Accelerated Inclusive Mathematics Project

YuMi Deadly Maths

AIM Module N4

Year B, Term 3

Number:

Percent, Rate and Ratio

Prepared by the YuMi Deadly Centre
Queensland University of Technology
Kelvin Grove, Queensland, 4059

ACKNOWLEDGEMENT

The YuMi Deadly Centre acknowledges the traditional owners and custodians of the lands in which the mathematics ideas for this resource were developed, refined and presented in professional development sessions.

YUMI DEADLY CENTRE

The YuMi Deadly Centre is a research centre within the Faculty of Education at the Queensland University of Technology which is dedicated to enhancing the learning of Indigenous and non-Indigenous children, young people and adults to improve their opportunities for further education, training and employment, and to equip them for lifelong learning.

"YuMi" is a Torres Strait Islander Creole word meaning "you and me" but is used here with permission from the Torres Strait Islanders' Regional Education Council to mean working together as a community for the betterment of education for all. "Deadly" is an Aboriginal word used widely across Australia to mean smart in terms of being the best one can be in learning and life.

YuMi Deadly Centre's motif was developed by Blacklines to depict learning, empowerment, and growth within country/community. The three key elements are the individual (represented by the inner seed), the community (represented by the leaf), and the journey/pathway of learning (represented by the curved line which winds around and up through the leaf). As such, the motif illustrates the YuMi Deadly Centre's vision: *Growing community through education*.

The YuMi Deadly Centre (YDC) can be contacted at ydc@qut.edu.au. Its website is http://ydc.qut.edu.au.

DEVELOPMENT OF THE AIM MODULES

The Accelerated Inclusive Mathematics (AIM) modules were originally developed from 2010 to 2013 as part of the Accelerated Indigenous Mathematics project funded under the Commonwealth Government's *Closing the Gap: Expansion of Intensive Literacy and Numeracy* program for Indigenous students. The project aimed to assist secondary schools with beginning junior-secondary Indigenous students who were at Year 2/3 level in mathematics by developing a junior-secondary mathematics program that accelerates the students' learning to enable access to mathematics subjects in the senior-secondary years and therefore enhance employment and life chances. The project developed three years of modules (Years A to C) that are vertical sequences of learning to take students from their ability level to their age level in mathematics. The YuMi Deadly Centre acknowledges the role of the Department of Education, Employment and Workplace Relations in the development of the AIM project and these modules.

Contents

Module Overview

This module covers numeration and applications with respect to three topics: percent, rate and ratio. This means it has some aspect of operations within it; however, for simplicity it is called a number module, Module N4 *Percent, Rate and Ratio*. In terms of numeration, the module is built around the same big ideas that were advocated for whole number, decimal number and common fraction numeration. The applications are multiplicative (involving multiplication and division) for all three of percent, rate and ratio with a focus on meaning more than computation. The module is based on the ideas developed in Modules N1, N2, N3, O1, O2 and O3.

Percent, rate and ratio have important applications in measurement and money. However, the money part of these applications will be left for Module O5 *Financial Mathematics* at the end of Year C.

Background information for teaching percent, rate and ratio

This section covers meanings and models (the nature of percent, rate and ratio, common models and the size model), and connections and big ideas for numeration and operations in relation to percent, rate and ratio.

Meanings and models

The nature of percent, rate and ratio

Percent, rate and ratio are **similar and yet different**. They can all be considered as **multiplicative comparison as follows.** There are three ways to compare numbers, for example, two numbers 4 and 20 can be compared: (a) by **number** – 20 is bigger than 4 (20 > 4); (b) by a**ddition** – 20 is 16 more than 4 (20 – 4 = 16); and (c) by **multiplication** – 20 is 5 times as much as 4 (20 = 5 × 4).

However, they have very **different notations**. Percent is based on decimals and common fractions, rate is largely based on decimals (and division), while ratio is based on or relates strongly to common fractions. Notation could be simplified by putting all three in decimal form, for example, 7.5% could be represented as 0.075, 5:2 could be represented as 0.4 (dividing second term by first term), and rate is normally a decimal.

However, this simplification is not widely used and so we have to deal with the three different notations, and we have to take into account that this **causes differences**. For example, because of their notation, two percents can be directly compared, as can two rates. However, this is not the case for ratio. Being part to part, ratio requires change to equivalent ratio for comparison (just like fractions) – it is not a simple extension. Obviously, a higher cement mixture would be formed from sand:cement of 5:3 rather than 5:2. However, for two different ratios without a common number, say 5:2 and 7:3, we need to obtain **equivalent ratios** where one number is in common. For example, 5:2 and 7:3 could be changed to a common first term of 35 giving 35:14 and 35:15, or a common second term of 6 giving 15:6 and 14:6. In both cases, the second ratio 7:3 gives the stronger cement mixture.

Percent is fairly straightforward in that it is an extension of fractions (it is a fraction whose denominator is always 100) and decimal (it is the number of hundredths). Rate and ratio are new and need a little more discussion. They are different in how their notation works but there is more than this: they have differences in where they are used. Ratio differs from rate in that ratio is comparison between like attributes. That is, 15 L to 6 L is ratio while 15 L for $6 is rate. Look at the table on the following page. The shaded areas are ratio.

	Length	Area	Volume	Mass	Time	Temp.	Money
Length							
Area							
Volume							
Mass							
Time							
Temp.							
Money							

Thus, rate and ratio are different in three ways:

1. **Ratio** compares the **same attributes** (for example, sand and cement in mass are 5:2 – that is 5 tonnes of sand is used with 2 tonnes of cement); **rate** compares **different attributes** (for example, the cost of petrol is $1.20 per litre which compares money to volume).

1. **Ratio** uses notation similar to fractions in that there are **two whole numbers** (but part to part, not part to whole – for example, sand to cement is 5:2); **rate** uses a **single number**, for example, $1.20 per litre. Rate could really be considered as a ratio with 1 as the second number (for example, $ to litres is $1.20:1 litre).

2. **Ratio problems** are worked out by using **proportion or equivalent ratio** (for example, if sand to cement is 5:2 and we wish to use 8 tonnes of cement, then we need 20 tonnes of sand as 20:8 is the same as 5:2); **rate problems** use **multiplication** (for example, if petrol is $1.20 per litre then 4 litres is $1.20 × 4 = $4.80).

