Accelerated Inclusive Mathematics Project

YuMi Deadly Maths

AIM Module A3

Year C, Term 2

Algebra:

Change and Functions

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ACKNOWLEDGEMENT

The YuMi Deadly Centre acknowledges the traditional owners and custodians of the lands in which the mathematics ideas for this resource were developed, refined and presented in professional development sessions.

YUMI DEADLY CENTRE

The YuMi Deadly Centre is a research centre within the Faculty of Education at the Queensland University of Technology which is dedicated to enhancing the learning of Indigenous and non-Indigenous children, young people and adults to improve their opportunities for further education, training and employment, and to equip them for lifelong learning.

"YuMi" is a Torres Strait Islander Creole word meaning "you and me" but is used here with permission from the Torres Strait Islanders' Regional Education Council to mean working together as a community for the betterment of education for all. "Deadly" is an Aboriginal word used widely across Australia to mean smart in terms of being the best one can be in learning and life.

YuMi Deadly Centre's motif was developed by Blacklines to depict learning, empowerment, and growth within country/community. The three key elements are the individual (represented by the inner seed), the community (represented by the leaf), and the journey/pathway of learning (represented by the curved line which winds around and up through the leaf). As such, the motif illustrates the YuMi Deadly Centre's vision: *Growing community through education*.

The YuMi Deadly Centre (YDC) can be contacted at ydc@qut.edu.au. Its website is http://ydc.qut.edu.au.

DEVELOPMENT OF THE AIM MODULES

The Accelerated Inclusive Mathematics (AIM) modules were originally developed from 2010 to 2013 as part of the Accelerated Indigenous Mathematics project funded under the Commonwealth Government's *Closing the Gap: Expansion of Intensive Literacy and Numeracy* program for Indigenous students. The project aimed to assist secondary schools with beginning junior-secondary Indigenous students who were at Year 2/3 level in mathematics by developing a junior-secondary mathematics program that accelerates the students' learning to enable access to mathematics subjects in the senior-secondary years and therefore enhance employment and life chances. The project developed three years of modules (Years A to C) that are vertical sequences of learning to take students from their ability level to their age level in mathematics. The YuMi Deadly Centre acknowledges the role of the Department of Education, Employment and Workplace Relations in the development of the AIM project and these modules.

Contents

Module Overview

This module, Module A3 *Change and Functions*, is the third in the sequence of algebra modules (see **Appendix C**). It explores how to represent everyday life in terms of change or transformation, in contrast to representing everyday life in terms of relationships. As we know from the big idea relationship vs change, all mathematics can be considered from two perspectives: as a relationship (e.g. 2 + 3 = 5) or a change or transformation (e.g. $2 \xrightarrow{+3} 5$). Module A1 *Equivalence and Equations* looked at algebra from a relationship view (e.g. equations and balance rule); this module looks at algebra from a change view (e.g. arrowmath and backtracking). The change view is a better start for functions while relationship builds equations and formulae.

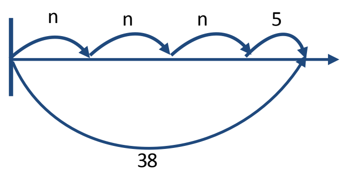
This module, therefore, studies the symbols, notation and rules for change and functions (including tables, arrowmath symbols, and graphs) and looks at the application of the change model to solutions of linear equations and percent, rate and ratio problems. In the long run, it returns to representing functions using equation notation. To get to this point requires studying: (a) change in unnumbered situations; (b) change in arithmetic situations (numbers and operations); (c) change in algebraic situations (numbers, operations and variables); and (d) change in multiplicative comparison situations (percent, rate and ratio).

Background information for teaching change and functions

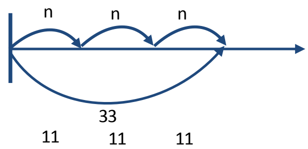
This subsection looks at two models for teaching change and functions (number line and function machine), and connections and big ideas for change and functions.

Number line model

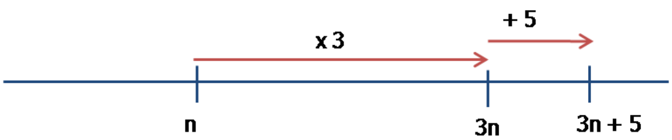
This is the first of the two major models that can help with change. As we saw in Module A1 *Equivalence and Equations*, operations can be represented on number lines by arrows. For example, the problem *I bought some $3 pies and a $5 chocolate, I spent $38 altogether, how many pies did I buy?* ($3n + 5 = n + n + n + 5 = 38$) can be represented on a double number line as in the diagram on the right.



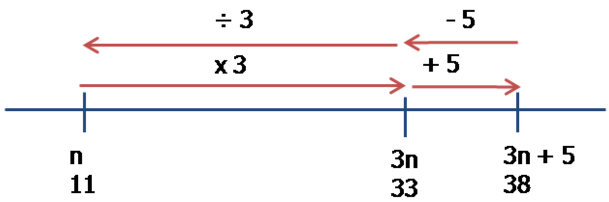
The number of pies (n) can be worked out by first crossing out the 5 which gives $3n = n + n + n = 33$ and then sharing the 33 evenly between the three n's as on the right. This gives the answer of 11 pies. This reflects the balance approach from equivalence and equations and does not use backtracking (inverse).



However, there is another way to represent the problem on a number line – this way shows operations as changes on the line, as shown below.

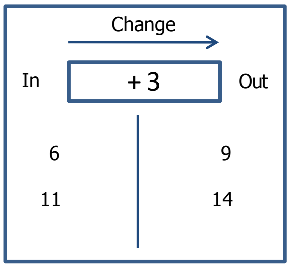


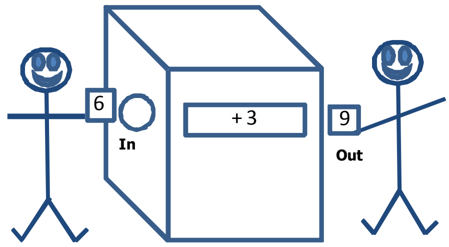
Backtracking is going back along the line and undoing what has been changed, as seen below.



Function machine model

This is the major model. A "machine" is constructed. It can be a whiteboard, or blackboard or a box with holes in it and takeout cards, for example:





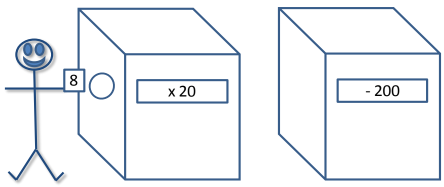
Function machines operate as follows.

1. Situations are described: *I sold small prints for $20 each, I paid $200 to rent the site, how much money did I make?*

2. Situations are translated to activities on an unknown or variable, e.g. P is the number of prints so:

$$P \xrightarrow{\times 20} \xrightarrow{-200}$$

3. These are translated to two function machines and the changes are acted out by students with numbers on cards.



4. Examples are put on an Input–Output table (shown on right) starting with input numbers, and students act out the change.

Input	Middle	Output
8	160	−40
10	200	0
20	400	200

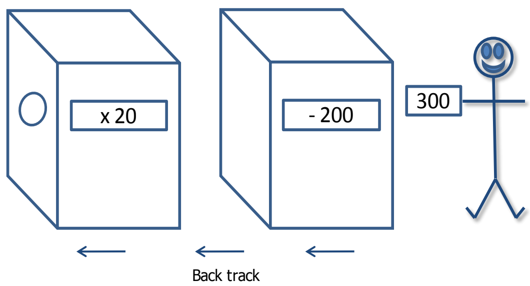
5. Numbers are then put into the middle or output and students act out how to find the other sections of the table.

Input	Middle	Output
?	240	?
?	?	100

6. A variable amount, say n, is considered and discussed at the function machines and responses included.

Input	Middle	Output
n	$20 \times n$	$20 \times n - 200$
n	$20n$	$20n - 200$

7. Then different examples (including variables) are given at output and students **backtrack** to find input using the function machines and recording on Input–Output table.



Input	Middle	Output
22	440	240
25	500	300
$\frac{m+200}{20}$	$m + 200$	m

8. The changes are reversed, and the inverse operations and inverse sequence is found and written in arrowmath notation, as shown on right.

$$P \xrightarrow{\times 20} \xrightarrow{-200} m$$

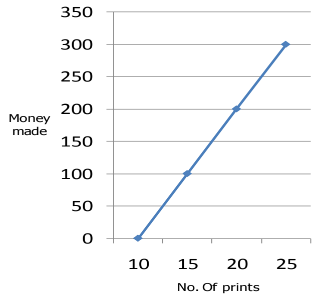
$$P \xleftarrow[\div 20]{} \xleftarrow[+200]{} m$$

9. Then the change and inverse change are written as equations, as shown on right (it takes time to get used to how these two are connected).

$P \xrightarrow{\times 20} \xrightarrow{-200} m$ same as $20P - 200 = m$

$P \xleftarrow[\div 20]{} \xleftarrow[+200]{} m$ same as $\frac{m+200}{20} = P$

10. The change is then graphed as on right below.



11. Everything is reversed: a graph is drawn and students have to find equations (forward and inverse), change equations, use arrowmath notation, put examples on the Input–Output table, and construct the function machines.

Notes: (a) the 11 steps above are the final endpoint of teaching across P–9; only a few of these steps would be undertaken for young students; (b) if this approach is learnt, it makes such things as the inverse of a function easy. For example, a function $f(x) = 2x + 1$ can be considered in arrowmath notation as shown below left. Thus, the inverse function, $g(x)$, is this arrowmath notation in reverse (backtracked) as shown below right. Changing this backtracking to an equation, this means that the inverse function is $g(x) = (x - 1) \div 2$.

$$x \overset{\times 2}{\rightarrow} \overset{+1}{\rightarrow} f(x) \qquad\qquad g(x) \overset{\div 2}{\leftarrow} \overset{-1}{\leftarrow} x$$

Connections and big ideas

Connections

The major connections are as follows:

1. **Change and relationship**. This is the basis of this module. A relationship is that 3 and 2 are related to 5 by addition and this is shown by equation notation, for example, 3+2=5. However, this relationship can also be seen as a change, that is, 3 changes to 5 by adding 2 and this is shown by arrowmath notation, for example, as on right. In the long run, as said before, equations become the notation for both meanings as they are both just different ways to describe the same thing. This module takes a change approach (Module A1 *Equivalence and Equations* took a relationship approach).

   $$3 \xrightarrow{+\,2} 5$$

2. **An operation and its inverse (backtracking)**. The basis of the use and importance of change and functions is inverse operations: − is the inverse of +, and ÷ is the inverse of ×. We can use this to solve difficult problems: *I had $52, how many litres of petrol can I buy for this money at $1.64 per litre?*

   First, we interpret the problem as a *change from litres to dollars*, that is, 1 litre is changed to 1.64 dollars (or litres are changed to dollars by multiplying by 1.64).

