Accelerated Inclusive Mathematics Project

YuMi Deadly Maths

AIM Module G3

Year B, Term 4

Geometry:

Coordinates and Graphing

ACKNOWLEDGEMENT

The YuMi Deadly Centre acknowledges the traditional owners and custodians of the lands in which the mathematics ideas for this resource were developed, refined and presented in professional development sessions.

YUMI DEADLY CENTRE

The YuMi Deadly Centre is a research centre within the Faculty of Education at the Queensland University of Technology which is dedicated to enhancing the learning of Indigenous and non-Indigenous children, young people and adults to improve their opportunities for further education, training and employment, and to equip them for lifelong learning.

"YuMi" is a Torres Strait Islander Creole word meaning "you and me" but is used here with permission from the Torres Strait Islanders' Regional Education Council to mean working together as a community for the betterment of education for all. "Deadly" is an Aboriginal word used widely across Australia to mean smart in terms of being the best one can be in learning and life.

YuMi Deadly Centre's motif was developed by Blacklines to depict learning, empowerment, and growth within country/community. The three key elements are the individual (represented by the inner seed), the community (represented by the leaf), and the journey/pathway of learning (represented by the curved line which winds around and up through the leaf). As such, the motif illustrates the YuMi Deadly Centre's vision: *Growing community through education*.

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DEVELOPMENT OF THE AIM MODULES

Contents

Module Overview

As stated in Modules G1 and G2, human thinking has two aspects: verbal logical and visual spatial. Verbal logical thinking, associated in some literature with the left hemisphere of the brain, is the conscious processing of which we are always aware. It tends to operate sequentially and logically and to be language and symbol (e.g. number) oriented. On the other hand, visual spatial thinking, associated in some literature with the right hemisphere of the brain, can occur unconsciously without us being aware of it. It tends to operate holistically and intuitively, to be oriented towards the use of pictures and seems capable of simultaneous processing.

Our senses and the world around us have enabled both these forms of thinking to evolve and develop. To understand and to modify our environment has required the use of logic and the development of language and number plus an understanding of the space that the environment exists in, and an understanding of shape, size and position that enables these to be visualised (what we call geometry). Thus, as it is a product of human thinking that has emerged from solving problems in the world around us, mathematics has, historically and presently, two aspects at the basis of its structure: number and geometry. (*Note:* School geometry appears to be the strand which is not so confronting to non-Western cultures, and to be an area in which nearly all cultures have excelled, particularly with respect to the geometric aspects of art.)

This module looks at the teaching of the second of these bases, geometry. In particular it looks at position and then **coordinates** and then uses these coordinates to introduce linear **graphs**. This Module G3 *Coordinates and Graphing* follows on from Module G1 *Shape* and Module G2 *Euclidean Transformations*, and is followed by Module G4 *Projective and Topology* (see scope and sequence for modules in **Appendix B**).

Background information for teaching coordinates and graphing

Geometry provides unique understanding to number, algebra, measurement and statistics and graphs. In particular, this module provides the coordinate underpinnings of graphing that is a major part of algebra in the junior secondary years. In this subsection, we provide brief background knowledge on coordinates and graphs, covering connections and big ideas, content and definitions, and specific geometry pedagogies.

Connections and big ideas

It is important to see mathematics as a structure built around big ideas, connections and sequencing. The AIM *Overview* describes this structure and its importance, which is that it enables teachers to:

(a) *determine what mathematics is important to teach* (mathematics with many connections or based on big ideas is more important than mathematics with few connections or little use beyond the present);

(b) *link new mathematics ideas to existing known mathematics* (mathematics that is connected to other mathematics or based on the one big idea is easier to recall and provides options in problem solving);

(c) *choose effective instructional materials, models and strategies* (mathematics that is connected or based around a big idea can be taught with similar materials, models and strategies); and

(d) *teach mathematics in a manner that makes it easier for later teachers to teach more advanced mathematics* (by preparing the linkages to other ideas and the foundations for the big ideas the later teacher will use).

The major big ideas for this module are listed below. There are many more big ideas in mathematics and even in geometry but this list focuses on those important for coordinates and graphing.

1. **Change vs relationship**. Mathematics has three components – objects, relationships between objects, and changes/transformations between objects. Everything can be seen as a change (e.g. 2 goes to 5 by +3, similar shapes are formed by "blowing one up" using a projector) or as a relationship (e.g. 2 and 3 relate to 5 by addition, similar shapes have angles the same and sides in proportion or equivalent ratio). Geometry can be the study of shape or transformations between shapes (e.g. flips, slides and turns). This module is relationship.

2. **Interpretation vs construction**. Things can either be interpreted (e.g. what operation for this problem, what principles for this shape) or constructed (write a problem for 2+3=5; construct a shape of four sides with two sides parallel). This is particularly true of geometry – shapes can be interpreted or constructed – and very true of graphing.

3. **Parts vs wholes**. Parts (these can also be seen as groups) can be combined to make wholes (this can also be seen as a total), and wholes can be partitioned to form parts (e.g. fraction is part to whole, ratio is part to part; addition is knowing parts and wanting whole, division is partitioning a whole into many equal parts, the whole area can be partitioned into equal square units). In geometry, this big idea is particularly applicable to dissections and tessellations.

4. **Cartesian vs polar**. Position can be found by rows and columns related to axes or a starting point, a direction and a distance.

5. **Shape vs algebra**. Shapes can be placed within Cartesian coordinates, and then can be represented by algebraic formulae. This gives two ways of thinking about and representing graphs and shapes. In particular the algebraic formula for a straight line is $y = mx + c$ where m is slope and c is y-intercept.

6. **Graphing formulae and algorithms**. The properties of Cartesian coordinates mean that formulae and algorithms exist for slope, distance (uses Pythagoras's theorem), midpoints, equations of a straight line and other graphs, and finding solutions to equations.

Content and definitions

This subsection briefly describes the major content of this module.

Position and mapping

Initially, position can be defined in terms of everyday words such as "here", "there", "near", "far", "under", "over", "above", "below", etc. Then position is seen in terms of coordinates (see below): (a) Polar – where directions and distances are given to define a position; and (b) Cartesian – where numbers (and letters) are used to more formally determine position in terms of ordered pairs of numbers and/or letters (e.g. "row C seat 4", "2 blocks south and 1 block west", or an ordered pair of numbers).

Cartesian coordinates

start ● 140 degrees from North

25 metres
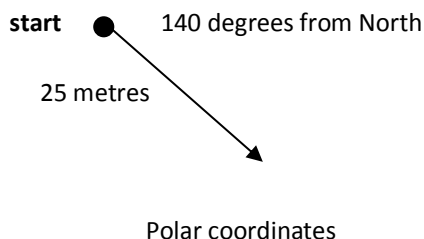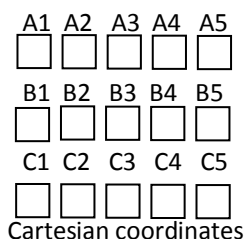
Polar coordinates

Cartesian and polar coordinate systems

Position is now extended to maps which can be based on:

(a) Cartesian coordinates, where position is in terms of two values or coordinates (e.g. theatres and street directories where the values are often given by a letter for the row and a number for the column, and latitude and longitude, where the values are both numbers); or

(b) Polar coordinates, where directions are shown from a starting point by directions and distances (e.g. hand-drawn or computer maps showing turns and distances, modern GPS systems which verbally/visually give directions).

Compass bearings and GPS

The special areas that need to be explored are direction in terms of: (a) north, east, west and south (i.e. compass bearings and smart phone directions), and the use of this knowledge with knowledge of map reading to enable learners to undertake orienteering and bush walking activities; and (b) latitude and longitude as this is the basis of GPS systems that find position on earth's surface.

Negative numbers and four quadrants

Another particular extension in later years is the use of negative numbers in coordinates; initially coordinates are all positive (as in the first diagram on right) and then there are four quadrants as the negative numbers are brought in (as in the second diagram on right).
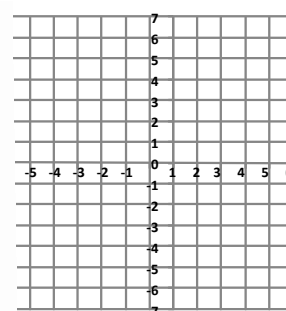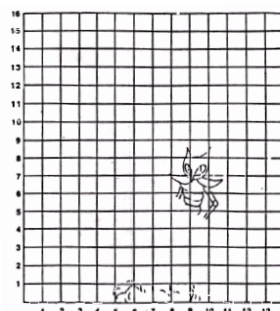
Line and other graphs

If two points are plotted on an axis, they can be joined by a straight line. This straight line reflects a sequence of points that relate linearly, in terms of equations of form, for example, $2x + 3$. Such lines, diagrammatically represented on the right, are defined by their slope (the rate determined by y increase divided by x increase or y increase across x increase of 1) and where they cut the y axis. The slope is usually represented by m and the y-intercept by c, thus the general equation of a straight line is $y = mx + c$. The line on the right has slope of 2 because y increases by 2 for each increase of x of 1, and cuts the x axis at 1. Thus the equation is $2x + 1$.
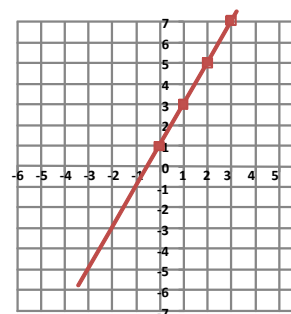
Nonlinear graphs are also formed from plotting points on axes, most noticeable the circle and the parabola.

Specific geometry pedagogies

Similar to the other strands of mathematics, geometry can be seen as a structure, as a language and as a tool for problem solving. Too often in the past teachers have focussed on the language aspect – developing the names for various shapes (such as prisms, polygons, cylinders), rules for relationships such as similarity (e.g. equal angles) and procedures for constructions (e.g. bisecting an angle). Yet some of the more interesting activities are associated with structure (e.g. Euler's formulae, the relationship between slides/turns and flips) and development of problem-solving skill (e.g. dissections, tessellations), particularly with respect to visual imagery.

Thus, geometry can be one of the most exciting and interesting sections of mathematics. It provides an opportunity for motivating learners that should not be missed. It can be colourful and attractive. Pattern and shape can be created and admired. Success can be enjoyed by the majority.

However, to allow the best development of geometric understandings, the following are important for effective sequencing of geometry teaching and learning.

1. The focus of geometry should be **from and to the everyday world** of the learner (as in the Reality–Abstraction–Mathematics–Reflection (RAMR) framework that is advocated in YDM).

2. There should be a **balance between** geometry experiences which enable learners to **interpret** their geometric world and **geometry processes, where problems are solved** with visual imagery, that is, geometry should be within a problem setting.

3. Learners' activity should be **multisensory** (using the students' bodies and actual physical materials and moving and transforming them – as in the body → hand → mind of the RAMR framework) and **structuring** (recording results on paper in words and pictures) – the "typical" geometry classroom would have groups, physical materials and pens and books ready to record, and there should also be opportunities for learners to display what they have made.

4. Teaching activities should move through **three levels of development** (based on **van Hiele levels**):

> the **experiential** level, at which learners learn through their own interaction with their environment (shapes are identified and named – e.g. this is a triangle);
>
> the **informal/analysis** level, in which certain shapes and concepts are singled out for investigation at an active, non-theoretical level (e.g. triangles have three sides and three angles); and
>
> the **formal/synthesis** level, where a systematic study is undertaken and relationships identified (e.g. interior angle sum of triangles is 180°).