Common models for percent, rate and ratio

For the most efficient and effective teaching of percent, rate and ratio, we need to use the similarities between percent, rate and ratio. If we can find common methods, we can do all three topics using the same approach. This may overcome the three topics often coming late in the curriculum and causing difficulties because of their notation.

The first and most important **commonality** is that percent, rate and ratio problems are all forms of **multiplicative relationship**. This relationship is between the amount and percentage for percent, the "per" attribute and the other attribute in rate, and the first and second amount in the ratio. The second and equally important commonality is that percent, rate and ratio all relate to **part-whole**. Percent are hundredths so they divide a whole into 100 equal parts, rate is a decimal formed by division which is also part-whole, while ratio is based on part-whole but really comes from comparing the part to the other part. Both of these commonalities can be used to develop a common approach to teaching percent, rate and ratio by developing models that cover both relationship and part-part-whole. This can be done by:

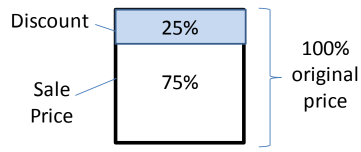
(a) *size diagrams (including part-part-whole)* representing size difference in terms of multiplication;

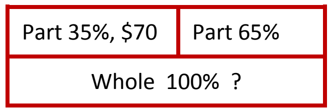
(b) *double number lines* where above and below the line represent different amounts/attributes but are equivalent in the way they multiplicatively relate (either through fractions, division or proportion); and

(c) *change diagrams* showing start, multiplier and finish.

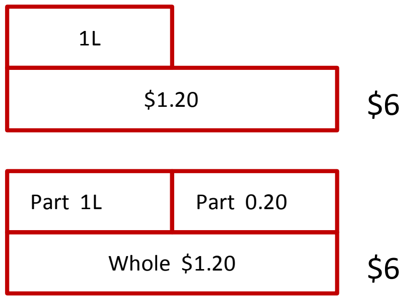
This module focuses on the **size models only**. All three models are briefly summarised for all three topics in **Appendix A**, including a description of how the double number line works. The application of the change model to percent, rate and ratio is part of Module A3 *Change and Functions* in Year C. The size models for percent, rate and ratio relate directly to the form of the topic. However, they can all be more abstractly represented by a part-part-whole diagram. This is summarised below.

1. **Size model for percent**. This model uses a picture (e.g. as on right) to represent the percentage as area in problems such as *How much for an $80 coat when discount is 25%*? You can see that, if the 100% is $80, the discount is 1/4 or $20, so the coat costs $60. This picture can also be a more detailed diagram with all 100 squares shown (and 25 shaded), a diagram like above right, or an even more abstract part-part-whole (P-P-W) diagram as on right. For the problem *if 35% is $70, what is the whole amount?*, the P-P-W diagram shows that the 35% is $70, which means $2 for each 1%, which means that the whole is $200.

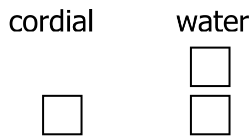


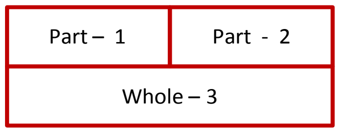


2. **Size model for rate**. For rate, the size model is a diagram with the “per” attribute as one. That is, $1.20/litre is as on right – in the first diagram, the 1 litre is compared to the $1.20, in the second diagram (P-P-W) the 1 L is one part and there is the other part and the $1.20 is the whole. Thus, if we have the problem *How many litres can I buy for $6 if petrol costs $1.20/litre?*, the $1.20 represents $6 so is × 5; and this means that the 1 litre is × 5 as well. Thus, the answer is 5 litres. *Note*: Rate is like fraction in that it is something that multiplies: it acts like a multiplicative operator or multiplier. Actually, multiplication is best understood as number by rate, e.g. 4 bags at 3 lollies/bag = 12 lollies.



3. **Size model for ratio**. For ratio, the size model can be in terms of area or sets. For the problem, *If cordial to water is 1:2, and we have 16 litres of water, how much cordial do we need?,* the set diagram is as on the right. For this diagram, if we have 1 cordial as unknown and 2 waters as 16 litres, so 8 for each square which means cordial is 8 litres. Ratio can also be understood in a P-P-W diagram. For the problem above, ratio is part to part (P-P) which here is 1:2. Thus, the 2 is 16 litres, so again the cordial is 8 litres. *Note*: Ratio normally differs from fraction in that fraction is part to whole while ratio is part to part.





Connections and big ideas

Connections

The connections between percent, rate and ratio and other parts of number and operations (and algebra) are strong.

1. **Percent.** Percent is connected to common fractions (part of whole when whole is 100 parts – 23% = 23/100) and decimal numbers (percent is hundredths – 23% is 0.23). It leads on to **interest**, financial mathematics, and to ways of comparing all changes (e.g. profits, losses, growth, recession, and so on). It has a major role in statistics and probability.

2. **Rate**. Rate is connected to decimal numbers (through notation) and to division (as it is multiplicative comparison between different attributes). It has a major role in physics and underlies calculus (differentiation determines rate of change). It has many applications in trades (e.g. marine in terms of speed and fuel usage, energy in terms of power use) and in finance (e.g. exchange rates).

3. **Ratio**. Ratio is connected to common fraction (P-P instead of P-W) and to division. Through proportion, it has a major role in all trades (e.g. construction, horticulture, hairdressing).

Numeration big ideas

For percent, rate and ratio numeration, the big ideas are similar to those for whole and decimal numbers and common fractions.

1. **Part-whole-group (includes Notion of a unit)**. The basis of number is the unit which is grouped to make large numbers and partitioned to make fractions. Thus everything can be seen as part, whole and group. This is particularly so for percent which is part of decimal numeration and common fraction. For rate and ratio, the relationship is more whole to whole and part to part, although ratio does imply a whole (e.g. sand to cement being 5:2 means 5 kg sand and 2 kg cement to make 7 kg of concrete). In notation, of course, rate is a decimal and ratio is similar to common fraction.

2. **Additive structure**. This applies to percent and to rate as a decimal number. However, all three, percent, rate and ratio, act multiplicatively in applications, so additive structure has little impact. Of course for percent and ratio, in notational form, counting follows a pattern, the **odometer pattern** – forwards from 0 to 9 and back to 0 as the number on left increases by 1, and backwards from 9 to 0 and back to 9 as number on left decreases by 1.