   Second, we backtrack this, and ×1.64 for L to $ becomes ÷1.64 from $ to L (as in the arrowmath on right). This means that the litres of petrol that can be bought is 52 ÷ 1.64 = 31.7 litres.

   $$\text{L (?)} \underset{\div 1.64}{\overset{\times 1.64}{\rightleftarrows}} \text{\$ (52)}$$

3. **Unnumbered to numbered (arithmetic) to variable (algebra) activity**. This module is an excellent example of the seamless transition of major ideas from unnumbered and early childhood activities without number to arithmetic situations and finally to algebraic situations, with applications to percent, rate and ratio. It also shows the importance of unnumbered before numbered activities when teaching big ideas.

Changes and backtracking

The two major ideas to be covered in change and functions deal with mathematical forms that describe change (such as functions) and with the major ideas that emerge from changes, namely:

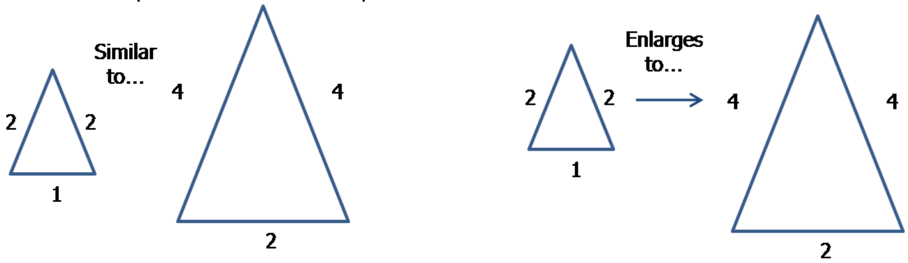
(a) changes that do not change anything (e.g. +0, ×1, a 360° turn) – **the identity principle;** and

(b) changes that reverse other changes (e.g. −6 reversing +6, ÷8 reversing ×8) – **the inverse principle and backtracking**.

Real-world situations can be translated into relationships or changes. For example, “2 joining 3 to make 5” is a relationship if considered as 2+3 = 5 and is a change if considered as on right.

$$3 \xrightarrow{+\,2} 5$$

Similarly, the two triangles below can be considered as a similarity relationship or a change due to projection that enlarges the first shape to the second shape.



Relationships are most often represented as equations and this form of notation is good for seeing equals as balance and for applying the balance principle (that there is a left-hand side and a right-hand side and they have to stay in balance).

Changes can also be represented as equations but it is easier to understand them if they are represented by arrowmath notation. For example, the situation, *I bought some $3 pies and a $5 chocolate, how much did I spend?* cannot be calculated because there is not enough information given. However, if we knew the number of pies, we could calculate the answer by multiplying this number by 3 and adding 5. Thus, the notation can be thought of as the equation n×3+5 (or 3n+5), or in arrows:

$$n \xrightarrow{\times 3} \xrightarrow{+5}$$

The arrowmath notation makes studying the change easy. First, changing forward, it is easy to work out what money will be paid for differing numbers of pies, for example, if the number of pies is 7, then the answer is $26.

$$7 \xrightarrow{\times 3} \xrightarrow{+5} 26$$

Second, by reversing the change, it is possible to find the number of pies if I paid $38 for the pies and chocolate (see below). We use the inverses of the operations and backtrack (as shown in the bottom arrows) to get the answer of 11 pies.

$$n \xrightarrow{\times 3} \xrightarrow{+5} \ 38$$

$$11 \xleftarrow{\div 3} 33 \xleftarrow{-5} \ 38$$

Big ideas

Thus the **big ideas** that are at the basis of this module are:

Concepts

(a) the concepts of the **four operations**, namely, addition, subtraction, multiplication and division; and

(b) the concepts of **line graphs** and the notations of **equations** and **arrowmath**.

Principles

(a) the notion of **identity** – the change that does not change anything;

(b) the notion of **inverse** – the change that undoes the previous change; and

(c) the notion of **backtracking** – that a sequence of changes can be undone by inverses in reverse sequence, that is, ×3 +2 can be undone by −2 ÷3.

Strategies

(a) **Interpreting as change** – the strategy of interpreting a problem as change and solving it forwards or by backtracking.

Pedagogies

(a) **Change vs transformations** – mathematics is relating and changing things and every change can be thought of as a relation and every relation can be thought of as a change; and

(b) **Interpretation vs construction** – situations can be interpreted or constructed.

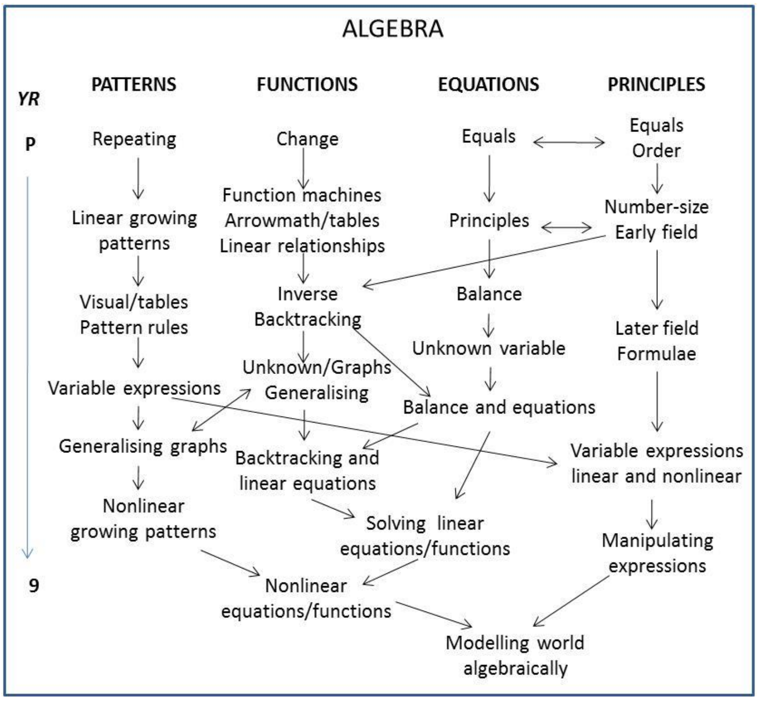
Sequencing for change and functions

This section briefly looks at the role of sequencing in algebra and in this particular module.

Sequencing in algebra

Overall sequence

The overall sequence for algebra is given in the figure below. It has four sections which are each AIM modules: Module A1 *Equivalence and Equations*, Module A2 *Patterns and Linear Relationships*, Module A3 *Change and Functions,* and Module O4 *Arithmetic and Algebra Principles*, with Module A4 *Algebraic Computation* covering some of the later activities.



It begins with **patterns** as training in the act of generalisation by finding pattern rules and relating to graphs. It then moves onto **functions**, starting from change rules in transformations, using real situations, tables and arrowmath notation before equations and graphs, solving for unknowns by the use of the balance rule. After this it moves to relationships that in arithmetic and algebra are represented predominantly by **equations**, solving them by the use of the balance rule. The sequence is completed by focusing on arithmetic and algebraic **principles** and extending these to methods such as substitution, expansion and factorisation.

Special features

There are three important aspects of sequencing for algebra because of its generalised nature and reliance on big ideas, particularly principle big ideas. These principles hold generally across mathematics in the building of big ideas.

1. **Unnumbered work before numbered**. Within each module, the sequencing will begin with unnumbered activities as these enable the big ideas to develop, move on to numbers and arithmetic situations and then move to generalised situations. YDM follows the view of the Russian mathematics educator Davydov that

to build big ideas like the *balance rule* requires **initially working in unnumbered situations** as numbers tend to result in students looking for answers not generalisations of concepts, processes (relationships/ changes) and strategies.

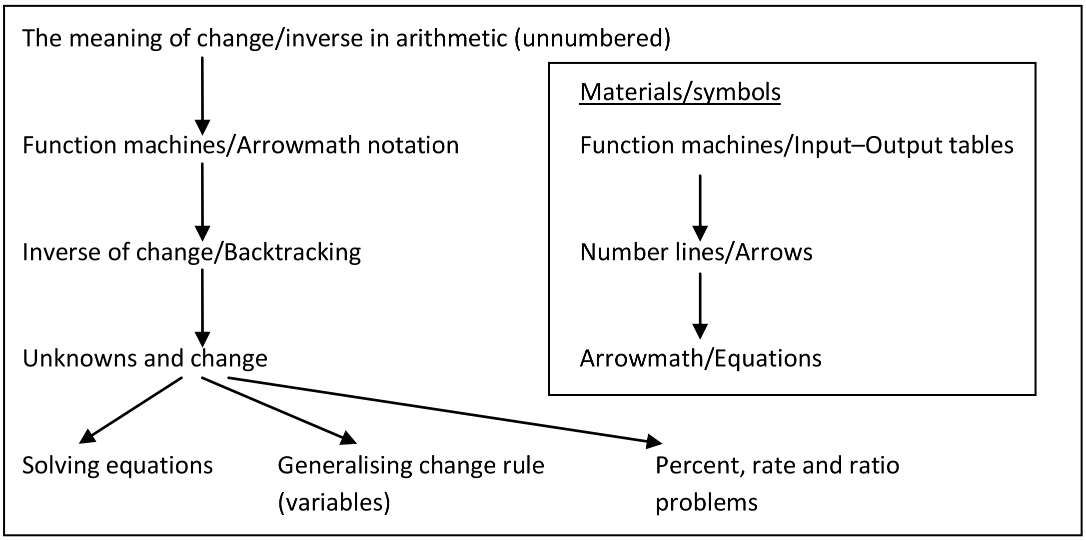
2. **Processes not answers**. The basis of algebra is things that hold for all numbers not particular answers. For 2+3 the processing is 2+3 (i.e. joining 2 things and 3 things) which gives answer 5. For x+3, the process and the answer are the same, that is, x+3. Thus, algebra has to be built around big ideas not computation. It should be noted that this has a consequence, that arithmetic does not teach two-operation processes well as the students simply do each process as it happens, for example, 2×5+3 becomes 2×5=10 and 10+3=13, two single steps not one double step. This means that time needs to be spent on teaching the processes of two-step operations (e.g. the inverse of 2×5+3 is −3 and ÷2 not the other way around).

3. **Separation to integration**. The sequence begins simply, in a separated manner, but by the time junior secondary is reached, the components are more integrated and connected to allow patterns, functions and equivalence all to be expressed in the same way (by equations), and for results to cover nonlinear as well as linear relationships and changes.

4. **Modelling as end point**. The overall end point of algebra is modelling as well as manipulation of symbolics. Computers and special calculators can do the manipulations to simplify and solve for unknowns – what is important, like in arithmetic, is to apply the knowledge to the world and solve problems – to model the world algebraically. This is important because most students cannot see the relevance of, say, $x + y = 7$ to their everyday world. Yet, with understanding it is very relevant. It could mean that you bought two things at a shop for \$7. Then the cost of the first thing (x) plus the cost of the second thing (y) is equal to \$7. This gives parameters in which thinking can be used. Suppose we were working in whole dollars. Then the first thing could cost \$1 and the second cost \$6, or \$2 and \$5, or \$3 and \$4, and so on.

Sequencing in this module

The **teaching sequence for change and functions** is shown in the figure below.



It is important that arrowmath and equations be related at the end so that both relationship and change ideas can be applied to algebraic situations using the same notational forms (the expression and the equation).