At the experiential level, learners should be allowed to learn through experience with materials, not the teacher's words. Shape can be labelled and described but not broken into its component parts. Learners should not be expected to be accurate in their statements.

At the informal level, experiences can include analysing shapes for their properties/principles and constructing shapes from their properties. The sub-concept approach discussed later would be appropriate here.

At the formal level, the focus should be on synthesis and relationships and principles such as congruence and similarity can be investigated, and formulae discovered. There should be no attempt at deductive proof and posing abstract systems.

Sequencing for coordinates and graphing

This section briefly looks at the role of sequencing in geometry and in this module.

Sequencing in geometry

By its very nature, geometry does not have the dominating sequential nature of arithmetic and much more teacher choice is available in determining appropriate teaching sequences. There are also many experiences in geometry not directly connected to the development of rules and general procedures but rather to the development of imagery and intuition and as such may not be recognised as important by teachers.

Thus, YDM has developed a sequence for geometry that is based on, but enriches, the geometry in the *Australian Curriculum: Mathematics*. It recommends that the study of geometry has:

(a) two overall approaches: relationship geometry and transformation geometry;

(b) two sections in relationship geometry: Shape (line, angle, 2D and 3D shape, and Pythagoras's theorem) and Coordinates and graphs (polar and Cartesian coordinates, line graphs, slope and y-intercept, and graphical solutions to unknowns); and

(c) two sections in transformational geometry: Euclidean (flips, slide, turns and congruence) and Projective and Topology (projections, similarity, trigonometry, perspective and networks).

The overall sequence for this is on the next page. The four sections are covered by four modules: G1 *Shape*, G2 *Euclidean Transformations*, G3 *Coordinates and Graphing*, and G4 *Projective and Topology* (see **Appendix B**).

```
                            GEOMETRY
              ↙                            ↘
        RELATIONSHIP                    TRANSFORMATION
       ↙          ↘                    ↙            ↘
   Shape    Coordinates          Euclidean      Projective
            and Graphing       Transformations  and Topology
```

Shape

3D concepts and experiences
↓
2D concepts and experiences
↓
Line-angle-path experiences
↓
2D properties and relationships
↓
Pythagoras
↓
3D properties and relationships

Coordinates and Graphing

Directions and polar coordinates
↓
Cartesian coordinates
↓
Directed-number axes and line graphs
↓
Slope and y-intercept
↓
Distance and midpoints
↓
Graphical methods to solve for unknowns

Euclidean Transformations

Flips-slides-turns and symmetry concepts
↓
Tessellation and dissection experiences
↓
Properties and puzzles
↓
Congruence
↓
Relationships, visual imagery and art

Projective and Topology

Visualisation experiences
↓
Projective experiences and properties
↓
Topology experiences and properties
↓
Similarity and scale
↓
Perspective and networks
↓
Trigonometry

Sequencing in this module

As can be seen above, this module will move from informal directions to polar coordinates, from Cartesian coordinates to graphs (with negative and positive numbered axes), and to applications. The various applications of the graphs to be covered are: (a) construction of line graphs (extending Module SP1 *Tables and Graphs*); (b) slope and *y*-intercept, from patterns (similar to what is in Module A2 *Patterns and Linear Relationships*); (c) distance through Pythagoras (applying a part of Module G1 *Shape*); (d) midpoints through a formula; (e) solutions for unknowns through plotting linear graphs (extending A1 *Equivalence and Equations*); and (f) nonlinear graphs.

Thus, the sections and units of this module are as follows:

Overview: Background information, sequencing and relation to Australian Curriculum

Unit 1: Directions and polar coordinates – experiencing directions and distance

Unit 2: Cartesian coordinates – experiencing finding position from a grid and axes

Unit 3: Axes and graphing – introducing four quadrants and plotting points to construct graphs

Unit 4: Distance and midpoints – how to teach these two skills

Unit 5: Graphical methods – finding solutions for unknowns

Unit 6: Nonlinear graphs – plotting and interpreting nonlinear graphs

Test item types: Test items associated with the six units above which can be used for pre- and post-tests

Appendix A: RAMR cycle components and description

Appendix B: AIM scope and sequence showing all modules by year level and term.

The modules are designed to provide resources – ideas to teach the mathematics – and sequences for teaching. Although the units are given in a recommended sequence, please feel free to change the sequence to suit your students. Each subsection should be taught using the RAMR cycle (see **Appendix A**) and, where possible, ideas will be given using the headings of the cycle.

Relation to Australian Curriculum: Mathematics

AIM G3 meets the Australian Curriculum: Mathematics (Foundation to Year 10)

Unit 1: Directions and polar coordinates
Unit 2: Cartesian coordinates
Unit 3: Axes and graphing
Unit 4: Distance and midpoints
Unit 5: Graphical methods
Unit 6: Nonlinear graphs

Content Description	Year	G3 Unit 1	2	3	4	5	6
Identify angles as measures of turn and compare angle sizes in everyday situations (ACMMG064)	3	✓	✓				
Create and interpret simple grid maps to show position and pathways (ACMMG065)	3	✓	✓				
Use simple scales, legends and directions to interpret information contained in basic maps (ACMMG090)	4	✓	✓				
Use a grid reference system to describe locations. Describe routes using landmarks and directional language (ACMMG113)	5	✓	✓				
Describe translations, reflections and rotations of two-dimensional shapes. Identify line and rotational symmetries (ACMMG114)	5		✓				
Continue and create sequences involving whole numbers, fractions and decimals. Describe the rule used to create the sequence (ACMNA133)	5			✓			
Investigate, with and without digital technologies, angles on a straight line, angles at a point and vertically opposite angles. Use results to find unknown angles (ACMMG141)	6	✓	✓	✓			
Investigate combinations of translations, reflections and rotations, with and without the use of digital technologies (ACMMG142)	6		✓	✓			
Introduce the Cartesian coordinate system using all four quadrants (ACMMG143)	6		✓	✓			
Investigate, interpret and analyse graphs from authentic data (ACMNA180)	7			✓			
Describe translations, reflections in an axis, and rotations of multiples of 90° on the Cartesian plane using coordinates. Identify line and rotational symmetries (ACMMG181)	7		✓	✓			
Given coordinates, plot points on the Cartesian plane, and find coordinates for a given point (ACMNA178)	7		✓	✓			✓
Solve simple linear equations (ACMNA179)	7			✓			
Plot linear relationships on the Cartesian plane with and without the use of digital technologies (ACMNA193)	7			✓			
Solve linear equations using algebraic and graphical techniques. Verify solutions by substitution (ACMNA194)	7			✓	✓		
Find the distance between two points located on a Cartesian plane using a range of strategies, including graphing software (ACMNA214)	9				✓		✓
Find the midpoint and gradient of a line segment (interval) on the Cartesian plane using a range of strategies, including graphing software (ACMNA294)	9				✓		
Sketch linear graphs using the coordinates of two points and solve linear equations (ACMNA215)	9			✓	✓	✓	
Explore the connection between algebraic and graphical representations of relations such as simple quadratics, circles and exponentials using digital technology as appropriate (ACMNA239)	10						✓
Solve simple quadratic equations using a range of strategies (ACMNA241)	10						✓

Unit 1: Directions and Polar Coordinates

There are two types of coordinates – polar and Cartesian. **Polar coordinates** give position based on starting positions, directions and distances. The system is based on finding the way with a compass (start at the corner of the house and walk 50 m in direction 75° from north) – it is like following a treasure map. **Cartesian coordinates** focus on position in relation to other objects or in a coordinate system (see below). It is like grid references for a street directory or latitude and longitude on a boat (they give position with two values related to axes). The diagrams below illustrate the two types of coordinates.

Cartesian coordinates   Polar coordinates
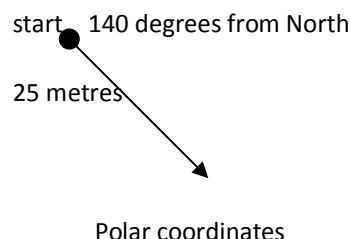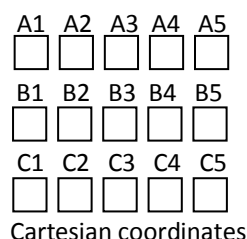
In this unit, we provide an overview of activities that focus on **position from a polar perspective**. As will become evident, these require strong development of language. As well as this, we provide a RAMR cycle lesson that encapsulates the major learning from this unit.

1.1   Overview of polar activities

Polar activities relate to direction and distance – they are like the GPS system, giving direction and distance.

Early teaching activities

1. **Play Polar "Mr Here"**. Obtain a small toy, hide the toy every day, provide students with instruction to find it that relate to direction (e.g. above the table, away from the blackboard, to the right of the mathematics books, and so on).

2. **Set up room to develop language**. Take students outside and photograph them doing polar position things "towards" something, "away from" something, "next to" something, and so on. Try to think of all the positional words you can (e.g. near, far, over, close, and so on). Take photographs of students doing these things, print them, attach words to them, and display around classroom.

3. **Giving and receiving directions**. Set up situations where students have to give directions to other students to walk through objects (e.g. turn left, walk ahead two steps, and so on). If safe, student following directions can be blindfolded but needs someone to walk with her/him.

4. **GPS game**. Use a mat to set up a town with shops, churches, and so on. Play the GPS game where one students acts as a car and the other as a GPS giving directions to travel from one place to the other.

Direction activities

1. **Treasure hunts**. Set up situations where students have to find things given a starting position, a direction and a distance (e.g. go to post with red circle on it, walk towards big tree, when travelled 15 stick lengths, look around for a rock, treasure will be under it).

2. **Constructing directional maps**. Draw plans for walking between familiar things (e.g. draw a map from here to the principal's office).

3. **Following directional maps**. Follow someone else's plan for getting from a start to a finish.

4. **Computers/GPS systems**. Give students experience with maps downloaded from internet to go to places. Give students experience following a GPS system for a car.

Orienteering activities

1. **Drop 20 cents**. Introduce pacing for distance in metres and compasses for direction in degrees from north and move onto orienteering type activities. Drop 20 cents on ground. Go forward 10 m, turn 120° using compass, repeat this twice more, how close are you to the 20 cent piece?

2. **Starting point**. Hide some things (letters for a word, numerals for a large number, or words and phrases for the answer to a joke) in the school yard. Give students a starting point and the angles or compass bearings and distances to these hiding places. Students follow these instructions to find the things.

3. **Shapes.** Give students shapes and compasses and ask them to make the shapes on the oval with stones or flags at the corners; move to instructions to travel from home to school and between towns (and so on).

4. **Orienteering track**. Get students to follow an orienteering track (short distance, pacing and angles, not running). Give them a start position and then distance and angle to the next position and so on until they get to finish point. At each position, have something they write into their directions that showed they got this far. Hide this something so have to look around for it so it is not visible as they walk up to the position.

1.2 Polar RAMR cycle: Make your own treasure map

This cycle sets the scene for a map from the reality of the students, abstracts ability to pace metres and find direction in terms of north and relate this to maps, mathematically prepares a treasure map for other students to follow, and reflects into reality to find situations where this type of map is used in our lives.