3. **Multiplicative structure**. Percent, rate and ratio are highly multiplicative, but in applications, this is because they act as multiplicative comparison. Of course, again, percent and rate as decimal numbers obey the multiplicative relationships between adjacent place value positions.

4. **Continuous vs discrete/Number line**. This relates percent, rate and ratio to number lines. This also appears to apply to all three, particularly in the way double number lines are able to represent their applications. Of course, again, percent and rate can be placed on a single number line. Differently, ratios can be placed on two number lines for comparison.

5. **Equivalence**. Sometimes a single quantity can be represented by more than one number. This is true for percent and rate in the same way it is true for decimals, with the added bonus that 23.5% is the same as 0.235. However, this big idea is a major part of ratio which is as complex and rich as common fractions with its equivalences. It is so powerful that a special name has been coined for equivalent ratios, and that is **proportion**.

Operation big ideas

For percent, rate and ratio applications, the big ideas are based on those for operations with whole and decimal numbers and common fractions. There are many of these and they are listed in the *Overview* booklet. The particular operation big ideas which contribute to percent, rate and ratio in a major way are below.

1. **Relationship vs change**. The models and applications will use both these approaches. Relationship is seen in size; double number line seems to be a combination of both.
2. **Part-part-total/whole**. This one appears to be applicable as a size model P-P-W.
3. **Identity/Inverse**. These two are applicable in how the models are used to solve the applications – changes are inversed, and identity for multiplication (1 and 100%) is often the point of first calculation.
4. **Commutativity** and **Associativity**. These two apply in calculation.
5. **Equivalence**. This is the basis of proportion, that is, 2:3 = 4:6 = 6:9 and so on, because 2:3 = 2x2:3x2 = 4:6, 2:3 = 2x3:3x3 = 6:9, 2:3 = 2x4:3x4: = 8:12, and so on.
6. **Triadic relationships**. This is **very powerful**, and is the basis of there being **three problem types** for each or percent, rate and ratio – see Units 1, 2 and 3.

