Accelerated Inclusive Mathematics Project

YuMiDeadly
Growing community through education

YuMi Deadly Maths

AIM Module N5

Year C, Term 1

Number:

Directed Number, Indices and Systems

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ACKNOWLEDGEMENT

The YuMi Deadly Centre acknowledges the traditional owners and custodians of the lands in which the mathematics ideas for this resource were developed, refined and presented in professional development sessions.

YUMI DEADLY CENTRE

The YuMi Deadly Centre is a research centre within the Faculty of Education at the Queensland University of Technology which is dedicated to enhancing the learning of Indigenous and non-Indigenous children, young people and adults to improve their opportunities for further education, training and employment, and to equip them for lifelong learning.

"YuMi" is a Torres Strait Islander Creole word meaning "you and me" but is used here with permission from the Torres Strait Islanders' Regional Education Council to mean working together as a community for the betterment of education for all. "Deadly" is an Aboriginal word used widely across Australia to mean smart in terms of being the best one can be in learning and life.

YuMi Deadly Centre's motif was developed by Blacklines to depict learning, empowerment, and growth within country/community. The three key elements are the individual (represented by the inner seed), the community (represented by the leaf), and the journey/pathway of learning (represented by the curved line which winds around and up through the leaf). As such, the motif illustrates the YuMi Deadly Centre's vision: *Growing community through education*.

The YuMi Deadly Centre (YDC) can be contacted at ydc@qut.edu.au. Its website is http://ydc.qut.edu.au.

DEVELOPMENT OF THE AIM MODULES

The Accelerated Inclusive Mathematics (AIM) modules were originally developed from 2010 to 2013 as part of the Accelerated Indigenous Mathematics project funded under the Commonwealth Government's *Closing the Gap: Expansion of Intensive Literacy and Numeracy* program for Indigenous students. The project aimed to assist secondary schools with beginning junior-secondary Indigenous students who were at Year 2/3 level in mathematics by developing a junior-secondary mathematics program that accelerates the students' learning to enable access to mathematics subjects in the senior-secondary years and therefore enhance employment and life chances. The project developed three years of modules (Years A to C) that are vertical sequences of learning to take students from their ability level to their age level in mathematics. The YuMi Deadly Centre acknowledges the role of the Department of Education, Employment and Workplace Relations in the development of the AIM project and these modules.

Contents

Module Overview

This module, Module N5 *Directed Number, Indices and Systems*, is the final in the Number modules. It follows on from modules on whole numbers, decimal numbers, common fractions, percent, rate and ratio (see **Appendix B**). It looks at different mathematics topics in number that lead to mathematics and mathematical notation, with a major role in higher levels of science and algebra, namely:

(a) **Number types** – different forms of number based on shapes, patterns, and operations;

(b) **Directed number** – positive and negative numbers, how they can be modelled and the rules by which they are added, subtracted, multiplied and divided;

(c) **Indices and scientific notation** – numbers that are represented as multiples of same numbers using index notation and the rules by which they are multiplied, divided and raised to higher powers, plus the application of the index notation to powers of 10 to enable very large numbers and very small numbers to be represented; and

(d) **Irrationals and rationals, and number systems** – exploring irrational and rational numbers and describing number system structure for whole and real numbers.

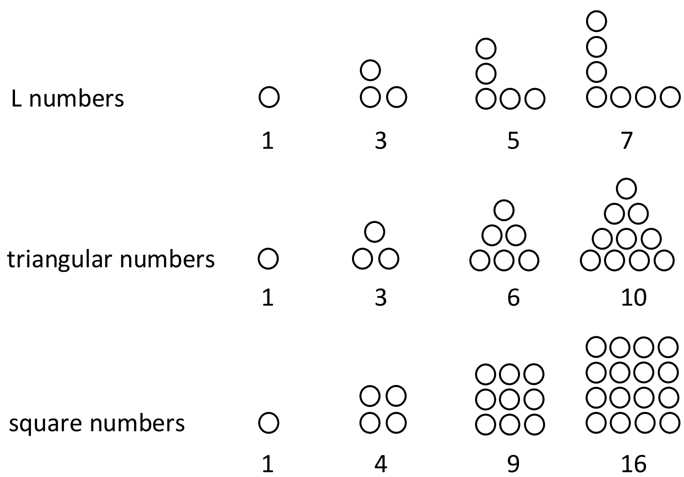
Background information for teaching directed number, indices and systems

This section covers the definitions and properties of the types of numbers in this module, plus discussion of connections and big ideas.

Definitions and properties for components of this module

This module covers mathematics for which there is: (a) a rich history and for which many opportunities exist for exploration; (b) a focus on patterns and the rules by which they continue; and (c) a focus on notation to enable very large or very small numbers to be represented. The major components are as follows.

1. **Shape numbers**. This covers numbers formed from the shapes that can be made from examples of them. Particular examples are those below – L numbers (all odd), triangular numbers (where the next number is the sum of all previous numbers – e.g. 1, 3, 6, 10, 15), and square numbers.



2. **Magic numbers**. This covers arrays/arrangements of counting numbers which add to the same answer along rows, columns and diagonals. A particular type is magic squares. *Example:* the 3 × 3 magic square on right adds to 15 along all rows, columns and diagonals.

8	1	6
3	5	7
4	9	2

3. **Factors**. Study of the factors of numbers (the numbers that divide into the given number) leads to primes and composites; abundant, perfect and deficient numbers; and so on.

   Examples:

   29 is a prime – no factors except itself and 29

   27 is a composite – factors are 1, 3, 9, and 27

   12 is abundant – factors other than 12 are 1, 2, 3, 4, 6 – add to 16 > 12

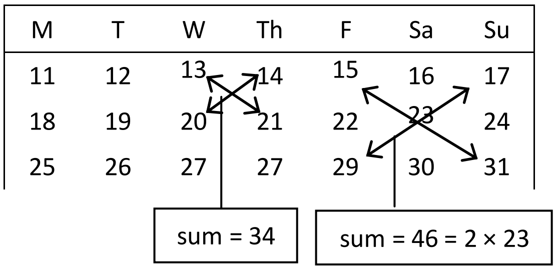
   6 is perfect – factors other than 6 are 1, 2, 3 – add to 6

   8 is deficient – factors other than 8 are 1, 2, 4 – add to 7 < 8

4. **Patterns**. Strong number patterns exist in many forms of number displays that have a pattern structure.

   Examples:

   Calendars – anywhere on a calendar, diagonals always add to the same amount.

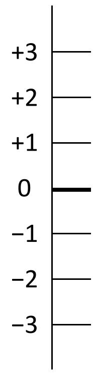


   Pascal's triangle – a triangle of numbers

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

   Fibonacci sequence – next number is sum of the previous two numbers: 1 1 2 3 5 8 13 and so on

5. **Directed numbers**. Positives (e.g. +6) and negatives (e.g. −7) best modelled by vertical line.



   (a) (+a) + (+b) is addition

   (b) (+a) + (−b) and (+a) – (+b) are both subtraction

   (c) (+a) – (−b) = (+a) + (+b) is addition

   (d) (+a) × (+b) = +ab

   (e) (+a) × (−b) = (−a) × (+b) = −ab

   (f) (−a) × (−)b = +ab

6. **Indices**. When the same number is multiplied many times, index notation shows this by a small superscript; negative superscripts are division.

   Examples:

   (a) $6 \times 6 \times 6 = 6^3$ (b) $7 \times 7 \times 7 \times 7 \times 7 = 7^5$ (c) $6^0 = 1$

   (d) $\frac{1}{6} = 6^{-1}$ (e) $\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{6^3} = 6^{-3}$

   Rules:

   (a) $6^a \times 6^b = 6^{a+b}$ regardless of what a, b are and what number 6 is

   (b) $(6^a)^b = 6^{a \times b}$ regardless of a, b, and 6

7. **Scientific notation**. Numbers are written as powers of 10.

   Examples:

   $1 = 10^0$, $10 = 10^1$, $100 = 10^2$, $1000 = 10^3$, ... , $1\,000\,000 = 10^6$, ... and so on

   $\frac{1}{10} = 10^{-1}$, $\frac{1}{100} = 10^{-2}$, ... , $\frac{1}{1\,000\,000} = 10^{-6}$

   Can be used to represent numbers, e.g.

   $6\,748\,000\,000 = 6.748 \times 10^9$ $\qquad$ $0.000\,000\,049 = 4.9 \times 10^{-8}$

8. **Rationals, irrationals and reals**. Reals are all numbers, rationals are reals or decimals that are common fractions, while irrationals are decimals that are not fractions.

   Examples:

   (a) Rationals are fractions and decimals that repeat or have fixed length, for example:

   $2.375 = 2\frac{3}{8} = \frac{19}{8}$ $\qquad$ $0.333333 \text{ and so on} = \frac{1}{3}$

   Rationals can be counted.

   (b) Irrationals are decimals that continue forever with no pattern (such as $\sqrt{2}$ or π). They cannot be counted.

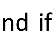
Connections and big ideas

In terms of **connections**, the activities that make up number types, directed numbers and index notation are either (a) at the point of connection between number and other strands, e.g. number and shape, number and operations (factors); or (b) a result of connection between number and operations (e.g. numbers and multiplication/division lead to indices being conceived).

This results in patterns/rules and notation that are: (a) the end product of mathematics study (e.g. scientific notation, the real number system, irrationals); (b) from the earliest thinking of mathematicians (e.g. triangular numbers, perfect numbers) or both (e.g. rationals and irrationals).

Many of the ideas in the module relate to multiplicative structure in one of two ways: (a) as the results of multiplication and division of numbers (prime numbers, indices); and (b) as the results of the multiplicative relationship in place value (e.g. the real number system).

In terms of **big ideas**, the topics and activities in this module represent some powerful ideas as follows.

1. **Shape numbers** represent one of the first integrations of two strands (number and geometry). This integration has been very important across the life of mathematics – great progress was made in mathematics once algebra and geometry were joined to bring number/function and shape together. This resulted in the circle becoming both shape,  , and equation/graph, $x^2 + y^2 = r^2$.