Reality

Discuss following directions. Have they had to give directions to another? Can they draw a rough map in terms of direction and distance to get around the school or somewhere else in the local area?

Prepare a map or a list of directions for the students to follow. Look up on the Internet for road directions to drive from one place to the other in a car.

Abstraction

Body

Set up a 10 metre distance between two lines and get students to pace this distance. Get them to calculate how many of their normal steps to 10 m. Get them to practise walking distance in m – measure some distances and get them to see how accurate they are in walking those distances.

Use Silva compasses or smart phones. If not available, make up a rough compass from a circular protractor (as on right) on a cardboard rectangle. Always align arrow with north (pick something like the side of a building or a tree in the distance which is close to north. Then follow direction given by angle 0 to 360° when arrow is pointing north.
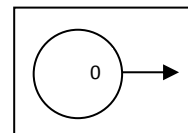
Teach students to find and walk distance – give them some directions to follow, so they can become used to doing it.

Practise finding things when distance and direction given. Set up a map for them to follow (give starting point and then distances and angles to follow).

Hand

Translate the walking activity to the drawing of a map giving distances and angles and vice versa. Use a ruler (10 m = 1 cm) and a protractor to draw the maps.

Mind

Encourage the students to imagine walking around a map in their minds. Give them verbal directions from the classroom when they have their eyes shut and see if they can guess where they will end up.

Mathematics

Language/Symbols

Ensure that students have all the language associated with directions – north, south, east and west; near, far, above, below, over, under, left, right, and so on. Take photos of students doing this and put on board with the words. Ensure students can use straight lines and angles, use a ruler and a protractor.

Practice

Make up a treasure map for students to follow. When they seem able to do this, get groups of students to make up own maps for other groups to follow.

Connections

Connect this work to angle and distance and the measurement of both.

Reflection

Validation

Ask students to make up directions for their neighbourhoods and draw maps.

Applications

Find maps in the world the students could interpret – maybe elders could provide directions.

Extension

Flexibility. Show a variety of types of maps.

Reversing. Do this by going from map to walking and vice versa.

Generalising. Ask students to describe the features of maps in terms of what gives what.

Unit 2: Cartesian Coordinates

Cartesian coordinates relate to finding position on a grid, such as a location in a street directory or a seat in a theatre. Position is usually given in terms of a coordinate pair, often a letter and a number for a grid position on a map, or two number values as in latitude and longitude. Initially coordinates are all positive, but this is extended to negative values in later years.

2.1 Overview of Cartesian activities

This section covers activities and games plus plotting and finding points with positive and negative numbers.

Early activities

1. **Play Cartesian "Mr Here" or Treasure hunt**. Take the small toy and hide it every day, but provide students with instruction to find it that relate to position re other things (e.g. next to the red box, in front of the whiteboard, and so on).

2. **Set up room**. Take students outside and photograph them being "above" something, "under" something, "in front of" something, "in the middle of" something, and so on. Try to think of all the positional words you can that relate to fixing position re other objects in terms of being in line. Take photographs, attach word to them and display around classroom.

3. **Drawing diagrams/maps of familiar areas you are in**. Ask students to draw a map of their classroom. Discuss what they produce and how it relates to the room (often not exact in terms of position in room but usually OK in terms of what next to – e.g. one student put all desks in a corner around the teacher's desk, leaving half the room with nothing in it).

4. **Drawing diagrams/maps of familiar areas**. Ask students to draw a map of their house from memory and discus what they produce.

5. **Constructing Cartesian maps**. Draw maps that have enough features that students can use them to find things (e.g. draw a map of the junior school).

6. **Following Cartesian maps**. Follow someone else's Cartesian map to find where something is.

7. **Set up room for coordinates**. Put chairs in rows and columns and labels on wall (e.g. A, B, C, and so on, for rows, and 1, 2, 3, and so on, for columns – as on right). Students are given coordinates by which to find their chair and table for the lesson. These arrangements can be changed, as can whether letters or numbers are used. The coordinates can also be used to find things.

8. **Setting up with Maths Mat**. Set up the Maths Mat with coordinates. Have students discover and plot points by standing on the appropriate square.
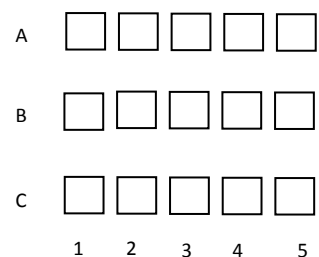
Coordinate games and activities

1. **Bear Pits**. This game can be played in small groups. You will need the playing board on right and a spinner with numbers 1, 2, 3, 4 as shown below (use a paper clip for the pointer); coloured marker per player. Each player starts at the point of origin (the shaded square) and spins the spinner twice – the first spin determines how many squares **across** you can move; the second spin determines how many squares **up** you can move. Place your marker on the region/square you landed on. If you land on a bear pit, you must go back to the start. The first player to reach either Edge 1 or Edge 2 is the winner.
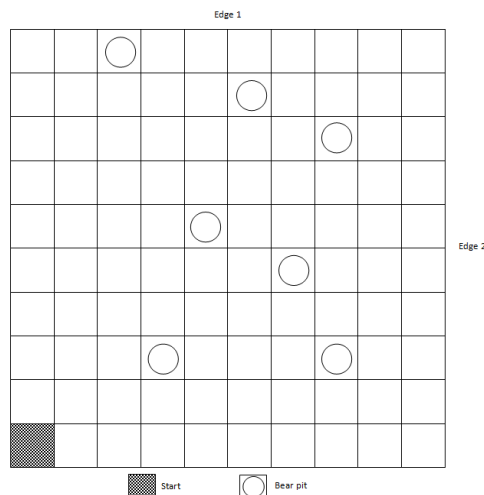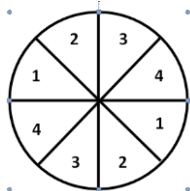
2. **Seek and Destroy** (a variation of Battleships). This is a game for two players. The objective of the game is to seek out and destroy your opponent's space station and missile destroyers without falling into the black hole. If you fall into the black hole, the game is over. When you've had one game each, try to destroy your opponent's space station and missile destroyers in a set number of calls. (a) Each player marks off a 6 × 6 section of the square grid to be their "war zone". Then sit back-to-back so that you can't see each other's grid. (b) On your grid, you are to place one space station (four points joined), two missile destroyers (two points joined per destroyer) and one black hole. The points can be horizontal, vertical or diagonal but they must form a straight line. An example of a war zone is shown at right. (c) One player begins the game by calling a pair of coordinates (e.g. 3, 2). On the war zone shown, this would be a miss and the other player gets a turn. If it had been a direct hit, then you would get another turn. It takes four hits to destroy the space station and two hits to destroy a missile destroyer.
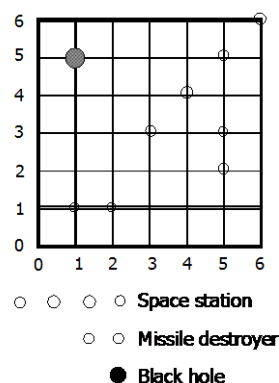
What solution strategies did you develop? That is, did you guess each time or did you refine your "guesses" after the first two or three turns? How is the grid in Seek and Destroy different from the grid in Bear Pit? How are the two grids the same?

Plotting and finding points

1. **Drawings**. Plot the points below on the grid. Join consecutive points (A, B, C etc.) with straight lines.

   A (6, 2); B (4, 2); C (4, 4); D (6, 6); E (6, 13); F (7, 15); G (8, 13); H (8, 6); I (10, 4); J (10, 2); K (8, 2); L (8, 1); M (6, 1), A (6, 2).

   Make up your own shape; draw it on a grid with corners where two grid lines intersect. Translate points to coordinates and provide coordinates to other students to draw the resultant shape.

2. **Maps and directories**. Collect maps and directories. Set up activities where students have to use coordinates to find positions. Provide positions and ask the students for coordinates.

3. **Construct maps**. Start with a grid. Construct/create your own map for this grid (say towns and roads). Set questions from the map that other students could answer.

   Construct a map of the school with a grid that covers the map. Ask students to find their way around this map by giving directions in terms of coordinates.

Negative numbers

1 Bring in plotting with negative numbers on the x axis. Set up mat from −5 to +5 and 0 to 6 and plot points. Set up a variation of Bear Pits and Seek and Destroy with positive and negative numbers on x axis. Do plotting points and other activities. Begin with students standing in the coordinate points.

2 Move on to four quadrants with positive and negative numbers in each axis as on right. Once again, repeat the games and activities to make students familiar with the four quadrants.
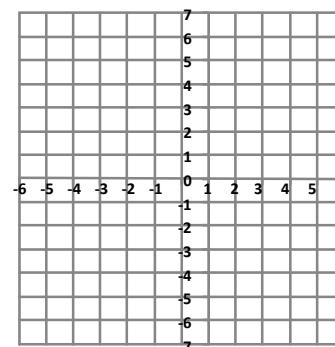
2.2   Cartesian RAMR cycle: Directed axes and flips-slides-turns

This activity covers activities above and then extends coordinates to flips, slides and turns.

Reality

Consider the reality of a mirror – student on one side and image on other. Consider a mirror along a wall, with people and images on each side – we can place the person with a + distance and the image with a – distance. Similarly many sports can be considered in this way – rugby league or soccer is 0 on the halfway line and positive/negative on each side; what about tennis?

Set up games where you have a grid with two sides, + and −. People move on each side trying to stop one person getting through; only one person can move and one member of a team has to get behind the other team; players can move sideways but not back. It is like a game of chequers or chess but with the pieces seen in terms of + and − in relation to the halfway line.

Abstraction

Body

1. Use mat to set up a grid with positive and negative numbers on the x axis; use rope to break mat into two halves and place numbers on grid lines.

2. Place numbers along x and y axes. Plot points on either side with students. Position students at coordinates. Go both directions: teacher states position and student moves to position; student stands in a position and states coordinates.

3. Repeat this for four quadrants. Use rope to break the mat into four quadrants. Once again, put positive and negative numbers on axes, and repeat: teacher gives coordinates → students stand in position AND teacher positions the students → students state coordinates.

4. Plot points with students to make shapes including plotting straight lines. Discuss slope of lines – relate to equation of line. Do many examples.

Hand

1. When doing above with body – replace students with a disc in their position and copy this onto a drawing of the mat (use graph paper) – once again, coordinate → position and position → coordinate.

2. Repeat all this for much larger coordinate situations – say 20×20 squares so quadrants all go from 0 to +10 or −10.

Mind

1. Imagine a coordinate system as large as needed and plot points on it in the mind.

2. Get students to shut eyes and draw out what they see with a finger in front of them.

Mathematics

Language/Symbols

1. Use the ordered pairs (−2,5) to repeat the work above.

2. Ensure all language is understood (e.g. axes).

Practice

1. Ensure that there are practices where bodies and pens are used to plot and determine coordinate points.

2. Ensure there are practices that reinforce the relation between symbols and position so it is well understood, particularly in four quadrants.

Reflection

Validation

1. Get students to look for coordinates in their lives – particularly with opposite sides where one side could be called negative.

Application (flips, slides and turns by coordinates)

1. Set up mat so there is a positive and negative side – one axis −5 to 5 and the other 0 to 6.

2. Use the flip-slide-turn work from G2, and the students' bodies on a grid (or mat) to show flips, slides and turns. Use a rope or band to divide the grid into two – label the axes.