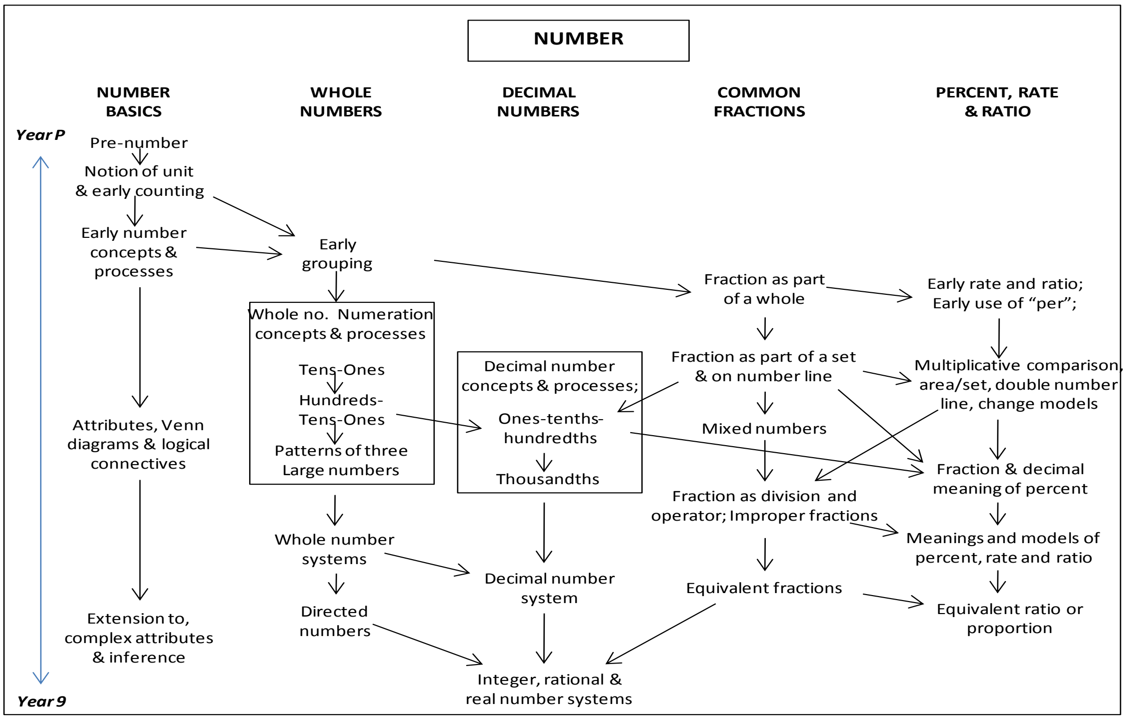
Sequencing for percent, rate and ratio

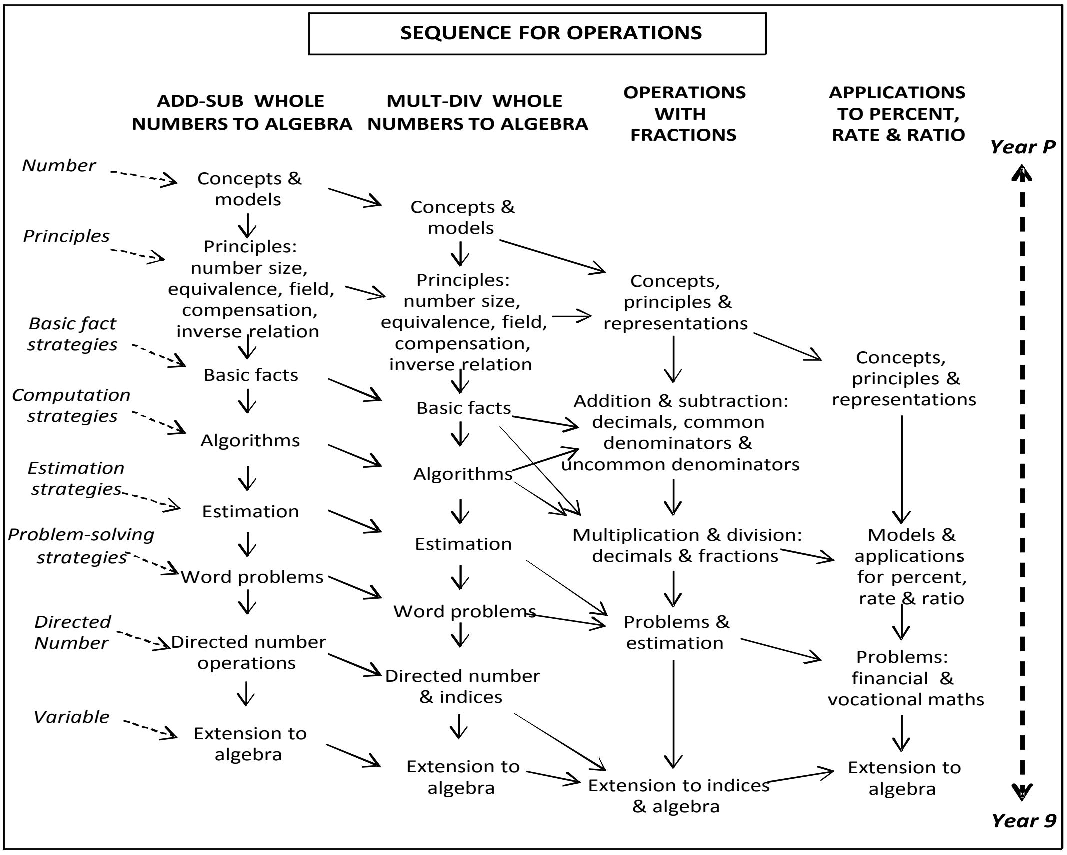
This section briefly looks at sequencing with respect to number and operations and sequencing in this module.

Sequencing in number and operations

Percent, rate and ratio have a place in the number sequence and in the operations sequence. This is diagrammatically illustrated on the next page. For Number, the five big ideas of Modules O1, O2 and O3 apply, namely, notion of unit, additive structure, multiplicative structure, discrete vs continuous/number line, and equivalence. For operations, it focuses on the models and the applications.

Due to the nature of the module scope and sequence, the financial applications are to be left to Module O5 *Financial Mathematics*. Due to difficulties with the first two Operation modules O1 and O2, principles and estimation were left to Module O4 in particular and other later modules where appropriate. This means that, for percent, rate and ratio, the two sections in Operations on applications and algebra are coalesced into one unit with rich tasks because the financial mathematics material is left to Modules O4 and O5.

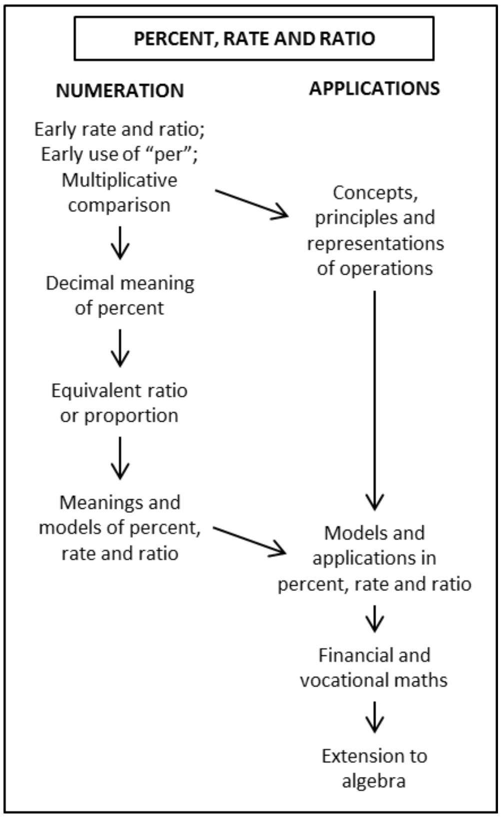




Sequencing in this module

Although the three concepts can be integrated, the booklet spends a unit on each of percent, rate and ratio separately for ease of understanding. Because this unit covers both numeration and operations, the activities cover the major big ideas of numeration and the major models and strategies for applications.

The sequence for the module is summarised in the figure on right. The major points are as follows.



1. The figure has two components – percent, rate and ratio as numeration and percent, rate and ratio as applications.
2. Early use of "per", rate and ratio and understanding use of multiplicative comparison and part-part-whole lead into later numeration and applications.
3. Numeration precedes applications (as seems obvious), setting up means and models which are then used in applications.
4. As part of numeration, percent as a decimal and equivalence as proportion sets up the methods used in the models in applications.
5. The concepts, principles and representations for operations are visited early to lead into the models used in applications. Common methods for all three topics are shown even though symbols and relationship to fractions are different.
6. Only one model, size, will be used in the units. Applications will be diminished to non-financial situations as financial mathematics is left for O5. Thus the last two possible units will be combined into one unit, Unit 4, Applications, Extensions and Rich Tasks and cover applications in vocations, extensions to algebra and rich tasks.

To meet the sequence above the module structure will be as follows.

Overview: Background information, sequencing and relation to Australian Curriculum

Unit 1: Percent – ideas for building an early foundation in "per", percent as a fraction and as a decimal, percent applications, and percent investigations

Unit 2: Rate – a major component of early multiplication, rate as a concept and its relation to ratio, rate applications, and rate investigations

Unit 3: Ratio – early ideas of ratio and its relation to fractions, ratio concept and relation to rate, equivalent ratio (proportion), ratio applications, and ratio investigations

Unit 4: Applications, extensions and rich tasks – vocational applications, estimation, extensions to algebra (variables and formulae), and rich tasks

Test item types: Test items associated with the four units above which can be used for pre- and post-tests

Appendix A: Other models for teaching percent, rate and ratio applications

Appendix B: RAMR cycle components and description

Appendix C: AIM scope and sequence showing all modules by year level and term.

Similar to other modules in Year B, the teaching ideas in the percent, rate and ratio module are given as sequences of activities, often numbered to show order. It is left to participating teachers to translate the idea sequences to RAMR cycles (see **Appendix B**). Similar to some other modules (e.g. M1 to M3), Units 1, 2 and 3 are sequences of teaching activities within units on different topics. Only Unit 4 shows growth beyond and across these topics.