The **sequence of activities** that can be developed in this strand **relates to inverse** and includes the following:

(a) developing the notion of change and inverse of change (backtracking) in **unnumbered** situations;

(b) extending the notions of change and inverse to **numbers and operations** (first with addition and subtraction, second with multiplication and division, and third with more than one operation);

(c) introducing drawings (number lines and function machines), tables and **arrowmath** notation to describe changes and inverses;

(d) relating change and inverse (**backtracking**) to real-world situations and vice versa;

(e) generalising change and inverse and using this to **introduce variables and algebraic expressions and equations** (including conversions between arrowmath and equation notations);

(f) interpreting real-world problems in terms of change and using backtracking to **solve for unknowns**; and

(g) representing generalised change with **graphs** and relating real-world situations, arrowmath and equation notation, and graphs and change, to graphs in all directions.

This module is therefore composed of the following sections:

Overview: Background information, sequencing, and relation to Australian Curriculum

Unit 1: Very early change and function activities

Unit 2: Early to middle change and function activities

Unit 3: Later change and function activities

Unit 4: Application to linear equations

Unit 5: Application to multiplicative comparison

Test item types: Test items associated with the five units above which can be used for pre- and post-tests

Appendix A: Application of change model to percent, rate and ratio problems

Appendix B: RAMR cycle components and description

Appendix C: AIM scope and sequence showing all modules by year level and term.

The units in this module are designed to provide a basic structure to the teaching sequence including exemplar (but not compulsory) activities. It is up to individual teachers to adapt/adopt the materials presented to suit the particular needs of their students. The units in this module are designed to be taught to students using the RAMR cycle (see **Appendix B**). In this module, this will be done by showing which parts of units meet the parts of the cycle – thus, Reality, Abstraction, Mathematics, and Reflection will be part of most headings in units.

Relation to Australian Curriculum: Mathematics

AIM A3 meets the Australian Curriculum: Mathematics (Foundation to Year 10)						
Unit 1: Very early change and function activities Unit 2: Early to middle change and function activities Unit 3: Later change and function activities Unit 4: Application to linear equations Unit 5: Application to multiplicative comparison						
Content Description	**Year**	**A3 Unit**				
		1	**2**	**3**	**4**	**5**
Use equivalent number sentences involving addition and subtraction to find unknown quantities (ACMNA083)	**4**	✓	✓			
Use equivalent number sentences involving multiplication and division to find unknown quantities (ACMNA121)	**5**			✓		
Continue and create sequences involving whole numbers, fractions and decimals. Describe the rule used to create the sequence (ACMNA133)	**6**	✓	✓			
Explore the use of brackets and order of operations to write number sentences (ACMNA134)				✓		
Recognise and solve problems involving simple ratios (ACMNA173)	**7**					✓
Introduce the concept of variables as a way of representing numbers using letters (ACMNA175)				✓	✓	
Create algebraic expressions and evaluate them by substituting a given value for each variable (ACMNA176)					✓	✓
Extend and apply the laws and properties of arithmetic to algebraic terms and expressions (ACMNA177)					✓	✓
Given coordinates, plot points on the Cartesian plane, and find coordinates for a given point(ACMNA178)					✓	
Solve simple linear equations (ACMNA179)					✓	✓
Investigate, interpret and analyse graphs from authentic data (ACMNA180)					✓	
Solve a range of problems involving rates and ratios, with and without digital technologies (ACMNA188)	**8**					✓
Plot linear relationships on the Cartesian plane with and without the use of digital technologies (ACMNA193)					✓	
Solve linear equations using algebraic and graphical techniques. Verify solutions by substitution (ACMNA194)					✓	
Sketch linear graphs using the coordinates of two points and solve linear equations (ACMNA215)	**9**				✓	

Unit 1: Very Early Change and Function Activities

Change and functions covers operations as change and is designed as a precursor to functions. The sequence for teaching it is to move from unnumbered activities to numbered activities, addition and subtraction to multiplication and division, and one operation to more than one operation. It introduces input/output tables and arrowmath notation. It develops inverse of one operation, inverse of a sequence of operations, and backtracking to solve for unknowns. It relates arrowmath notation to equations, and graphs the results, relating all aspects in all directions. It continually relates symbols and models back to everyday situations so that it can model reality. Finally, it allows change to be generalised into a rule using algebraic notation, thus introducing variable and leading to function, and shows how change can help students solve percent, rate and ratio problems.

The early years are the time when unnumbered activities are used to get across the language of change (the word change itself, plus input and output) and the notion of what change and reversing change are.

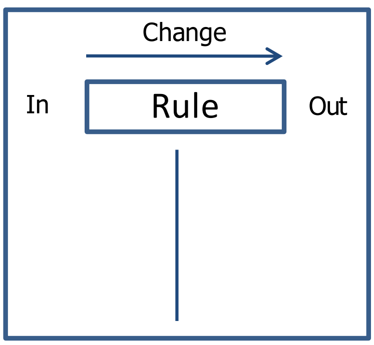
1.1 Reality – Introduce notion of change

Discuss with students how things change (e.g. we clean things, we grow things, the weather gets hotter, things get moved around, we change clothing, change hair colour, and so on). Discuss how we can have before and after – before we wash our hair, after we have washed our hair, before we put on the dress, after we put on the dress. Take photographs – use these as before and after discussion points, e.g. what happened here? Look at relationship patterns – what goes with what? For example, shoes with feet, shirts on bodies, hats on heads, and so on. Focus on how before and after have to be related in some way.

Play games like “switch” (or its commercial form, UNO). If a spade is put down, you have to follow it. To change it you can put the same value on top in a different suit, and so on. Play snakes and ladders – things change if you land on a snake or a ladder.

1.2 Abstraction – Unnumbered change activities

Set up a function machine that will have an input and output and a rule for change; see example on right for whiteboard. Give students a small copy of the board to record their changes (doing this with a small chalkboard was very successful in one school).



Whiteboard

Choose an unnumbered change and put this in the RULE box. Discuss what input and output are – act out some changes. Have students walk in on the left with a picture of a thing to be changed and stick this on input. Discuss what it could change to. Give students a picture of this change to put on RHS. Get students to record on their recording sheets or chalkboards the input and output (the in and the out).

Examples of unnumbered change include:

- lower case to capital letter (e.g. input h and output H);
- “cook it” (e.g. input a picture of a potato and output a picture of chips);
- “wash it” (e.g. dirty car to clean car);
- “wear it” (e.g. hand to glove, foot to shoe);

- add “at” (e.g. b to bat, fl to flat);
- add “ing” (e.g. r to ring , s to sing); and
- move first letter of word to end of word.

Involve students in bringing out picture cards (or potatoes or letters or whatever is relevant for the RULE) and working out what the output will be. Get students to discuss what is happening, and encourage students to think of things to change and even to think of changes.

Note: Attribute logic blocks or pattern blocks can also be used – changes can be blue to red, large to small or triangle to square. The problem here is that attributes that are not mentioned should not change.

1.3 Mathematics – Inverse of unnumbered activities

Use the function machine from 1.2 above to look at the notion of inverse. Make up a matching set of pictures before and after cooking (e.g. pasta in a packet to spaghetti bolognaise, and so on). Organise students to come out front and pick an input card. Get students to show the picture to the class and stick it on the input side of the table. Discuss what would be on the output side. Select the likely picture from output cards.

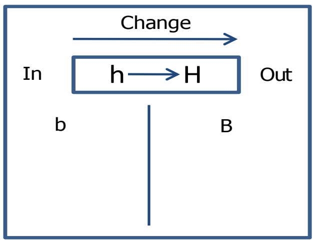
After doing this for some time, ask students to select an output card and stand on the RHS. Discuss what the input card could be. Repeat this as often as required. Initially, get students to think what input could give this output, e.g. *We have mashed potato, what could we cook to get this?* But after a while, get them to “think backwards”, e.g. *What do we get when we “uncook” the mashed potato?*

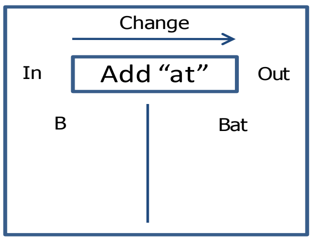
This is easier done with, for example, the “add *at*” RULE. Here, input of *s* goes to output of *sat* – we have “added” the “at”. When we look at an output of *rat*, it is fairly straightforward to consider removing, or “taking away”, the “at” to find an input of *r*. Similarly, input and output makes *h* a capital *H* if “capitalise” is our change, and we can think of output to input as “uncapitalising”, e.g. *R* to *r*. Practise with many activities.

1.4 Reflection – Extending change

As reversing (and flexibility) have already been uncovered, there are two ways we can begin extending what we have done in the above.

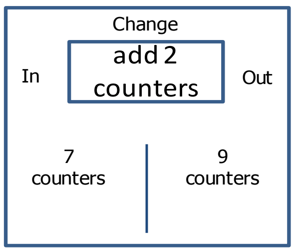
1. **Consider two changes** – one after the other (e.g. two function machines together), for example:





   Now students can move through two changes – first change from *b* to *B* and second change from *B* to *Bat*. Can also try to reverse both changes, e.g. if we ended with *Fat*, then this goes back to *F* and then *f*. (It should be noted that changes like *e* to *Eat* could be challenging as well as *fl* to *Flat*.)

2. **Consider bringing in number**. This could be done initially by adding two extra counters or removing three counters from plastic bags with counters, or using input and output cards showing sets of counters, as for the function machine on right. For this function machine, an input of 4 counters would give an output of 6 counters, while an output of 9 counters comes from an input of 7 counters.



Unit 2: Early to Middle Change and Function Activities

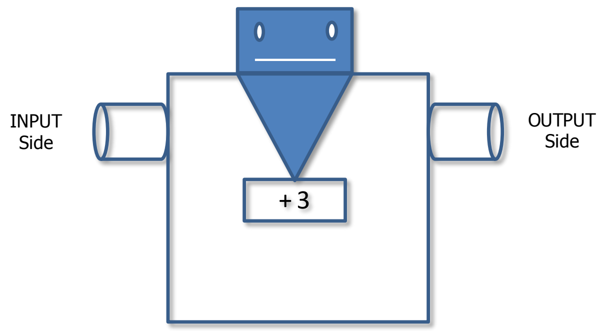
In the early to middle years of primary, the idea is to move change and functions to numbered activities. This is similar to what happened with equivalence and equations in Module A1, and with measurement in Modules M1, M2 and M3. The numbered activities begin with one operation, addition first, then subtraction and then multiplication and division. This also involves presentation of change with formal symbols for the first time.

This unit focuses on the first move from unnumbered activities, that is, activities for one of addition and subtraction, extending this to multiplication and division. It also begins the teaching of the arrowmath notation.

2.1 Reality/Abstraction – One operation (addition or subtraction)

Discuss with students how addition and subtraction could be thought of as change. For example: *What happens if I have two more toys or dollars – what if I start with 3 toys or 3 dollars? What does 7 toys become if you give 3 toys to a friend?*

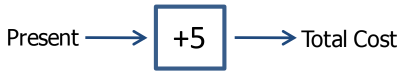
Set up a function machine that adds or subtracts a small number. The whiteboard function machine is still excellent but for this we will move onto the "robot" function machine. Basically it is a large box (in which students can stand) with a small "head", two openings (one on each side), a rule hung around the neck or from the top (if there is no "head"). See example on right.



