any price. To solve this, we have to know how to substitute in order to make sense of this algebra in particular situations; for example, if the price is $95, John pays 95 − 10 = $85. We also need to be able to solve equations because we will want to reverse this process; for example, if John pays $77, this means that $p - 10 = 77$ and therefore p, the marked price, is $87.

2. Second, the big difference between algebra and arithmetic is that an arithmetic expression, such as 2 × 3 + 4, shows process in its symbols (multiplying 2 and 3 and adding 4) but its product, or answer, in a single number (which for 2 × 3 + 4 is 10). An algebraic expression such as $2x + 4$ also shows its process in its symbols (multiplying 2 and x and adding 4) but its product or answer remains as $2x + 4$. Thus, for algebra, **process and product are the same**, which means that we have to rely, in algebraic computation, mostly on the understanding of process that comes from arithmetic, not the understanding of getting answers. This process includes substitution and finding answers for unknowns but also how to simplify, expand and factorise.

Problems and methods for algebraic computation

This module will, therefore, provide the following methods for computation.

1. **Substitution**. This is replacing a variable in an expression with a number or with another expression. For example, $2x + 4 = 2 \times 5 + 4 = 14$ when we substitute x with 5, and $2x + 4 = 2(3y - 7) + 4 = 6y - 14 + 4 = 6y - 10$ when we substitute $3y - 7$ for x.

2. **Simplification**. This is where we simplify expressions or equations by combining components that can be combined. For example, $3x + 6 + 4x - 2$ can be simplified to $7x + 4$ because $3x + 4x = 7x$ and $6 - 2 = 4$; and $2y - 3x = y + 2x - 5$ is $2y - y = 2x + 3x - 5$ is $y = 5x - 5$ because of the balance rule and the fact that $2y - y = y$ and $2x + 3x = 5x$.

3. **Expansion**. This involves multiplying and dividing out expressions; for example, $y = x(x + 2)$ is $y = x^2 + 2x$ because of the distributive law where x multiplied by $(x + 2)$ is x multiplied by x and x multiplied by 2 which is $x^2 + 2x$.

4. **Factorisation**. This is the inverse of expansion and reduces expressions to multiplication or division; for example, $y = 3x + 6 = 3(x + 2)$ and $y = x^2 + 5x + 4 = (x + 4)(x + 1)$. This is most useful when $y = 0$ because then one of the multiples has to equal 0.

The module will also look at how variables as unknowns can be found by simultaneous equations (two unknown variables and two equations) and from quadratic equations (which have x^2 along with x and a constant).

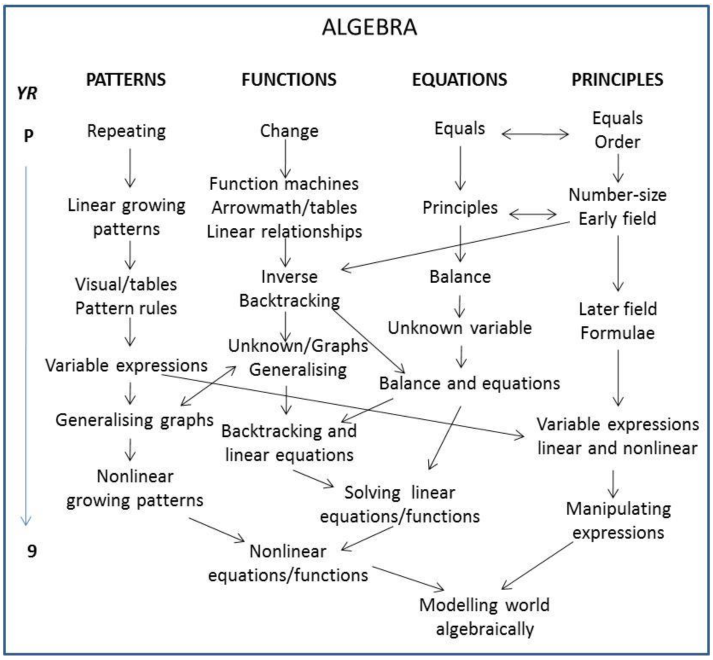
Connections and big ideas

The major connections and big ideas for algebraic computation are the same as those for operations, which are listed in detail in Module O4, *Arithmetic and Algebra Principles* and thus are not repeated here. Please refer to Module O4 for a comprehensive list of the big ideas relating to the concepts of operations and the major strategies for computation; the connections between operations/algebra and the other mathematics strands as well as the connections between topics within operations/algebra; and the properties of operations in terms of global/teaching big ideas, field properties and their extension, and equals properties. These big ideas apply in algebra as well as arithmetic.

Sequencing for algebraic computation

This section briefly looks at the role of sequencing in algebra and the role of sequencing in this module.

Sequencing in algebra



Overall sequence

The overall sequence for algebra is given in the figure on the right. It has four sections which are each AIM modules: Module A1 *Equivalence and Equations*, Module A2 *Patterns and Linear Relationships*, Module A3 *Change and Functions*, and Module O4 *Arithmetic and Algebra Principles*. This module, A4 *Algebraic Computation*, covers more advanced activities that are involved in the last sections of each of the four areas, particularly the last sections of Equations and Principles. The module covers material that underlies Year 10 algebra (e.g. quadratics and simultaneous equations) as well as the Junior Secondary topics of substitution, simplification, expansion and simple factorisation.

The overall sequence in the figure above begins with **patterns** as training in the act of generalisation by finding pattern rules and relating to graphs. It then moves on to **functions**, starting from change rules in transformations, using real situations, tables and arrowmath notation before equations and graphs, solving linear equations for unknowns by the use of backtracking. After this it moves to relationships that in arithmetic and algebra are represented predominantly by **equations**, solving linear equations by the use of the balance rule. The sequence is completed by focusing on arithmetic and algebraic **principles** and extending these to methods such as substitution, simplification, expansion and factorisation and to solutions for simultaneous equations and quadratics (which are the focus of this module).

Special features

In Module A3 we raised three important aspects of sequencing for algebra. Not all apply here but they are worth repeating. It should be noted that they generally hold across mathematics in the building of big ideas.

1. **Unnumbered work before numbered**. Unnumbered activities better enable big ideas to develop, so it is useful to start with these move on to numbers and arithmetic situations and then move to generalised and algebra situations. This will not apply strongly to this module as we do little early primary work.

2. **Processes not answers**. The basis of algebra is things that hold for all numbers not particular answers, thus algebra has to be built around processes and big ideas not computation. This means that time needs to be spent on teaching arithmetic processes and big ideas such as the Field properties so that they can be translated from arithmetic to algebra computation. This is important to this module.

3. **Separation to integration**. Early primary algebra is separated but by the time junior secondary is reached, the components are more integrated and connected (e.g. all algebra is expressed as equations) and to cover nonlinear as well as linear. This module covers ideas that make use of all the algebra understandings.

4. **Modelling as end point**. The overall end point of algebra is modelling as well as manipulation of symbols. This aspect of algebra is a major focus of this module. Computation will related to real situations wherever possible.

Sequencing in this module

The **sequence of activities** that can be developed in this strand **relate to operations** (and back to arithmetic) and includes the following:

(a) *substitution* – replacing variables with actual numbers to find the value of expressions (e.g. in linear expressions, $2x + 3y = 23$ if $x=4$ and $y=5$/in quadratics, $3x^2+1 = 28$ if $x=3$);

(b) *simplification* – using principles and processes of operations to simplify expressions (e.g. in linear expressions, $5x + 4 – 2x = 3x + 4$ as $5x – 2x = 3x$/in quadratics $3x^2 + 2x – x^2 – 1 = 2x^2 +2x – 1$ as $3x^2 – x^2 = 2x^2$);

(c) *expansion* – using the principles and processes of operations to expand out multiplications and divisions (e.g. the products of 4 and $3x+1$ and x and $2x–4$ are $12x+4$ and $2x^2–4x$ using the distributive law);

(d) *factorisation* – reversing the above process to find simple factors of expressions (e.g. $4x + 8 = 4(x+2)$) and more complicated factors (e.g. $3x^2+2x=x(3x+2)$ and $2x^2–9x+4 = (2x–1)(x–4)$);

(e) *simultaneous equations* – solving for two unknowns using two simultaneous equations (e.g. the problem *12 heads 30 feet, how many ducks, how many rabbits* can be “modelled” by $d+r=12$ and $2d+4r=30$, then $d=12–r$ can be substituted into second equation to give $24–2r+4r=30$ which is $2r=6$, $r=3$ and $d=9$, which is 3 rabbits and 9 ducks); and

(f) *quadratics* – solving for unknowns in quadratics is based on changing the quadratic so it equals zero, reducing/factorising the quadratic to two linear expressions one of which has to be zero which means that x has to be one of the values that makes the linear factors zero (e.g. relationship is $2x^2+4=9x$, this is changed to $2x^2–9x+4 = 0$, thus factors $(2x–1)(x–4) = 0$, thus $x=1/2$ or $x=4$).

This module is therefore composed of the following sections:

Overview: Background information, sequencing and relation to Australian Curriculum

Unit 1: Substitution and simplification

Unit 2: Expansion and factorisation

Unit 3: Solving for unknowns in simultaneous equations

Unit 4: Solving for unknowns in quadratics

Test item types: Test items associated with the four units above which can be used for pre- and post-tests

Appendix A: RAMR cycle components and description

Appendix B: AIM scope and sequence showing all modules by year level and term.