Relation to Australian Curriculum: Mathematics

AIM N4 meets the Australian Curriculum: Mathematics (Foundation to Year 10)					
Unit 1: Percent Unit 2: Rate	Unit 3: Ratio Unit 4: Non-financial applications, extensions and rich tasks				
Content Description	**Year**	**N4 Units**			
		1	**2**	**3**	**4**
Make connections between equivalent fractions, decimals and percentages (ACMNA131)	6	✓	✓		✓
Investigate and calculate percentage discounts of 10%, 25%, and 50% on sale items, with and without digital technologies (ACMNA132)		✓			✓
Connect fractions, decimals and percentages and carry out simple conversions (ACMNA157)	7	✓			✓
Find percentages of quantities and express one quantity as a percentage of another, with and without digital technologies (ACMNA158)		✓			✓
Recognise and solve problems involving simple ratios (ACMNA173)				✓	✓
Solve problems involving the use of percentages, including percentage increases and decreases, with and without digital technologies (ACMNA187)	8	✓			✓
Solve a range of problems involving rates and ratios, with and without digital technologies (ACMNA188)			✓	✓	✓
Solve problems involving direct proportion. Explore the relationship between graphs and equations corresponding to simple rate problems (ACMNA208)	9		✓	✓	✓

Unit 1: Percent

Percent is a fraction out of 100 and a decimal that is hundredths. It represents already known ideas when it is first studied. However, it uses a new word "per" instead of fraction or decimal words and this makes for difficulties. This unit looks at the teaching of percent. It covers early work on percent, percent as fraction and decimal, and applications of percent to problem types using the size model (P-P-W).

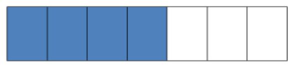
1.1 Early percent ideas

Percent is an extension of previous work but the use of "percent" can make it seem to be something new and unrelated to previous work. So we bring this word in earlier – by using the word "per" with simple fractions. These early activities reflect one of the bases of the AIM program, of laying a foundation for later acceleration.

Activities

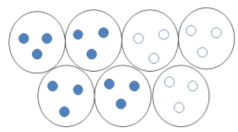
A. Looking at fractions

1. Shade fractions of strips. For example, 4/7 is 4 shaded out of 7 as on right:



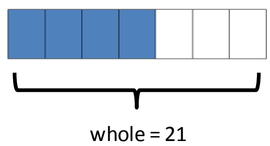
   Shade: (a) 3/8 (b) 1/5 (c) 6/11

2. Shade fractions as set model. For example, 4/7 is as shown on right:



   – that is, it is the set of objects divided equally into 7 parts and 4 out of these parts gives the fraction.

3. Find fractions of things, e.g. 4/7 of 21. Think of this as a fraction as on the right where the 7 is the whole. If the 7 is 21, then each part is 3 so 4/7 = 12.

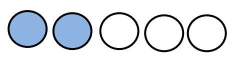


4. Find:

   (a) 1/5 of 25 (b) 6/11 of 33 (c) 3/8 of 20 (you may have to have ½ pieces).

B. Introducing "per"

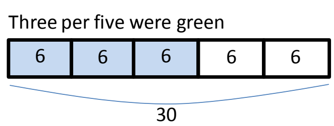
1. Introduce the language "2 per 5" for fractions like 2/5. Draw pictures to show 2/5 as on right. Say, we could say this fraction as 2 per 5 as it is 2 shaded out of a whole of 5. State that 2 per 5 means that for every 5, we take 2.



2. Count using this language: 1/5, 2/5, 3/5, 4/5 and so on becomes 1 perfive, 2 perfive, 3 perfive, and so on, up to 5 perfive which is 1 (as 100 percent is 1).
3. Draw (a) 3 per 5, and (b) 4 per 5. State what these mean in own language.
4. Repeat the above steps for (a) 3 per 7, (b) 4 per ten, and (c) 7 per 20.
5. What other drawings could we use for 2 per 5 or one of the other examples?

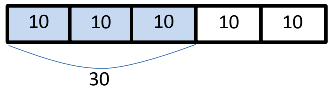
C. Doing applications with early "per"

1. Consider the problem, *Three per five were green. How many were green if there were 30 items?*
2. Draw 5 circles or squares or divide a strip into 5 parts. Show the "per 5" by shading the parts of 5 that are required. An example of a drawing or diagram is shown (green is shown by shading) on right.



3. Place information from the problem on the drawing (as shown above right). The 30 items total means 6 in each box which means 18 were green.

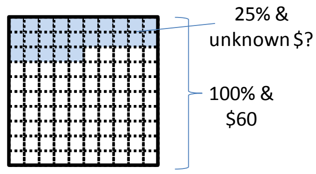
Note: If the problem is the other way around, that is, *Three per five were green. How many were there if 30 items were green?*, then the diagram would be as on right at this level. Thirty green items means 10 per square which is a total of 50 items.



4. Solve the following problems the same way:
   (a) Four per 5 were red, how many were red if there were 60 items?
   (b) Four per 5 were red, how many items were there if 60 were red?
   (c) 7 per 10 were boys, how many children if there were 35 boys?
   (d) 12 per 20 were large cars, how many small cars if there were 80 cars?

D. Extending to "per cent" or "per 100"

1. Consider problem *find 25% of $60*.
2. Draw 25% on a 10×10 grid. Record all pertinent problem information on grid as on right.

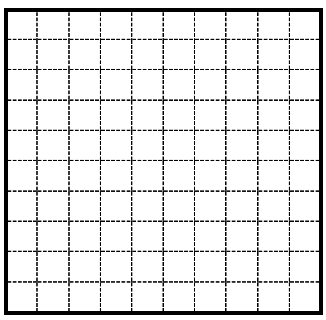


3. Solve the problem by working out what 1% is [the 100% is $60 and the 25% is the unknown dollars – the $60 for the whole 10×10 grid means each row is $6 and each square is 60c or $0.60. So 25% is 25 × 0.60 = $15].
4. How would you solve *25% is $60, how much is the total*?

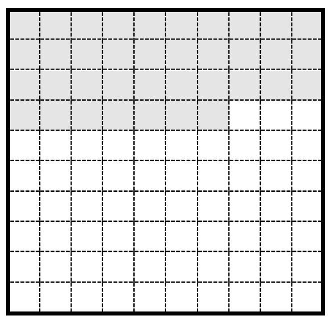
1.2 Percent (%) as a fraction and a decimal

Percent means per centum or per hundred and is a special type of fraction which is always parts per hundred (e.g. 37% is $^{37}/_{100}$) as in a common fraction, or hundredths (e.g. 37% = 37 hundredths = 0.37) as in decimal fraction. The best way to think of this is as follows.