Two card sets are made – numbers 1 to 20 for Input cards and 1 to 30 for Output cards. Students, in twos, bring an Input card to LHS of robot and place it in the opening. For the machine on the right, students inside add 3 and push Output card out RHS opening to be picked up by student. Other students have a calculator to check that correct change has been made (e.g. 6 to 9) and a worksheet on which to record Input and Output numbers.

The following is a sequence of activities found useful.

1. Give students a **real-world problem** that adds/subtracts a small number. For example, *It costs $5 to have a present wrapped. What is the total cost of present and wrapping?* Discuss what we can do with this. [We cannot get answer as is but there are two things that can be done: (a) if given the present's price, we can work out the total cost, e.g. present is $36, total cost $41; and (b) if given the total cost, we can work out the present cost, e.g. total cost is $24, present cost $19.]
2. Get students to consider problem as a **change** – ask what is the operation? and then draw a function machine as on right.



3. Act out change with the **function machine**. Organise a student to go into robot with Output cards. Give other students Input card numbers and ask them to bring them out front, in turn, to Input and then collect a changed card at Output.
4. Fill in **Input–Output table**. Get students to follow the function machine activity with a calculator, checking calculations and filling in Input–Output tables. Ask students to complete tables without watching a student at the front use the function machine.

2.2 Mathematics – One operation (addition and subtraction)

1. **Practice**. Do the above for a variety of additions and subtractions.

2. **Reverse** the change. Teacher directs a student to collect an Output card without showing Input. Ask class what was the Input card. Walk the student backwards from Output to Input as you are doing this. Discuss options and how to find this inverse number. Teacher provides a series of Input and Output numbers for students to fill in on their Input–Output tables. Have large numbers as part of this.

3. Develop **inverse**. Teacher leads discussion on quick ways to find the inverses and encourages students to see that –5 gives inverse of +5.

4. Connect to **arrowmath notation**. Students are directed to write both changes as arrowmath diagrams using examples (as on right). It is useful to get students to think of operations as change.

   $6 \xrightarrow{+5} 11$
   Change

   $8 \xleftarrow{-5} 13$
   Inverse change

5. Get students to think of travelling from one place to another – travelling distances. Use arrowmath to represent the travel – calculate the distances, as below.

   Fredtown $\xrightarrow{+126}$ Sueburg $\xrightarrow{+209}$ Roncity

6. Adapt this and build the idea of arithmetic excursions, travelling from number to number by actions or operators, as below. Call this "arithmetic excursions". Introduce that you can have subtraction as well as addition.

$$2 \xrightarrow{+7} 9 \xrightarrow{+26} 35 \xrightarrow{-2} 33 \xrightarrow{+61} 94$$

2.3 Reflection – One operation (all operations)

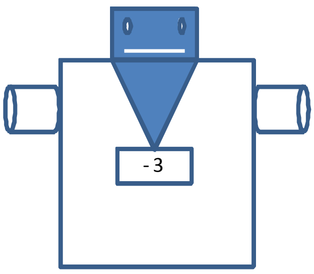
1. **Generalise** the change and its reverse. Choose a student and ask to go to Input. Tell other students that this student has a number to Input in but does not know what it is. Get class to discuss what Output would be. Do the same for any Output number – what would be the Input? Give students a variety of large numbers to say what the change would be. Get students to write their rules in language. Repeat this for the inverse change. Move onto symbols (but do not push for accuracy or for everyone getting the answer). Ask what the Output would be if Input was *n*? Ask what the Input would be if Output was *k*? See if students can write *n* + 5 and *k* – 5.

2. **Reverse everything**. Give students a generalisation of a change (can give it as language or as examples of numbers). Ask the students to represent the change and its inverse, with examples, use arrowmath notation, fill in an Input–Output table for some values, draw the change as a function machine, and create a problem for it. For example:

   Input n $\longrightarrow$ N – 3 Output

   $6 \xrightarrow{-3} 3$
   Change

   $11 \xleftarrow{+3} 8$
   Inverse change



Input	Output
5	2
10	7
18	15
	84
112	

3. **Extend to multiplication and division**. Repeat what was done for addition and subtraction. The function machine is set up for changes like ×5 and ÷4. The Input and Output cards have to be especially

constructed. For change ÷4, Input cards are 4, 8, 12, 16 and so on to 80, while Output cards are 1 to 20. A student is put in the robot function machine. Other students, in turn, bring up Input cards and receive Output cards, and later bring up Output cards in order to work out the inverse that gives the Input. Practise with many examples. The idea is to cover steps as below (using example of ÷4):

(a) starting with a real-world problem;

(b) drawing and setting up a function machine for this problem;

(c) filling in an Input–Output table;

(d) drawing arrowmath diagrams (forward and reverse);

(e) conceptualising inverse as ×4;

(f) generalising change in language and using variables, e.g. $n \div 4$ for Input n, and $k \times 4$ for Output k; and

(g) reversing the whole process – go from, say, $n \times 3$ right through to real-world problem.

4. **Extend arrowmath to multiplication and division**. Do the following activities as “arrowmath excursions”, using calculators to:

(a) go from 2 to 62 by 4 changes;

(b) go from 68 to 1001 by 7 changes;

(c) go from 687 to 23 going through 2099 on the way;

(d) make a long journey from 3679 to 9763 passing through 2 000 001 on the way; and

(e) go from 654 to 268 by multiplications only.

Work hard to encourage students to use calculators without first working out in their head where they are going. Let them understand that making a number larger is achieved through adding, multiplying and dividing by a fraction; similarly, making a number smaller is achieved by subtracting, dividing and multiplying by a fraction. If they get a decimal, just subtract it on the next move.

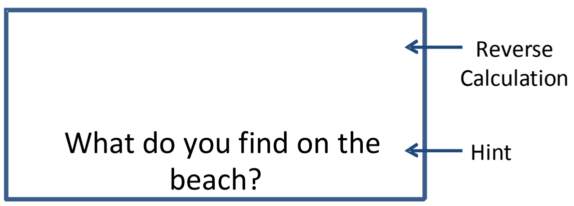
Extra activities

(a) Reverse arrowmath excursions and use arrowmath to study inverse, as below.

$$6 \xrightarrow{\times 8} 48 \xrightarrow{+64} 112 \xrightarrow{\div 4} 28$$

$$6 \xleftarrow{\div 8} 48 \xleftarrow{-64} 112 \xleftarrow{\times 4} 28$$

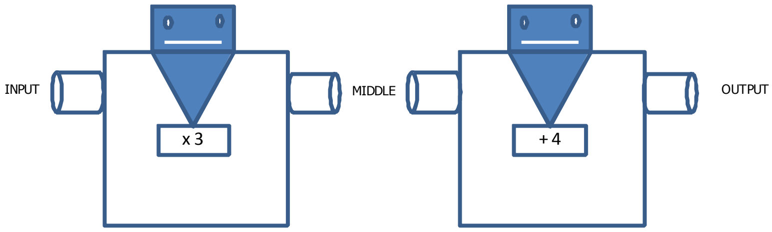
(b) Use the inverse of change and relation to equations to make up **“talking calculator”** activities. Discover which numbers upside down form letters (e.g. 0 is O, 1 or 7 is L, 3 is E, 4 is h, 5 is S, 8 is B). Make up a number which upside down is a word (like “shells”). Take this number, make changes to it with an arrowmath excursion and follow these with calculator. Reverse the sequence of changes and write as an equation – it should equal the original number but leave this place blank (i.e. do not show the original number as the answer to the reverse calculation). Google “talking calculator” for more examples.



Unit 3: Later Change and Function Activities

In the final years of P–9, change and function activities are used to generalise, introduce variable and algebraic expressions, draw graphs, and introduce functions. It is also the time that two operations are used together (using two function machines in sequence) and the term “backtracking” becomes important.

This repeats the types of activities from Unit 2 but with two function machines and two operations in sequence as below. This means that there are three columns in the Input–Output Table and three generalisations – see below for example (circled item is starting point).



Input	Middle	Output
7	21	25
64	192	196
n	$n \times 3$	$n \times 3 + 4$
p	$3p$	$3p + 4$
$\frac{k-4}{3}$	$k - 4$	k

This means that there are three columns in the Input–Output table and three generalisations, see above right for example (circled item is starting point).

Inverse is important here, as is the arrowmath, as it shows that, as well as inversing all operations, the order of the operations is also reversed, e.g.

$$6 \underset{1^{st}}{\overset{\times 3}{\rightarrow}} 18 \underset{2^{nd}}{\overset{+4}{\rightarrow}} 22 \qquad\qquad 11 \underset{2^{nd}}{\overset{\div 3}{\leftarrow}} 33 \underset{1^{st}}{\overset{-4}{\leftarrow}} 37$$

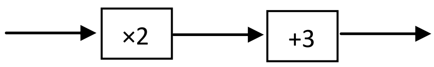
Once again it is important to go both ways: (a) from real-world problems to drawing to table to arrowmath diagrams to generalisation; and (b) then reverse from a generalisation to arrowmath to table to drawing to real-world problem. Do not be tempted to miss the real-world problem; **it is crucial to relate the function machine to everyday life**.

Note: For middle years, do not expect or require students to successfully generalise for a variable n. Students go through the following stages in generalising:

- Not being able to generalise
- Quasi-generalisation – doing it for any number given
- Saying it in language
- Writing it with letters

3.1 Reality/Abstraction – Two operations

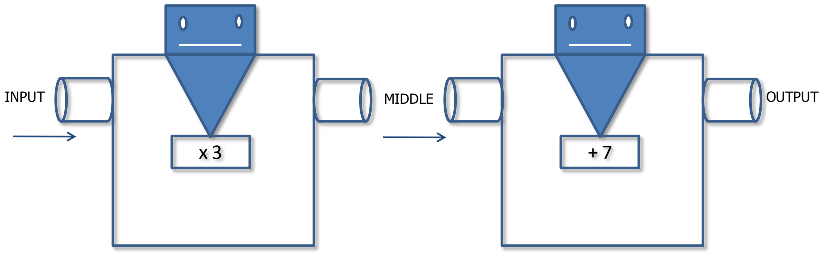
Discuss change problems in which there are two operations. For example: *Two busloads of children came to sport, three other children came by car, how many children?* or *I bought two bottles of drink and a chocolate for three dollars, how much did I spend?* [both of these are ×2 +3]. Encourage the students to change the problem to a drawing of two function machines as on right, then to dress the two function machines with these operations. From then on, repeat and extend the activities from Unit 2 as follows.



1. Begin with a **two-operation problem.** *I bought every student a Child's Meal for $3 and myself a Burger Meal for $7. How much did I spend*?

2. Discuss what can be done with this problem – **what change means.** Encourage students to realise that if they know how many students, they can work out how much was spent; and if they know how much was spent, they can work out how many students. Note that these things can be expressed as changes: forward – students to spending; and backward – spending to students.