3. Have three or four students make simple 3- to 5-sided shapes one side (holding on to corners of a 3- to 5-sided shape) and have three or four other students (with help from class) on the other side making the shape using the three different changes – sliding across the centre line (so shape and image is same distance away from line on each side), reflecting about line, and rotating 180°.

4. Repeat this for another group of students and another shape but, this time, use the labels on the lines to give the coordinates of the first shape and have students say where students should go for the image by **stating the coordinates**.

5. Repeat the above again but getting the students to state the coordinates before the shape is made, and to state the coordinates of corners to show flips-slides-turns.

6. Copy the original shape and its slide, flip and turn onto graph paper – write in the coordinates of corners. Repeat this activity by only using paper – not the mat.

7. **Computers**. Repeat the above on computer. Place a grid on a computer, draw simple shapes, use the mouse and the software to slide shape along a line that starts and ends on coordinates, flip shape about a line that starts and ends on coordinates, and rotate shape 90°, 180° and 270° about a centre which is a coordinate. Write down the coordinates.

8. **Quadrants**. Repeat the above but for the mat/grid divided into four sections using x and y axes (the coordinates can now be negative). Make shapes in one quadrant (top right is normal). Use coordinates to slide the shape across axes, flip the shape about axes, and rotate the shape 90°.

Extension

Flexibility. Use a variety of grids and shapes.

Reversing. We have done this in the coordinates ←→ position changes (and see generalising below).

Generalising. Write down the starting and finishing coordinates, and **compare differences** in these coordinates for flips, slides and turns – use these comparisons to suggest **coordinate rules** for these slides, flips and turns – then reverse and flip, slide and turn only with coordinate rules and use these to draw the flips, slides and turns and see that the rules hold.

Unit 3: Axes and Graphing

In this unit, we look at plotting linear equations to give line graphs and the role of slope and y-intercept. These are related to growing and fixed parts of patterns in Module A2 and ratio in Module N4. This relates to defining the line by $y = mx + c$ where m is the slope (y change for an x change of 1) and c is the y-intercept (where the line crosses the y axis).

Finally the requirement that there be only two coordinate points to define any line is shown. This should be considered in relation to Module G1 *Shape*, particularly lines and angles. (*Note*: It is important that information in this unit is elicited as discoveries, NOT told.)

A line graph is a straight line. Therefore it only needs two points to define it.

To plot a graph:

(a) a Cartesian coordinate system is drawn (as on right);

(b) points on this line are determined (source could be patterns or function machine as below, or could be two points):

y	1	3	5	7	9	11
x	0	1	2	3	4	5

(c) these points are plotted and a line drawn through them (as on right);

(d) since the points above have a fixed part of 1 and increase 2 for each change of 1 in x, the graph is a representation of $y = 2x + 1$.
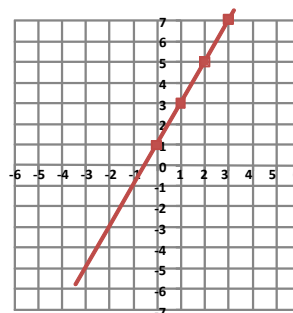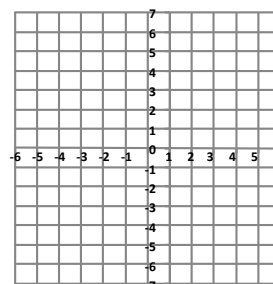
3.1 Overview of line graph activities

These activities cover plotting line graphs, relating line graphs to linear patterns from Module A1, and ways of defining line graphs.

Plotting line graphs

1. **Kinaesthetic activity**. Take students out to the school yard and line them up on a line. Number them in order from 0 to whatever. Take a linear equation (say, $2x + 1$) and state to each student that they have to double the number given to them, add one, and take that many steps (see diagrams below).

$$2x + 1$$

double plus 1
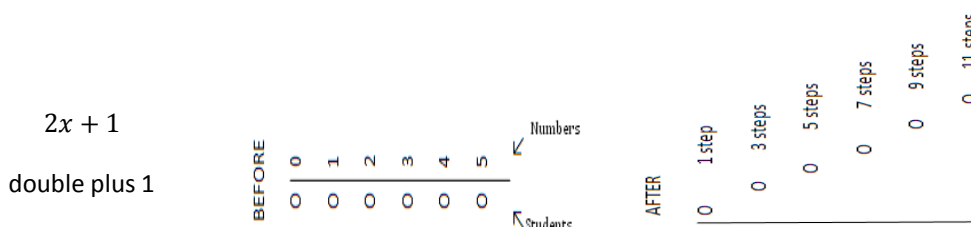
When they have finished, discuss that they have made the graph of linear equation $y = 2x + 1$ with their bodies. (*Note:* Train students to take the same-size steps.)

2. Next, place students on a line (preferably on a large grid) so that they can make lines with negative slopes, that is, examples like $1 - 2x$. Here, of course, the students would have to start at the top of the grid, or along a line (not a wall) so they can step forward or back.

   It is best if you can do this with actual students on a grid as shown on right (using a mat or a grid painted on the school yard). Students need to stand on intersections of lines so they imitate a graph.
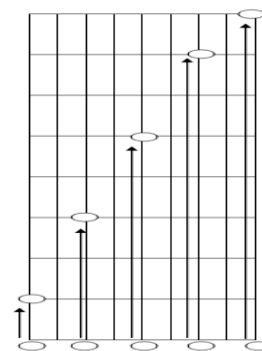
3. **Formalising kinaesthetic activity**. Get students to hold a rope so they can see that they have made a straight line. Then students can replace themselves with a token and draw a copy of the graph on graph paper to see what they have acted out. Finally, students should be encouraged to imagine the change in their mind (so completing body → hand → mind).

Relating to patterns (Module A2)

1. **Relating patterns to linear equations**. Start with a pattern, e.g. oo x, oo xx, oo xxx, oo xxxx, and so on. Determine fixed and growing parts, determine position rule [$n + 2$] and plot graph. Rename position as x and pattern value as y, redraw graph and rewrite the rule ($y = x + 2$). This is the linear equation or function that the graph now represents. Repeat this activity if necessary.

2. **Extend line graphs to four quadrants.** Continue lines into negative coordinates.

3. **Reverse the situation**. Starting with a linear equation e.g. $y = 3x - 2$, draw a table of x and y values as on right and fill in the table for x = 0, 1, 2, 3 and so forth. Draw the graph. Go backwards and forwards with other examples: graph → linear equation (interpreting) and linear equation → graph (plotting).

y				
x	0	1	2	3

4. **Look for a relationship**. Relationships between patterns and graphs of linear equations can be found by relating solutions to different patterns, looking for similarities and differences. Provide examples of patterns that have fixed growing parts (pattern rule is a linear equation) and are related in a way that enables relationships with regard to graphs and functions to be seen. In other words look at patterns with changing growing parts and the same fixed part (e.g. patterns that lead to $x + 1$, $2x + 1$, $3x + 1$, and so on), and patterns with the same growing part and changing fixed parts (e.g. patterns that lead to $2x$, $2x + 1$, $2x + 2$, and so on).

   Study patterns and their pattern rule and identify information on a table as below.

 | PATTERN | COMPOSITION | POSITION RULE | GRAPH CHARACTERISTICS |
   |---|---|---|---|
   | (a) | Grows by 2, fixed part of 1 | $2n+1$ | Slope 2, y-intercept 1 |

   This leads to the **general rule for slope**: if the growing part is p, the pattern rule starts with pn and the graph has slope p. If the fixed part is c, this means that the **function for a linear equation** is as follows:

   $f(x) = mx + c$ where m is the slope (and the growing part if from a linear growing pattern) and c is the y-intercept and the constant part for the zero term ($n = 0$).

Defining a line graph

1. **Two points are all that is needed**. Relook at patterning activities. Use table but plot only two points. Draw a line between them continuing on each side.

   Plot extra points and show that they fit on the line and are not needed (as shown below). Repeat for different two points – show that the line is the same.

| Plot two points | Draw graph | Plot other points |
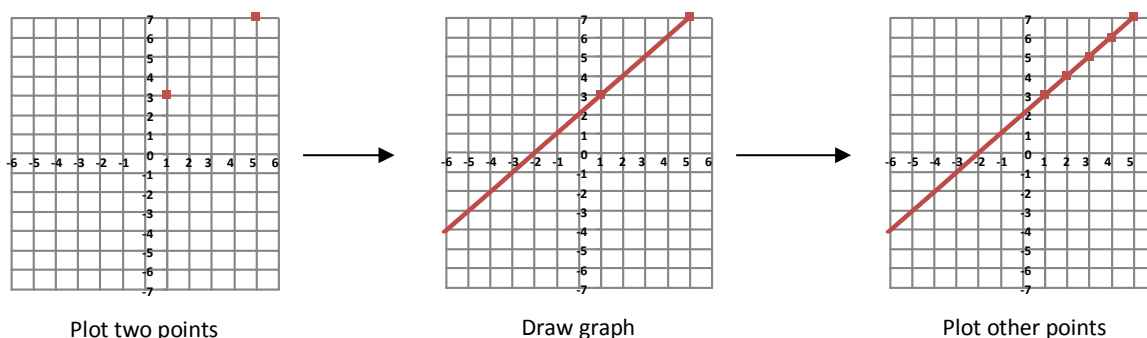
2. **Finding y-intercept**. Continue line to cut y axis – this y value is the y-intercept.

3. **Finding slope through rate**. Pick two points – find the y-difference and divide by the x-difference – this gives slope which is a rate (i.e. y-difference per x-difference). Of course, we can simply find the y-difference for x-difference of 1.

y-difference 2   Slope = y-difference $\div$ x-difference $= \frac{2}{4} = \frac{1}{2}$

x-difference 4
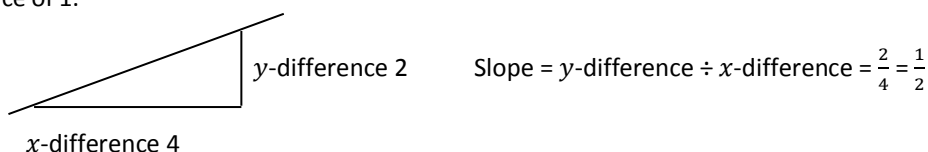
Note: This is an unusual calling of rate in one way. Normally when multiplicatively comparing one thing against the other, rate is when attributes are different (e.g. km/h or $/kg) while ratio is used when attributes are the same (e.g. concrete is 5:2, that is, 5 kg of sand to 2 kg of cement). This slope is distance per distance but they are of different axes.

4. **Fraction and negative slope**. Use fraction slopes and negative slopes to draw lines – practise a variety of these.

5. **Equation $y = mx + c$**. Construct a line graph from a pattern using a table of values. Find slope m and y-intercept c. Plot points for $y = mx + c$ and show the graph is the same. Repeat this for negative and fraction slopes.

3.2   Graphing RAMR cycle: Slope, y-intercept and line graphs

This cycle shows how slope and y-intercept define a line graph.