Because the ideas in this module are often new for teachers as well as students, the sections in each unit have two parts: one providing information, and the second discussing methods of teaching the ideas. These methods are described but not always placed in a RAMR framework. However, in teaching the ideas, the RAMR framework should be used (see **Appendix A**). It is assumed that all activities will be related to the real world of the students as far as possible and that teaching will be active, involving the students acting out situations.

Relation to Australian Curriculum: Mathematics

AIM A4 meets the Australian Curriculum: Mathematics (Foundation to Year 10)					
Unit 1: Substitution and simplification Unit 2: Expansion and factorisation	Unit 3: Solving for unknowns in simultaneous equations Unit 4: Solving for unknowns in quadratics				
Content Description	**Year**	**A4 Unit**			
		1	**2**	**3**	**4**
Explore and describe number patterns resulting from performing multiplication (ACMNA081)	**4**	✓	✓	✓	✓
Use equivalent number sentences involving addition and subtraction to find unknown quantities (ACMNA083)		✓	✓	✓	✓
Use estimation and rounding to check the reasonableness of answers to calculations (ACMNA099)	**5**			✓	✓
Describe, continue and create patterns with fractions, decimals and whole numbers resulting from addition and subtraction (ACMNA107)				✓	✓
Use equivalent number sentences involving multiplication and division to find unknown quantities (ACMNA121)			✓	✓	✓
Continue and create sequences involving whole numbers, fractions and decimals. Describe the rule used to create the sequence (ACMNA133)	**6**				✓
Explore the use of brackets and order of operations to write number sentences (ACMNA134)		✓	✓		✓
Apply the associative, commutative and distributive laws to aid mental and written computation (ACMNA151)	**7**	✓	✓		
Create algebraic expressions and evaluate them by substituting a given value for each variable (ACMNA176)		✓	✓	✓	✓
Extend and apply the laws and properties of arithmetic to algebraic terms and expressions (ACMNA177)		✓	✓	✓	✓
Carry out the four operations with rational numbers and integers, using efficient mental and written strategies and appropriate digital technologies (ACMNA183)	**8**			✓	✓
Extend and apply the distributive law to the expansion of algebraic expressions (ACMNA190)		✓	✓		✓
Factorise algebraic expressions by identifying numerical factors (ACMNA191)					✓
Simplify algebraic expressions involving the four operations (ACMNA192)					✓

Unit 1: Substitution and Simplification

The first algebraic computation processes are substitution and simplification. Substitution is a replacement of the variable in an equation or expression by something else – a number or a new expression. Such substitutions are part of algebraically modelling a real-world situation and are part of how formulae are used (e.g. *What is the area of a rectangular wall 5 m by 3 m?* Area = Length × Width so substituting for L and W, we have A (area) = 5 × 3 = 15 m^2), so using formulae is a good way to start this topic.

Simplification is an extension of arithmetic patterns with the idea of variable as "any number". If 2 threes plus 5 threes is 7 threes, we can use this to show that 2 of any number plus 5 of the same number is 7 of that number – this language can be replaced with symbols to show that $2x + 5x = 7x$.

There are important features of arithmetic and algebra that must be considered when looking at the concepts and skills of substitution, simplification, expansion and factorisation:

(a) they hold for division and subtraction as well as multiplication and addition;

(b) subtraction is the inverse of addition but this does not affect activities based on distribution as the crucial component is multiplication (and division); and

(c) division is the inverse of multiplication and so the effect is that expansion for division is similar to factorisation for multiplication. For example, $3 + 9a = 3(1 + 3a)$ is factorisation which is similar to expansion for division $(3 + 9a) \div 3$; in both we have to find the common factor.

Thus, the units in this module focus on multiplication and addition and rely on inverses to give insight into what happens to subtraction and division. That is, if something works for multiplication and addition, it also works for adding a negative (subtraction as inverse of addition) and multiplying by reciprocal (division as inverse of multiplication).

1.1 Substitution

The following sequence of steps could be used to introduce this idea.

Step 1: Substituting into everyday situations (language → writing → student-chosen symbol)

1. Set up real-life situations. For example, *I have $3 more than you*. This statement could be simply related to questions like *If you have $7, how much do I have?*. This would mean no symbols, just language representations.

2. Ask students to write down what you said, for example, *Student plus $7 = teacher*. Then expand discussion – what if the student had $20, $130, $247?

3. Finally, let students choose something to stand for the unknown and use a more symbolic representation, for example, ☖ + 3 = teacher's amount. What if ☖ = $14?

It is important to find things in the life of the students for which such relationships make sense, for example, *John runs twice as fast as Jack, The big truck can carry three times the sheep of the smaller truck, In the netball game, we doubled their score*.

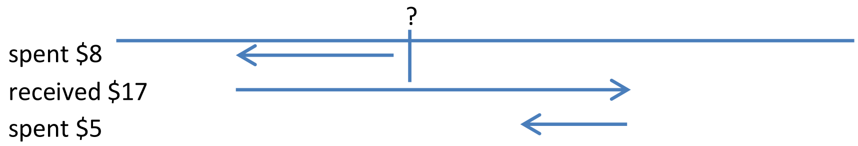
Step 2: Substituting into unknowns

1. Set up situations in arithmetic where there are unknowns, for example, *I am a number, I have been multiplied by 3, and 11 has been added, what am I now?* or *I went to the shop and bought lunch and a drink. The drink cost $4, how much did I spend?* In these situations, the students cannot answer the

question because they do not have all the information. So, we ask, *What information could it be? What if it was …?, could we get an answer now?*, and *What would it be?*

This is a great way to build backtracking too – because we can ask what else could we do with the problem *I went to the shop and bought lunch and a drink. The drink cost $4, how much did I spend?* other than make up amounts for the lunch and then using these to work out how much was spent. Students will volunteer that if they knew the total amount spent, they could work out the cost of the lunch. If the students volunteer $12, they are solving $x + 4 = 12$.

2. Use a number track, number ladder or number line and act out what has happened, for example, *My mother gave me some money, I spent $8, my father gave me $17, and I spent $5, how much money do I have left?* The starting point can be recognised as an unknown (say with a question mark), this can be placed on a number line, and the spending and other actions put in as arrows, as shown below.



Now if the starting point is $20, the line can be used to find how much is left.

Notes: (a) If using a number track or ladder, get students to act out what has happened by substituting a starting point and walking forward and backward along the track/ladder. (b) Tracks, ladders and number lines also help backtracking because if we substitute the answer, we can reverse the movements to get the start.

3. If we are developing algebra as a language by relating it to real-world situations, then we would be translating real-world situations into variables. For example, *Three buses all had the same number of people, 7 more got on the buses, how many were there on the buses now?* would be translated into $3x + 7$. Now the substitutions made in the story (e.g. the buses each had 30 people) can be made in relation to the variable x.

Note: We can also use symbols other than letters for unknowns in the beginning of this substep, for example, the translation could be 3? + 7 or 3☖ + 7.

Step 3: Substituting into formulae or other early algebra situations

When formulae begin to appear in measurement, they can be introduced in terms of substitution.

1. Start with formula as language, for example, the area of a rectangle is length multiplied by width. Then provide examples of rectangles with different lengths and widths and show how these are substituted into the formula, for example, width 4 m and length 7 m means that the area is 7 multiplied by 4 which is 28 square metres.

2. Move on to formulae as symbols, for example, $A = L \times W$ and $V = \pi r^2 h$, and then simply substitute into the letters to find areas or volumes, and so on. Stress to students, when they are doing their perimeter, area and volume applications, that they are substituting a number or measure for the variable.

Note: The methods advocated for introducing algebra (such as patterning) also have formula-like situations (e.g. the n^{th} term is $3n + 2$) which we substitute into to find the value of a particular term (e.g. the 27^{th} terms is $3 \times 27 + 2 = 83$).

Step 4: Substituting into algebraic expressions and equations

Finally, we move on to the substitutions that are part of algebraic computation.

1. The first is substituting numbers and measures, for example, $3y - 2 = 3 \times 4 - 2 = 10$ if substituting 4 for y; $3y - 2m = 3 \times 7m - 2m = 19m$ if substituting $7m$ for y; $y^2 - 5y + 7 = 4 \times 4 - 5 \times 4 + 7 = 3$ if substituting 4 for y, and $A = \pi r^2 = 113$ m^2 to the nearest square metre if substituting 6m for r.

2. The second is substituting expressions into other expressions or equations, for example, $3y - 2 = 21$ means that $3(2a + 5) - 2 = 21$ which means $6a + 15 - 2 = 21$ which means $6a = 8$, if substituting $2a + 5$ for y. This type of substitution is a common occurrence in simultaneous equations.

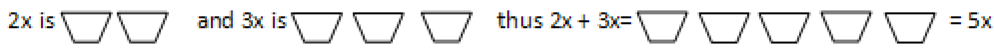
1.2 Simplification

Simplification is the reverse or inverse of expansion which makes things longer. Simplification enables algebraic expressions to be computed to simpler forms, for example, $2x + 3x = 5x$ and $(x + 1)(x + 2) = x^2 + 3x + 2$. Simplification is based on the laws of equals (reflexive, symmetric and transitive) and laws of operations (commutative, associative and distributive). Basically it enables us to add and multiply variables in the same manner as we do numbers. The following four basic simplifications are given in the order they could be presented, each with a variety of methods that could be used to introduce them. The methods for subtraction are similar to addition and the methods for division are similar to multiplication.