3. Work out operations used and construct/draw function machines, e.g. robots, as below.



Input	Middle	Output
3	9	16
11	33	40
156	468	475

4. **Do examples and record** on Input–Middle–Output tables as above. Get students in robots with cards to act out what happens. Other students check with calculators as well as record on tables – **important that all students are active in recording and checking**. Note that in the table, the circled numbers are given and the rest have to be worked out and filled in.

5. Look at **reversing process**. Again act this out, check with calculators and record on Input–Middle–Output tables. Again, the circled numbers are given and the rest are worked out and filled in by students.

Input	Middle	Output
5	15	22
9	27	34
18	54	61

   Discuss with students until they see that reversing involves the inverse of the operators, e.g. ×3 goes to ÷3 and +7 goes to –7. Also ensure students see that order of operations is reversed, e.g. ×3+7 goes to –7÷3. Get students to stand on the Output side with say 22 on a card and **walk them backwards** to the Middle and the Input side showing the inverses as the students walk backwards.

   Introduce the term "backtracking" and make sure that this process of backtracking is formalised.

6. Record forward and backward (reverse) as **arrowmath**, e.g.

$$9 \xrightarrow{\times 3} 27 \xrightarrow{+7} 34 \qquad\qquad 18 \xleftarrow{\div 3} 54 \xleftarrow{-7} 61$$

3.2 Mathematics/Reflection – Two operations

1. **Practise** the two-problem, two-operation change activities – making sure students can go in both directions.

2. **Connect** the two-operation problems to two one-operation problems.

3. **Generalise** the forward and backward changes by using letters and requiring students to complete Input–Middle–Output tables as on right. Again, the circled letters are starting points – the rest are worked out.

Input	Middle	Output
n	3n	3n + 7
p	3p	3p + 7
$\frac{k-7}{3}$	k - 7	k

4. **Reverse everything** – start with a generalisation, e.g. $2n + 3$, and work through arrowmath, chart and function machine to a real-world problem. **Spend a lot of time on equation to problem** – e.g. what is a story for $2n - 3$? What about $\frac{n}{4} + 3$?

5. **Play what's my number** – say *I am a number, I have been divided by 5 and 2 subtracted, I am now 9, what was I?* **Relate to movement** to reinforce this – get students to step forward, one step as they divide by 5

and one step as they subtract 2, then move backwards for backtracking, step back for +2 and second step backwards for ×5. **Act out the arrowmath!**

6. See if students can extend to **backtracking to solve real-world problems** as follows.
   (a) Start with problem – *Each team member is to carry 3 litres of water and a truck is to carry 25 litres. How many in the team if there are 58 litres of water to be carried?*
   (b) Put this into an arrowmath: $? \xrightarrow{\times 3} \xrightarrow{+25} 58$
   (c) Reverse this (backtrack) to show there are 11 members: $11 \xleftarrow{\div 3} 33 \xleftarrow{-25} 58$
   (d) Use a worksheet where one of the four columns below is filled in and the rest are completed by students.

Real-world Problem	Forwards Arrowmath	Backwards Arrowmath	Solution

Extra activities

1. Use arrowmath notation to study properties for a variety of numbers. Some examples of activities are below. (*Note*: These examples use ? as "any number".)
   (a) Does the change on the left give the same Output for given Input as the change on the right?

$$? \xrightarrow{\times 4} \xrightarrow{\times 3} ? \qquad\qquad ? \xrightarrow{\times 3} \xrightarrow{\times 4} ?$$

   (b) Does the change on the left give the same Output for given Input as the change on the right? Relate this answer to (a) – what does this mean?

$$? \xrightarrow{\times 4} \xrightarrow{+3} ? \qquad\qquad ? \xrightarrow{+3} \xrightarrow{\times 4} ?$$

2. Importantly, there is a need to begin relating arrowmath notation to equations, as below.

$$\text{Change} \quad 6 \xrightarrow{\times 8} 48 \xrightarrow{+64} 112 \xrightarrow{\div 4} 28 \quad \text{is} \quad \frac{6\times 8+64}{4} = 28$$

$$\text{Inverse} \quad 6 \xleftarrow{\div 8} 48 \xleftarrow{-64} 112 \xleftarrow{\times 4} 28 \quad \text{is} \quad \frac{28\times 4-64}{8} = 6$$

Unit 4: Application to Linear Equations

As is evident in Unit 3, when students find the generalisation to changes with two operations, it is in the form of linear equation and a linear function. This unit explores this relationship between two operations, function machines and linear relationships.

4.1 Reality/Abstraction – Backtracking and solutions to linear equations

1. **Look at real-world situations in terms of change** and discuss them in terms of how they give rise to problems, as follows.

   (a) Give problem: *4 teams of players plus 9 adults got on buses. How many on the bus?* Discuss whether we can work this out and why [encourage students to realise we cannot work anything out because we need to know how many in a team].

   (b) Discuss what we could do [encourage students to realise that, if we know the number in each team, we can work out the number on the bus – this is called **substitution**; and, if we know the number on the bus, we can work out the number in each team – this is called **solving for an unknown**] – trial some of the things they offer. (*Note*: It is also worth discussing what would happen if teams were different in size.)

   (c) Practise substitution for various problems, then say we are going to learn solving for an unknown.

2. **Relate real-world problems to arrowmath and equations**, as follows.

   (a) Translate problem/story to arrowmath, e.g. *6 fishers caught 5 fish each, they gave 16 away and were left with 14*.

   $$6 \xrightarrow{\times 5} 30 \xrightarrow{-16} 14$$

   (b) Reverse these arrowmaths and translate these to equations, e.g. forward $6 \times 5 - 16 = 14$ and reverse $\frac{14+16}{5} = 6$.

   $$6 \xleftarrow{\div 5} 30 \xleftarrow{+16} 14$$

   (c) Start to replace numbers in stories and in arrowmath with unknowns as on right, e.g. *I bought a box of chocolates each and a $6 cake each for 4 students and it cost me $88, how much did the box of chocolates cost?* Then write it as an equation, e.g. $(C + 6) \times 4 = 88$

   $$C \xrightarrow{+6} ? \xrightarrow{\times 4} 88$$

   (d) Solve these by backtracking: 88 ÷ 4 = 22 and 22 − 6 = 16, so $16 is the price of the box of chocolates.

   $$C \xleftarrow{-6} ? \xleftarrow{\div 4} 88$$

3. **Translate** real-world problems to equations **and solve** by thinking backtracking, as follows.

   (a) Problem *There were 4 teams of players plus 9 adults got on buses. This made 57 people on the buses. How many in each team?*

   (b) Translate to arrowmath and equation (have to identify unknown and give it a letter):

   $$? \xrightarrow{\times 4} \xrightarrow{+9} 57$$

   $$t \times 4 + 9 = 57 \quad \text{or} \quad 4t + 9 = 57$$

   (c) Backtrack to get *t* (see below), i.e. t = 57 − 9 = 48 and 48 ÷ 4 = 12

   $$t \xleftarrow{\div 4} \xleftarrow{-9} 57$$

   (d) Translate this back to the problem, i.e. there are 12 players in each team.

4. Finally, go from equations to answers **by thinking** backtracking, e.g.

Equation: $2x + 5 = 17 \longrightarrow x = 17 - 5 \div 2 = 6$

Thinking: $x \xrightarrow{\times 2} \xrightarrow{+5} 17$ $\qquad$ $6 \xleftarrow{\div 2} 12 \xleftarrow{-5} 17$

(However, note that **the crucial part** of this is the **translation from symbols to real-world situations and back again** – so start and end with the problem.)

4.2 Mathematics/Reflection – Functions and graphing

This extends the above activities to graphing and functions. The steps are as follows.

1. **Practise backtracking and solving problems**.

   (a) Start with a **problem:** *Five fishermen each caught the limit of fish. They gave 7 fish away. This left them with 33 fish. What was the limit?*

   (b) Translate to **function machine** (i.e. translate problem to changes) and identify change operations.

$$\longrightarrow \boxed{\text{x } 5} \longrightarrow \boxed{-7}$$

   (c) Complete Input–Middle–Output **table**, identify reverse (inverses), and write change as **arrowmath** and **equations** and use backtracking (or reverse equation) to **solve** the problem for an unknown (in this example we use ? for unknown, but could use a letter).

$$? \xrightarrow{\times 5} \xrightarrow{-7} 33 \qquad 8 \xleftarrow{\div 5} \xleftarrow{+7} 33$$

$$5? \; -7 = 33 \qquad ? = \frac{(33+7)}{5} = 8$$

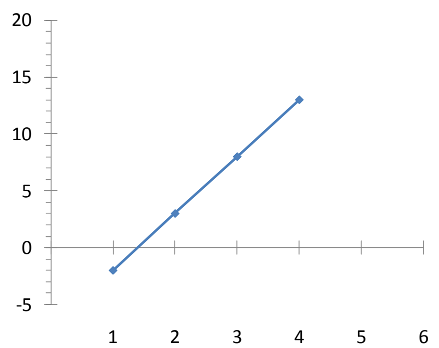
2. **Generalisation**. Generalise forward and backward change, e.g. Input *n* gives Output 5*n* – 7. Output *k* gives Input $\frac{k+7}{5}$. Practise this also.

3. **Reverse from and to this point**. Start from generalisation, e.g. start with ? × 3 – 5 (or $3n - 5$), fill in an Input–Output table, write arrowmath equations, draw the function machine and then develop a real-world problem that leads to this generalisation via a function machine. Go both ways.

4. **Extend to graphing**. Start from a problem, construct function machine, table, arrow equations and generalisation.

   (a) Fill in an Input–Middle–Output **table**. Use table points to plot a **graph** (as below and on right for $5n - 7$)

Input	Middle	Output
1	5	-2
2	10	3
3	15	8
4	20	13



(b) Do this a few times and relate the equations to the slopes and y-intercepts of the graph. For the example $3n - 5$, the slope is 3 and the graph cuts the y axis at −5.

(c) Notice the generalisation: equation $y = mx + c$ gives graph with slope m and y-intercept c.

5. **Reverse** by going from graph to problem and then problem to graph again.

6. **Extend to functions**.

(a) A function $y = 4x - 7$ can be considered as a change from x to y; for this example, the change is $\times 4 - 7$.

(b) This can be written in arrowmath as below:

$$x \xrightarrow{\times 4} \xrightarrow{-7} y$$

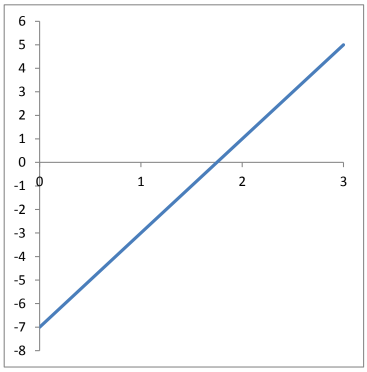
(c) **New notation** can then be introduced to represent function as f with $f(x)$ to denote variable being used, that is:

$f(x) = 4x - 7$ is the same as

$$x \xrightarrow{\times 4} \xrightarrow{-7} f(x) \text{ or } y$$

(d) This leads to **tables and graphs** as on right.