Reality

Discuss local situations which are linear relationships that can be represented by line graphs. Look at different patterns. For example:

- My boat can travel for 3 hours on its fuel tank before I need to refill, and 2 hours on a can of fuel, how long can I travel on 1, 2, 3 (and so on) cans of fuel? [$2x + 3$]

- I bought a $3 icecream for everyone plus a $5 chocolate. How much for 1 person, 2, 3, and so on? [$3x + 5$]

- I had $13 and I paid $2 per hour to be there, how much money did I have left or how much do I have to go into debt? [$-2x + 13$]

- X   XOO   XOOOO   XOOOOOO   and so on [$2x + 1$]

Act out these problems with materials representing the objects in the situation.

Abstraction

Body

Use students on grids representing different numbers to act out the line graph (see kinaesthetic activity for plotting line graphs in section 3.1 above). Copy the resulting line graphs.

Hand

Use results for different numbers to make a table and plot points using this table. Use the points to draw a line graph. Calculate the rate (m) and the y-intercept (c), extending the graph if necessary. Now represent the graph with an equation $y = mx + c$ putting in the values of m and c. Use the equation to plot the graph again and to see that it represents the line determined from the situation.

Prepare tables for all the situations with all x values present and in order from 1. Look at the difference between the elements, and the value for 0 (may have to extrapolate that by extending the graph). Show that the slope is the difference and the y intercept is the 0 value.

Ensure students practise examples with negative and fraction slopes.

Mind

Have students imagine graphs with various slopes (fraction and negative) and y-intercepts. Draw the graphs in air with fingers while eyes shut. Be able to imagine all types.

Mathematics

Language/symbols and practice

Ensure words such as slope etc. are understood, as is $y = mx + c$. Practise relationship between lines and $y = mx + c$, going from line graph → equation and equation → line graph.

Connections

Ensure connections are made between line graphs (m and c) and pattern rules, and also between line graphs and function machine rules.

Reflection

Validation/Application

Ensure that students can see the relationships in Mathematics above. Undertake applications as follows.

1. **Spaghetti Bridge**. Materials: spaghetti, foam cup, paper clip for hook. Use spaghetti, one strand, between two desks. Suspend the cup with an opened-out paper clip hooked over the spaghetti. Add nails (or similar) one at a time to see when it collapses. Repeat for two strands of spaghetti, then three, and so on. What will they hold? How many strands would you need to support a mobile phone (or similar)? Create a chart to record the results and graph the outcomes.

2. **Heart rates**. Take pulse before starting. Do step-up exercises for two minutes then take pulse again. Continue to record the pulse every two minutes until the heart rate has fallen back to normal. Record on a chart and plot the graph. Discuss the slow-down rate. (*Note*: Athletes drop back to normal very quickly.)

Extension

Flexibility. Do many forms of linear situations and for many slopes.

Reversing. Make sure you go from situation to line graph and reverse and line graph to equation and reverse.

Generalising. Make sure all students can generalise a line graph to an equation.

Unit 4: Distance and Midpoints

Once we have plotted line graphs, we can find distance along lines from one coordinate point to a second and halfway along a line from one coordinate to a second. The formulae are:

1. **Distance**. Distance between two coordinates points (x_1, y_1) and (x_2, y_2) is the square root of the sum of the x-difference squared and the y-difference squared:

$$\text{Distance} = \sqrt{(x\text{-difference})^2 + (y\text{-difference})^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

2. **Midpoint**. The midpoint between two coordinate points (x_1, y_1) and (x_2, y_2) is the x and y average:

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

4.1 Overview of distance activities

This subsection looks at some activities to find and to apply the formulae above.

Distance between two points

1. **Parallel to axes**. Find distances where lines are parallel to the axes – this is just the x-difference or the y-difference.

2. **Pythagoras**. Recap Pythagoras's theorem and use this with the line – this gives the formula.

$$\text{distance} = \text{hypotenuse} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
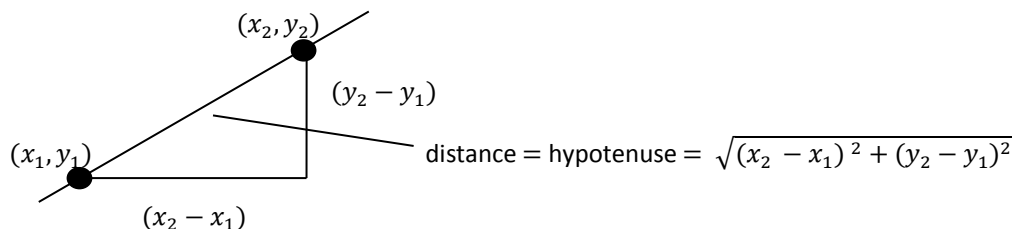
3. **Application**. Find various distances using Pythagoras's theorem for a variety of slopes – including fraction and negative.

4. **Problems**. Given midpoint and one point, find the other end of a line segment.

Midpoints

1. **Discovery for simple examples.** Draw a line between two points, use ruler to halve it and write down coordinates of the midpoint. Discover the rule for halfway.

2. **Application**. Use a ruler to find halfway and check it is the midpoint using the formula.

3. **Extension**. Extend to finding one-third points and quarter points along lines. What is the general rule?

4.2 Distance RAMR cycle: Distance and midpoints

This looks at finding distance and midpoint formulae for straight-line graphs.

Reality

Discuss the point of no return where a plane is better flying on than turning back. Discuss how we find these and how they are related to distance. Look at plotting travel on a grid like in radar – how do we know the point of no return?

Act this out – give two positions outside – students pace out distances – students re-pace to mark halfway. Try this on a grid.

Abstraction

Body

Start with mat and simple examples to find distances, half distances and midpoints.

Hand – Stage 1: Distance

The shortest distance between two points is a straight line and this line is a graph of a linear equation. This distance can be found by using Pythagoras's theorem as below.

1. Plot two points (say, $x=1$, $y=1$ and $x=4$, $y=7$) as on right. (*Note: Should do this first on the mat or outside with the students' bodies.*)

2. Draw a line between the points (and ongoing, as shown on right).

3. Make a right-angle triangle with the sides parallel to the x and y axes as shown on right.

4. The sides parallel to the x and y axes have lengths (here, 4–1=3 for x and 7–1=6 for y).

5. Using Pythagoras's theorem, $d^2 = 4^2 + 7^2$ therefore $d = \sqrt{16 + 49}$.

6. Repeat this for other points; ask students for pattern or generalisation. Encourage or elicit that: (a) distance is calculated by Pythagoras's theorem; (b) it is based on the x-difference and the y-difference; and (c) the actual distance is:

$$\sqrt{(x\text{-difference})^2 + (y\text{-difference})^2}$$

7. With advanced students, generalise so that if two points are (x_1, y_1) and (x_2, y_2) then

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

8. Practise whatever generalisation students come up with.

Hand – Stage 2: Midpoints

1. Gather students around a large grid. Stand students on two points (see O's on graph on right). Join them with a rope (line).

2. Get a student to stand halfway (can take the rope and fold in half to find this point but must keep direction), as shown by X on right.

3. Discuss where this midpoint is in terms of coordinates; students should
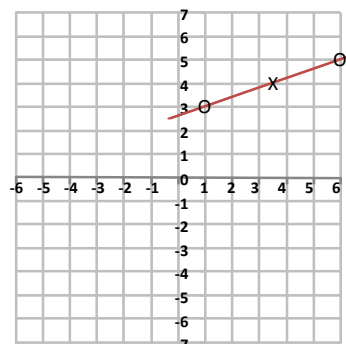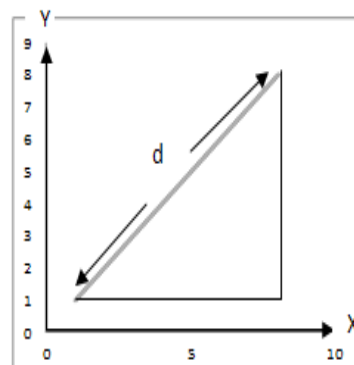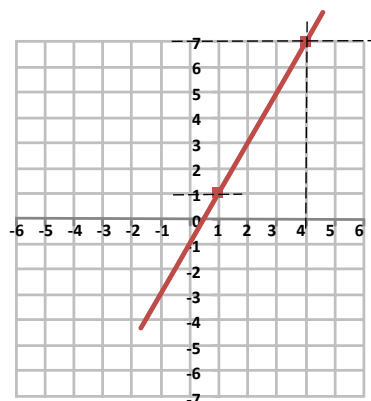
justify answers. For example, points are (1,3) and (6,5) – halfway is (3½,4) because this is halfway for x's and for y's.

4. Get students to draw this on a graph (place tokens on the grid to help). Repeat for other examples, some simpler and some harder.

Mathematics

Try to elicit a pattern, but do not insist on correct answers. Be happy with "the midpoint is the point halfway between the x's and halfway between the y's".

If looking for formality, start with easier examples such as (4,2) and (8,8). Discuss how to find halfway for x's of 4 and 8. Show that this is 6 which is 2 more than 4 and 2 less than 8. Show that this can be found by the average of the two points, that is, adding the two points and dividing by 2 ($\frac{4+8}{2} = 6$).

Make sure the connection to Pythagoras's theorem is obvious for the distance calculation.

Reflection

Two things here: generalisations and applications. Flexibility is in the examples. Reversing has been covered throughout the activities by ensuring we go from midpoint to end points and end points to midpoint.

Generalisations

1. Generalise distance so that if two points are (x_1, y_1) and (x_2, y_2) then
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

2. Try to get to the midpoint formula: that is, midpoint between (x_1, y_1) and (x_2, y_2) is
$$\text{Midpoint} = (\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$$

3. Practise the generalisation students come up with; try to get them to see it formally but the important thing is that they see it informally.

Applications

1. Give distance, direction (slope) and one end point – students have to find other end point.

2. Give midpoint and one end point – students have to find the other end point.

3. Find one-third and quarter points and reverse.

4. A freeway and two towns are positioned as on right. Where will the turn-off be placed so the new roads (dotted line) are as short as possible (and therefore the least expensive)?
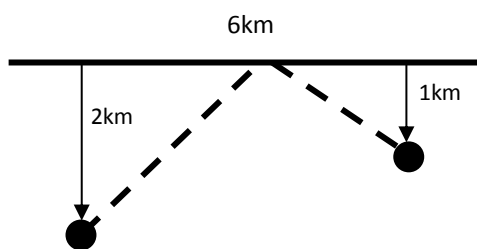
Unit 5: Graphical Methods

Equations such as $3x + 7 = 22$ and $4x + 3 = x + 9$ can be solved by backtracking ($3x + 7 = 22 \rightarrow x = \frac{(22-7)}{3} = 5$) and the balance rule ($4x + 3 = x + 9 \rightarrow 4x + 3 - 3 = x + 9 - 3 \rightarrow 4x = x + 6 \rightarrow 4x - x = x + 6 - x \rightarrow 3x = 6 \rightarrow \frac{3x}{3} = \frac{6}{3} \rightarrow x = 2$), as described in Module A1 *Equivalence and Equations*. However, they can also be solved by plotting points and seeing where lines cross; for example, $3x + 7 = 22$ where $y = 3x + 7$ crosses at $y = 22$; $4x + 3 = x + 9$ where $y = 4x + 3$ crosses at $y = x + 9$.

5.1   Overview of graphical methods activities

There are two things to do – build expertise with the plotting approach, and use applications.

Plotting techniques

These have two parts: (a) determining what graphs have to be drawn, and (b) finding where they cross.