Step 1: Adding/subtracting like variables – example $2x + 3x = 5x$

1. **Use reality**. Think up a story for this. For example, *All boxes have the same number of lollies, x. I bought 2 boxes of x lollies and then 3 boxes of x lollies. How many lollies did I buy altogether?* Obviously it is five boxes worth of lollies which is $5x$.

2. **Use materials**. Think of a cup as variable (can hold an unknown number of counters) and counters as numbers. Then,



3. **Use patterns**. Calculate the following and show they are equal:

LHS	RHS
2 threes plus 3 threes	5 threes
2 eights plus 3 eights	5 eights

Continuing in this way it is evident that 2 of any number plus 3 of the same number = 5 of that number, which means that $2x + 3x = 5x$ as x stands for any number.

4. **Use arrays**.

X X	X X X		X X X X X	
X X	X X X		X X X X X	
X X	X X X	2 sixes and 3 sixes	X X X X X	5 sixes
X X	X X X		X X X X X	
X X	X X X		X X X X X	
X X	X X X		X X X X X	

This holds for any height array thus $2x + 3x = 5x.$

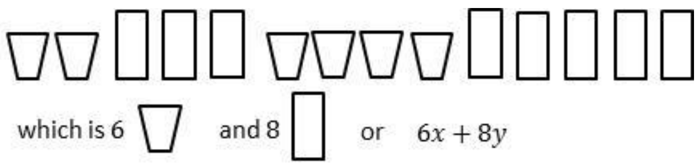
5. **Use inverse**. $5x = (2 + 3)x = 2x + 3x$ because $5 = 2 + 3$ and the distributive law shows that $(2 + 3)x = 2x + 3x$. Reversing this relationship (allowed by the symmetric law of equals) we have that $2x + 3x = 5x$ and this can be used to introduce it.

Step 2: Adding/subtracting more than one like variable – $2x + 3y + 4x + 5y$

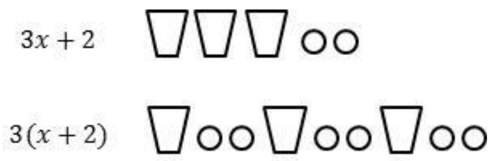
In this one we return to the meaning of addition which is to add like things. This can be seen using the following methods.

1. **Use reality**. A story for the above is *I bought 2 bottles of drink for \$$x$ each and 3 pies for \$$y$ each, then I bought 4 more bottles of drink for \$$x$ each and 5 more pies for \$$y$ each. How much did I pay?* Obviously this will be $2 + 4$ \$$x$ and $3 + 5$ \$$y$, thus $2x + 3y + 4x + 5y = 2x + 4x + 3y + 5y = 6x + 8y$.

2. **Use materials**. We need two different cups to represent x and y, then $2x + 3y + 4x + 5y$ is



Note: The use of cups and counters can help with early algebra. For instance, many students confuse $3x + 2$ with $3(x + 2)$ but with cups and counters, as on right, it is easy to see the difference. The cups and counters for $3(x + 2)$ which equals $3x + 6$ are as on right and the difference between it and $3x + 2$ above it can be seen.



3. **Patterns**. As usual with patterns, start in arithmetic, look at many examples and then extend to algebra. We can then see, for example, that 2 threes plus 3 fives plus 4 threes plus 5 fives is 6 threes plus 8 fives. This holds for any two groups (e.g. 50s and 75s). Thus, 2 any numbers plus 3 another any number plus 4 first number plus 5 second number is 6 any number plus 8 second any number. That is, $2x + 3y + 4x + 5y = 6x + 8y$.

4. **Arrays**. Drawing the arrays, we would see 2 arrays of same height (2 and 4 columns) and two other arrays with same but different height (3 and 5 columns). This means that we have 6 of one column height and 8 of the other, and so $2x + 3y + 4x + 5y = 6x + 8y$. (*Note:* It is possible for x and y to be the same but this should be discussed later.)

5. **Inverse**. Using distributive and associative laws, $6x + 8y = (2 + 4)x + (3 + 5)y = 2x + 4x + 3y + 5y = 2x + 3y + 4x + 5y$. Reversing by using the symmetry law gives $2x + 3y + 4x + 5y = 6x + 8y$.

6. **Laws**. Direct application of the laws also works: $2x + 3y + 4x + 5y = 2x + 4x + 3y + 5y$ (by the commutative and associative laws) = $(2 + 4)x + (3 + 5)y$ (by the distributive law) = $6x + 8y$.

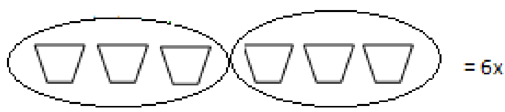
7. **Algorithm setting out**. Compare to 23 + 45 and extend the vertical algorithm. This is based on the same laws as 6 above but it is more user friendly:

$$\begin{array}{rl} 23 & \\ +45 & \\ \hline 8 & \text{add ones} \\ 60 & \text{add tens} \\ \hline 68 & \end{array} \qquad\qquad \begin{array}{rl} 2x + 3y & \\ +4x + 5y & \\ \hline 6x & \text{add } x\text{'s} \\ 8y & \text{add } y\text{'s} \\ \hline 6x + 8y & \end{array}$$

Step 3: Multiplication of number and variable – example $2 \times 3x = 6x$

1. **Use reality**. A story for $2 \times 3x = 6x$ is *I bought 2 lots of 3 boxes of lollies, how many lollies did I buy altogether?* Obviously there are 6 boxes of lollies equivalent to $6x$.

2. **Use materials**. $2 \times 3x = 6x$ is represented by cups as on right.



3. **Use patterns**.

   2 lots of 3 fours = 6 fours
   2 lots of 3 eights = 6 eights
   2 lots of 3 hundreds = 6 hundreds
   2 lots of 3 any numbers = 6 same numbers
   $2 \times 3x = 6x$

4. **Use arrays**.

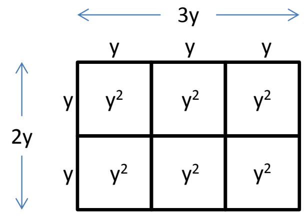
3 fives		3 fives		6 fives
o o o		o o o		o o o o o o
o o o		o o o		o o o o o o
o o o		o o o	equals	o o o o o o
o o o	and	o o o		o o o o o o
o o o		o o o		o o o o o o

   We can see from the above that 2 lots of 3 fives is 6 fives. As this holds for any height array, it means $2 \times 3x = 6x$.

5. **Use inverse**. $6 = 2 \times 3$, therefore $6x = (2 \times 3) \times x = 2 \times 3x$ from the associative law that $(a \times b) \times c = a \times (b \times c)$. Reversing, we have $2 \times 3x = 6x$.

Step 4: Multiplication of variables – example $\mathbf{2y \times 3y = 6y^2}$

1. Use reality. Consider an area 2 units by 3 units. This gives area of 6 square units. If unit is y then $2y \times 3y = 6y^2$



2. **Use patterns**. An area of 2 cm by 3 cm = 6 cm^2, and an area 2 m by 3 m is 6 m^2, thus (continuing this pattern) $2y \times 3y = 6y^2$.

3. **Use area model**. A rectangle of $2y$ by $3y$ is divided as on right. The area formed by multiplication is six lots of y^2. Thus, $2y \times 3y = 6y^2$.

4. **Use laws (direct or reverse)**. $2y \times 3y = 2(y \times 3y)$ by associative law $= 2(3y \times y)$ by commutative law $= 2(3 \times (y \times y))$ by associative law $= (2 \times 3) \times (y \times y)$ by associative law $= 6 \times y^2 \ = \ 6y^2$.

   Note: $4 \times 4 = 4^2$, $9 \times 9 = 9^2$, $157 \times 157 = 157^2$, thus $y \times y = y^2$.

Step 5: Practice

Use the rules above to calculate the following:

(a) $a(a+2) - 3$

(b) $3(b-4) + 2(2b+1)$

(c) $y(y-3) + 2y(3y-1)$

Unit 2: Expansion and Factorisation

This unit looks at expansion and factorisation where factorisation is the reverse or inverse of expansion (e.g. $2(y+1) = 2y+2$ expansion and $2y+2 = 2(y+1)$ factorisation). This consideration has three components as follows:

1. **Expansion and factorisation are based on the distributive principle**. The distributive principle is that multiplication (and division) acts across all components of addition (and subtraction), that is,

   $a \times (b+c) = (a \times b) + (a \times c)$ and $\frac{p-q}{r} = \frac{p}{r} - \frac{q}{r}$

   The effect of the distributive principle can be seen in the difference between 43 + 2 and 43 × 2 (as on right).

   $$\begin{array}{r} 43 \\ +\,2 \\ \hline 45 \end{array} \qquad \begin{array}{r} 43 \\ \times\,2 \\ \hline 86 \end{array}$$

2. **Expansion, factorisation and the distributive law are best seen with respect to array and area model**. Thus, we will use this across arithmetic and algebra.