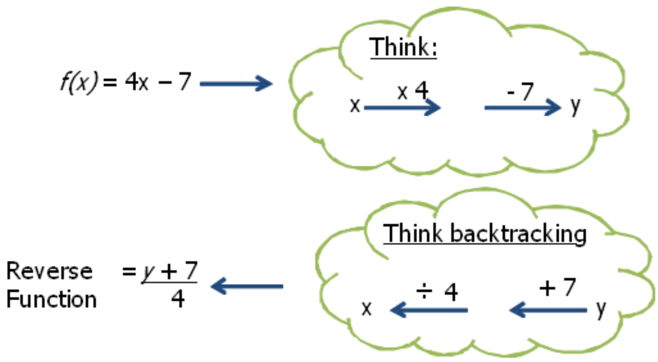
x	***f(x) or y***
0	-7
1	-3
2	1
3	5



7. **Use to find inverse function**. A function, like an equation, can be used to construct Input–Output tables and to draw graphs. Backtracking can be used to find the inverse function as on right.

Thus, if we stay with x as variable for all functions and label the inverse function as g, we have inverse function for $f(x) = 4x - 7$ being the backtracking :

$$g(x) = \frac{x+7}{4}$$

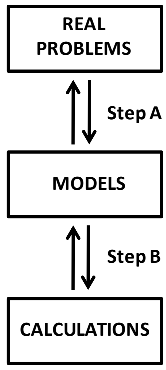


8. **Reverse everything** by starting from function or graph and using these to construct arrowmath notation for change and reverse change, Input–Output table, drawing of function machine, and story (real-world situation). Then go back the other way.

Unit 5: Application to Multiplicative Comparison

The teaching objective for this unit is to show that the function machine can be made into a very strong mental model for percent, rate, and ratio problems and applications. The way this is done is to: (a) translate real-world problems to change and backtracking mental models; and (b) use models to calculate answers.

Thus, we focus our teaching on two steps:



- Step A: lessons and worksheets that **relate real problems to models as diagrams**, and the reverse; and
- Step B: lessons and worksheets that **use the models to determine the calculations** and the reverse (the actual calculations can be done with calculators).

This is a two-step process, namely, **real problems ←→ models** and **models ←→ calculations**, as on right.

5.1 Reality/Abstraction – Translating problem to model

1. **Percent**

   (a) *Start with percent problems*. Consider two types of percent problems: (a) *What is 45% of $80?* and (b) *45% is $80, what is total?*

   (b) *Use a function machine to develop the model*. Get students to put each problem, in turn, onto a function machine. To do this, the first question is *What is the change when finding 45%?* For this we need to know that 45% is × 0.45 and that 0.45 = 45%.

   (c) *Dress the function machine*. Input (money) $\longrightarrow \boxed{\times 0.45} \longrightarrow$ Output (percentage). Translate this to arrowmath.

   $$\text{Amount/Input} \xrightarrow{\times 0.45} \text{Percentage/Output}$$

   (d) *Trial inputs*. For example, an input of $200 would mean an output of $200 × 0.45 or $90 [this seems correct as 45% of $100 is $45 and there is double this which makes $90].

   (e) *Trial outputs*. For example, an output of $200 would mean *what × 0.45 equals $200* and this would require backtracking. Thus, the answer is found by dividing $200 by 0.45; this would mean that the input is $444.44. Thus the function machine–arrowmath model from (d) seems to work. Tell students, **we will call it the change model**.

   (f) *Use the arrowmath model to solve the problems*. Translate problems to the arrowmath model as follows and calculate forward or backward depending on where the unknown is.

What is 45% of $80?	*45% is $80, what is total or 100%?*
$\$80 \xrightarrow{\times 0.45}$ Percentage?	Amount? $\xrightarrow{\times 0.45} \$80$
(forward) $\$80 \times 0.45 = \$36$	$\frac{\$80}{0.45} = \$177.78$ (backtracking)

2. **Rate**

(a) Consider two rate problems, e.g. *Bananas are $4.50 a kg, what is the cost of 3 kg of bananas?* and *Bananas are $4.50 a kg, how many kg of bananas for $10?*

(b) Use the same model as used for percent problems – the change model.

(c) Dress the function machine – the change is from kg of bananas to $ of money and the multiplier is 4.50.

(d) Use the function machine ideas to solve the problems, either forward or backtracking. The diagrams for the problems are as follows:

Bananas are $4.50 a kg, what is the cost of 3 kg of bananas?	*Bananas are $4.50 a kg, how many kg of bananas for $10?*
$3\text{ kg} \xrightarrow{\times 4.50} \$?$	$?\text{ kg} \xrightarrow{\times 4.50} \$10$
(forward) $3 \times\ 4.5 = \$13.50$	$\frac{\$10}{4.5} = 2.22$ kg (backtracking)

3. **Ratio**

(a) Consider two types of problems: *Sand and cement are 5:3, how much cement for 15 kg of sand?* and *Sand and cement are 5:3, how much sand for 15 kg of cement?*

(b) Use the same model as used for percent problems – the change model.

(c) Dress the function machine – the change is from sand to cement and the multiplier is a number which changes 5 to 3 which is $^3/_5$ or 0.6.

(d) Use the function machine ideas to solve the problems, either forward or backtracking. The diagrams are as below:

Sand and cement are 5:3, how much cement for 15 kg of sand?	*Sand and cement are 5:3, how much sand for 15 kg of cement?*
$15\text{ kg} \xrightarrow{\times ^3/_5} ?\text{kg}$	$?\text{kg} \xrightarrow{\times ^3/_5} 15\text{ kg}$
(forward) $15\ \times {}^3/_5 = 15 \times 0.6 = 9\text{ kg}$	$15 \div {}^3/_5 = 15 \div 0.6 = 25$ kg (backtracking)

5.2 Mathematics/Reflection – Using model for calculation

1. Formalise the **change model**.

$$\text{Start (S)} \xrightarrow{\times \text{ Multiplier } (\times\text{M})} \text{Finish (F)}$$

2. Study this model; encourage students to discover how to get answers.

	F unknown	this means	S × M (forward)
	S unknown	this means	F ÷ M (backtracking)
Note also that if	M unknown	this means	F ÷ S

3. Apply this to problems.

(a) **Percent**

What is 45% of $80? $\$80 \xrightarrow{\times 0.45} \text{Percentage?}$ Finish is unknown so percentage is: $\$80 \times 0.45 = \$36$	*45% is $80, what is total or 100%?* $\text{Amount?} \xrightarrow{\times 0.45} \$80$ Start is unknown so have to reverse, so divide, so amount is: $\frac{\$80}{0.45} = \$177.78$ (backtracking)

(b) **Rate**

Bananas are $4.50 a kg, what is the cost of 3 kg of bananas? $3\text{ kg} \xrightarrow{\times 4.50} \$?$ Finish is unknown, so: $3 \times 4.5 = \$13.50$	*Bananas are $4.50 a kg, how many kg of bananas for $10?* $?\text{ kg} \xrightarrow{\times 4.50} \$10$ Start is unknown, so: $\frac{\$10}{4.5} = 2.22\text{ kg}$

(c) **Ratio**

Sand and cement are 5:3, how much cement for 15 Kg of sand? $15\text{ kg} \xrightarrow{\times 3/5} ?\text{ kg}$ Finish is unknown, so: $15 \times {}^{3}/_{5} = 15 \times 0.6 = 9\text{ kg}$	*Sand and cement are 5:3, how much sand for 15 kg of cement?* $?\text{ kg} \xrightarrow{\times 3/5} 15\text{ kg}$ Start is unknown, so: $15 \div {}^{3}/_{5} = {}^{15}/_{0.6} = 25\text{ kg}$

For a full set of problem types, see Appendix A.

Test Item Types

This section presents instructions and the test item types for the subtests associated with the units. These will form the bases of the pre-test and post-test for this module.

Instructions

Selecting the items and administering the pre-post tests

This section provides an item bank of test item types, constructed around the units in the module. From this bank, items should be selected for the pre-test and post-test; these selected items need to suit the students and may need to be modified, particularly to make post-test items different to pre-test items. The purpose of the tests is to measure students' performance before and after the module is taught. The questions should be selected so that the level of difficulty progresses from easier items to more difficult items. In some modules this will follow the order of the units and subtests, and in other modules it will not, depending on the sequencing across the module and within units. The pre-test items need to allow for students' existing knowledge to be shown but without continual failure, and the post-test items need to cover all the sections in a manner that maximises students' effort to show what they can do.

In administering the pre-test, the students should be told that the test is not related to grades, but is to find out what they know before the topic is taught. They should be told that they are not expected to know the work as they have not been taught it. They should show what they know and, if they cannot do a question, they should skip it, or put "not known" beside questions. They will be taught the work in the next few weeks and will then be able to show what they know. Stress to students that **any pre-test is a series of questions to find out what they know** before the knowledge is taught. They should do their best but the important questions come at the end of the module. For the **post-test**, the students should be told that **this is their opportunity to show how they have improved**.

For all tests, **teachers should continually check to see how the students are going.** Items in later subtests, or more difficult items within a particular subtest, should not be attempted if previous similar items in earlier subtests show strong weaknesses. Students should be allowed to skip that part of the test, or the test should be finished. Students can be marked zero for these parts.

Information on the change and function item types

The change and function item types are divided into five subtests, one for each of the five units in this module. Units 1, 2, 3 and 4 represent a sequence of change and function knowledge from early childhood through to Year 9. Unit 5 covers applications to multiplicative comparison but this is at a similar level to Unit 4 on backtracking and graphs. Thus the five units can be considered to be in sequence.

This means that the pre-test should focus on the items in the early subtests, moving up the subtests until students can no longer do the items. It also means that the post-test must cover all the later subtest items. However, it is important that sufficient content is included in the pre-test to ensure that: (a) teaching begins where students are at; (b) what is missed out is because the students cannot answer the questions; and (c) the pre-test provides both achievement level and diagnostic information. It is also important that sufficient content is included in the post-test to ensure that: (a) what is not included is because students can do this; (b) what is included will give the level of achievement at the end of the module; and (c) legitimate comparisons can be made between pre- and post-tests in terms of effect.

Subtest item types

Subtest 1 items (Unit 1: Very early change and function activities)

1. Circle the changes that <u>can</u> be reversed:

   flat tyre → inflated tyre

   apple → apple pie

   water → ice

   egg → omelette

   break up jigsaw → piece together jigsaw

2. Fill in the empty boxes in the table. The first one has been done for you.

	Input	Change	Output
	 potatoes	$\xrightarrow{\text{cook it}}$	 chips
(a)	 eggs	$\xrightarrow{\text{cook it}}$	
(b)		$\xrightarrow{\text{cook it}}$	stewed apricots
(c)	s	$\xrightarrow{\text{add “at”}}$	
(d)		$\xrightarrow{\text{add “at”}}$	flat

3. Fill in the empty boxes:

   (a) $7 \xrightarrow{double} \boxed{} \xrightarrow{+4} \boxed{}$

   (b) $\boxed{} \xrightarrow{double} \boxed{} \xrightarrow{+4} \boxed{26}$

Subtest 2 items (Unit 2: Early to middle change and function activities)

1. Fill in the empty boxes in the table. The first one has been done for you.

	Story	One-step function machine
	Story: *The sandwich was $3 more than the drink. How much was the sandwich?*	input → $\boxed{+3}$ output →
(a)	Story: *The price was reduced by $2. What was the new price?*	input → $\boxed{}$ output →
(b)	Story: *The coat cost twice as much as the pants. How much did the coat cost?*	input → $\boxed{}$ output →
(c)	Story:	input → $\boxed{+7}$ output →
(d)	Story:	input → $\boxed{\div 11}$ output →

2. Complete the function machine, the Input–Output table, and the arrowmath for this story:
Each taxi had 5 people in it. How many people altogether?