2. **Irrationals/rationals** grew out of the Pythagoreans' desire to show every number was a common fraction. However, when they analysed $\sqrt{2}$ they found if they let $\sqrt{2} = \frac{p}{q}$, a common fraction in cancelled down form, then $2 = \frac{\mathrm{p}^2}{\mathrm{q}^2}$. This meant that $\frac{\mathrm{p}^2}{\mathrm{q}^2}$ is even, which is only possible if p is even and q is even which means they are not in cancelled down form. This is a contradiction, so $\sqrt{2}$ is not a common fraction. This argument is an early version of the big idea of proof by **contradiction**.

3. **Magic numbers** were important in the development of computers. Computers find solving magic squares difficult and slow to do because they have to check all possibilities. Humans "prune" the tree diagram of options using their knowledge/intuition (e.g. 5 is middle of 1 to 9 so 5 is in middle of magic square) and so

do them faster and easier than computers. This allowed computer scientists to realise that the use of knowledge and "sloppiness of intuition" were strengths in human thinking.

4. **Most of the activities** reflect the big ideas of number and operations. For example:

   (a) magic numbers reflect addition and subtraction big ideas like compensation (e.g. if a row has a 1, it must also have a large number);

   (b) patterns and factors reflect all operations big ideas as one uses operations to solve them;

   (c) directed numbers and indices reflect all the multiplication and division big ideas of inverse and inverse of inverse; and

   (d) scientific notation and the real number system reflect the numeration big ideas of wholes and decimals.

Sequencing for directed number, indices and systems

This section briefly looks at the role of sequencing in acceleration of learning and its importance in this project, then looks at the role of sequencing in this particular module.

Sequencing of components

In terms of **sequencing**, their study in school should represent the period in history from where they were discovered. This means the following order:

Shape numbers
Magic numbers
Factors
Patterns
Rationals/Irrationals
Directed numbers
Indices
Scientific notation and number systems

Interestingly, although rationals and irrationals relate back to Pythagoras, their study in terms of what infinity means is very modern (e.g. rationals are countably infinite while irrationals are uncountably infinite – and therefore more infinite).

Sequencing in this module

This module consists of the following sections:

Overview: Background information, sequencing, and relation to Australian Curriculum

Unit 1: Number types and patterns – shape numbers, magic numbers, factors and patterns (e.g. triangular numbers, magic squares, perfect numbers, Pascal's triangle, Fibonacci sequence, and so on)

Unit 2: Directed number – positive and negative numbers and calculation properties

Unit 3: Indices and scientific notation – specialised notation for representing multiples of the same number, extensions to negative and fractional indices, scientific notation where that number is 10 and calculation properties for all of these

Unit 4: Rationals, irrationals and number systems – difference between rationals and irrationals, application to infinity and setting up the real number system

Test item types: Test items associated with the four units above which can be used for pre- and post-tests

Appendix A: RAMR cycle components and description

Appendix B: AIM scope and sequence showing all modules by year level and term.

Although stages of the RAMR cycle are sometimes alluded to, this module gives teaching information as lists of activities and not as full RAMR cycles. The units outline ideas. As well, the patterning and puzzle basis of many of the ideas in this module make reality difficult. However, teachers should wherever possible translate their teaching into the four stages of the cycle.

Because much of the content of early units is intrinsically interesting and enjoyable as puzzles, there also may be a tendency not to spend time on the later units. However, in terms of usefulness for careers with high stakes (e.g. engineering), the later units with their focus on exploring notations for large numbers and systems are important.

Relation to Australian Curriculum: Mathematics

AIM N5 meets the Australian Curriculum: Mathematics (Foundation to Year 10)					
Unit 1: Number types and patterns Unit 2: Directed number	Unit 3: Indices and scientific notation Unit 4: Rationals, irrationals and number systems				
Content Description	**Year**	**N5 Unit**			
		1	**2**	**3**	**4**
Investigate index notation and represent whole numbers as products of powers of prime numbers (ACMNA149)	**7**			✓	
Compare, order, add and subtract integers (ACMNA280)			✓		
Use index notation with numbers to establish the index laws with positive integral indices and the zero index (ACMNA182)	**8**			✓	
Carry out the four operations with rational numbers and integers, using efficient mental and written strategies and appropriate digital technologies (ACMNA183)			✓		
Investigate the concept of irrational numbers, including π (ACMNA186)					✓
Apply index laws to numerical expressions with integer indices (ACMNA209)	**9 to 10**			✓	
Express numbers in scientific notation (ACMNA210)				✓	
Extend and apply the index laws to variables, using positive integer indices and the zero index (ACMNA212)				✓	
Define rational and irrational numbers and perform operations with surds and fractional indices (ACMNA264)	**10**				✓

Unit 1: Number Types and Patterns

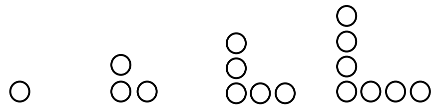
This unit consists of activity with the first four areas of this module: shape number, magic number, factors and patterns.

1.1 Shape numbers

One of the earliest explorations of numbers was with respect to shape. We will now explore some examples.

1. **Letter-shaped numbers**

   (a) Make these L shaped numbers with counters:

   (b) Make the next three in the sequence. Write down the number of each L [1. 3. 5. 7. 9, 11, 13]. What numbers are being made? [odd numbers]

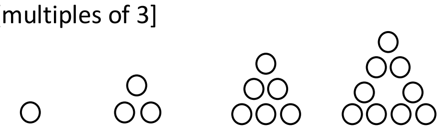
   (c) What characteristic of the L shape ensures that these numbers are being made? [two same length lines with a common point] Are there other shapes that make the same numbers? [Yes – T, X, V, etc.]

   (d) What sort of numbers form N's or W's? [N – multiples of three; W multiples of 4]

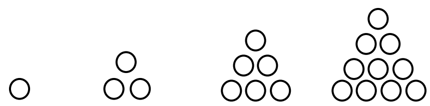
   (e) Make up your own numbers (you may have to have a rule about lengths of sides)!

2. **Shape numbers**

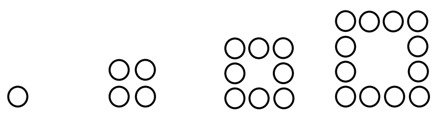
   (a) Make these triangle-shaped numbers with counters –and construct two more [1, 3, 6, 9, 12, 15]. What numbers do they make? [multiples of 3]

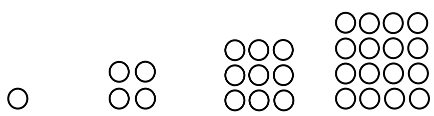
   (b) What is the difference if we fill in the triangle as below? These are called triangular numbers. What are the triangular numbers now? [1, 3, 6, 10, 15, 21, 28] What is the pattern for working out the next one? [next number is previous number plus number of shape]. Can you write the numbers as a sum of other numbers? [Yes – 4th term is 1+2+3+4; 5th term is 1+2+3+4+5]

   (c) What about square-shaped numbers?

   (d) What about filled in square-shaped numbers (these are called square numbers)?

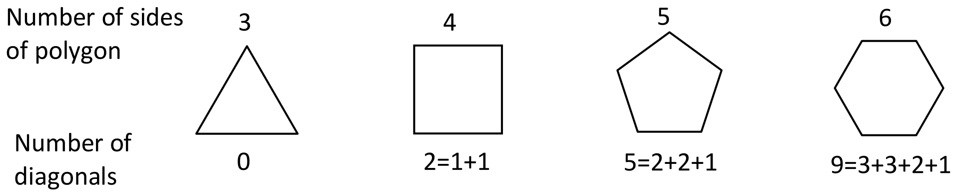
   

   (e) What about hexagonal-shaped numbers?

3. **Problems**

These shape numbers are important in many problems. Here is one: What is the pattern for the number of diagonals in a polygon – a straight line sided shape?

(a) Draw the four shapes below. Put in all the diagonals. Are the numbers below correct? Why is the addition given? [The diagonals are best done a corner at a time, the first two corners always have diagonals from them 3 less than the number of sides, and the other corners reduce by 1 as move to them.]



(b) Fill in a table with headings: *Number of sides*, *Drawing*, *Number of diagonals*. What is the number of diagonals for a 10-sided shape? 100-sided shape? [10-sided = 7+7+6+5+4+3+2+1, 100-sided = 97+97+96+95+ ... +3+2+1]

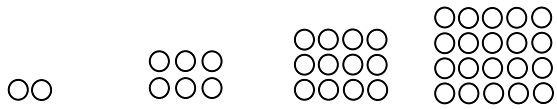
(c) How does the pattern relate to triangular numbers? [10-sided shape's diagonals are 7 + 7th triangular number.]

(d) Find other problems/puzzles where the solution is triangular numbers.

4. **Relationships**

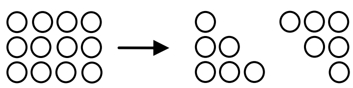
Shape numbers give rise to relationships. For example:

(a) Look as this type of rectangle.



(b) What is number for the 10th rectangle; for any rectangle, n? [10th = 11×10; n^{th} = (n+1)×n = $n(n+1)$]

(c) Divide the 3rd rectangle to form two triangles. Can this be done for all rectangles? What do you get? [Two triangular numbers = one rectangular number] What does this mean for the triangular number? [A triangular number is half the rectangular number]



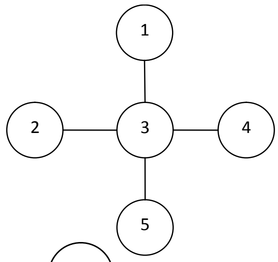
(d) Can you use this relationship to find the value of the 3rd triangular number? The 10th triangular number? The 100th triangular number? [3rd triangular number = 4×3/2 = 6; 10th = 11×10/2 = 55; 100th = 101×100/2 = 5050]

(e) Can you use this to find the formula for 1+2+3+...+n? [1+2+3+ ... +n = $\frac{n(n+1)}{2}$]

1.2 Magic numbers

This section has a large history – the first recorded instance of a magic square is about 2000 BC in China on an engraving of a turtle. Some activities:

1. **Magic lines**

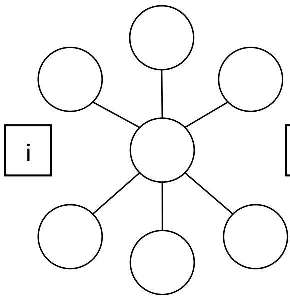


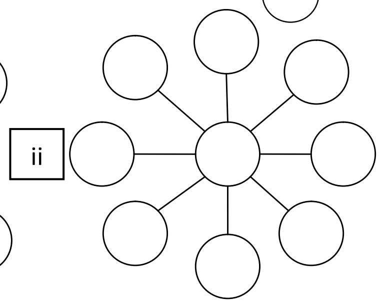
   (a) The numbers 1 to 5 can be placed in the shape on right so all lines add to 9.