(a) Use a 10×10 grid as below. The 10×10 grids enable 37% to be represented as hundredths and therefore both a common fraction and a decimal number.



37%



(b) Use a place value chart and change unit from ones to hundredths (making the decimal point at the hundredths) as below.

Thousands	Hundreds	Tens	Ones	Tenths	Hundredths	Thousandths
			0•	1	7	5

Thousands	Hundreds	Tens	Ones	Tenths	Hundredths	Thousandths
			o	1	7•	5

0.175 = 17.5%

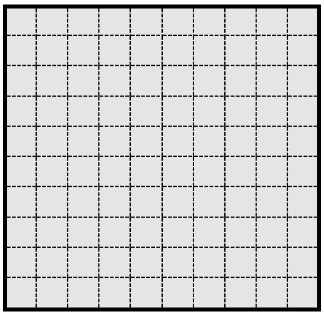
Activities

A. Percents and hundredths

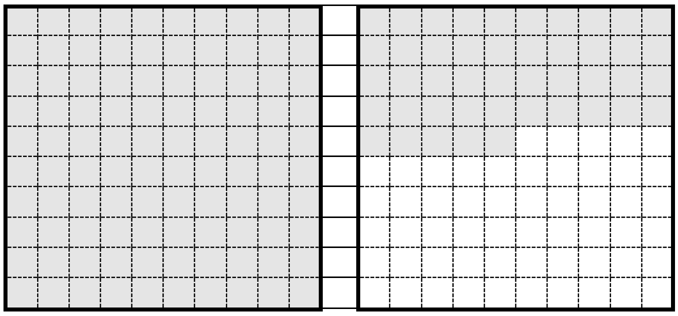
1. Obtain sheets of 10×10 grids. Draw 40 hundredths. Consider 40 hundredths as 4 rows out of 10 rows. In terms of rows, what is it as a fraction? What relationship does this show?
2. Shade 6 tenths of a 10×10 grid – break the 100 squares into 10 equal parts (e.g. 10 rows) and shade 6 of them. How many hundredths? What relationship does this show?
3. Shade 10×10 grids to show (a) 8 tenths = 80 hundredths, and (b) 30 hundredths = 3 tenths.
4. Shade 10×10 grids to show 34 hundredths = 3 tenths and 4 hundredths = 0.34. What about 0.27 = 2 tenths and 7 hundredths = 27 hundredths?

B. Percents as fractions and decimals

1. Obtain sheets of 10×10 grids. Shade 100% or 100 hundredths. This shows that 100% is 1 and that percentages over 100 are greater than 1. See example below.
2. Shade 10×10 grids to show that profit of 37% is 137% = 1.37. Repeat this to show 68% growth is 168% = 1.68.
3. Shade a 25% loss on 10×10 grids. Show that this means that percent reduces to 75% or 0.75.
4. Shade to show that 46% loss leaves 54% and 60% discount leaves 40% or 0.40.



100% = 1



145% = 1.45

C. Percents as decimals

1. Obtain a Thousands to Thousandths Place Value chart. Mark in 3 ones, 6 tenths, 7 hundredths and 4 thousandths. What is this number as ones? The maths mat is very effective for this activity.
2. Make the unit hundredths as in the RHS diagram below. This is done by moving the decimal point to the hundredths position. With % as the unit, what is the number as a percentage?

Thousands	Hundreds	Tens	Ones	Tenths	Hundredths	Thousandths
			3•	6	7	4

Thousands	Hundreds	Tens	Ones	Tenths	Hundredths	Thousandths
			3	6	7•	4

3.674 = 367.4%

3. Draw diagrams to show that: (a) 4.25 = 425%, (b) 0.068 = 6.8%, (c) 30% = 0.3, and (d) 6% = 0.06.

D. Conversions between percents, decimals and fractions as decimals

1. Construct a number line and label one end 0 and the other end 1. Make up decimals, fractions and percentages examples on paper (include ones that are the same). Give randomly to students. Students place the numbers on the line, discussing where they go, and why they are the same thing.

2. Discuss rules for changing from one notation to the other two.

1.3 Percent applications–problems

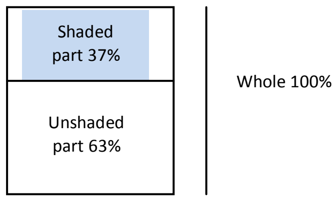
There are three problem types for percent (triadic big idea):

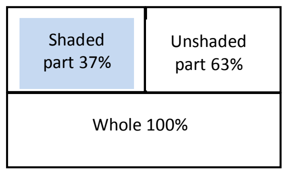
Type 1: Amount is unknown (e.g. find 45% of $675)

Type 2: Percentage is unknown (e.g. 45% is $675, how much is the total?)

Type 3: Percent is unknown (e.g. what % is the $675 of $1500?).

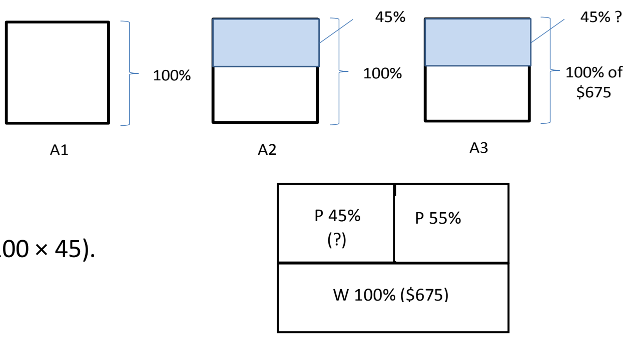
To prevent errors, we teach an approach/model that will enable students to do all three types. The size/P-P-W model is the simplest. It can be done using a 10×10 grid, a square to represent the grid, or a P-P-W model. Shading 37% on a grid represents the three components – the shaded part is 37%, the unshaded part is 100-37 = 63%, and the whole is 100%. It is not necessary for diagrams to actually show the hundredths – any diagram that shows relative size can be used. Thus, percent can be represented by a plain square without the 10×10 grid (see right top). It can also be represented by a part-part-whole diagram (see right bottom). The more abstract models are particularly useful when percents in the problems are complicated, such as 17.45%. These percents are better suited to diagrams which do not show all the 100 squares. The shaded part and the other part need not show relative size – they can be the same size rectangles.