(a) Function machine: input → $\boxed{}$ output →

(b) Input–Output table:

Input	Output	Input	Output
3		16	
11			40
6			420
	35	600	

(c) Arrowmath:

Forward $\boxed{8} \xrightarrow{\boxed{}} \boxed{}$ Inverse $\boxed{} \xleftarrow{\boxed{}} \boxed{40}$

3. Complete the function machine, the Input–Output table, and the arrowmath for this story.

Each person paid $3 to get inside. How much money do they each have left?

(a) Function machine: $\xrightarrow{\text{input}} \boxed{} \xrightarrow{\text{output}}$

(b) Input–Output table:

Input	Output	Input	Output
4			14
12			64
34			187

(c) Arrowmath:

Forward $\boxed{13} \xrightarrow{\boxed{}} \boxed{}$ Inverse $\boxed{} \xleftarrow{\boxed{}} \boxed{10}$

Subtest 3 items (Unit 3: Later change and function activities)

1. Draw the double function machine, complete the table, and give the arrowmath for this story:

   I bought the 5 children the same meal each, and a $6 meal for me. How much did I spend?

   (a) Function machine: $\longrightarrow \boxed{} \longrightarrow \boxed{} \longrightarrow$

   (b) Input–Middle–Output table:

Input	Middle	Output	Input	Middle	Output
3					31
7					46
12			n		
	20				k

   (c) Arrowmath:

   Forward $\boxed{6} \xrightarrow{\boxed{}} \xrightarrow{\boxed{}} \boxed{}$ Inverse $\boxed{} \xleftarrow{\boxed{}} \xleftarrow{\boxed{}} \boxed{36}$

2. Fill in the empty boxes in the table. The first one has been done for you.

	Story	Forward arrowmath	Backward arrowmath	Solution
	I bought 3 pies and a $4 drink. What was the cost of each pie if I spent $19?	cost of each pie $\xrightarrow{\times 3} \xrightarrow{+4}$ 19	cost of each pie $\xleftarrow{\div 3} \xleftarrow{-4}$ 19	cost of each pie = $5
(a)	4 children paid to enter. They got a $9 discount altogether. What was the cost per child if, after the discount, it cost $15 altogether?			
(b)		Tom's money $\xrightarrow{\div 2} \xrightarrow{+5}$ 23		
(c)			Sue's money $\xleftarrow{+6} \xleftarrow{\times 2}$ 11	
(d)				bus fare = $8

Subtest 4 items (Unit 4: Application to linear equations)

1. Fill in the empty boxes in the table. The first one has been done for you.

	Forward arrowmath	Forward equation	Backward arrowmath	Backward equation
	$3 \xrightarrow{\times 4} \xrightarrow{-2} 10$	$3 \times 4 - 2 = 10$	$3 \xleftarrow{\div 4} \xleftarrow{+2} 10$	$(10 + 2) \div 4 = 3$
(a)	$4 \xrightarrow{\times 7} \xrightarrow{+5} 33$			
(b)		$\frac{12}{4} + 1 = 4$		
(c)			$36 \xleftarrow{+6} \xleftarrow{\times 2} 15$	
(d)				$\frac{(18-3)}{5} = 3$

2. Complete the following through to a graph and function.

Story: *Fishermen all caught 4 fish each. They gave 3 away. How many fishermen were there?*

(a) Function machine: $\longrightarrow \boxed{} \longrightarrow \boxed{} \longrightarrow$

(b) Input–Middle–Output table:

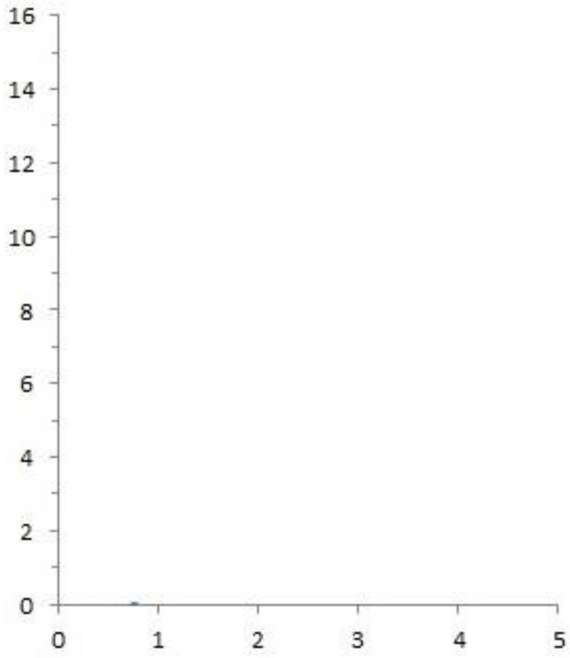
Input	Middle	Output	Input	Middle	Output
3					25
8			n		
	16				k

(c) Arrowmath and equation:

Forward $\boxed{9} \xrightarrow{\boxed{}} \xrightarrow{\boxed{}} \boxed{}$ Inverse $\boxed{} \xleftarrow{\boxed{}} \xleftarrow{\boxed{}} \boxed{33}$

Equation $\boxed{}$ $\boxed{}$

(d) Fill in the Input–Middle–Output table and use the table points to draw a graph.



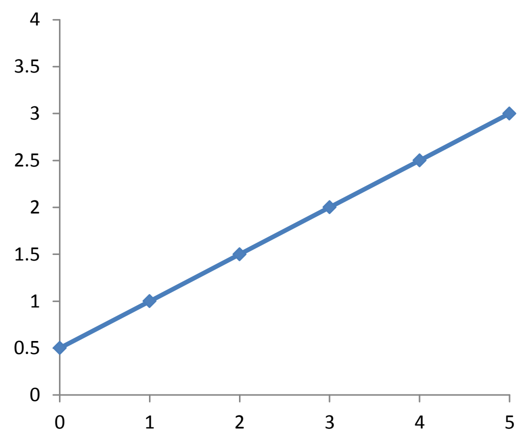
Input	Middle	Output
1	4	1
2	8	
3		
4		

(e) Give the equation of the function for this graph: $f(x) =$ ____________

3. Using the graph shown on the right,

(a) Write the function for the graph:

$f(x) =$ ____________



(b) Fill in the forward and inverse arrowmath and equation:

Forward $\boxed{} \xrightarrow{\boxed{}} \xrightarrow{\boxed{}} \boxed{}$

Inverse $\boxed{} \xleftarrow{\boxed{}} \xleftarrow{\boxed{}} \boxed{}$

Equation $\boxed{}$ $\boxed{}$

(c) Complete an Input–Middle–Output table of values:

Input	**Middle**	**Output**
1		
2		
3		
4		

(d) Complete the function machine: $\longrightarrow \boxed{} \longrightarrow \boxed{} \longrightarrow$

(e) Write a story to fit the equation:

Subtest 5 items (Unit 5: Applications to multiplicative comparison)

1. Relate the problems to change by filling in the empty space in each row of the table (some have been done for you). **Do not calculate answers**.

	Problem	Change
Done for you	What is 67% of $95?	$\$95 \xrightarrow{\times 0.67} \$\ ?$
(a)	What is 25% of $80?	
(b)	Profit of 16% means selling $800 painting for how much?	
(c)	Discount of 25%, sale price is $96, what is original price?	
Done for you	Sand to cement is 7:2, how much sand for 10 tonnes of cement?	$?\ \text{sand} \xrightarrow{\times ^2/_7} 10\ \text{t cement}$
(d)	Cordial to water is 2:5, how much water for 20 L of cordial?	
(e)	Chemical to water is 25 mL : 1000 mL, how much chemical for 10,000 mL of water?	
Done for you	Air speed was 74 km/hr, how many km in 12 hrs?	$12 \xrightarrow{\times 74} ?\ \text{km}$
(f)	Petrol is $1.60/L, how many L for $70?	
(g)	Running at 12 km/hr, Jane went how many km in 3 hours?	

2. Relate the problems to change to get the answers. You may have to backtrack. The first two are done for you.

	Problem	Change	Backtrack	Answer
	Find 63% of $240.	$\$240 \xrightarrow{\times 0.63} ?$		$\$240 \times 0.63$
	Find amount when 63% is $240.	$\$? \xrightarrow{\times 0.63} \$240$	$\$? \xleftarrow{\div 0.63} \$240$	$\frac{\$240}{0.63}$
(a)	A painting was sold for $360 which was a profit of 20%. How much was the painting bought for?			
(b)	The ratio of sand:cement is 7:3. How much cement for 49 tonnes of sand?			
(c)	Potatoes cost $4.80/kg. How many kg of potatoes could you buy for $60?			

Appendix A: Change Models and Percent, Rate and Ratio Problem Types

Percent problem types

The three types of percent problems are:

Type 1: Percentage amount of 100% unknown (e.g. find 45% of \$675)

Type 2: Total (100%) unknown (e.g. 45% is \$675, how much is the total?)

Type 3: Percent unknown (e.g. what % is the \$675 of \$1500?)

The aim is for students to be able to go from problem to answer on their own. Thus, they need to be able to do two things **without the assistance** of the teacher:

(a) take a percent problem and turn it into a change diagram with numbers and unknown correctly placed on diagram; and

(b) use the diagram to solve the problem.

Therefore in this appendix, we will look at the steps for each of these things, once in detail and then in summary form for each of the three problem types.

In change model we have start, finish and multiplier, where start is always the 100%.

Problem Type 1:	***Find 45% of \$675***	
Step A1:	Draw the outline of the change model	$\text{START} \xrightarrow{\text{x Multiplier}} \text{FINISH}$
Step A2:	Place in the 100% and the multiplier, here 45% or ×0.45	$100\% \xrightarrow{\text{x } 0.45} ?$
Step A3:	Put in amounts and unknowns	$100\%\ \$675 \xrightarrow{\text{x } 0.45} ?$
Step B1:	Determine whether ? is placed for × or ÷ (here ×)	$\$675 \xrightarrow{\text{x } 0.45} ?$
Step B2:	Calculate ?	
Step B3:	Give answer – the answer for what is 45% of \$675 = \$303.75	$\$675 \xrightarrow{0.45} 675\times0.45 = 303.75$
Problem Type 2:	***45% is \$675, what is the total?***	
Steps A1–A3:	Draw and fill in information on diagram	$? \xrightarrow{\times 0.45} 675$
Steps B1–B3:	Calculate answer (work out whether this is × or ÷)	
	The total is \$675 ÷ 0.45 = \$1500	
Problem Type 3:	***What % is \$675 of \$1500?***	
Steps A1–A3:	Draw and fill in information on diagram	$1500 \xrightarrow{\times ?} 675$
Steps B1–B3:	Calculate answer (work out whether this is × or ÷)	
	The % is 675 ÷ 1500 = 0.45 = 45%	

Rate problem types

Here we repeat the previous percent section for rate applications, but only using summaries. The three problem types are:

Type 1: First attribute unknown (e.g. petrol is \$1.64/L, how much in \$ for 52L?)