   (b) Place the numbers 1 to 7 in (i) so that all lines add to the same number.

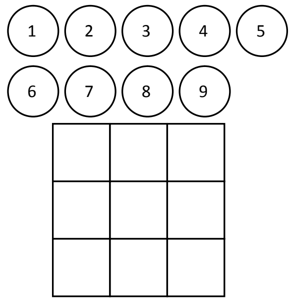
   (c) Place the numbers 1 to 9 in (ii) so that all lines add to the same number.

   (d) Make up your own magic lines problem.





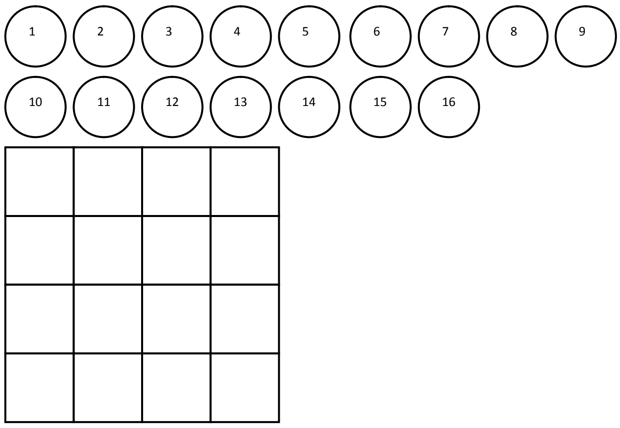
2. **Magic squares**



   (a) On right is a 3×3 magic square. Place all the number in the square so all rows, columns and diagonals add to the same number (15) – it helps to make a larger copy of the square and copies of the circular numbers and move them around.

   (b) Do the same to the 4×4 magic square.

   (c) Look up "magic squares" on the Internet and see what else there is to do.



1.3 Factors

This is a major area of mathematics that goes on to the big topic in higher mathematics of number theory. Numbers are studied in relation to their factors. This was one of the earliest areas of maths. Some examples are below – look up the Internet for more examples.

1. **Primes and Composites**

   An interesting study is the factors of a number: 7 has only two (1 and 7); 9 has three (1, 3 and 9), 6 has four (1, 2, 3 and 6), and 12 has six (1, 2, 3, 4, 6 and 12). (*Note*: A factor is something which divides into a number or which equals the number when multiplied by another factor.)

   - If only two factors, numbers are **prime** (e.g. 7);
   - If more than two factors, numbers are **composite** (e.g. 9, 6 and 12).

A number is the multiple of its factors (e.g. 6 = 1×6 and = 2×3), but factors can appear more than once and, in this case, can use indices (e.g. 72 = 2×2×2×3×3 = $2^3 \times 3^2$ – see later in this module).

(a) One way to find the primes is to **remove the composites.** This is done by removing all multiples of smaller primes, e.g. 2, 3, 5 and so on (*Note*: 4s are removed with the 2s). Look up Sieve of Eratosthenes on the Internet. Use a 100 board – go through and remove all multiples of 2 by crossing them out, then remove in order, multiples of 3, 5, and 7 What is left is the primes.

Why can we stop at 7?

(b) **Prime factors and factors of factors**. If a factor of a number is a composite (e.g. 6 is a factor of 24 and 6 is a composite), then the prime factors of that factor are factors of the original number and, in fact, any factor of a factor is a factor of the original number (e.g. 6=2×3 so 2 and 3 are factors of 24 – 2×12=24 and 3×8=24).

This can be shown by examples, but how can it be shown to be always true?

2. **Perfect and other numbers**

Factors can also be used to define other numbers, for example:

- *Perfect numbers*: factors other than numbers add to numbers, e.g. factors of 6 other than 6 are 1, 2 and 3. 1+2+3=6; 6 is perfect.
- *Abundant numbers*: factors other than number add to more than number, e.g. factors of 12 other than 12 are 1, 2, 3, 4 and 6, 1+2+3+4+6=16 which is greater than 12; 12 is abundant.
- *Deficient numbers*: factors other than number add to less than the number, e.g. factors of 8 other than 8 are 1, 2 and 4, 1+2+4=7; 7 is less than 8; 8 is deficient.

(a) What is the next perfect number after 6? What is the next abundant number after 12?

(b) Are there more abundant numbers under 100? What are they?

(c) Look up a list of perfect numbers on the Internet. Look up amicable numbers on the Internet, what are they? What are the first two?

3. **Other interesting number stuff**

(a) Highest common factors: 24 and 36 have factors in common. 24=2×2×2×3 and 36=2×2×3×3. Thus 2×2×3=12 is the highest common factor of 24 and 36. What are the highest common factors of:

(i) 18 and 14 (ii) 68 and 51?

(b) Factors lead to other interesting activities. One of these we use in geometry is Modulo Art. Modulo Art is based on Modulo arithmetic. What is Modulo arithmetic? Look it up on the Internet!

(c) There are some famous conjectures in maths related to number theory: Goldbach's Conjecture, Fermat's Last Theorem. Look into these (they are hard to understand).

1.4 Patterns

There are many patterns in number structure. Here are three:

1. **Calendar**

(a) Take a calendar as below. Take any square of numbers. Do the diagonals always add to the same number as each other? For 2×2, 3×3, 4×4 squares? Why is this?

(b) Look up other calendar tricks.

M	T	W	T	F	S	S
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	1	2	3	4	5

2. **Pascal's triangle**

This is a triangle that adds previous numbers (see below).

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

(a) What patterns do you see?

(b) Can you find a sloped line of 1s, of counting numbers, of the triangular numbers?

(c) Make an upside down triangle starting with 1 3 3 1 and ending with 20. What do you see?

(d) Add the rows, is there a pattern there?

3. **Fibonacci Sequence**

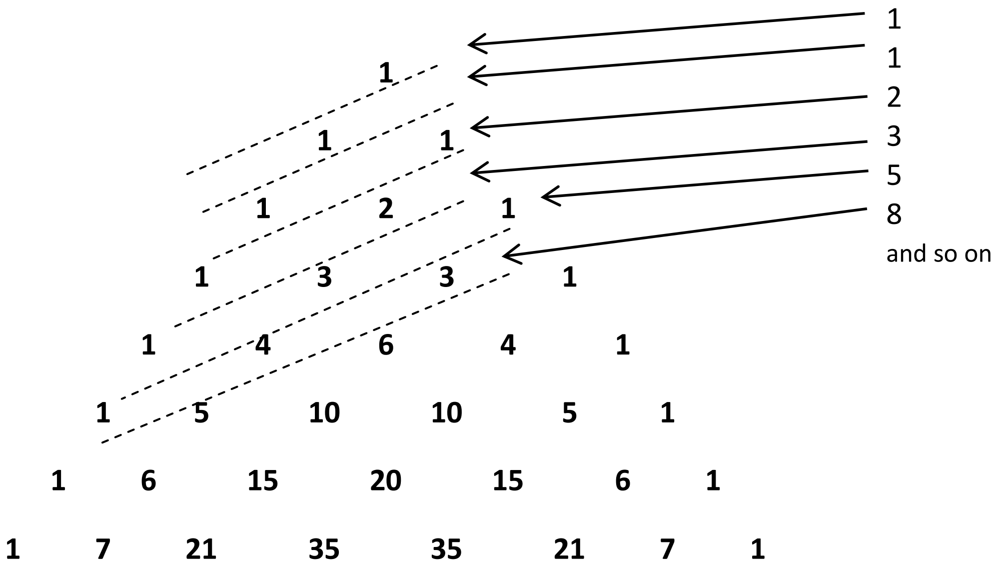
Here it is: 1 1 2 3 5 8 13 21

(a) How is this sequence formed? What patterns do you see?

(b) Can you get it from Pascal's triangle? Look at the numbers along the dotted sloped lines shown on the next page (hint: add them up along the lines).

(c) Look it up on the Internet – what are Pascal's triangle and Fibonacci sequences used for?

The diagram below shows Pascal's triangle becoming a Fibonacci sequence.



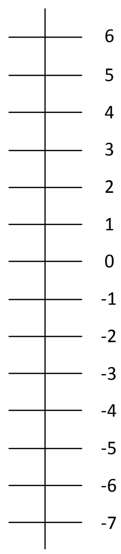
Unit 2: Directed Number

This unit looks at positive and negative numbers, how they can be modelled and taught and how they function with respect to addition, subtraction, multiplication and division.

2.1 Teaching directed number

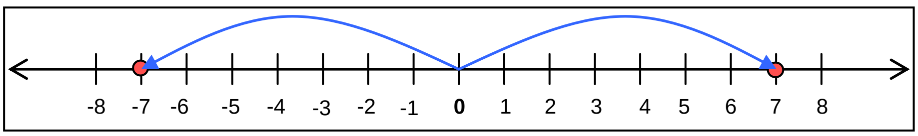
There is a **mathematics joke**: A mathematician is a person who knows that if she sends three people into an empty room and five walk out, she has to send in two more people to make it empty again. This is funny because we know that this is true for arithmetic (i.e. $0+3-5+2=0$) but it makes no sense. However, if we change the story to temperature: A mathematician is a person who knows that if he raises the temperature in a room that is zero degrees Celsius by 3 degrees and then lowers it by 5 degrees, he has to raise the temperature by 2 degrees to make the room zero again. Now the story makes sense. The moral of this is that, although theoretically there are negative numbers, the reality they come from is measures that go below zero where 0 is not nothing but the point between positive and negative.