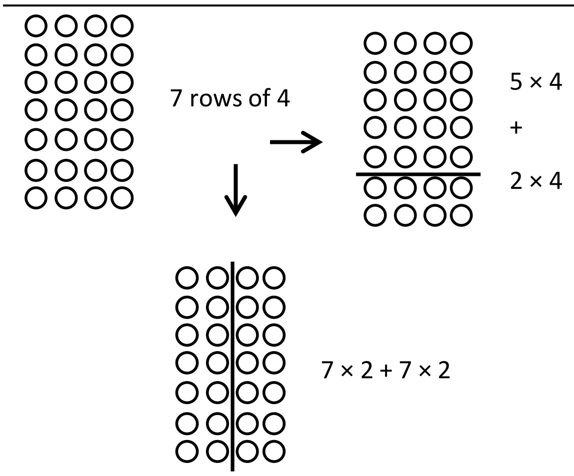
3. **If we do not change syntax, there is a seamless sequence from concept of operation through to algebra operations**. This includes the separation method of algorithms.

This seamless sequence in point 3 above is an important part of this unit. The unit will focus on it in terms of (a) the array/area model for multiplication and the laws, particularly the distributive principle; and (b) the sequence from arithmetic use of the distributive principle and the separation strategy to algebra use of the distributive principle and the separation strategy.

2.1 Array/area model of multiplication and distributive principle

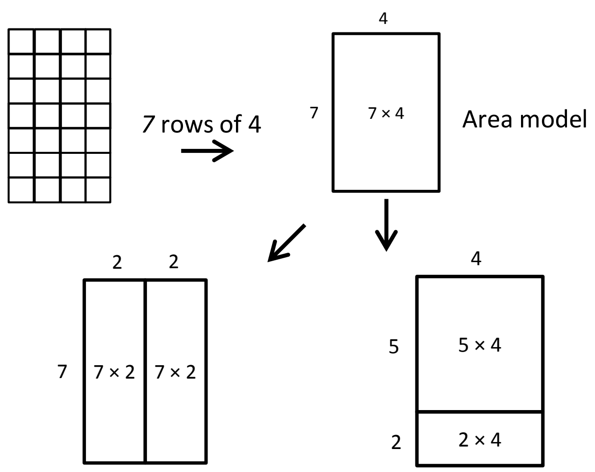
The distributive principle is best seen on the area model of multiplication. The area model is an extension of the array model (see below). However, it is different to the array model in that it is length × length = area where the array model is number of rows by number of columns as area. This means that for a rectangle that is 3 m by 5 m, the area model is 3 m × 5 m = 15 m^2, while the array model is 3 rows by 5 square metres or 5 m^2 per row (still equalling 15 m^2).

Arrays 7 × 4



Two versions of the distributive principle

Area 7 × 4



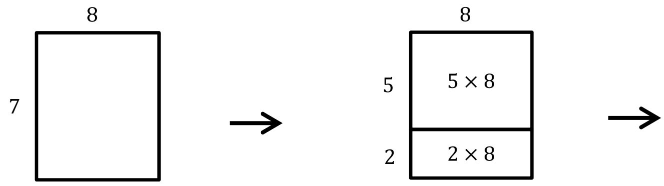
Two versions of the distributive principle

The steps to build expansion are as follows.

Step 1: Arithmetic – return to arithmetic and look at an example of expansion:

Basic facts

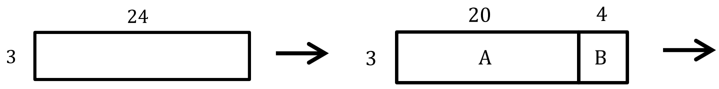
7×8



$5 \times 8 + 2 \times 8 = 56$

Algorithms 2 digit × 1 digit

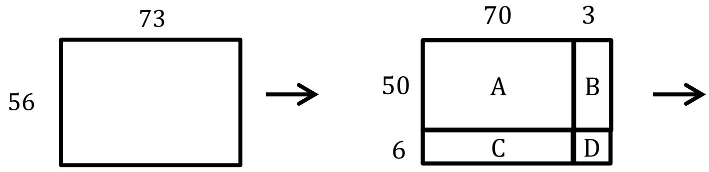
3×24



3×20 A
$+ 3 \times 4$ B

Algorithms 2 digit × 2 digit

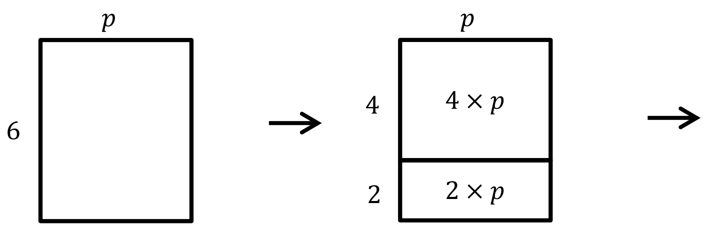
56×73



50×70 A
$+ 50 \times 3$ B
$+ 6 \times 70$ C
$+ 6 \times 3$ D

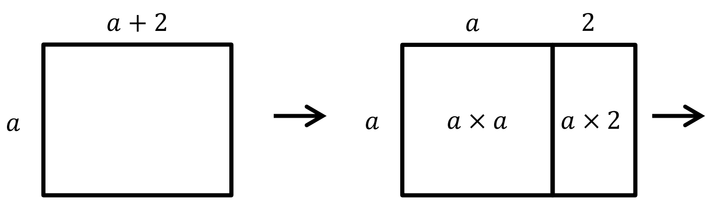
Step 2: **Relate this to algebra** – use the area model (also tiles):

$6p = 6 \times p$



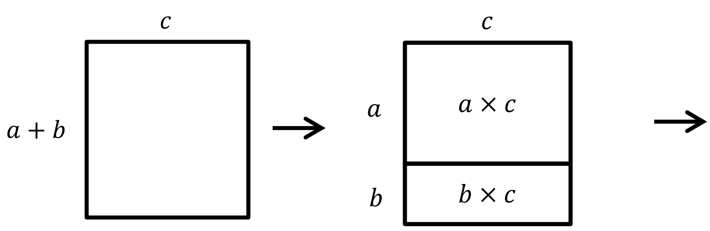
$4 \times p + 2 \times p = 4p + 2p$

$a \times a + 2$



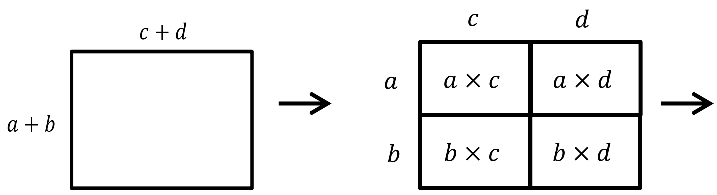
$a \times a + a \times 2 = a^2 + 2a$

$(a + b) \times c$



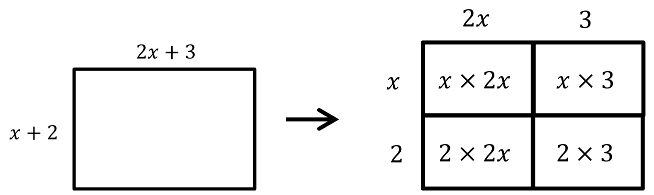
$(a + b) \times c = ac + bc$

$(a + b) \times (c + d)$



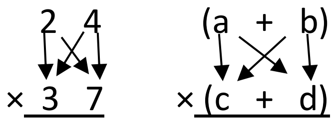
$ac + ad + bc + bd$

$(x+2)(2x+3) =$
$(x+2) \times (2x+3)$



$\rightarrow$ $2x^2 + 3x + 4x + 6$
$= 2x^2 + 7x + 6$

Step 3: **Notice pattern and follow it** – the pattern is the multiplication algorithm pattern when using the separation strategy – see below.



$\longrightarrow$ (a+b)(c+d) = ac + ad + bc + bd

2.2 Expansion sequence arithmetic to algebra

The basis of the seamless transition from expansion in arithmetic to expansion in algebra is to initially set out algebraic computation in the same way (the same syntax) as arithmetic expansion. Thus, to reinforce the information in section 2.1, show the expansion in vertical format, relate to the separation strategy form of the algorithm, and show the translation from arithmetic with numbers to algebra with variables (or letters).

Step 1: Addition and subtraction

2-digit plus 2-digit addition to equivalent algebra

24		a+2b
+32	⇨	+3a+4b
6		6b
50		4a
56		4a+6b

Activity – repeat the above translation for simple 2-digit subtraction using the separation strategy (the traditional strategy for arithmetic)

Step 2: Multiplication and division

2-digit by 1-digit multiplication to equivalent algebra

24			a+b	
× 3		⇨	× a	
12	3×4		ab	a×b
60	3×20		a^2	a×a
72			a^2+ab	

2-digit by 2-digit multiplication to equivalent algebra

24			$x+2$	
× 37		⇨	$\times\ 2x+3$	
28	7×4		6	3×2
140	7×20		$3x$	$3 \times x$
120	30×4		$4x$	$2x \times 2$
600	30×20		$2x^2$	$2x \times x$
888			$2x^2 + 7x + 6$	

Activity – repeat the above for 2-digit divide 1-digit separation (standard) division procedure.