1. **Simple methods**. Consider $3x - 2 = 13$:

    (a) determine the two graphs – this is based on the equation having two sides – the two graphs are the sides of the equation ($y = 3x - 2$ and $y = 13$); and

    (b) draw the lines and see where the line graph $y = 3x - 2$ crosses 13 ($y = 13$) – see diagram on right.

    Do a few examples and also solve the equation by backtracking to check that the same answers are calculated as from the graph.
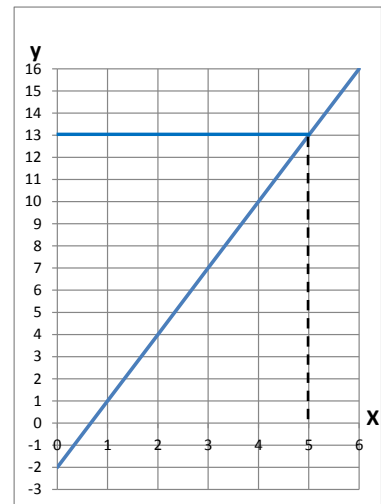
2. **More complex methods**. Consider $3x - 2 = 2x + 2$:

    (a) determine the two graphs – this again is based on the equation having two sides – the two graphs are the sides ($y = 3x - 2$ and $y = 2x + 2$); and

    (b) draw the lines and see where the two line graphs cross.

    Do a few examples. Also solve them by the balance rule and check that the answers are the same.

3. **Reversing**. Give a point and find/draw two graphs that cross at that point.

Applications

For these, do not start with equations and line graphs; start with a problem which has to be translated to equations and line graphs.

1. **Knotted rope**. Provide several ropes of different thickness each with three knots in them. Task is to work out (without untying the knots) how long each rope is.

    Clue: Tie more knots and graph the resulting length each time, so you are recording the effect of the knot.

    (a) Make a table or chart:

No. of knots	3	4	5
Length of rope	340	240	140

    and so on

(b) Plot the points as a graph and work backwards to find the length of rope at zero knots, i.e. the full length of the rope; length at zero knots will be the y-intercept.

Note: It can be useful to cut the ropes (different thicknesses) all at the same length (e.g. 960 cm) so you (the teacher) know the y-intercept (length of the rope before the rope is knotted) and so that there are different line graphs of different slope but with the same y-intercept. Also, all the graphs will have a negative slope.

(c) Discuss the implications of this - could you extend the graph in either direction? Why or why not?

2. **Cars**. Tom started 40 km closer than Fred. Tom travelled at 80 km/hour and Fred at 100 km/hour. When does Fred catch Tom?

(a) Let x be number of hours that Fred drives. At this point, Fred has travelled $100x$ and Tom has travelled $80x + 40$.

(b) Plot these graphs and see where they cross.

(c) Check answer by substitution.

5.2   Graphical methods RAMR cycle: Solving two-sided equations

This cycle looks at how to solve problems with unknowns for linear situations by graphing and finding where graphs cross.

Reality

Problem

Start with problems that relate to students' lives, e.g. NRL or netball. For example: *The "space shots" ball (a game where payers line up and shoot three points as fast as they can) Team A was losing 8 to 13 to Team B at half time, but scored at 3 points per minute in the second half while Team B only scored at 2 points per minute. Answer these questions:*

(a) *At what time in the second half did Team A pass a score of 26?*

(b) *At what time in the second half did Team A pass Team B?*

(c) *If halves are 10 minutes, what was the final score?*

Students need to see that Team A's score in the second half is 3m+8 and Team B's is 2m+13 where m is number of minutes. Then (a) is answered by seeing when 3m+8 = 26; (b) is answered by seeing when 3m+8 = 2m+13; and (c) is answered by the scores at 10 minutes. Students need to show how this can be done by drawing line graphs.

Acting out

This could be solved by simply acting out the scores at each minute in the second half:

Minutes	0	1	2	3	4	5	6	7	8	9	10
Team A	8	11	14	17	20	23	26	29	32	35	38
Team B	13	15	17	19	21	23	25	27	29	31	33

Abstraction

Body–hand–mind

Act out the problem first, then with materials and pen and paper; check what is done, and then imagine what is done.

Part One: Simple equations

1. **Determine the two graphs**. Make up problems like in Reality above – e.g. *You have $5 and you earn $8 per hour, how many hours to earn enough to buy a $69 phone?* Go through and show that the two equations are $y = 8x + 5$ and $y = 69$.

    One way to do this is to make a table of hours (e.g. 0, 1, 2, 3 and so on) versus money (e.g. $5, $13, $21, $29 and so on). Look at the differences to obtain the growing part (which is 8) and fixed part (which is 5) and remember that this gives $8n + 5$ in patterns or $y = 8x + 5$ in graphs. The other side is the target and this means getting to $69. Students need to show that such targets are a horizontal line, in this case at 69 or $y = 69$.

2. **Use line graphs to solve the problem**. Construct appropriate axes (y axis 0 to 70 and x axis 0 to 10). Draw the graphs. See where they cross. Then check to see that this is the correct time by calculating when you get to $69 from the table.

3. Redo these steps for two other problems – try "space shots" and "cars" above.

Part Two: Complex equations

1. **Determine the two graphs**. Make up more complex problems (extend the simple ones): *You have $5 and you earn $8 per hour, Bob has $20 but only earns $5 per hour. When do you have the same money as him? How many hours longer does Bob have to work for his $69 phone than you have to work?*

    Repeat the process of Part One – you will need to do chart/table for both you and Bob. The two equations are $y = 8x + 5$ and $y = 5x + 20$.

2. **Use line graphs to solve the problem**. Draw axes, draw graphs, and see where they cross. Check results. See when Bob gets to or over $69 by seeing the point on the x axis where his graph cuts the line for $y = 69$.

3. Redo these steps for new examples (e.g. space shots and cars).

Mathematics

Language/symbols

Ensure students understand equations (e.g. that = means "same value as" not "where to put the answer").

Practice

Separately practise Steps 1 and 2 above for the two parts, then practise both and all parts together. Make sure students can go: story → equation → graph **and** graph →equation → story.

Connections

Two crucial activities:

1. **Reversing**. Really push hard to ensure that students are able to go from real-world situations to equations to graphs and vice versa. **Ensure reversing is done** – get students to make up situations for equations and graphs that you give the students.

2. **Backtracking/Balance**. Draw strong **connections** between action of graphs and backtracking and balance as ways to solve equations for unknowns/problems of this type.

Reflection

Validation

Get students to find problems in their world.

Applications

We have done this across the cycle but maybe there is more to be done (e.g. rich tasks with more than one problem).

Extension

Flexibility. Check all different situations for equations.

Reversing. This is very important as mentioned in Mathematics above (go from story → equation → graph **and** graph → equation → story).

Generalising. Ensure students can begin to see where the unknown is being used in a general sense.

Unit 6: Nonlinear Graphs

As stated in the Algebra modules, the focus up to Year 9 is on linear graphs. However, other graphs are possible and should be pre-empted in junior secondary years.

6.1 Plotting nonlinear graphs

Begin with a nonlinear expression such as $x^2 + 2$. Start to plot points where (x, y) is such that $y = x^2 + 2$.

1. Determine the nonlinear expression: $x^2 + 2$

2. Use it to relate x and y coordinates: $y = x^2 + 2$

3. Construct a table of points:

x	−5	−4	−3	−2	−1	0	1	2	3	4	5
y	27	18	11	6	3	2	3	6	11	18	27

4. Plot the points on a graph.

5. Draw a smooth curve through the points.

Challenge: What happens if x and y are related by $x = y^2 + 2$? Draw it. Try plotting these two other examples: $x^2 − 1$ and $1 − 2x^3$.

6.2 Relationships in plotting nonlinear graphs and functions

The most common extension of linear graphs is quadratics. There are relationships between the equation and the shape of the graph. Can you find them?

1. Plot and find the rules for the following quadratic examples for $x = −5$ to 5 (use technology if available):

   (a) $y = x^2 − 1$ (b) $y = 2x^2 + 1$ (c) $y = 2 − 3x^2$ (d) $y = x^2 + 2x + 1$ (e) $y = 3x + 1 − 2x^2$

   Look at the quadratic graphs and develop some relationships between the graphs, their shape and the coefficients of x^2 and the constant number. What would $ax^2 + bx + c$ look like?

2. The next level of equations or functions is cubics. These have an x^3. Try plotting these three cubics:

   (a) $x^3 + 1$ (b) $1 − 2x^3$ (c) $\frac{1}{2}x^3 − 2x^2 + 3x + 2$

   What patterns have you found between these forms of equation? What about the coefficient and the constant term?

3. Finally, there are the exponentials. Plot the graph of $y = 2^x$ for x from −5 to +5. What does the graph look like? How would we change the graph of 3^x below left to the symmetrical one on the right? (What is its equation?)

$y = 3^x$

?
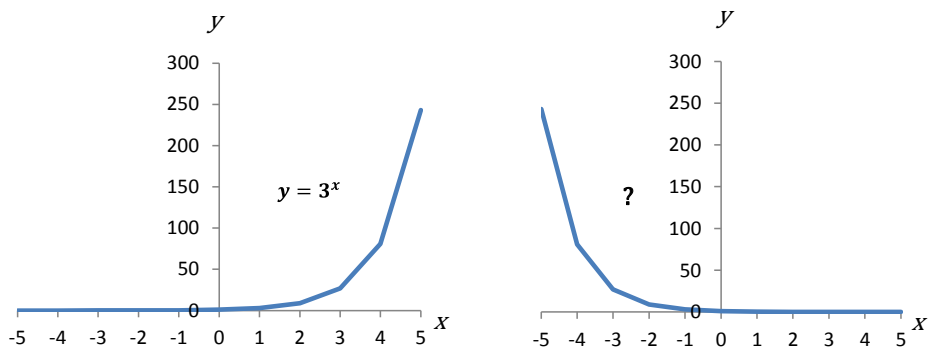
4. Reversing is always important in maths, so look at the six graphs below and say what characteristics they have (there could be two options). That is, are they squares, cubics, exponentials or logarithmic? What could their coefficients be?

1.

2.

3.

4.

5.

6.
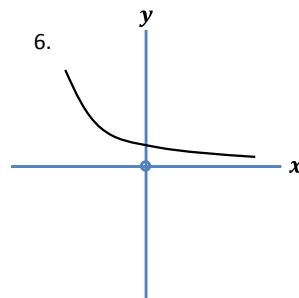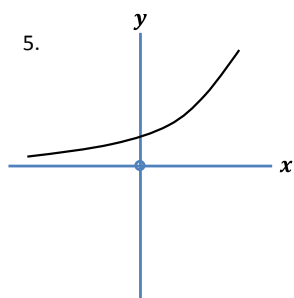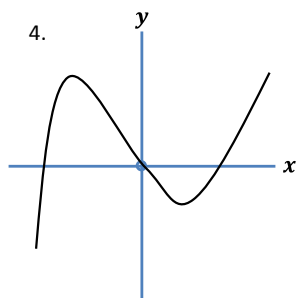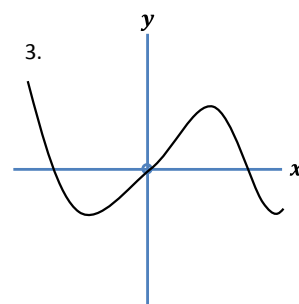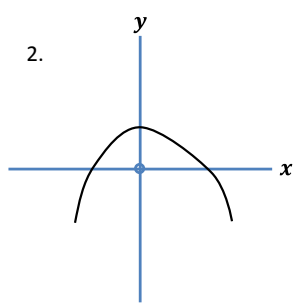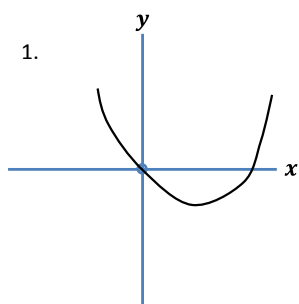
Test Item Types

This section presents instructions and the test item types for the subtests associated with the units. These will form the bases of the pre-test and post-test for this module.