Thus to introduce the idea of directed numbers, we have to use measures, e.g. temperature (above and below zero), height (above and below sea level), and buildings (floors above and below ground level); and money, e.g. assets and debts (liabilities). We have to get across the idea that some numbers **can be both positive and negative** – they can be something that is less than zero as well as more than zero.



Directed numbers are numbers that are used to represent situations that involve both quantity and direction. The directions are opposites and are described as being either positive or negative. Directed numbers allow the whole number system to be extended beyond zero in a negative direction. The concept of negative numbers makes most sense in relation to money and measures. A person can spend more money than they earn and the result will be a negative quantity or an amount of money will need to be earned to get the person back to zero. Temperatures can fall to below zero. Directed whole numbers are also referred to as integers.

When introducing directed numbers to students it is helpful to refer to them in terms of them being opposite to the whole numbers students know. It is important to make the distinction between negative numbers and the operation of subtraction. For example the number that is opposite 7 is negative 7 and is written as −7. A number line is an effective tool for demonstrating how these numbers are opposites. The numbers are the same distance or are the same number of jumps in each direction from zero.



Often a vertical number line is particularly effective for modelling directed numbers as it matches contexts where negative numbers occur in everyday life, e.g. thermometers, above and below sea level, and even the levels of a building above and below ground level (a lift going up and down).

Introducing directed whole numbers

We introduce negative numbers through two measures – money and height – in the following RAMR activity.

Reality

Discuss with students where we have negatives. Try to find a local instance (e.g. diving – above and below water level). Also discuss local possibilities – students can discuss what happens when we spend more than we have in our credit card, or if we borrow more than we can pay back immediately.

Abstraction

Body

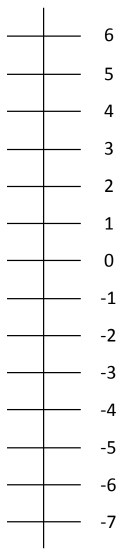
Get students to experience a local instance of negatives. If lacking local instances, set up experiences. Students can experience negative temperatures in a freezer. There are also negatives in buildings with basements – going down below ground level.

Hand

Start with money – look at a film star or someone famous – look at assets and debts – what they have and what they owe. An example is below:

> The Courier Mail *reported on 27 April 2009 the following about the net worth of Kylie Minogue: "Kylie Minogue is singing the recession blues, having lost a sizeable chunk of her wealth in the global financial crisis. The 2009 rich list of musos compiled by* The Sunday Times *revealed that the pop princess has seen her net worth drop by $12.25 million from $83.5 million. She also recently sold property in Melbourne worth $1 million.*

What is her net worth now? What does it mean to have this net worth? What sort of assets could Kylie have to have this net worth? Make a list of possible assets. Kylie will also have debts. What kind of debts? Make a list of possible debts.



Is it possible for someone to have more debts than assets? Look at this example of Ron: Assets – car $23,000, bank balance $2,000. Debts – car loan $15,000, owes uncle $5,000, credit card debt $13,000. What would be his net worth in thousands of dollars? What does this mean?

Move onto another measure – say a lift in a large building 28 floors up and 6 basements down. Start on ground level – go up 8 floors, down 5 floors, up 2 floors and down 7 floors. Where are we? What could we call that floor?

Develop a vertical line with positive and negative numbers on it as on right. This can be a large one on the floor with tape, a reasonably sized one on board or small ones for each student. **The mat is an excellent material for this**.

Use this vertical number line to act out instances of negative: 0 can be all even with assets and debts, the ground floor of a building, 0 degrees centigrade in temperature, present day in a time machine, and so on. Act out stories and end up with positive and negative numbers.

Mind

Picture vertical line in mind, shut eyes and act out stories in the mind.

Mathematics

Language and symbols/Practice

Practise situations which end up in negative numbers – relate symbols to situation; e.g. you take $500 on credit card to a sale, spend $700 – what is your situation on the card? [−$200]

Connections

Ensure negative numbers are related to appropriate measures and any other topic in mathematics. Is it possible to have negative fractions? Is it possible to have negative area (what about the area needed for the display and the area available – display could be bigger so not enough area so negative area?)

Reflection

Validation/Application. Apply understanding of negative numbers back into students' lives.

Flexibility. Do a poster on negative numbers – where they are in the world. (*Note:* Some sports have negative numbers – full backs in gridiron have rushing yards and on a bad day this is negative.)

Reversing. Very important – provide students with a negative number and a context and they have to make up the story. This is reversing what we have so far done.

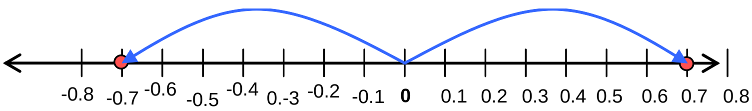
Generalising. This is students understanding when a vertical number line goes from 0 upwards or in both directions with negative numbers.

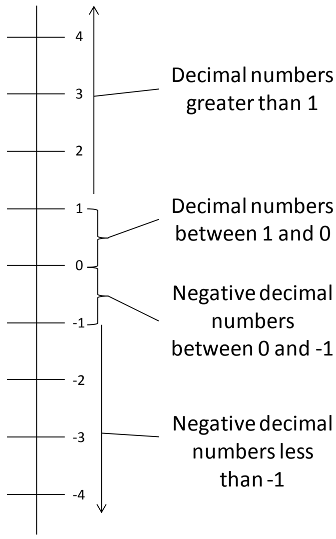
Changing parameters. Is it possible to need numbers to go horizontally positive and negative as well as vertically?

Directed decimal and fraction numbers

Decimals

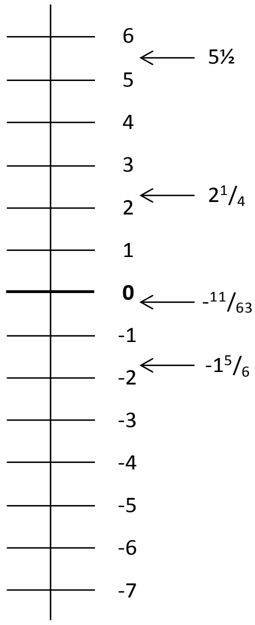
As with whole numbers, directed numbers can be used to represent situations that involve both quantity and direction involving decimals. The directions are opposites and are described as being either positive or negative. Directed numbers allow the decimal number system to be extended beyond zero in a negative direction. An example of the use of negative decimals can be with temperature where the temperature drops a whole and decimal degrees below zero.





As with whole numbers, directed numbers are opposite to the corresponding positive decimal. It is important to make the distinction between negative numbers and the operation of subtraction. For example, the number that is opposite 0.7 is negative 0.7 and is written as −0.7. A number line is an effective tool for demonstrating how these numbers are opposites. The numbers are the same distance or are the same number of jumps in each direction from zero.

There is a potential difficulty with horizontal PVC as above. It is useful to show the diagram on the right. Students sometimes confuse PV with directed number as a horizontal line is commonly used for PV – some students start to see decimals as similar to negative numbers and feel that negative numbers start after the ones.



Fractions

Directed number applies to common fractions as it applies to whole and decimal numbers. The vertical number line is still the most useful for representing directed fractions. There are the positives (above zero) and the negatives (below zero) and these can be fractions, as the figure on the right shows.

Once again, measures are the best to introduce this idea – however, not many measures are common fractions and not decimals. In fact, time and angle are the only ones that could provide some variety – weeks and days are sevenths, hours and minutes are sixtieths, and angle is 360ths. With some creativity, hours could be quarters and we could have a countdown – as they say in Houston 4, 3, 2, 1, 0, −1, −2, and so on. We could do this with ¼ hours.

2.2 Directed number addition and subtraction

As we saw in the earlier modules, the application of numbers to continuous entities (such as length or temperature) changed the nature of number and allowed 0 to become other than nothing (e.g. the first point on a ruler). It also allowed number lines to be bi-directional and to go into the negative (e.g. a lift going up and down through floors and basements; a thermometer showing temperatures above and below zero). Now we look at how to do operations with such numbers.

Prerequisites

Ensure that the students have the following understandings for this topic, that is, directed number and the operations of addition and subtraction.

Directed number

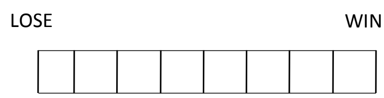
1. Start with developing ideas of opposites (up, down; hot cold; large, small; happy, sad; multiplication, division; add, subtract; plus, minus; positive, negative). Discuss $^{-}1$ is opposite $^{+}1$; $^{-}4$ is opposite $^{+}4$, etc.
2. Teach the language: $^{-}1$ is negative 1, not minus 1; $^{+}1$ is positive 1, not plus 1; $^{-}4$ is negative 4, not minus 4; $^{+}4$ is positive 4, not plus 4.
3. Relate language to notation: $^{-}1$ is also (−1); $^{-}3$ is also (−3). (*Note*: The brackets allow the sign to be similar to a subtraction minus sign.)
4. Ensure students know that if there is no sign the number is positive; e.g. 6 on its own represents $^{+}6$ or (+6).
5. Relate to real world using examples relevant to students, e.g. temperature on thermometers; lifts in building with underground car parks; football penalties, etc. (*Note*: Not all students will be familiar with such examples, depending on where they live and their experiences.)
6. Relate to vertical number line. Also ensure students know that line in many instances is horizontal – ensure students do not confuse place value with directed number when using horizontal line.

Operations

1. Recap what addition and subtraction actually mean.
2. Act addition and subtraction out with counting numbers (e.g. 6−2 is up 6 and back 2).
3. Repeat this for a horizontal number line (e.g. 6−2 is 6 to right and 2 to left).
4. It is important to do these prerequisites in earlier years to prepare for directed number and to rebuild ideas at the time of doing directed number.

Addition and subtraction with directed number

Introduction activity



Game. Materials – number line like on right; two ends – one good and the other not good (we'll call ours WIN/LOSE); dice with F1, F2, F3, B1, B2, B3 (F – forward, B – backward); dice with W, W, W, L, L, L (W – facing towards WIN, L – facing towards LOSE). The Mathematics Curriculum Teaching Program (MCTP) version of this with **Shark and Pirate** is excellent.

Set up a story like someone seeing if they can win or lose in some way which involves walking along a number track – this person will win or lose depending on throws of dice. Throw the two dice – W or L die gives direction facing, other die gives the number of steps forward or back. The player starts in the middle of the track.