Steps 1 and 2 show that algebraic expansion is an extension of the expansion used in the traditional algorithm which we have called the separation algorithm and which is based on the distributive law or principle. Thus we see that if we maintain syntax, it is easy to see that addition and multiplication of algebra is the same as addition and multiplication of arithmetic but without carrying/renaming.

Step 3: Negative numbers and variables

The above translation can also be done for negatives, if we allow, for instance, 2 tens and 8 ones to be 3 tens and −2 ones as below. This can also be acted out on graph paper if we use the negative columns as a fold over to reduce the positive columns.

2 8		3 −2		a+2		3b−4	
× 7		× 7		× 7		× 7	
1 4 0	(7×20)	2 1 0	(7×30)	7a	(7×a)	21b	(7×3b)
5 6	(7×8)	−1 4	(7×−2)	14	(7×2)	−28	(7×−4)
1 9 6		1 9 6		7a+14		21b−28	

It can also work for examples like 3 7 × 2 9. We simply assume that the first digit is tens and the second is ones and we allow the ones to be negatives. Since 37 is 40 − 3 and 29 is 30 − 1, then 3 7 can be represented as 4 −3 and 29 as 3 −1 and the multiplication as 4 −3 × 3 −1. The arithmetic multiplication can then be translated to multiplying (a−2) by (2a−3) as follows. Note here that it does not matter whether we move L → R or R→ L.

37		4 −3			a −2	
× 29	=	× 3 −1			× 2a −3	
600		1200	40×30	⇒	$2a^2$	a×2a
270		− 40	40×−1		−3a	a×−3
140		− 90	−3×30		−4a	−2×2a
63		3	−3×−1		6	−2×−3
1073		1073			$2a^2 - 7a + 6$	

As stated before, the negative operations can also be shown with graph paper using the area model and folding back the negatives.

2.3 Factorisation as inverse to expansion

Factorisation is the reverse of expansion. Up to Year 9 it is only necessary to factorise for numbers, for example, $2 + 4x = 2(1 + 2x)$. However, the inverse understanding is a powerful way to view all factorisations so we include variable factors. We focus on three ways to teach factorisation.

1. **Indirectly using inverse**. Use examples like those below to become familiar with expansions and various multiplications and think about their inverse (only need examples ** for up to Year 9):

Expansion	**Inverse**
$2(a + b) = 2a + 2b$ **	$2a + 2b = 2(a + b)$ **
$3(3a + 5b) = 9a + 15b$ **	$9a + 15b = 3(3a + 5b)$ **
$5(2x + 3) = 10x + 15$ **	$10x + 15 = 5(2x + 3)$ **
$a(a + b) = a^2 + ab$	$a^2 + ab = a(a + b)$
$a(2a + 3b) = 2a^2 + 3ab$	$2a^2 + 3ab = a(2a + 3b)$
$p(x + y) = px + py$	$px + py = p(x + y)$

In this way, students become familiar with factorisation as an inverse of expansion.

Step 1: The first step is to identify what is common in these inverse examples – they have something in each part that is a factor (e.g. a in $a^2 + ab$, 3 in $9a + 15b$, and so on). Then reinforce this by setting up exercises to determine if there is a factor, for example:

Tick the ones that have a common factor.
Write the factor.

Expansion	Yes/No	Common factor
$3a + 4$		
$6a + 2$		
$3a + ab$		

Step 2: The second step is to realise that the factor can be taken out of all parts (check by expanding), for example:

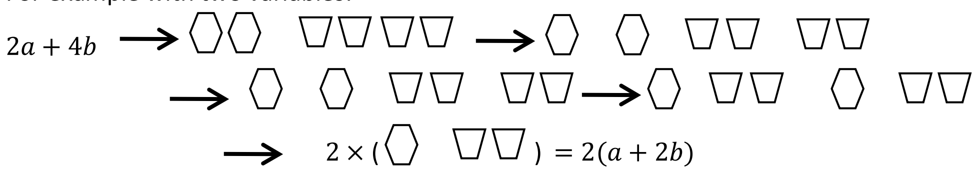
$3x + 6$ → 3 is a factor → $3x + 6 = 3(x + 2)$ because $3x = 3 \times x$ and $6 = 3 \times 2$.

2. **Materials (cups/counters and area model)**. Use cups and counters as follows.

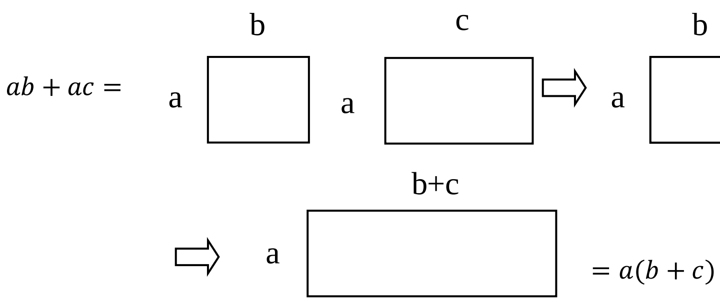
   Step 1: For numerical factorisations, use a different type of cup for each variable.

   (a) For example with one variable:

   (b) For example with two variables:



   Step 2: For variable factorisation, use the area model.



3. **Use of formulae and technology**. Factorisation is not the difficult or important task it used to be. For example, there are formulae for solving quadratics and there are calculators and apps that can solve equations for you just by entering them. In fact, like algorithms in primary school, solving equations can be done with technology. This means the important algebraic skills to develop in students is how to model problems and how to know what equations they have to solve.

   Unit 4 on quadratics looks into this in more detail.

Unit 3: Solving for Unknowns in Simultaneous Equations

With regard to solving equations, linear equations can be solved in two ways that have been covered in earlier modules, as follows:

1. By **backtracking** if there are not variables on both sides; for example, $3x - 2 - 10$. This equation can be represented as a change (using arrowmath) and then reversed as shown below.

   $$x \xrightarrow{\times 3} \xrightarrow{-2} 10 \qquad\qquad 10 \xrightarrow{+2} 12 \xrightarrow{\div 3} 4$$

   Thus, since the variable side of the equation is to multiply by 3 and subtract 2, backtracking means that the number side has 2 added and divides by 3, to give the answer of 4.

2. By using **the balance rule** if there are variables on both sides; for example $3x - 2 = 2x + 5$. This equation can be changed by subtracting $2x$ from both sides and then adding 2 to both sides. This gives $x - 2 = 5$ (subtracting $2x$) and then $x = 7$.

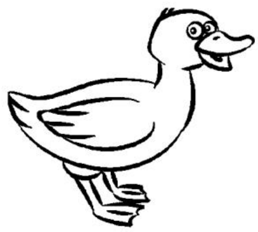
However, there are problems which have two unknowns. These require two equations to be able to solve for both unknowns. This form of modelling and solution finding is called simultaneous equations. This unit looks at the problems (problems with two unknowns) that arise for them and how these are solved.

3.1 Problems with two unknowns

Activity 1: Duck and rabbit problem

The problem is: *There are 52 feet and 17 heads, how many ducks, how many rabbits?* The solution is shown by questions [with the answers in square brackets].

1. Solve this by guess and check:

   (a) Make a guess for one of the animals. [Say 11 rabbits]

   (b) Work out how many feet and heads this would give. [11 rabbits is 44 feet plus 6 ducks which is 12 feet, and a total of 56 feet]

   (c) Make a better guess. [We need less rabbits so reduce the number to 10 and try again.]

   (d) Keep going until you get the right number of heads and feet. [Nine rabbits and eight ducks will give the answer.]

2. Let us rethink the problem:

   (a) Use letters for the number of ducks and rabbits – for example: d for **number of** ducks and r for **number of** rabbits.

   (b) What do we know about the numbers d and r? (Hint: What do 52 feet and 17 heads mean in terms of d and r?) [$d + r = 17$ and $2d + 4r = 52$]

   (c) Write what you know about d and r as equations. Can you relate d and r? [Yes, $d = 17 - r$]

3. Let’s solve it with the letters:

   (a) How can we solve equations? We need one variable. Can we use the relation between d and r to do this? [$d = 17 - r$]

   (b) Can you substitute into one variable so that there is only one variable in an equation? [Substitute $d = 17 - r$ into $2d + 4r = 52$ → $2(17 - r) + 4r = 52$ → $34 - 2r + 4r = 52$ → $34 + 2r = 52$ → $2r = 18$ → $r = 9$]

   (c) Solve the equation. Did you get the same answers for that letter? What about the other letter? [$r = 9$ means $d = 8$ since $r + d = 17$]

   Note: This is called the **substitution method** for solving simultaneous equations.

Activity 2: Other problems

60 cents 40 cents

1. A small ice block costs 40 cents; a small chocolate costs 60 cents. I spend \$6.20 on 12 items, how many ice blocks and how many chocolates did I buy?

   (a) Solve this by guess and check.

   (b) Solve this by the substitution methods. Remember, write letters for numbers to set the problem up. Use letters for the number of ice blocks and the number of chocolates. Then reduce to one letter.

2. 26 young children, 64 wheels, how many bicycles, how many tricycles?

   (a) Solve this by guess and check.