Instructions

Selecting the items and administering the pre-post tests

This section provides an item bank of test item types, constructed around the units in the module. From this bank, items should be selected for the pre-test and post-test; these selected items need to suit the students and may need to be modified, particularly to make post-test items different to pre-test items. The purpose of the tests is to measure students' performance before and after the module is taught. The questions should be selected so that the level of difficulty progresses from easier items to more difficult items. In some modules this will follow the order of the units and subtests, and in other modules it will not, depending on the sequencing across the module and within units. The pre-test items need to allow for students' existing knowledge to be shown but without continual failure, and the post-test items need to cover all the sections in a manner that maximises students' effort to show what they can do.

In administering the pre-test, the students should be told that the test is not related to grades, but is to find out what they know before the topic is taught. They should be told that they are not expected to know the work as they have not been taught it. They should show what they know and, if they cannot do a question, they should skip it, or put "not known" beside questions. They will be taught the work in the next few weeks and will then be able to show what they know. Stress to students that **any pre-test is a series of questions to find out what they know** before the knowledge is taught. They should do their best but the important questions come at the end of the module. For the **post-test**, the students should be told that **this is their opportunity to show how they have improved**.

For all tests, **teachers should continually check to see how the students are going.** Items in later subtests, or more difficult items within a particular subtest, should not be attempted if previous similar items in earlier subtests show strong weaknesses. Students should be allowed to skip that part of the test, or the test should be finished. Students can be marked zero for these parts.

Information on the coordinates and graphing item types

As for other modules, the coordinates and graphing item types are divided into six subtests to match the six units in the module. The six units are in sequence from easy to hard or Prep to Year 9. Thus, the pre-test should start at Subtest 1 item types and continue through the subtests in sequence until it reaches where students can no longer do the items and there is continuous failure. The post-test should include all subtests.

Subtest item types

Subtest 1 items (Unit 1: Directions and polar coordinates)

1. (a) Draw a map showing how you would walk from your Maths classroom to the school office.

   (b) Add in compass points at the top of your map.

   (c) Write instructions to tell someone (over the phone) how to get there.

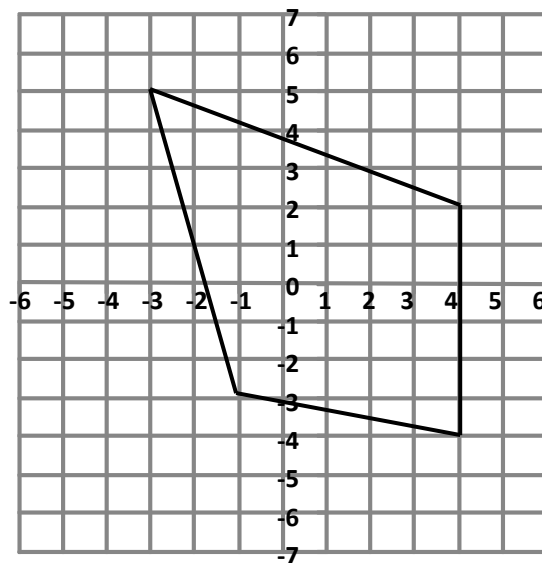## Subtest 2 items (Unit 2: Cartesian coordinates)

1. Plot these points on the coordinate system, and then join consecutive points with straight lines.

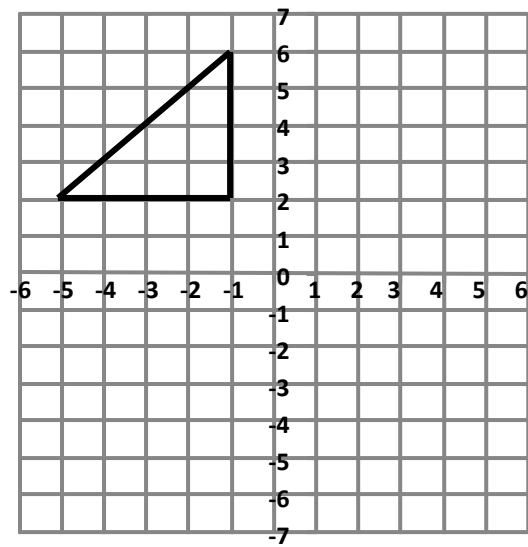   A (6, 2)   B (5, 5)   C (3, 5)   D (1, 2)
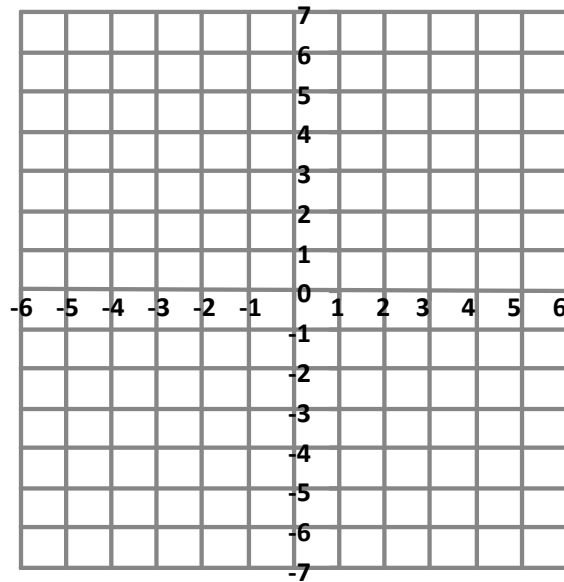
2. Write the coordinates for this shape.

    (_____,_____);   (_____,_____);   (_____,_____);   (_____,_____)
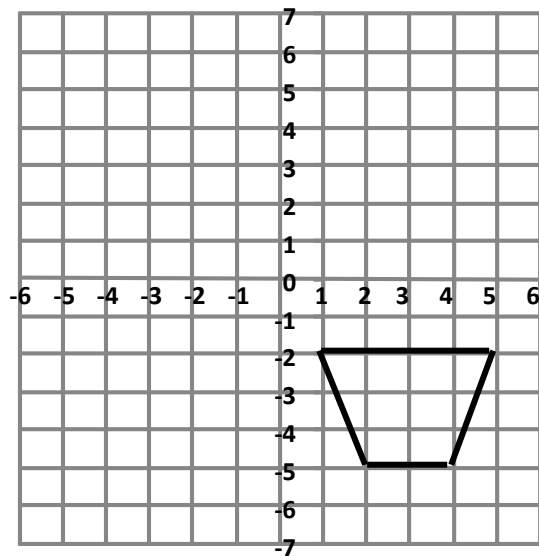
3.   (a)   <u>Slide</u> this shape into another quadrant.

      (b)   Write the new coordinates for the shape.

            (_____,_____);   (_____,_____);   (_____,_____)

4.   (a)   <u>Flip</u> this shape into another quadrant.

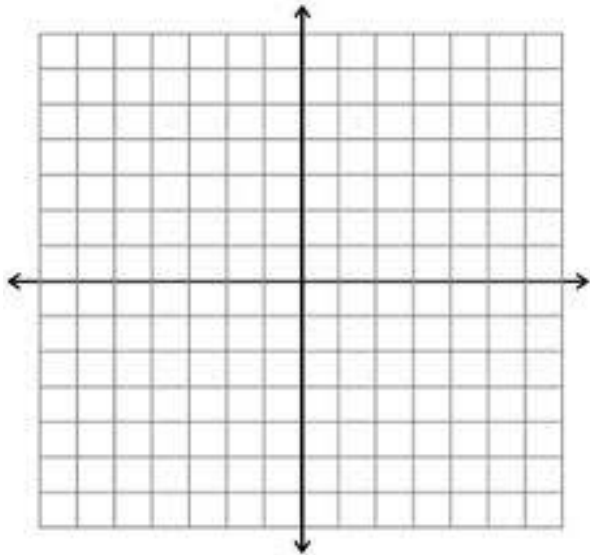   (b)   Write the new coordinates for the shape.

            (_____,_____);   (_____,_____);   (_____,_____);   (_____,_____)

Subtest 3 items (Unit 3: Axes and Graphing)

1. The points on a line are determined by this table:

y	−2	−1	0	1	2
x	0	1	2	3	4

(a)   Plot these points and draw a line through them.

(b)   Find the slope.

(c)   Find the y-intercept.

(d)   Find the equation for this line graph.

2. What is the slope and the y-intercept for these functions?
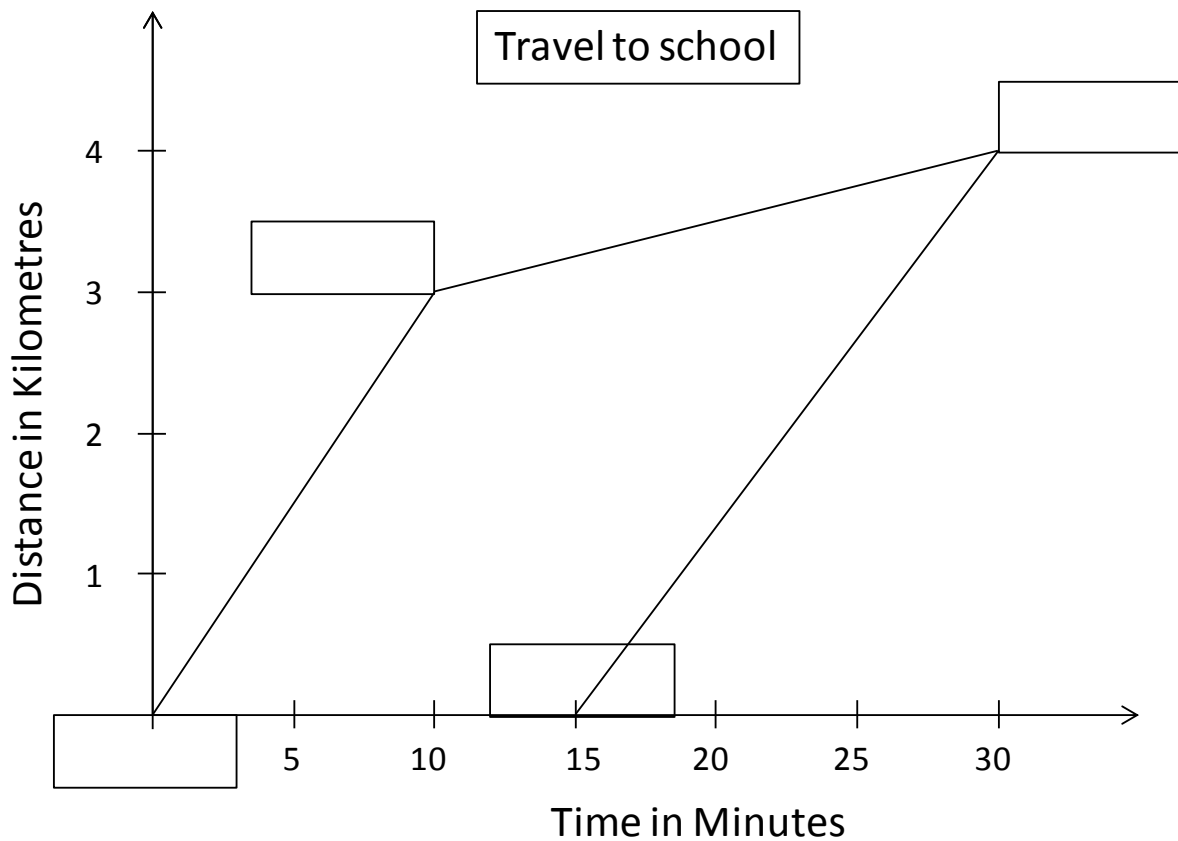
(a) $y = 2x + 3$ Slope: _____ y-intercept: _____

(b) $y = -2x - 3$ Slope: _____ y-intercept: _____

(c) $y = \frac{1}{4}x + \frac{3}{4}$ Slope: _____ y-intercept: _____

Challenge questions

3. Jake used his scooter to travel to school. He met his friend on the way and they walked together. His sister, Kia, left after him and went to school on her bike. They all arrived together.