Type 2: Second attribute unknown (e.g. petrol is \$1.64/L, how many L for \$52?)

Type 3: Rate is unknown (e.g. what is the rate when paying \$85.28 for 52L?)

There are still the two steps, A and B, to teach.

As for all rates, the starting number is always the second attribute (the "per" attribute) and the finishing number is the first attribute. For all problems below, the rate is \$1.64/L (the price of petrol).

Problem Type 1: ***How much for 52L?***

Steps A1–A3: Draw and fill in information on diagram

Steps B1–B3: Calculate ? (work out whether × or ÷)

The cost of 52L is 52 × 1.64 = \$85.28

$$\mathrm{L}\ (52) \xrightarrow{\times 1.64} \$\ (?)$$

Problem Type 2: ***How many L for \$52?***

Steps A1–A3: Draw and fill in information on diagram

Steps B1–B3: Calculate ? by inversing ×1.64

The litres for \$52 is 52 ÷ 1.64 = 31.7L

$$\mathrm{L}\ (?) \underset{\div 1.64}{\overset{\times 1.64}{\rightleftarrows}} \$\ (52)$$

Problem Type 3: ***Rate if pay \$85.28 for 52L?***

Steps A1–A3: Draw and fill in information on diagram

Steps B1–B3: Calculate ? by division

The rate is 85.28 ÷ 52 = \$1.64/L

$$\mathrm{L}\ (52) \xrightarrow{\times ?} \$\ (85.28)$$

Ratio problem types

Here we again repeat the process from the previous two sections but for ratio (but again using only summaries). The three problem types are:

Type 1: The second amount is unknown (e.g. red and yellow colouring are mixed in ratio 4:7, how many mL of yellow for 280 mL of red?)

Type 2: The first amount is unknown (e.g. red and yellow colouring are mixed in a ratio 4:7, how many mL of red for 280 mL of yellow?)

Type 3: The ratio is unknown (e.g. red and yellow colouring are mixed, 280 mL of red and 490 mL of yellow, what is the ratio?)

There are still two parts (A and B) that have to be taught, that is A to draw the diagram, and B to solve it.

For ratio, the starting number for change is the first number or attribute and the finishing number is the second number or attribute. The multiplier is the fraction, second number ÷ first number. The ratio for the first two problem types is 4 red: 7 yellow.

Problem Type 1: How much yellow for 280 mL of red?

Steps A1–A3: Draw and fill in information on diagram

$$\begin{matrix}\text{Red}\\ \text{280mL}\end{matrix} \xrightarrow{\times 7/4} \begin{matrix}\text{Yellow}\\ \text{? mL}\end{matrix}$$

Steps B1–B3: Calculate ? by realising whether × or ÷

Answer: amount of yellow is 280 × 7 ÷ 4 = 490 mL

Problem Type 2: How much red for 280 mL of yellow?

Steps A1–A3: Draw and fill in information on diagram

$$\begin{matrix}\text{Red}\\ \text{? mL}\end{matrix} \xrightarrow{\times 7/4} \begin{matrix}\text{Yellow}\\ \text{280mL}\end{matrix}$$

Steps B1–B3: Calculate ? by realising this is ÷ as reversing the arrow

Answer: amount of red is $280 \div \frac{7}{4} = 280 \times \frac{4}{7}$ = 160 mL

Problem Type 3: What is ratio for 280 mL of red and 490 mL of yellow?

Steps A1–A3: Draw and fill in information on diagram

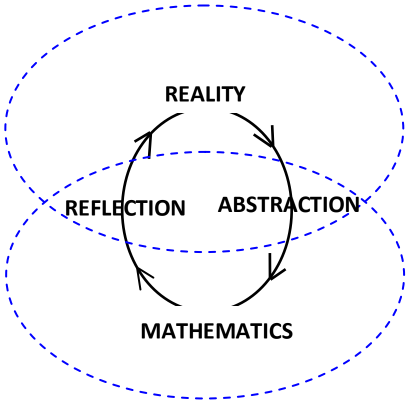
$$\begin{matrix}\text{Red}\\ \text{280mL}\end{matrix} \xrightarrow{\times\, ?/??} \begin{matrix}\text{Yellow}\\ \text{490mL}\end{matrix}$$

Steps B1–B3: Calculate ? by realising this is cancelling down situation for $\frac{490}{280} = \frac{7}{4}$

Answer: red and yellow are in ratio 4:7

Appendix B: RAMR Cycle

AIM advocates using the four components in the figure on right, reality–abstraction–mathematics–reflection (RAMR), as a cycle for planning and teaching mathematics. RAMR proposes: (a) working from reality and local culture (prior experience and everyday kinaesthetic activities); (b) abstracting mathematics ideas from everyday instances to mathematical forms through an active pedagogy (kinaesthetic, physical, virtual, pictorial, language and symbolic representations, i.e. body → hand → mind); (c) consolidating the new ideas as mathematics through practice and connections; and (d) reflecting these ideas back to reality through a focus on applications, problem solving, flexibility, reversing and generalising (see figure on right).



The innovative aspect of RAMR is that the right half develops the mathematics idea while the left half reconnects it to the world and extends it. For example, whole-number place value built around the **pattern of threes** where hundreds-tens-ones applies to ones, thousands, millions, and so on, can be easily extended to metrics by considering the ones to be millimetres, the thousands to be metres and the millions to be kilometres.

Planning the teaching of mathematics is based around the RAMR cycle if it is deconstructed into components that are applied to a mathematical idea. By breaking instruction down into the four parts and taking account of the pedagogical approaches described above, the cycle can lead to a structured instructional sequence for teaching the idea. The table below briefly outlines how this can be done. Prerequisite mathematical ideas are considered in the Reality and Mathematics components of the cycle, while extensions and follow-up ideas are considered in the Reflection component.

REALITY

- **Local knowledge**: Identify local student cultural-environmental knowledge and interests that can be used to introduce the idea.
- **Prior experience**: Ensure existing knowledge and experience prerequisite to the idea is known.
- **Kinaesthetic**: Construct kinaesthetic activities, based on local context, that introduce the idea.

ABSTRACTION

- **Representation**: Develop a sequence of representational activities (physical to virtual to pictorial materials to language to symbols) that develop meaning for the mathematical idea.
- **Body-hand-mind**: Develop two-way connections between reality, representational activities, and mental models through body → hand → mind activities.
- **Creativity**: Allow opportunities to create own representations, including language and symbols.

MATHEMATICS

- **Language/symbols**: Enable students to appropriate and understand the formal language and symbols for the mathematical idea.
- **Practice**: Facilitate students' practice to become familiar with all aspects of the idea.
- **Connections**: Construct activities to connect the idea to other mathematical ideas.

REFLECTION

- **Validation**: Facilitate reflection of the new idea in terms of reality to enable students to validate and justify their new knowledge.
- **Applications/problems**: Set problems that apply the idea back to reality.
- **Extension**: Organise activities so that students can extend the idea (use reflective strategies – *flexibility*, *reversing*, *generalising*, and *changing parameters*).

Appendix C: AIM Scope and Sequence

Yr	Term 1	Term 2	Term 3	Term 4
A	**N1: Whole Number Numeration** Early grouping, big ideas for H-T-O; pattern of threes; extension to large numbers and number system **N2: Decimal Number Numeration** Fraction to decimal; whole number to decimal; big ideas for decimals; tenths, hundredths and thousandths; extension to decimal number system	**O1: Addition and Subtraction for Whole Numbers** Concepts; strategies; basic facts; computation; problem solving; extension to algebra **M1: Basic Measurement (Length, Mass and Capacity)** Attribute; direct and indirect comparison; non-standard units; standard units; applications	**O2: Multiplication and Division for Whole Numbers** Concepts; strategies; basic facts; computation; problem solving; extension to algebra **M2: Relationship Measurement (Perimeter, Area and Volume)** Attribute; direct and indirect comparison; non-standard units; standard units; applications and formulae	**G1: Shape (3D, 2D, Line and Angle)** 3D and 2D shapes; lines, angles, diagonals, rigidity and properties; Pythagoras; teaching approaches **SP1: Tables and Graphs** Different tables and charts; bar, line, circle, stem and leaf, and scatter graphs; use and construction
B	**M3: Extension Measurement (Time, Money, Angle and Temperature)** Attribute; direct and indirect comparison; non-standard units; standard units; applications and formulae **N3: Common Fractions** Concepts and models of common fractions; mixed numbers; equivalent fractions; relationship to percent, ratio and probability	**G2: Euclidean Transformations (Flips, Slides and Turns)** Line-rotation symmetry; flip-slides-turns; tessellations; dissections; congruence; properties and relationships **O3: Common and Decimal Fraction Operations** Addition, subtraction, multiplication and division of common and decimal fractions; models, concepts and computation	**A1: Equivalence and Equations** Definition of equals; equivalence principles; equations; balance rule; solutions for unknowns; changing subject **N4: Percent, Rate and Ratio** Concepts and models for percent, rate and ratio; proportion; applications, models and problems	**SP2: Probability** Definition and language; listing outcomes; likely outcomes; desired outcomes; calculating (fractions); experiments; relation to inference **G3: Coordinates and Graphing** Polar and Cartesian coordinates; line graphs; slope and y-intercept; distance and midpoints; graphical solutions; nonlinear graphs
C	**A2: Patterns and Linear Relationships** Repeating and growing patterns; position rules, visual and table methods; application to linear and nonlinear relations and graphs **N5: Directed Number, Indices and Systems** Concept and operations for negative numbers; concept, patterns and operations for indices; scientific notation and number systems	**A3: Change and Functions** Function machine; input-output tables; arrowmath notation, inverse and backtracking; solutions for unknowns; model for applications to percent, rate and ratio **G4: Projective and Topology** Visualisation; divergent and affine projections; perspective; similarity and trigonometry; topology and networks	**O4: Arithmetic and Algebra Principles** Number-size, field and equivalence principles for arithmetic; application to estimation; extension to algebra; simplification, expansion and factorisation **SP3: Statistical Inference** Gathering and analysing data; mean, mode, median, range and deviation; box and whisker graphs; large data sets, investigations and inferences	**A4: Algebraic Computation** Arithmetic to algebra computation; modelling-solving for unknowns; simultaneous equations, quadratics **O5: Financial Mathematics** Applications of percent, rate and ratio to money; simple and compound interest; best buys; budgeting and planning activities

Key: N = Number; O = Operations; M = Measurement; G = Geometry; SP = Statistics and Probability; A = Algebra.

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