   (b) Solve it by substitution – remember to create letters for the numbers, set up the equations and substitute to get one letter.

3. It costs \$49 for 2 adults and 5 children to get into the fair and it costs \$27 for 1 adult and 3 children to get into the fair. What is the price of an adult ticket and what is the price of a child ticket?

   (a) Solve this by guess and check.

   (b) Solve it by substitution – remember to create letters for the numbers, set up the equations and substitute to get one letter (use a calculator).

3.2 Simultaneous equations solution method

1. Consider Activity 2, problem 1, being solved by substitution.

   (a) It has the following equations (substituting i for **number of** ice blocks and c for **number of** chocolates):

   $$i + c = 12$$

   $$40i + 60c = \$6.20 = 620$$

   (b) It is solved by realising that:

   $i = 12 - c$
   $40i + 60c = 40(12 - c) + 60c = 480 - 40c + 60c = 620$
   Therefore: $480 + 20c = 620$
   $20c = 140$
   $c = 7$ chocolates
   $i = 12 - 7 = 5$ ice blocks

2. The equations can be solved another way (this method is called **whole equations**):

   (a) Write down both equations:

   $i + c = 12$ A
   $40i + 60c = 620$ B

   (b) Multiply equations so one variable is the same then subtract so one variable disappears:

A × 60	$60i + 60c = 720$
B as is	$\underline{40i + 60c = 620}$
Subtract equations	$20i + \ \ 0 \ = 100$
	$i = 5$
	$c = 7$

3. Repeat method (2) above (whole equations) for:

   (a) Activity 2, problem 2 (t – number of tricycles, b – number of bicycles)

   $t + b = 26$
   $3t + 2b = 64$

   (b) Activity 2, problem 3 (a – amount of adult ticket, c – amount of child ticket)

   $2a + 5c = \$49$

   $a + 3c = \$27$

4. Solve the following by simultaneous equations using any of the methods (substitution or whole equations) above.

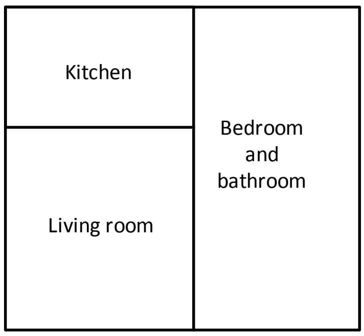
   (a) Of the 73 Year 6 students at Henry State School, there are seven more girls than boys. How many girls? How many boys?

   (b) In this two-digit number, the tens digit is greater than the ones digit. The sum of the two digits is 11 and the product is 28. What is the two-digit number?

   (c) Fred gave James a box of chocolates containing 30 small bite-size Mars Bars, KitKats and Cherry Ripes. There were twice as many Cherry Ripes as KitKats, there were three times as many Mars Bars as Kit Kats. How many of each chocolate?

   (d) **Challenge**. Sue threw a die four times. The numbers were different each time. The product was 48, the sum was 13. What numbers did Sue throw?

   (e) **Challenge**. A three-room apartment followed these specifications. The apartment was designed as on the right (not to scale):

   - area of living room was 42 m^2
   - shorter side of kitchen was 3 m
   - shorter width of bedroom and bathroom was 5 m
   - the entire area of the apartment was 110 m^2

   Work out the specifications (length and width) for each room and the total apartment.

5. **Challenge**. Operating on whole equations was the method used in the 1960s and 1970s, while substitution is the method currently used today. Why have the methods changed?

Unit 4: Solving for Unknowns in Quadratics

As discussed in other units, linear is the major focus of equations up until Year 9. However, there are nonlinear forms that can be considered (e.g. quadratics, cubics, exponentials). This unit will look at one of these, quadratics, from two perspectives: (a) the role of the balance rule; and (b) solutions to quadratics beyond the balance rule.

4.1 The balance rule and quadratics

In algebra, linear equations involve variables. However, these variables are simply x or y. They do not involve indices such as squares as in quadratics, as can be seen below.

Linear equations	Quadratics
$2x + 1 = 11$	$x^2 + 1 = 5$
$\frac{x}{4} - 2 = x$	$2x^3 - x^2 = 45$
$2x + 3y - 4 = 2x - y + 8$	$3x = 2x^2 + 7x - 8$
$3x + 2 = 4x - 7$	$2x - x^2 + 7 = 54$

However, some features are still the same for nonlinear equations:

1. Nonlinear equations are still two expressions equivalent to each other, e.g. $x^2 - 2x = 3x - 6$.
2. Nonlinear expressions are still symbolic of describing things in real life, e.g. x^3 is the volume of a cube. $6x^2$ is the surface area of the same cube, and $x^3 + 7x^2$ is the volume of a cube of side x plus a square prism with an end length of x and a height of 7.
3. Equivalence for nonlinear equations is still "the same value as" and still follows the rules of reflexivity, symmetry and transivity.

Using balance rule

This means that the balance rule applies to nonlinear equations – whatever you do to one side, you should do to the other. There are, however, more possibilities in terms of what you can do. In linear equations we could add, subtract, multiply and divide. Now we can also do square root, cube, cube root, make exponentials and reverse the exponentials (logarithm) – see examples in right column of the table above.

We only have to check that the change is "well-defined" and only gives one answer. This is mostly true but for square roots, there are two possibilities: positive and negative. For example, using the balance rule to solve $2x^2 + 6 = 38$ gives two answers, +4 and −4, as the following shows:

$$\boxed{2x^2 + 6 = 38}$$

subtract 6 from both sides (keeps balance)

$$\boxed{2x^2 = 32}$$

divide both sides by 2

$$\boxed{x^2 = 16}$$

square root both sides

$$\boxed{x = \pm 4}$$

Keeping the above in mind, the balance rule can solve nonlinear equations in cases where all the unknowns are on one side and simple (only having one variable with one coefficient).

When balance rule cannot get all the way to the answer

$x^2 + 3x + 2 = 29 + 2x$ cannot be solved by the balance rule alone:

$$\boxed{x^2 + 3x + 2 = 29 + 2x}$$
$$\Big\downarrow \text{ subtract } 2$$
$$\boxed{x^2 + 3x = 27 + 2x}$$
$$\Big\downarrow \text{ subtract } 2x$$
$$\boxed{x^2 + x = 27}$$

To be solved, this would need to be made equal to zero and the expression factorised or the rule for determining unknown for a quadratic used (see section 4.2).

Challenge

Get students to make up nonlinear examples that can be calculated by the balance rule alone and solve them. For example:

(a) $x^3 + 2y - 4 = x^3 + y - 10$

(b) $3x + x^2 - 2 = x^2 + 15$

4.2 Going beyond the balance rule for quadratics (challenge)

One of the reasons the balance rule alone cannot solve quadratics is that the x^2 and the x often mean that there are two types of x that end up on one side of the equation, e.g. $2x^2 - 3x = 7$. However, the balance rule plus two other pieces of information can be combined to develop a way of solving quadratics. The two other pieces of information are:

(a) quadratic expressions are multiples of two linear expressions, e.g. $2x^2 - 3x = x(2x - 3)$ and $x^2 - 5x - 6 = (x + 1)(x - 6)$; and

(b) the product of two expressions can only equal zero if one of the expressions equals zero, e.g. $(x + 1)(x - 6) = 0$ means $x = -1$ or $x = 6$.

Factorisation method

Putting all of this together, we can solve quadratics by using the following steps to factorise the quadratic:

Step 1	Use the balance rule to make one side equal to zero.	$x^2 - 3x = 6x - 4 - x^2$ $\downarrow$ add x^2 $2x^2 - 3x = 6x - 4$ $\downarrow$ add 4 $2x^2 - 3x + 4 = 6x$ $\downarrow$ subtract $6x$ $2x^2 - 9x + 4 = 0$
Step 2	Determine a way to make the left-hand side equal to a product of two linear expressions.	$-4 \times -1 = 4$ $(2 \times -4) + (1 \times -1) = -9$ $\downarrow$ $2x^2 - 9x + 4 = (2x - 1)(x - 4)$ $\downarrow$ $(2x - 1)(x - 4) = 0$

Step 3	Work out the x's that will make the two linear expressions equal to zero. (Two answers are $x = \frac{1}{2}$ or $x = 4$)	$(2x-1) = 0 \longrightarrow x = \frac{1}{2}$ $(x-4) = 0 \longrightarrow x = 4$
Step 4	Check by substitution.	$x = \frac{1}{2}:\ x^2 - 3x = 6x - 4 - x^2$ $\frac{1}{4} - \frac{3}{2} = 3 - 4 - \frac{1}{4} \longleftarrow$ correct $x = 4:\ 16 - 12 = 24 - 4 - 16$ $4 = 4 \longleftarrow$ correct

Quadratic formula method

A formula has been constructed to find the two answers, no matter what the quadratic:

If $ax^2 + bx + c = 0$ then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Use either factorisation (as in the table above) or the quadratic formula to determine the expressions below:

(a) $3x^2 + 4x - 7 = 2x^2 + 7x - 9$

(b) $2x^2 - x + 6 = 6 - 2x^2 - 4x$

(c) $3x^2 + 5x - 7 = 6 - 4x - x^2$

Note: Algebra tiles can be used as a physical material to assist with the balance rule part of the above process.