As game is played (player can change), start to record the moves informally (e.g. L–F2, L–B3, W–B1 and so on), discuss movements (e.g. L–B3 actually moves towards WIN), change recording to formal (e.g. W is +, L is – , F1 is $^{+}1$ and B2 is $^{-}2$, and so on; and call middle 0 with to the right (to WIN) positive and to left (to LOSE)negative. Record positions as well as movements; e.g. starting at $^{-}3$ and getting a L and B2 ends at $^{-}1$ is recorded as

$$^{-}3 - {}^{-}2 = {}^{-}1 \quad \textbf{or} \quad -3 - (-2) = -1$$

Acting out operations

Number line. Movement of units to either the left or right (or up and down – depending on form of line). The number gives the number of units moved, + before number means facing to right, and – before number means facing to left. Operation of addition is moving forward and operation of subtraction is moving backwards, e.g.:

> Sentence +7 + (–6) {or 7 + $^{-}6$} means you start at 7, 6 means 6 units of movement, + means you move forward, and the – before the 6 means you face to left. This gives starting at 7 facing left and moving 6 units forward to +1.
>
> Sentence –1 – (–3) means that you start at –1, – means move back, – before 3 means facing left, and 3 means 3 units. This gives starting at –1, facing left and moving 3 units backward to +2.

This can be done using students in class by drawing a line on floor, or using tape, with clearly marked numbers and divisions. Student walks to starting number, faces left or right and moves forward or back depending on operation signs and numbers.

Patterns

Used to justify rules for subtracting integers. First with known simple subtractions and discuss patterns when students have done answers. Students to observe and identify patterns. You can also use the number line. For example 7 – 9, place your finger at 7, and draw an arrow that is 9 units long towards the left. Arrow should stop at (−2). Do the same using a negative number as a starting point, like (−6) − 3. Start at (−6) and move 3 units to the left. Again the pattern works – have student observe the answers, and then continue the pattern. Finally, justify subtraction of a negative number to give a positive number; that is two negatives turn into a positive.

6 − 3 = 3	(−3) + 2 = −1	5 − 3 = 2	(−4) – 2 = −6
6 − 4 = 2	(−3) + 1 = −2	5 − 2 = 3	(−4) – 1 = −5
6 − 5 = 1	(−3) + 0 = −3	5 − 1 = 4	(−4) – 0 = −4
6 − 6 = 0	(−3) − 1 = −4	5 − 0 = 5	(−4) – (−1) = −3
6 − 7 = −1	(−3) − 2 = −5	5 − (−1) = 6	(−4) – (−2) = −2
6 − 8 = −2	... and so on	5 − (−2) = 7	(−4) – (−3) = −1
... and so on		5 − (−3) = 8	(−4) – (−4) = 0
		... and so on	(−4) – (−5) = 1
			(−4) – (−6) = 2
			... and so on

Game: "HIT ME" card game

Take a deck of playing cards and remove jokers and face cards (leave only aces and 2–10). Make red positive, black negative. The idea is to get to ZERO in shortest number of deals (7 is maximum). Deal each player a single card face up. Each player is then dealt another card face down which only the player can look at. If player wants another card he/she says "hit me" (max five "hit me" requests). When everyone either rests or reaches maximum "hit me"s, all cards are turned over. Whoever reaches ZERO is winner and dealer for next game OR whoever has value closest to zero is winner.

Using mathematical structure

If, during the school years before the advent of directed number, students can gain an understanding of the structure of number and operations (the structure called "field" by mathematicians), they can do directed number **as a consequence** of their understanding of the structure. For example, the field structure means there is no – operation, 7−3 is 7+(−3) where −3 is the inverse of +3 and undoes +3. Thus 7+(−3) is just 7−3=4. This also means that 7−(−3) is adding the inverse of the inverse of 3 which is 7+3=10.

2.3 Directed number multiplication and division

This is an extension of the work done above on addition and subtraction of directed numbers. Since directed numbers are associated with number lines, multiplication will be repeated jumps forward along the line and division repeated jumps backward.

Prerequisites

Ensure that prerequisites are known, that is, that students understand:

- idea of opposites;
- modelling directed numbers on number lines;
- use of language, symbols and conventions; and
- reality problems and situations for directed number.

Also ensure that students understand the operations:

- what multiplication and division mean; and
- how multiplication and division operate on number line (in particular).

Multiplying and dividing teaching methods

1. **Multiplication on number line**. Set up +2 as being 2 forward or to right and –3 as 3 backward or to left, and 4× or +4 as 4 jumps facing forward/right and –5× as 5 jumps facing backward/left. Thus:
   - 3×(–2) is 3 jumps facing forward with each jump 2 backwards, this gives 6 backward, that is, 3×(–2)=–6; and
   - (–3)×(–2) or –3(–2) is 3 jumps of 2 backward facing backward – this give 6 forwards, that is, (–3)×(–2)=+6.

2. **Division on number line**. Set up ÷2 as number of 2 jumps forward, but facing either forward or backward, to give initial jump and ÷-2 as number of 2 jumps backward, but facing either forward or backward, to give initial jump. Facing forward means the number of jumps is positive, facing backwards means the number of jumps is negative, Thus:
   - 6 ÷ –2 is how many 2 jumps forward will give 6 backward; this can only be done if facing backward, so answer is 3 jumps forward but facing backwards, so 6 ÷ –2=-3; and
   - –6 ÷ –2 is how many jumps backward will give 6 backward which is 3 facing forward so –6 ÷ –2 = 3.

3. **Acting out**. Use the setting up above to act out with students on large number line, or counters on small number line, the various multiplication and division options.

4. **Patterns**. The complexity of multiplication and division on number line means that patterns can be the best way to teach the multiplication and division rules for directed numbers, as follows.

4 × 2 = 8	–4 × 2 = –8	2 ÷ 1 = 2	2 ÷ –1 = –2
4 × 1 = 4	–4 × 1 = –4	1 ÷ 1 = 1	1 ÷ –1 = –1
4 × 0 = 0	–4 × 0 = 0	0 ÷ 1 = 0	0 ÷ –1 = 0
4 × –1 = –4	–4 × –1 = 4	–1 ÷ 1 = –1	–1 ÷ –1 = 1
4 × –2 = –8	–4 × –2 = 8	–2 ÷ 1 = –2	–2 ÷ –1 = 2
... and so on	... and so on	... and so on	... and so on

Using structure

If students have acquired knowledge of operations as a structure, then 3 × (−4) would be 3 lots of inverse of 4 which is inverse of 12, so 3 × (−4) = −12. However for (−3) × (−4) it is necessary to work off a pattern. We can show (−1) × (−1) is +1 and that (−n) = (−1) × n. Thus (−3) × (−4) is (−1) × 3 × (−1) × 4 = 12.

Unit 3: Indices and Scientific Notation

This unit looks at setting up the notation for advanced number work in mathematics and science – indices and scientific notation.

3.1 Indices

This involves representing numbers by using indices for repeatedly multiplying the same number, for example:

- $7 \times 7 = 7^2$, $7 \times 7 \times 7 = 7^3$, $7 \times 7 \times 7 \times 7 = 7^4$;
- by patterns, $7 = 7^1$, $1 = 7^0$, $\frac{1}{7} = 7^{-1}$, $\frac{1}{7} \times \frac{1}{7} = 7^{-2}$, $\frac{1}{7} \times \frac{1}{7} \times \frac{1}{7} = 7^{-3}$, and so on; and
- $72 = 2 \times 2 \times 2 \times 3 \times 3 = 2^3 \times 3^2 = 2^3 3^2$.

To introduce these ideas:

1. **Patterns for index sequence**. These show that the notation is reasonable:

Four 7s multiplied together	$= 7^4$	$= 7 \times 7 \times 7 \times 7$
Three 7s multiplied together	$= 7^3$	$= 7 \times 7 \times 7$
Two 7s multiplied together	$= 7^2$	$= 7 \times 7$
One 7 multiplied on own	$= 7^1$	$= 7$
Identity 1 is	$= 7^0$	$= 1$
One 7 dividing	$= 7^{-1}$	$= \frac{1}{7}$
Two 7s dividing	$= 7^{-2}$	$= \frac{1}{7 \times 7}$

... and so on.

2. **Introduction by investigation**. Set up an investigation that gives indices. For example – if we can go up stairs 1, 2, 3 or any number of steps at a time, how many different ways can we go up 1 step, 2 steps, 3 steps, ... any number of steps (pattern)?:

STEPS	WAYS	STEPS	WAYS
1	1	4	8 or 2×2×2
2	2	5	16 or 2×2×2×2
3	4 or 2×2	... and so on	

The result of this investigation is that n steps means 2×2 ... ×2 $n-1$ times; we can use this to introduce the notation 2^{n-1}. There is a similar result from the old story of someone placing a grain of rice on the first square on a chessboard, two grains on the second square, four grains on the third square and continue doubling until you get to the 64th square – how many grains on the 64th square?

This approach makes the introduction of the indices more palatable.

Another problem/puzzle is the **Tower of Hanoi** – this also develops powers of 2 as the number of pieces in the tower increases.