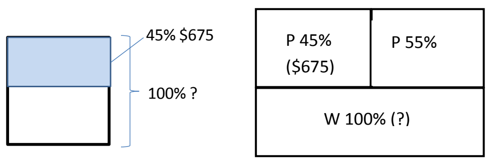


The use of these models for the three problem types is as below.

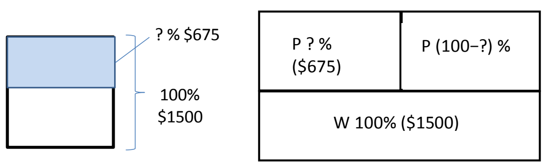
1. **Problem type 1 – Find 45% of $675.** The steps are: draw the 100% square, mark in the 45%, put in amounts and unknown (as in A1, A2 and A3 on right), find some way of getting from 100% to 45%, and use this to calculate. The simplest way is to find the value of 1% (the unitary approach) by ($675 ÷ 100) and then the value of 45% ($675 ÷ 100 × 45). This gives the answer of $303.75.



2. **Problem Type 2 – 45% is $675, what is the total?** The steps are: draw and fill in the diagram as on right, and calculate by finding the value of 1% and using this to find the unknown (here 100%). Answer is 675 ÷ 45 × 100 = $1500.



3. **Problem Type 3 – What % is $675 of $1500?** The steps again are: draw and fill in the diagram as on right, and calculate by finding the value of $1 (since unknown is a %) and using this to find the unknown (here $675). $1500 is 100%, so $1 is 100 ÷ 1500 and $675 is 100 ÷ 1500 × 675 = 45% which is the answer.



To do the above, students need to be able to go from problem to answer on their own. Thus, they need to be able to do two things **without the assistance** of the teacher. (*Note*: Also need to **reverse** these steps – to go from solution to a problem.)

Step A Take a problem and turn it into a diagram based on size with numbers and unknown correctly placed on diagram; and

Step B Use the diagram to solve the problem (one way to do this is the unitary method).

Activity

Use one of the above models to find the following:

(a) What is 78% of $184?

(b) If 78% is $184, how much is total?

(c) John has to pay $126 of the bill of $319. How much % is John paying?

(d) Frank makes a profit of 68% selling at $434. How much did he pay?

(e) The car was $14,680 at the 35% discount price. What was its original price?

Unit 2: Rate

This unit covers rate, including early rate ideas, rate as division, and applications of rate to problem types using the size models (drawings and P-P-W).

2.1 Early rate ideas

Rate can be introduced early because it is used throughout early mathematics without being recognised. For example, *3 bags of lollies with 4 lollies in each bag* is really 3 × a rate because 4 lollies/bag is a rate. In fact, some mathematics educators argue that all simple multiplications are number × rate (e.g. 3 bags × 4 lollies/bag = 12 lollies).

Activities

A. Introducing early rate

1. Consider the problem *The apples are packed 6 apples/bag, then how many apples in 2 bags?* Draw what it would look like.
2. Change the problem into "number × rate = total" where the number relates to bags, the rate relates to apples in each bag, and the total to the total number of apples.
3. Repeat the above two steps for the following problems:
   (a) There were 5 cases with 4 bottles in each case, how many bottles?
   (b) The kangaroo jumped 4 times, each jump was 3 metres, how many metres?

B. Reversing

1. Make up a story for each of the following: (a) 3 bags × 5 bananas per bag, and (b) 7 bottles at $5 per bottle.
2. Make up your own stories and matching number × rate = total equations.

C. Changing directions (another form of reversing)

1. Consider the problem *The apples are packed 6 apples/bag, then how many bags for 18 apples?* Draw the problem.
2. Solve the problem – use the drawing.
3. Can we reverse the number × rate = total to solve it?

Note: We should spend time in the early years on activities such as this, using simple diagrams to act out the problems as we have above. However, before this, **act out the problems with real-life and physical materials** (e.g. unifix instead of pictures of squares). The best way is to find real-life materials to act out situations that are real to the students.

2.2 Concept of rate

There are five interesting features of the concept of rate.

1. **Rate as comparison**. Rate is comparison between two different attributes – like speed (km/hour) or cost ($/kg or $/L).

2. **Rate compares with one**. Rate is also a comparison where the second attribute has only one of its attributes. An example could be one hour – then we have speed of 68 km in that hour or 68 km/hour or fuel use of 64 L in that hour (64 L/hr).

3. **Rate is worked out by division**. For example:

   (a) if I drove 320 km in 4 hours, then 320 ÷ 4 = 80 km, so 80 km/hr;

   (b) if 3.5 kg bag costs $14, then 14 ÷ 3.5 = 4 so $4/kg.

   This is why the nomenclature for rate is a division sign, e.g. $20/hr.

4. **Rate is the basis of multiplication**. When we start multiplication, we multiply a number by a rate. For example, 3 bags with 4 lollies in each bag is 3 bags × 4 lollies/bag = 12 lollies. Similarly, 5 rows of 7 objects is 5 rows × 7 objects/row.

   Thus, rate is a component of multiplication, that is, any multiplication can be thought of as number × rate.

5. **The rate has the "per" the same attribute as the number**. For example, 5×3 is 5 rows of 3 objects = 5 rows × 3 objects/row. The rows are with the number and with the "per". Looking at rate, this means that if the cost is $4/kg for apples, then 5 kg of apples is going to be 5 kg × $4/kg giving $20. This means that if we have the right number and rates, the answer is to multiply. For example, if we have 6 hours and 85 km/hr we multiply 6×85 and get 510 km.

Activity

Use one of the above ideas to do the following:

(a) Cost of beans is $6.85/kg. How much for 7.5 kg?

(b) John runs 17 km/hr. How far could he run in 3 hours?

(c) The boat uses 3.7 L of fuel every km, how many litres for 250 km?

2.3 Rate applications–problems

Rate is a triad – it has the first attribute, the second attribute, and the rate. For 85 km/hour, the km is the first attribute, the hours are the second attribute, and 85 is the rate. Thus, there are three problem types as below (the types have one of the triads as an unknown).