4.3 Measurement formulae activities

Many quadratics are related to area and volume activities, e.g. area of a circle, and volume of a cylinder. This section looks at (a) measurement formulae as a way of introducing the unknown and variables, and (b) algebra applications with quadratic formulae.

Using formulae to introduce variable

So far in the AIM Algebra modules, we have seen how the following can be used to define variable and to introduce algebraic expressions mostly in linear form, e.g. $2n + 3$ or $\frac{n}{2} - 4$, but also in nonlinear form, e.g. $n^3 + 1$:

(a) using function machines and backtracking, and equivalence equations and the balance rule to introduce equation, equations with unknowns, pre-algebraic expressions and equations, and the notion of the unknown (i.e. n or x can be a symbol for an unknown number) as a precursor to variable; and

(b) using patterns (finding the position rule) and function machines (finding change rules) to introduce the notion of variable (i.e. n or x stand for any number) and algebraic expressions.

However, there is a third way to introduce unknown and variable, and algebraic expressions and equations:

(c) using formulae from measurement, geometry, probability and statistics.

For example, we can study rectangles drawn on square grid paper and see that a 4 by 3 rectangle has 12 square units. This enables us to see that the area of a rectangle is length × width. After meaning has been developed, the long written relationship is changed to a formula using letters, that is, $A = L \times W$. In this way A, L and W are introduced as unknowns and variables.

Interestingly, formulae are not restricted to linear forms. For example, area of a circle is πr^2, volume of a cube is L^3, number of diagonals in an n-sided polygon is $\frac{n(n-3)}{2}$, and number of different outcomes for throwing a dice is 2^n.

Applications with measurement formulae

Obviously formulae are used to determine the answers to the relationship being studied. This means that formulae are used in **substitution**. For example, the volume of a cylinder is $\pi r^2 h$. A common activity therefore is to work out volumes for given radii and heights. For example, if a radius (r) = 2 m and height (h) = 4 m, then the volume of the cylinder is $\pi \times 2^2 \times 4 = 16\pi$ cubic metres or m³.

However, the difficult problems in using formulae usually involve changing the subject of the formula. This requires using the balance rule. For example: *The tradesperson has to build a cylindrical tank with a diameter of 4 m to hold 120 m³. How high does the tank have to be?*

Step 1	Write formula	$V = \pi r^2 h$
Step 2	Substitute	$120 = \pi \times 2^2 h$
Step 3	Change subject to h by using the balance rule	$\pi 2^2 h = 120$ $\downarrow$ divide by $\pi 2^2$
Step 4	Calculate answer	$h = \frac{120}{4\pi}$ m

It should be noted that this change of subject could be done without numbers; for example, change the subject of $V = \pi r^2 h$ to h.

Step 1	Write formula	$V = \pi r^2 h$
Step 2	Put h on the left-hand side of the equation	$\pi r^2 h = V$ $\downarrow$ divide by π
Step 3	Use the balance rule	$r^2 h = \frac{V}{\pi}$ $\downarrow$ divide by r^2
Step 4	Have a new formula	$h = \dfrac{V}{\pi r^2}$

The tradesperson has to build a cylindrical tank with a height of 5 m to hold 120 m³. What is the diameter?

Step 1	Write formula	$V = \pi r^2 h$
Step 2	Put r on the left-hand side of the equation	$\pi r^2 h = V$ $\downarrow$ divide by h
Step 3	Use the balance rule	$\pi r^2 = \frac{V}{h}$ $\downarrow$ divide by π $r^2 = \dfrac{V}{\pi h}$ square root $r = \sqrt{\dfrac{V}{\pi h}}$
Step 4	Calculate diameter (assuming only positive square roots)	$D = 2\sqrt{\dfrac{V}{\pi h}}$

Most work in Years 7–9 is with linear equations. However, it is useful to begin work with nonlinear equations such as quadratics (which have a variable squared).

Test Item Types

This section presents instructions and the test item types for the subtests associated with the units. These will form the bases of the pre-test and post-test for this module.

Instructions

Selecting the items and administering the pre-post tests

This section provides an item bank of test item types, constructed around the units in the module. From this bank, items should be selected for the pre-test and post-test; these selected items need to suit the students and may need to be modified, particularly to make post-test items different to pre-test items. The purpose of the tests is to measure students' performance before and after the module is taught. The questions should be selected so that the level of difficulty progresses from easier items to more difficult items. In some modules this will follow the order of the units and subtests, and in other modules it will not, depending on the sequencing across the module and within units. The pre-test items need to allow for students' existing knowledge to be shown but without continual failure, and the post-test items need to cover all the sections in a manner that maximises students' effort to show what they can do.

In administering the pre-test, the students should be told that the test is not related to grades, but is to find out what they know before the topic is taught. They should be told that they are not expected to know the work as they have not been taught it. They should show what they know and, if they cannot do a question, they should skip it, or put "not known" beside questions. They will be taught the work in the next few weeks and will then be able to show what they know. Stress to students that **any pre-test is a series of questions to find out what they know** before the knowledge is taught. They should do their best but the important questions come at the end of the module. For the **post-test**, the students should be told that **this is their opportunity to show how they have improved**.

For all tests, **teachers should continually check to see how the students are going.** Items in later subtests, or more difficult items within a particular subtest, should not be attempted if previous similar items in earlier subtests show strong weaknesses. Students should be allowed to skip that part of the test, or the test should be finished. Students can be marked zero for these parts.

Information on the algebraic computation item types

The algebraic computation item types have been divided into four subtests, one for each of the four units in this module. The four units are in sequence but all units are from the junior secondary years and above – this module covers advanced mathematics – which is why it is Year C Term 4 in the scope and sequence. Units 1 and 2 cover substitution, simplification, expansion and factorisation. Units 3 and 4 cover simultaneous equations and quadratics.

Therefore, the pre-test will be difficult – nearly everything from each of the units will be difficult for students. However, sufficient content must be covered in the pre-test to ensure that everything students know for this module is revealed. To assist this, each unit has a starting section dealing with primary ideas that could underlie the secondary ideas, and each subtest has an early item type. Thus, the pre-test should at least contain these early item types and the post-test all item types in all subtests.

Subtest item types

Subtest 1 items (Unit 1: Substitution and simplification)

1. Work out the following with the help of a number line (if needed).

   (a) My father gave me some money. I spent \$6, received \$12 from an uncle then spent \$5 and a further \$8. If my father gave me \$30, how much did I have left?

   (b) I chose a number, I multiplied it by 3, subtracted 11 and then added 14. If the number was 42, where did I end up?

2. Calculate the following (use a calculator).

   (a) The area of a triangle is $\frac{L \times W}{2}$. If L = 5 m and W = 4 m, what is the area of the triangle?

   (b) The volume of a cylinder is $\pi r^2 h$. If π = 3.14, r = 2 m and h = 4 m, what is the volume of the cylinder?

3. Substitute for x as indicated.

   (a) Calculate $4x + 3$ if $x = 7$

   (b) Calculate $\frac{x^2}{4} - 1$ if $x = 6$

4. Substitute for x and simplify.

(a) $3x - 2$ when $x = y - 4$

(b) $2x + 1$ when $x = \frac{y}{2} + 1$

5. Simplify the following.

(a) $3p + 5p - 2p + 1 =$

(b) $3x + y + 4x + 2y =$

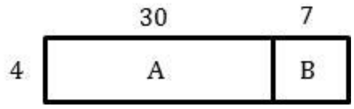
(c) $2 \times 5a - b + a + 3b =$

(d) $b \times b + 6b - 11 + 3b =$

Subtest 2 items (Unit 2: Expansion and factorisation)

1. Complete the following using the area model.

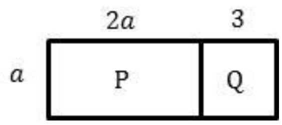
(a) 4×37 is as shown on the right:



A = ________________ B = ________________

4×37 = ________________

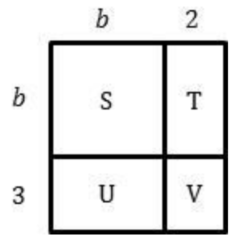
(b) $a \times (2a + 3)$ is as shown on the right:



P = ________________ Q = ________________

$a \times (2a + 3)$ = _____________________

(c) $(b + 3)(b + 2)$ is as shown on the right:



S = ________________ T = ________________

U = ________________ V = ________________

$(b + 3)(b + 2)$ = ___________________________

2. Complete the following – use the method in the algorithm.

(a)

$$\begin{array}{r} 32 \\ \times\ 4 \\ \hline 8 \\ 120 \\ \hline 128 \\ \hline \end{array} \quad \Longrightarrow \quad \begin{array}{r} P\ +3 \\ \times\quad 4 \\ \hline \\ \\ \hline \\ \hline \end{array}$$

(b)

$$\begin{array}{r} 28 \\ \times 19 \\ \hline 72 \\ 180 \\ 80 \\ 200 \\ \hline 532 \\ \hline \end{array} \quad \Longrightarrow \quad \begin{array}{r} 3\ -2 \\ \times\ 2\ -1 \\ \hline 2 \\ -\ 30 \\ -\ 40 \\ 600 \\ \hline 532 \\ \hline \end{array} \quad \Longrightarrow \quad \begin{array}{r} a\ +2 \\ \times\ 2a\ -3 \\ \hline \\ \\ \\ \\ \hline \\ \hline \end{array}$$

3. Expand:

(a) $(x+3)(x+2)$

(b) $(2x-3)(2x+5)$

4. Factorise:

(a) $3a-3b$

(b) $6p+9q$

(c) $4a-6b$

(d) $3ab+12b$

(e) x^2+3x+2

(f) $2a^2+5a+2$

Subtest 3 items (Unit 3: Solving for unknowns in simultaneous equations)

1. Use **guess and check** to solve this problem:

   12 children, 30 wheels, how many bicycles, how many tricycles?