3. **Pattern rules for multiplication and division of indices**. Use pattern to get across the rules for multiplying and dividing indices, for example:

   $4^1 \times 4^1 = 4 \times 4 = 4^2$
   $4^2 \times 4^1 = 4 \times 4 \times 4 = 4^3$
   $4^3 \times 4^1 = 4 \times 4 \times 4 \times 4 = 4^4$
   $4^3 \times 4^2 = 4 \times 4 \times 4 \times 4 \times 4 = 4^5$ and so on generalisation: $4^p \times 4^q = 4^{p+q}$
   $4^3 \div 4^1 = 4 \times 4 \times 4 \div 4 = 4 \times 4 = 4^2$
   $4^3 \div 4^2 = 4 \times 4 \times 4 \div (4 \times 4) = 4 = 4^1$
   $4^3 \div 4^3 = 4 \times 4 \times 4 \div (4 \times 4 \times 4) = 1 = 4^0$
   $4^3 \div 4^4 = 4 \times 4 \times 4 \div (4 \times 4 \times 4 \times 4) = \frac{1}{4} = 4^{-1}$ and so on generalisation: $4^p \div 4^q = 4^{p-q}$

   The overall generalisation, therefore, is that: $n^p \times n^q = n^{p+q}$ and $n^p \div n^q = n^{p-q}$

4. **Pattern rules for indices of indices.** It is also possible to use patterns to develop the rules for indices of indices. For example:

   $(4^2)^1 = 4^2 = 4 \times 4 = 4^2$
   $(4^2)^2 = 4^2 \times 4^2 = 4 \times 4 \times 4 \times 4 = 4^4$
   $(4^2)^3 = 4^2 \times 4^2 \times 4^2 = 4 \times 4 \times 4 \times 4 \times 4 \times 4 = 4^6$
   $(4^3)^2 = 4^3 \times 4^3 = 4 \times 4 \times 4 \times 4 \times 4 \times 4 = 4^6$
   $(4^{-1})^{-1} = (\frac{1}{4})^{-1} = \frac{1}{1/4} = 4^1$
   $(4^{-2})^{-1} = (\frac{1}{4^2})^{-1} = \frac{1}{1/4^2} = 4^2$
   $(4^{-2})^{-3} = (\frac{1}{4^2})^{-3} = \frac{1}{1/4^2} \times \frac{1}{1/4^2} \times \frac{1}{1/4^2} = 4^6$ and so on generalisation: $(4^p)^q = 4^{p \times q}$
   even when p and q are negative.

   The overall generalisation, therefore, is that: $(n^p)^q = n^{p \times q}$; $(n^p)^{-q} = n^{p \times -q}$ and $(n^{-p})^{-q} = n^{p \times q}$

5. **Pattern rules for fractions with indices**. Whether numbers are fractions or whole numbers, even whether they are irrationals or rationals (common fractions), makes no difference – they still can have indices with the same meaning. For example:

   $(\frac{3}{5})^0 = 1$; $(\frac{3}{5})^1 = \frac{3}{5}$; $(\frac{3}{5})^2 = \frac{3}{5} \times \frac{3}{5} = \frac{3 \times 3}{5 \times 5}$; $(\frac{3}{5})^3 = \frac{3 \times 3 \times 3}{5 \times 5 \times 5}$; and so on; and
   $(\frac{3}{5})^{-1} = \frac{1}{3/5} = \frac{5}{3}$; $(\frac{3}{5})^{-2} = \frac{5}{3} \times \frac{5}{3} = \frac{5 \times 5}{3 \times 3}$; $(\frac{3}{5})^{-3} = \frac{5}{3} \times \frac{5}{3} \times \frac{5}{3} = \frac{5 \times 5 \times 5}{3 \times 3 \times 3}$

   Using the same techniques as in points 3 and 4 above, these results can be generalised to show that:

   $(\frac{3}{5})^p = \frac{3^p}{5^p}$, and $(\frac{3}{5})^{-q} = (\frac{5}{3})^q = \frac{5^q}{3^q}$, and $(\frac{a}{b})^p = \frac{a^p}{b^p}$, and $(\frac{a}{b})^{-q} = (\frac{b}{a})^q = \frac{b^q}{a^q}$

3.2 Scientific notation

Using indices for multiples of 10 is particularly important for the way it relates to place value (see sections 4.1 and 4.3 in Unit 4).

1. **Introduce powers for 10**. List the PV positions – put beside them how many 10s have to be multiplied – introduce the index as a symbol to show number of 10s:

Ten	10	10	10^1	
Hundred	100	10×10	10^2	
Thousand	1000	10×10×10	10^3	
Ten thousand	10000	10×10×10×10	10^4	
Hundred thousand	100000	10×10×10×10×10	10^5	
Million	1000000	10×10×10×10×10×10	10^6	and so on

2. **Introduce powers of 10 for decimal place values**. List the decimal PV positions and repeat 1 above:

Tenth	1/10	1/10	10^{-1}
Hundredth	1/100	1/10×1/10	10^{-2}
Thousandth	1/1000	1/10×1/10×1/10	10^{-3}
Ten thousandth	10000	1/10×1/10×1/10×1/10	10^{-4}
Hundred thousandth	100000	1/10×1/10×1/10×1/10×1/10	10^{-5}
Millionth	1000000	1/10×1/10×1/10×1/10×1/10×1/10	10^{-6}
and so on			

3. **Introduce power of 10 for one**. Discuss what one would be – use patterns to show that it is one index less than 10^1 and one index more than 10^{-1} so it is 10^0:

   10^3 10^2 10^1 $10^?$ 10^{-1} 10^{-2} 10^{-3}

4. **Using powers for numbers**. Look at numbers in terms of multiplying and dividing by 10. Use a mat with place values along its length. Look at numbers in various PV position and make these powers of 10 as follows. Utilise the flexibility of the decimal point (e.g. 35 = 35 ones, 3.5 tens or 0.35 hundreds). Make sure students know the number of zeros to form 3 million, 12 billion, 45 millionths, 3 thousandths, and so on.

   420 = 42 tens = 42×10^1

   420 = 4.2 hundreds – 4.2×10^2

   This is the secret of scientific notation – the numerals are written between 1 and 10 – that is, 2.35, 2.8, 7.0345 or 9.991 and then put with a power of 10 to give the same value as the number, for example:

   3 245 000 = 3.245 millions = 3.245×10^6

   0.000 0776 = 7.76 hundred thousandths = 7.76×10^{-5}

   Scientific notation is most useful for very large or very small numbers, for example:

   89 200 000 000 000 000 000 000 000 = 8.92×10^{25}

   0.000 000 000 000 000 000 000 000 000 000 000 048 76 = 4.876×10^{-35}

Unit 4: Rationals, Irrationals and Number Systems

This unit covers the setting up of the number system for whole numbers, the exploration between rationals and reals, and the combination of these two into the real-number system.

4.1 The whole number system

Most countries today use the Decimal Number System, called "decimal" in English because it has a base of 10 and 10 digits (0, 1, 2, 3, 4, 5, 6, 7, 8, 9). The word *decimal* came from the Roman word *decem* which means 10. The whole numbers in the decimal number system do not stop but go on to infinity. Two of the big ideas, **multiplicative structure** and **the pattern of threes**, form the basis of being able to take this learning further to understand the whole number system.

Once students have been introduced to the concept of exponents, usually by exploring square numbers and using the index notation to represent squares, they can see the connection between the exponents of 10 and the structure of the whole number system. A table like in the RAMR activity below will help show the connection between the place values, the value, the multiplicative structure and the exponential notation.

When the pattern-of-threes structure is included, numbers can be considered as just powers of 10 or as pattern-of-threes powers of 10. Thus the system can be considered as:

H	T	O	H	T	O	H	T	O	H	T	O
Billions			Millions			Thousands			Ones		
10^{11}	10^{10}	10^9	10^8	10^7	10^6	10^5	10^4	10^3	10^2	10^1	10^0

OR

H	T	O	H	T	O	H	T	O	H	T	O
Billions			Millions			Thousands			Ones		
10^9			10^6			10^3			10^0		
10^{11}	10^{10}	10^9	10^8	10^7	10^6	10^5	10^4	10^3	10^2	10^1	10^0

Reality

Get students to look at PVCs to millions – use pattern-of-threes PVC – can students see pattern that enables this to go on forever? What would be the next pattern of three to add? The one after that? How could we think of it so it goes on forever?

Abstraction

Body

Set up pattern of threes on the mat. Students walk – H-T-O ones, then H-T-O thousands, then H-T-O millions – where do we walk to next?

Hand

Use the powers of 10 from section 3.2 and the mat to make up a PVC that is just the PV positions and a second PVC that is pattern of threes – replace names with indices, what have we got? Can we see the pattern?

HB	TB	B	HM	TM	M	HTH	TTH	TH	H	T	O
10^{11}	10^{10}	10^{9}	10^{8}	10^{7}	10^{6}	10^{5}	10^{4}	10^{3}	10^{2}	10^{1}	10^{0}

H	T	O	H	T	O	H	T	O	H	T	O
BILLIONS			MILLIONS			THOUSANDS			ONES		
10^{9}			10^{6}			10^{3}			10^{0}		

Mind

Think of place values in the mind going up by a factor of ×10 – think of the indices increasing.

Mathematics

Language and symbols/Practice

Get students to describe and draw the system and show how it continually gets bigger – get them to look up the names for bigger and bigger numbers.

Connections

Relate this system to metrics, for example:

H	T	O	H	T	O	H	T	O	H	T	O
1000 km			km			m			mm		
10^{9} mm			10^{6} mm			10^{3} mm			10^{0} mm		

Reflection

Validation/Applications

Look in the world for whole number systems that students use, for example:

H	T	O	H	T	O	H	T	O	H	T	O
GIGABYTES			MEGABYTES			KILOBYTES			BYTES		

Extension

Flexibility. Do a poster showing many examples of systems of numbers – on kilos, megas, gigas and so on.

Reversing. Give a number, 64 gigabyte say, and students have to find where to put this on a diagram of the system. Then reverse, give a position and students have to make up a situation. That is:

number situation → system AND system → number situation

Generalising. Can the students see the two forms of the system in their mind? Can they make up their own system? What happens when we move two places in multiplicative structure? We are at 10^6, we multiply by 100, we are now at 10^8. Can students generalise – 2 places is 10^2, 3 places is 10^3?

Changing parameters. What if we changed base (e.g. seconds, minutes, hours)? What would the Mayan base 20 system look like? What would the Babylonian base 60 system look like?

4.2 Rationals and irrationals

From common fractions emerge the concept of **rationals** or rational numbers. A rational number can be represented by a fraction which is a whole number “over” a whole number (the name comes from “ratio”).

Irrationals

The Pythagoreans believed that whole numbers were the way the creator created the world. They were not worried about common fractions because they were made from whole numbers and were therefore rational (logical). However, they ran up against a problem with square root of 2 ($\sqrt{2}$). If it is assumed that $\sqrt{2}$ is a common fraction $\frac{p}{q}$ fully cancelled down, then $2 = \frac{p^2}{q^2}$. This means that p^2 had to be even which means that p had to be even, but this means that q had to be even so that $\frac{p \times p}{q \times q} = 2$, which means that p and q are not cancelled down. Thus, $\sqrt{2}$ is not a common fraction, which means it is something else (and ungodly), therefore it is **irrational**. Many more irrationals were then found, such as *pi*.