Travel to school

Label the points on the graph:

(a)   Home for Jake and Kia

(b)   Where Jake met his friend

(c)   School

4. (a)   Why is there a bend in Jake's graph? _____

    _____

    (b)   How long did it take for Jake to get to school? _____

    (c)   How long did it take for Kia to get to school? _____

Subtest 4 items (Unit 4: Distance and midpoints)

1. Joshua is at the shops (3,3) and Selena is at the netball courts (7,1).

    Selena said to Joshua "Meet you halfway, so we can share the load of the groceries".

    Use the grid to label the shops (3,3) and netball courts (7,1) and find the midpoint, where Selena and Joshua will meet.
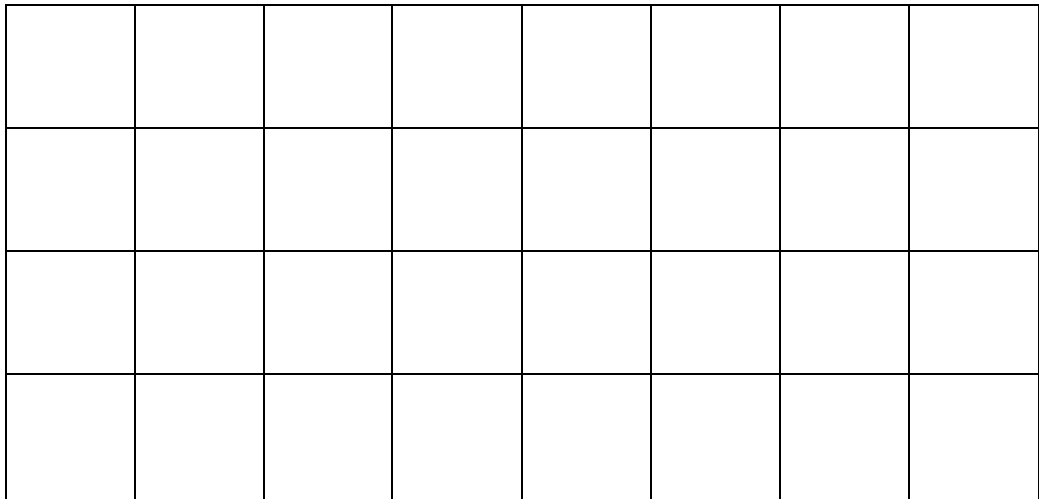
2. For the line graphs going through the points shown in the table, find the slope, midpoint and equation of the graph.

	Points	Slope	Midpoint	Equation of graph
(a)	(1,3) and (4,7)			
(b)	(−2,−1) and (4,3)			
(c)	(−3,7) and (7,−3)			

Subtest 5 items (Unit 5: Graphical methods)

1. Use graphical methods to solve for x.

    (a) $3x + 2 = 11$

   (b) $3x + 2 = 4x - 2$

    (c) $2x + 5 = 7x - 35$

    (d) $-4 - 3x = 8 - 5x$

Subtest 6 items (Unit 6: Nonlinear graphs)

1.   For the following nonlinear equations:

(i)   plot points from −3 to +3; and

(ii)   draw graphs on graph paper (4 quadrants)

(a) $3x^2 + 2$

(b) $2x - x^2$

(c) $13 + x - x^3$

2.   Complete activity 4 in section 6.2 (p. 28).

Appendix A: RAMR Cycle

AIM advocates using the four components in the figure on right, reality–abstraction–mathematics–reflection (RAMR), as a cycle for planning and teaching mathematics. RAMR proposes: (a) working from reality and local culture (prior experience and everyday kinaesthetic activities); (b) abstracting mathematics ideas from everyday instances to mathematical forms through an active pedagogy (kinaesthetic, physical, virtual, pictorial, language and symbolic representations, i.e. body → hand → mind); (c) consolidating the new ideas as mathematics through practice and connections; and (d) reflecting these ideas back to reality through a focus on applications, problem solving, flexibility, reversing and generalising (see figure on right).
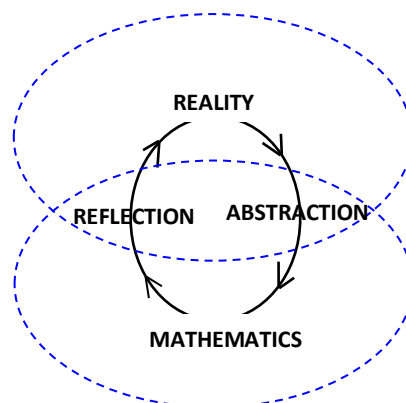
The innovative aspect of RAMR is that the right half develops the mathematics idea while the left half reconnects it to the world and extends it. For example, whole-number place value built around the **pattern of threes** where hundreds-tens-ones applies to ones, thousands, millions, and so on, can be easily extended to metrics by considering the ones to be millimetres, the thousands to be metres and the millions to be kilometres.

Planning the teaching of mathematics is based around the RAMR cycle if it is deconstructed into components that are applied to a mathematical idea. By breaking instruction down into the four parts and taking account of the pedagogical approaches described above, the cycle can lead to a structured instructional sequence for teaching the idea. The table below briefly outlines how this can be done. Prerequisite mathematical ideas are considered in the Reality and Mathematics components of the cycle, while extensions and follow-up ideas are considered in the Reflection component.

REALITY
• **Local knowledge**: Identify local student cultural-environmental knowledge and interests that can be used to introduce the idea. • **Prior experience**: Ensure existing knowledge and experience prerequisite to the idea is known. • **Kinaesthetic**: Construct kinaesthetic activities, based on local context, that introduce the idea.

ABSTRACTION
• **Representation**: Develop a sequence of representational activities (physical to virtual to pictorial materials to language to symbols) that develop meaning for the mathematical idea. • **Body-hand-mind**: Develop two-way connections between reality, representational activities, and mental models through body → hand → mind activities. • **Creativity**: Allow opportunities to create own representations, including language and symbols.

MATHEMATICS
• **Language/symbols**: Enable students to appropriate and understand the formal language and symbols for the mathematical idea. • **Practice**: Facilitate students' practice to become familiar with all aspects of the idea. • **Connections**: Construct activities to connect the idea to other mathematical ideas.

REFLECTION
• **Validation**: Facilitate reflection of the new idea in terms of reality to enable students to validate and justify their new knowledge. • **Applications/problems**: Set problems that apply the idea back to reality. • **Extension**: Organise activities so that students can extend the idea (use reflective strategies – *flexibility*, *reversing*, *generalising*, and *changing parameters*).

Appendix B: AIM Scope and Sequence

Yr	Term 1	Term 2	Term 3	Term 4
A	**N1: Whole Number Numeration** Early grouping, big ideas for H-T-O; pattern of threes; extension to large numbers and number system **N2: Decimal Number Numeration** Fraction to decimal; whole number to decimal; big ideas for decimals; tenths, hundredths and thousandths; extension to decimal number system	**O1: Addition and Subtraction for Whole Numbers** Concepts; strategies; basic facts; computation; problem solving; extension to algebra **M1: Basic Measurement (Length, Mass and Capacity)** Attribute; direct and indirect comparison; non-standard units; standard units; applications	**O2: Multiplication and Division for Whole Numbers** Concepts; strategies; basic facts; computation; problem solving; extension to algebra **M2: Relationship Measurement (Perimeter, Area and Volume)** Attribute; direct and indirect comparison; non-standard units; standard units; applications and formulae	**G1: Shape (3D, 2D, Line and Angle)** 3D and 2D shapes; lines, angles, diagonals, rigidity and properties; Pythagoras; teaching approaches **SP1: Tables and Graphs** Different tables and charts; bar, line, circle, stem and leaf, and scatter graphs; use and construction
B	**M3: Extension Measurement (Time, Money, Angle and Temperature)** Attribute; direct and indirect comparison; non-standard units; standard units; applications and formulae **N3: Common Fractions** Concepts and models of common fractions; mixed numbers; equivalent fractions; relationship to percent, ratio and probability	**G2: Euclidean Transformations (Flips, Slides and Turns)** Line-rotation symmetry; flip-slides-turns; tessellations; dissections; congruence; properties and relationships **O3: Common and Decimal Fraction Operations** Addition, subtraction, multiplication and division of common and decimal fractions; models, concepts and computation	**A1: Equivalence and Equations** Definition of equals; equivalence principles; equations; balance rule; solutions for unknowns; changing subject **N4: Percent, Rate and Ratio** Concepts and models for percent, rate and ratio; proportion; applications, models and problems	**SP2: Probability** Definition and language; listing outcomes; likely outcomes; desired outcomes; calculating (fractions); experiments; relation to inference **G3: Coordinates and Graphing** Polar and Cartesian coordinates; line graphs; slope and y-intercept; distance and midpoints; graphical solutions; nonlinear graphs
C	**A2: Patterns and Linear Relationships** Repeating and growing patterns; position rules, visual and table methods; application to linear and nonlinear relations and graphs **N5: Directed Number, Indices and Systems** Concept and operations for negative numbers; concept, patterns and operations for indices; scientific notation and number systems	**A3: Change and Functions** Function machine; input-output tables; arrowmath notation, inverse and backtracking; solutions for unknowns; model for applications to percent, rate and ratio **G4: Projective and Topology** Visualisation; divergent and affine projections; perspective; similarity and trigonometry; topology and networks	**O4: Arithmetic and Algebra Principles** Number-size, field and equivalence principles for arithmetic; application to estimation; extension to algebra; simplification, expansion and factorisation **SP3: Statistical Inference** Gathering and analysing data; mean, mode, median, range and deviation; box and whisker graphs; large data sets, investigations and inferences	**A4: Algebraic Computation** Arithmetic to algebra computation; modelling-solving for unknowns; simultaneous equations, quadratics **O5: Financial Mathematics** Applications of percent, rate and ratio to money; simple and compound interest; best buys; budgeting and planning activities

Key: N = Number; O = Operations; M = Measurement; G = Geometry; SP = Statistics and Probability; A = Algebra.