   Show all working.

2. Use a **simultaneous equation** method to solve this problem:

   15 children, 42 wheels, how many bicycles, how many tricycles?

   Show all working.

3. Solve these two problems by a **simultaneous equation** method. Show all working.

   (a) There were 26 students in the class. There were 4 more girls than boys. How many boys were in the class? How many girls were in the class?

   (b) Amy and Ben had their birthdays in July. Ben's birthday was 3 days later than Amy's. Together the dates of their birthdays added to 33. What was Amy's birthday date?

Subtest 4 items (Unit 4: Solving for unknowns in quadratics)

1. Use the balance rule and square root to solve for x for the following equations. Show all working.

   (a) $2x^2 + 3x - 5 = \ +3x + 3$

   (b) $2x - x^2 = 2x - 25$

2. (a) Use $x^2 + 3x + 2 = (x + 2)(x + 1)$ to solve for x for the following equation. Show all working.

   $2x^2 + 3x - 6 = \ x^2 - 8$

   (b) Use $x^2 - 5x + 6 = (x - 3)(x - 2)$ to solve for x for the following equation. Show all working.

   $3 - 8x - x^2 = 3x - 2x^2 - 3$

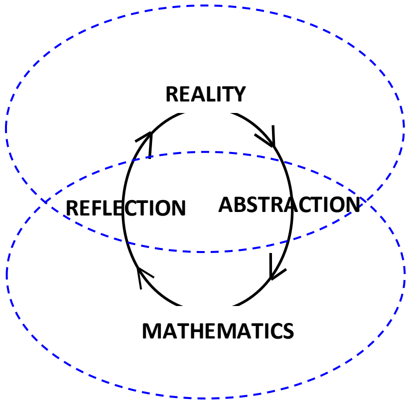
3. **Challenge**. Use the formula: If $x^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to solve the following. Use a calculator if needed.

   (a) $3x^2 - 7x + 2 = 0$

   (b) $4x^2 - 5x + 6 = 6x + 3 - 2x^2$

Appendix A: RAMR Cycle

AIM advocates using the four components in the figure on right, reality–abstraction–mathematics–reflection (RAMR), as a cycle for planning and teaching mathematics. RAMR proposes: (a) working from reality and local culture (prior experience and everyday kinaesthetic activities); (b) abstracting mathematics ideas from everyday instances to mathematical forms through an active pedagogy (kinaesthetic, physical, virtual, pictorial, language and symbolic representations, i.e. body → hand → mind); (c) consolidating the new ideas as mathematics through practice and connections; and (d) reflecting these ideas back to reality through a focus on applications, problem solving, flexibility, reversing and generalising (see figure on right).



The innovative aspect of RAMR is that the right half develops the mathematics idea while the left half reconnects it to the world and extends it. For example, whole-number place value built around the **pattern of threes** where hundreds-tens-ones applies to ones, thousands, millions, and so on, can be easily extended to metrics by considering the ones to be millimetres, the thousands to be metres and the millions to be kilometres.

Planning the teaching of mathematics is based around the RAMR cycle if it is deconstructed into components that are applied to a mathematical idea. By breaking instruction down into the four parts and taking account of the pedagogical approaches described above, the cycle can lead to a structured instructional sequence for teaching the idea. The table below briefly outlines how this can be done. Prerequisite mathematical ideas are considered in the Reality and Mathematics components of the cycle, while extensions and follow-up ideas are considered in the Reflection component.

REALITY

- **Local knowledge**: Identify local student cultural-environmental knowledge and interests that can be used to introduce the idea.
- **Prior experience**: Ensure existing knowledge and experience prerequisite to the idea is known.
- **Kinaesthetic**: Construct kinaesthetic activities, based on local context, that introduce the idea.

ABSTRACTION

- **Representation**: Develop a sequence of representational activities (physical to virtual to pictorial materials to language to symbols) that develop meaning for the mathematical idea.
- **Body-hand-mind**: Develop two-way connections between reality, representational activities, and mental models through body → hand → mind activities.
- **Creativity**: Allow opportunities to create own representations, including language and symbols.

MATHEMATICS

- **Language/symbols**: Enable students to appropriate and understand the formal language and symbols for the mathematical idea.
- **Practice**: Facilitate students' practice to become familiar with all aspects of the idea.
- **Connections**: Construct activities to connect the idea to other mathematical ideas.

REFLECTION

- **Validation**: Facilitate reflection of the new idea in terms of reality to enable students to validate and justify their new knowledge.
- **Applications/problems**: Set problems that apply the idea back to reality.
- **Extension**: Organise activities so that students can extend the idea (use reflective strategies – *flexibility*, *reversing*, *generalising*, and *changing parameters*).

Appendix B: AIM Scope and Sequence

<table>
<tr><th>Yr</th><th>Term 1</th><th>Term 2</th><th>Term 3</th><th>Term 4</th></tr>
<tr><td rowspan="2">A</td><td>N1: Whole Number Numeration
Early grouping, big ideas for H-T-O; pattern of threes; extension to large numbers and number system</td><td>O1: Addition and Subtraction for Whole Numbers
Concepts; strategies; basic facts; computation; problem solving; extension to algebra</td><td>O2: Multiplication and Division for Whole Numbers
Concepts; strategies; basic facts; computation; problem solving; extension to algebra</td><td>G1: Shape (3D, 2D, Line and Angle)
3D and 2D shapes; lines, angles, diagonals, rigidity and properties; Pythagoras; teaching approaches</td></tr>
<tr><td>N2: Decimal Number Numeration
Fraction to decimal; whole number to decimal; big ideas for decimals; tenths, hundredths and thousandths; extension to decimal number system</td><td>M1: Basic Measurement (Length, Mass and Capacity)
Attribute; direct and indirect comparison; non-standard units; standard units; applications</td><td>M2: Relationship Measurement (Perimeter, Area and Volume)
Attribute; direct and indirect comparison; non-standard units; standard units; applications and formulae</td><td>SP1: Tables and Graphs
Different tables and charts; bar, line, circle, stem and leaf, and scatter graphs; use and construction</td></tr>
<tr><td rowspan="2">B</td><td>M3: Extension Measurement (Time, Money, Angle and Temperature)
Attribute; direct and indirect comparison; non-standard units; standard units; applications and formulae</td><td>G2: Euclidean Transformations (Flips, Slides and Turns)
Line-rotation symmetry; flip-slides-turns; tessellations; dissections; congruence; properties and relationships</td><td>A1: Equivalence and Equations
Definition of equals; equivalence principles; equations; balance rule; solutions for unknowns; changing subject</td><td>SP2: Probability
Definition and language; listing outcomes; likely outcomes; desired outcomes; calculating (fractions); experiments; relation to inference</td></tr>
<tr><td>N3: Common Fractions
Concepts and models of common fractions; mixed numbers; equivalent fractions; relationship to percent, ratio and probability</td><td>O3: Common and Decimal Fraction Operations
Addition, subtraction, multiplication and division of common and decimal fractions; models, concepts and computation</td><td>N4: Percent, Rate and Ratio
Concepts and models for percent, rate and ratio; proportion; applications, models and problems</td><td>G3: Coordinates and Graphing
Polar and Cartesian coordinates; line graphs; slope and y-intercept; distance and midpoints; graphical solutions; nonlinear graphs</td></tr>
<tr><td rowspan="2">C</td><td>A2: Patterns and Linear Relationships
Repeating and growing patterns; position rules, visual and table methods; application to linear and nonlinear relations and graphs</td><td>A3: Change and Functions
Function machine; input-output tables; arrowmath notation, inverse and backtracking; solutions for unknowns; model for applications to percent, rate and ratio</td><td>O4: Arithmetic and Algebra Principles
Number-size, field and equivalence principles for arithmetic; application to estimation; extension to algebra; simplification, expansion and factorisation</td><td>A4: Algebraic Computation
Arithmetic to algebra computation; modelling-solving for unknowns; simultaneous equations, quadratics</td></tr>
<tr><td>N5: Directed Number, Indices and Systems
Concept and operations for negative numbers; concept, patterns and operations for indices; scientific notation and number systems</td><td>G4: Projective and Topology
Visualisation; divergent and affine projections; perspective; similarity and trigonometry; topology and networks</td><td>SP3: Statistical Inference
Gathering and analysing data; mean, mode, median, range and deviation; box and whisker graphs; large data sets, investigations and inferences</td><td>O5: Financial Mathematics
Applications of percent, rate and ratio to money; simple and compound interest; best buys; budgeting and planning activities</td></tr>
</table>

Key: N = Number; O = Operations; M = Measurement; G = Geometry; SP = Statistics and Probability; A = Algebra.

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