It is possible to show that all fractions, when divided out to make a decimal, are a **finite decimal** (end after a number of decimal place-value digits – e.g. $^3/_8$ = 0.375) or a **repeating decimal** ($^2/_7$ = 0.285714285714 repeating). Therefore, all the decimals which are *not finite and do not repeat* are irrationals. This means that there are an infinite number of rationals and an infinite number of irrationals.

The positive rationals can be written down in an order that shows that all will be included by advancing the denominator a counting number at a time.

1/1 →	2/1	3/1 →	4/1	5/1	...
1/2	2/2	3/2	4/2	5/2	...
1/3	2/3	3/3	4/3	5/3	...
1/4	2/4	3/4	4/4	5/4	...
1/5	2/5	3/5	4/5	5/5	...

... and so on.

These rationals can be counted in the order shown by the arrows, so they are **countably infinite**. It has been show that the irrationals are **uncountably infinite**. Both rationals and irrationals are always dense, meaning there is an infinite number of them between any two numbers. Of course, being uncountable, irrationals are denser.

Real number activities

In Modules N3 and O3, common fractions are shown to be a form of division. If we share three cakes among four people we get $^3/_4$ cake for each person (i.e. 3÷4 is the same as $^3/_4$). This idea of fraction as division can help us determine how rationals (fractions) and irrationals are represented as decimals.

Activity 1

1. Take a variety of common fractions (e.g. $^1/_2$, $^3/_4$, $^7/_8$, $^4/_9$, $^2/_7$, $^3/_{11}$) and divide them out – what kind of decimals do we get?

   We will see that in all cases, these common fractions will divide out as either:

   - terminating decimals (e.g. $^3/_4$ = 0.75), or
   - infinite decimals with a repeating part (e.g. $^2/_7$ = 0.285714285714285714285714 ... which is repeating 285714).

2. This process can be reversed – terminating decimals and repeating-part decimals can be seen to be common fractions, for example:
   - terminating decimal: e.g. $0.46 = \frac{46}{100}$ so fraction is $\frac{23}{50}$
   - repeating-part decimal: e.g. $0.\overline{4646}$ Let f be fraction, therefore:
     $100f - f = 46.\overline{4646} - 0.\overline{4646}$, thus
     $99f = 46$, and so fraction $= \frac{46}{99}$

   Sometimes the decimal does not start repeating until there has been a non-repeating part, for example, $0.45\overline{6666}$. Here, we need to be more creative in removing the non-repeating part to get the whole numbers, for example:

   $$f = 0.45\overline{6666} \qquad \begin{array}{rl} 1000f & = 456.\overline{6666} \\ -100f & = 45.\overline{6666} \\ \hline 900f & = 411 \end{array} \qquad \text{fraction} = \frac{411}{900}$$

   Thus, finite and repeating part decimals are all common fractions and are called **rationals** – the other ones are called irrationals.

3. Discuss decimals that are infinite but have no repeating part (examples of these are *pi*, $\sqrt{2}$, and so on). State these are not fractions and are called irrationals.

   The argument of why $\sqrt{2}$ is not a common fraction was important in history. It relates to whole numbers having a religious role and, therefore, the importance of all numbers being in some way whole numbers or at least having two whole numbers in their symbol. The argument that showed that $\sqrt{2}$ was not a fraction was likewise important as it was one of the first examples of proof by contradiction.

4. To follow the argument – say $\sqrt{2}$ was $\frac{p}{q}$ where $\frac{p}{q}$ are fully cancelled down– then squaring both sides gives $2 = \frac{p^2}{q^2}$, but this is impossible – it means that both p and q have to be even (can you work out why?) – thus, $\frac{p}{q}$ is not in fully cancelled down form – this contradiction means that $\sqrt{2}$ cannot be a fraction. Therefore, the irrationals are infinite decimals that do not repeat.

5. Discuss how these come together to form the real number system. Discuss how the rationals (which are all common fractions) are infinite in number and **countable** (can be put in an order and counted) and the irrationals are also infinite in number but **uncountable** (because they are not common fractions). Discuss whether this is a large infinity.

4.3 The real number system (decimals)

We have seen so far that decimal numbers are an extension of whole numbers – so the system should be as well. Some properties that appeared evident earlier are that, in terms of decimals, the decimal number system is: (a) symmetrical about ones; (b) bi-directional in relationship in that the place-value positions work left and right from the ones and ×10 will move the place value one place to the left and ÷10 will move the place value one place to the right; (c) exponential in structure and this structure extends to decimals; and (d) continuous across the decimal point.

The crucial element in teaching/developing the real number system is to facilitate students seeing the pattern in the structure of numbers, looking for things that hold true for all numbers. What is evident is that number systems are generalisations. Some activities to develop this understanding are as follows.

Place-value structure

Look at millions-to-thousandths as a PVC – expand out to millions-to-millionths – discuss the PV positions in terms of ×10 and ÷10, as below. (*Note*: O is one or ones, T is ten, H is hundred, Th is thousand, M is million, t is tenths, h is hundredths, th is thousandths, and m is millionths.)

HM	TM	OM	HTh	TTh	OTh	H	T	O	t	h	th	Tth	Hth	m

OR in pattern of threes

HM	TM	OM	HTh	TTh	OTh	H	T	O	Hth	Tth	Oth	Hm	Tm	Om

Indices

Discuss the pattern in terms of indices or powers of 10 as below.

10^8	10^7	10^6	10^5	10^4	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}

OR in pattern of threes

$H10^6$	$T10^6$	$O10^6$	$H10^3$	$T10^3$	$O10^3$	$H10^0$	$T10^0$	$O10^0$	$H10^{-3}$	$T10^{-3}$	$O10^{-3}$	$H10^{-6}$	$T10^{-6}$	$O10^{-6}$

Powers of 10

Discuss how the pattern on left and right of the one goes:

... 10^5 10^4 10^3 10^2 10^1 10^0 10^{-1} 10^{-2} 10^{-3} 10^{-4} 10^{-5} ...

OR

HTO 10^6 HTO 10^3 HTO 10^0 HTO 10^{-3} HTO 10^{-6}

Test Item Types

This section presents instructions and the test item types for the subtests associated with the units. These will form the bases of the pre-test and post-test for this module.

Instructions

Selecting the items and administering the pre-post tests

This section provides an item bank of test item types, constructed around the units in the module. From this bank, items should be selected for the pre-test and post-test; these selected items need to suit the students and may need to be modified, particularly to make post-test items different to pre-test items. The purpose of the tests is to measure students' performance before and after the module is taught. The questions should be selected so that the level of difficulty progresses from easier items to more difficult items. In some modules this will follow the order of the units and subtests, and in other modules it will not, depending on the sequencing across the module and within units. The pre-test items need to allow for students' existing knowledge to be shown but without continual failure, and the post-test items need to cover all the sections in a manner that maximises students' effort to show what they can do.

In administering the pre-test, the students should be told that the test is not related to grades, but is to find out what they know before the topic is taught. They should be told that they are not expected to know the work as they have not been taught it. They should show what they know and, if they cannot do a question, they should skip it, or put "not known" beside questions. They will be taught the work in the next few weeks and will then be able to show what they know. Stress to students that **any pre-test is a series of questions to find out what they know** before the knowledge is taught. They should do their best but the important questions come at the end of the module. For the **post-test**, the students should be told that **this is their opportunity to show how they have improved**.

For all tests, **teachers should continually check to see how the students are going.** Items in later subtests, or more difficult items within a particular subtest, should not be attempted if previous similar items in earlier subtests show strong weaknesses. Students should be allowed to skip that part of the test, or the test should be finished. Students can be marked zero for these parts.

Information on the directed number, indices and systems item types

The directed number, indices and systems item types are divided into four subtests, one for each of the four units in the module. The four units represent different topics. First, Unit 1 looks at number in terms of shapes, computations, factors and patterns and represents activity from the primary years which provides engagement and diversion and is not central to the development of number (although it represent topics that have attracted mathematicians for years and have been responsible for advancements in number theory). Second, Unit 2 looks at directed number (negative numbers) and how to represent and compute with these numbers. Third, Unit 3 looks at indices, their computation principles, and how they lead to scientific notation (representing numbers by powers of 10). Fourth, Unit 4 uses scientific notation, work covered in Modules N1, N2 and N3, and meanings of rationals and irrationals to represent number systems. Units 2, 3 and 4 represent junior secondary number ideas.

Thus, the pre-test should be made up of test item types from Subtest 1 and the simpler ideas from Subtests 2, 3 and 4, while the post-test should cover all four subtest item types. *Note:* Always read the questions to the students and explain any contextual information as long as this does not direct to answer.

Subtest item types

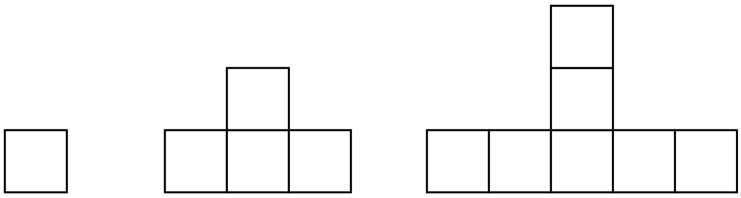
Subtest 1 items (Unit 1: Number types and patterns)

1. Draw a line from each number to the word that it matches:

4	**Triangular number**
5	**Square number**
6	**Prime number**

2. *Growing patterns*

These blocks make up a pattern:



___blocks	___blocks	___blocks	___blocks
Stage 1	**Stage 2**	**Stage 3**	**Stage 4**

(a) Draw the next stage in the pattern.

(b) Write in the number of blocks for each stage.

(c) Describe the pattern in words: ____

____

3. (a) Describe what **perfect** numbers are: ____

____

Give an example of a perfect number and explain why it is perfect: ________________

____

(b) Describe what **abundant** numbers are: ____

____

Give an example of an abundant number and explain why it is abundant: ___________

____

4. (a) Draw the first 4 triangular numbers.

(b) Draw the first 4 square numbers.

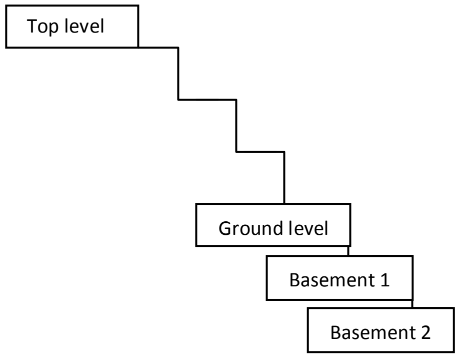
(c) Draw the first 4 rows of Pascal's triangle.

(d) Write the first 6 terms of the Fibonacci sequence.

Subtest 2 items (Unit 2: Directed number)

1. In Canberra the night time temperature was 2 degrees below zero (two degrees of frost). By midday the temperature had risen 14 degrees. What was the midday temperature? _____________

2. A shopping centre has four levels of shops and two levels of basement car parks.



Complete the following number sentences. (Ground level is called zero, so the top level is called level 3.)

(a) You are at level 3 and you go down two levels:

$3 + {}^{-}2 =$ __________

(b) From the top level go down three levels and then down another two levels:

$3 + {}^{-}3 + {}^{-}2 =$ __________

(c) From the ground level go up two levels then do the opposite of going down one level:

$2 - {}^{-}1 =$ __________

3. Complete the following:

(a) $4 - {}^{+}2 =$ __________

(b) $6 + {}^{-}4 =$ __________

(c) $8 - {}^{-}3 =$ __________

(d) ${}^{-}3 \times 2 =$ __________

(e) ${}^{-}4 \times {}^{-}3 =$ __________

(f) ${}^{-}8 \div {}^{-}2 =$ __________

4. **Challenge question**: A shipwreck on the ocean floor is 15 metres below sea level. A plane is circling above it at 150 metres above sea level. How far is the plane above the wreck? Draw a picture of the plane, the sea level and the shipwreck and write in the numbers.

Subtest 3 items (Unit 3: Indices and scientific notation)

1. Write these numbers in index form:

   (a) 2×2 __________

   (b) $7 \times 7 \times 7$ __________

   (c) 10 000 __________

2. What is the value of:

   (a) 3^2 __________

   (b) 10^{-1} __________

   (c) 4^0 __________

   (d) $2^4 \times 2^2$ __________

   (e) $5^5 \div 5^2$ __________

3. What is the rule for:

   (a) $3^p \times 3^q$ __________

   (b) $3^p \div 3^q$ __________

   (c) $(3^p)^q$ __________

4. Change these numbers to scientific notation:

   (a) 3 200 __________

   (b) 0.085 __________

Subtest 4 items (Unit 4: Rationals, irrationals and number systems)

1. (a) Describe what **rational** numbers are: ____

____

Give an example of a rational number and explain why it is rational: _______________

____

(b) Describe what **irrational** numbers are: ____

____

Give an example of an irrational number and explain why it is irrational: ____________

____

2. (a) What kind of decimals are rational? ____

(b) What fraction does 0.343434… (34 repeating) equal? Show all working.

(c) **Challenge**: What fraction does 0.4666666… (6 repeating) equal? Show all working.

3. Show the real number system place-value positions in powers of 10 on the diagrams below.

(a)

Ones

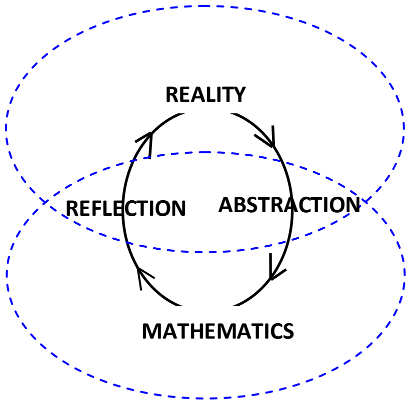
(b)

Ones

H	T	O	H	T	O	H	T	O	H	T	O	H	T	O

Appendix A: RAMR Cycle

AIM advocates using the four components in the figure on right, reality–abstraction–mathematics–reflection (RAMR), as a cycle for planning and teaching mathematics. RAMR proposes: (a) working from reality and local culture (prior experience and everyday kinaesthetic activities); (b) abstracting mathematics ideas from everyday instances to mathematical forms through an active pedagogy (kinaesthetic, physical, virtual, pictorial, language and symbolic representations, i.e. body → hand → mind); (c) consolidating the new ideas as mathematics through practice and connections; and (d) reflecting these ideas back to reality through a focus on applications, problem solving, flexibility, reversing and generalising (see figure on right).



The innovative aspect of RAMR is that the right half develops the mathematics idea while the left half reconnects it to the world and extends it. For example, whole-number place value built around the **pattern of threes** where hundreds-tens-ones applies to ones, thousands, millions, and so on, can be easily extended to metrics by considering the ones to be millimetres, the thousands to be metres and the millions to be kilometres.

Planning the teaching of mathematics is based around the RAMR cycle if it is deconstructed into components that are applied to a mathematical idea. By breaking instruction down into the four parts and taking account of the pedagogical approaches described above, the cycle can lead to a structured instructional sequence for teaching the idea. The table below briefly outlines how this can be done. Prerequisite mathematical ideas are considered in the Reality and Mathematics components of the cycle, while extensions and follow-up ideas are considered in the Reflection component.

<table>
<tr><td>REALITY
Local knowledge: Identify local student cultural-environmental knowledge and interests that can be used to introduce the idea.
Prior experience: Ensure existing knowledge and experience prerequisite to the idea is known.
Kinaesthetic: Construct kinaesthetic activities, based on local context, that introduce the idea.</td></tr>
<tr><td>ABSTRACTION
Representation: Develop a sequence of representational activities (physical to virtual to pictorial materials to language to symbols) that develop meaning for the mathematical idea.
Body-hand-mind: Develop two-way connections between reality, representational activities, and mental models through body → hand → mind activities.
Creativity: Allow opportunities to create own representations, including language and symbols.</td></tr>
<tr><td>MATHEMATICS
Language/symbols: Enable students to appropriate and understand the formal language and symbols for the mathematical idea.
Practice: Facilitate students’ practice to become familiar with all aspects of the idea.
Connections: Construct activities to connect the idea to other mathematical ideas.</td></tr>
<tr><td>REFLECTION
Validation: Facilitate reflection of the new idea in terms of reality to enable students to validate and justify their new knowledge.
Applications/problems: Set problems that apply the idea back to reality.
Extension: Organise activities so that students can extend the idea (use reflective strategies – flexibility, reversing, generalising, and changing parameters).</td></tr>
</table>

Appendix B: AIM Scope and Sequence

Yr	Term 1	Term 2	Term 3	Term 4
A	**N1: Whole Number Numeration** Early grouping, big ideas for H-T-O; pattern of threes; extension to large numbers and number system **N2: Decimal Number Numeration** Fraction to decimal; whole number to decimal; big ideas for decimals; tenths, hundredths and thousandths; extension to decimal number system	**O1: Addition and Subtraction for Whole Numbers** Concepts; strategies; basic facts; computation; problem solving; extension to algebra **M1: Basic Measurement (Length, Mass and Capacity)** Attribute; direct and indirect comparison; non-standard units; standard units; applications	**O2: Multiplication and Division for Whole Numbers** Concepts; strategies; basic facts; computation; problem solving; extension to algebra **M2: Relationship Measurement (Perimeter, Area and Volume)** Attribute; direct and indirect comparison; non-standard units; standard units; applications and formulae	**G1: Shape (3D, 2D, Line and Angle)** 3D and 2D shapes; lines, angles, diagonals, rigidity and properties; Pythagoras; teaching approaches **SP1: Tables and Graphs** Different tables and charts; bar, line, circle, stem and leaf, and scatter graphs; use and construction
B	**M3: Extension Measurement (Time, Money, Angle and Temperature)** Attribute; direct and indirect comparison; non-standard units; standard units; applications and formulae **N3: Common Fractions** Concepts and models of common fractions; mixed numbers; equivalent fractions; relationship to percent, ratio and probability	**G2: Euclidean Transformations (Flips, Slides and Turns)** Line-rotation symmetry; flip-slides-turns; tessellations; dissections; congruence; properties and relationships **O3: Common and Decimal Fraction Operations** Addition, subtraction, multiplication and division of common and decimal fractions; models, concepts and computation	**A1: Equivalence and Equations** Definition of equals; equivalence principles; equations; balance rule; solutions for unknowns; changing subject **N4: Percent, Rate and Ratio** Concepts and models for percent, rate and ratio; proportion; applications, models and problems	**SP2: Probability** Definition and language; listing outcomes; likely outcomes; desired outcomes; calculating (fractions); experiments; relation to inference **G3: Coordinates and Graphing** Polar and Cartesian coordinates; line graphs; slope and y-intercept; distance and midpoints; graphical solutions; nonlinear graphs
C	**A2: Patterns and Linear Relationships** Repeating and growing patterns; position rules, visual and table methods; application to linear and nonlinear relations and graphs **N5: Directed Number, Indices and Systems** Concept and operations for negative numbers; concept, patterns and operations for indices; scientific notation and number systems	**A3: Change and Functions** Function machine; input-output tables; arrowmath notation, inverse and backtracking; solutions for unknowns; model for applications to percent, rate and ratio **G4: Projective and Topology** Visualisation; divergent and affine projections; perspective; similarity and trigonometry; topology and networks	**O4: Arithmetic and Algebra Principles** Number-size, field and equivalence principles for arithmetic; application to estimation; extension to algebra; simplification, expansion and factorisation **SP3: Statistical Inference** Gathering and analysing data; mean, mode, median, range and deviation; box and whisker graphs; large data sets, investigations and inferences	**A4: Algebraic Computation** Arithmetic to algebra computation; modelling-solving for unknowns; simultaneous equations, quadratics **O5: Financial Mathematics** Applications of percent, rate and ratio to money; simple and compound interest; best buys; budgeting and planning activities

Key: N = Number; O = Operations; M = Measurement; G = Geometry; SP = Statistics and Probability; A = Algebra.

Accelerated Inclusive Mathematics Project

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