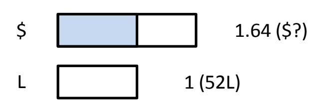
Type 1: First attribute unknown (e.g. petrol is $1.64/L, how many $s for 52 L?)

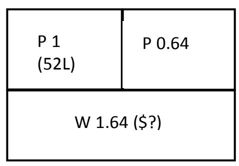
Type 2: Second attribute unknown (e.g. petrol is $1.64/L, how many Ls for $52?)

Type 3: Rate is unknown (e.g. what is the rate when paying $85.28 for 52 L?)

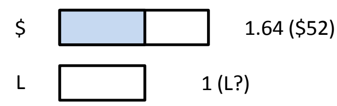
Thus, we look for a model that will enable us to work out how to do all these problem types. In this booklet, we focus on size models (drawings/P-P-W). There are also double number line (see Appendix B) and change (see *Change and Functions* module) models. *Note*: The models for rate are straightforward as one number is usually 1.

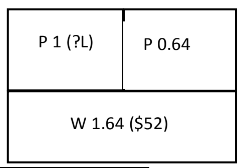
1. **Problem Type 1: Petrol is $1.64/L, how much for 52 L?** The steps are: draw diagram and mark in values (as on right), and find the unknown by calculating using the unitary method (already given – 1 is 52). So 1.64 is 1.64 × 52 = 85.28. Thus, 52L costs $85.28.





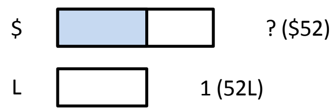
2. **Problem Type 2: Petrol is $1.64/L, how many L for $52?** The steps are: draw and fill in diagram (as on right), and find unknown by using unitary method. Here, 1.64 is 52,

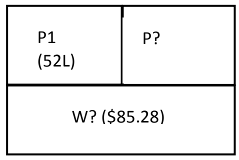




so 1 unit is 52 ÷ 1.64 = 31.7. So, fuel for $52 = 31.7 L.

3. **Problem Type 3: What rate if I pay $85.28 for 52L?** The steps are: draw and fill in diagram (as on right), and find unknown by using unitary method. Here, find cost of 1 L, this is 85.28 ÷ 52 = 1.64, so rate is $1.64/L.





To do the above, students need to be able to go from problem to answer on their own. Thus, they need to be able to do two things **without the assistance** of the teacher. *Note*: Also need to **reverse** these steps – to go from solution to a problem.

Step A Take a problem and turn it into a diagram based on size with numbers and unknown correctly placed on diagram; and

Step B Use the diagram to solve the problem (one way to do this is the unitary method).

Activity

Use one of the above models to do the following:

(a) The cost was $1.68 per kg. How much for 37 kg?

(b) The cost was $1.68 per kg. How many kg for $37?

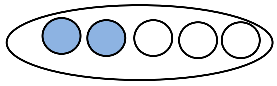
(c) 7.8 kg costs $24.70, what is cost per kg?

(d) 2-step: Ted travelled 720 km in 7.25 hours. How far would he get travelling at the same speed in 12.5 hours? How long would he take to do 840 km?

Unit 3: Ratio

This unit covers early ratio ideas, ratio and proportion (including proportion relationships and how these relate to double number line), and applications of ratio to problem types using the size models (drawings and P-P-W). Most activity with ratio is built around **proportion** which is equivalent ratio, that is, 2:3 = 4:6 = 6:9 = 8:12 = and so on. Like fractions, this is based on multiplying and dividing by the same amount, that is 2:3 = 8:12 because 8:12 = 2x4:3x4. As a consequence of this, 32:56 = 60:85 because dividing both parts of 32:56 by 8 gives 32/8:56/8 = 4:7 and dividing both parts of 60:85 by 15 also gives 60/15:85/15 = 4:7.

3.1 Early ratio ideas



Ratio is related to fraction. In the example on right, the fraction is 2/5. However, there are three components to the diagram (part – 2 counters; other part – 3 counters, and total – 5 counters). The fraction shaded in is 2 parts out of 5 parts as one whole = 2/5. The ratio is part to other part and is 2 to 3, or 2:3.

Thus, **introduce ratio when introducing fraction**. Always consider a fraction as part-whole but **also say there is a third component** (the unshaded circle – the other part). This will mean that a drawing of a fraction contains three things – **fraction 2/5 (part-whole), fraction 3/5 (other part-whole) and ratio 2:3 (part-part)**.

Activities

A. Introducing early ratio

1. Draw the fraction 2/3. Identify the part, the other part, and the whole.
2. Write two fractions and a ratio – part/whole, other-part/whole, and part:part.
3. Repeat the above for fractions: (a) 3/5, (b) 4/7, (c) 1/6, and (d) 7/11.
4. Draw two fractions for each of these ratios (a) 3:7, and (b) 5:8.
5. The fractions and ratio form a family – e.g. 1/4, 3/4, 1:3. Draw families for (a) 2/5, and (b) 6/11.

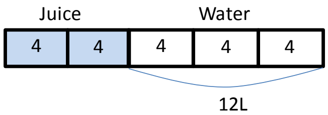
B. Experiencing ratio

1. This activity gets students to look at the strength of cordial. Need 12 glasses – 5 with normal strength cordial, 5 with water, 1 with strong cordial and 1 with weak mix. Its aim is to get across the early idea of ratio being not related to amount but mixture.
2. Get students to taste the weak and strong so they know the difference.
3. Use the other 10 to make up and compare different mixtures. For example:
   - 1 cordial 1 water to 1 cordial 3 water
   - 2 cordial 1 water to 1 cordial 2 water
   - 3 cordial 1 water to 2 cordial 1 water
4. Then do the same thing with equivalent mixes, e.g. 1 cordial 1 water to 2 cordial 2 water; 2 cordial 1 water to 4 cordial 2 water; and so on.
5. Get students to come up with a pattern for when the mixture is weak and strong.

C. Simple applications

1. Consider the problem *Juice is mixed with water 2:3, how much juice for 12 L of water?*

2. Draw the ratio/fraction diagram (as on right). Place all the information from the problem on it. Solve the problem from this information – the 12 litres of water mean that each square is 4, so there is 8 litres of juice.

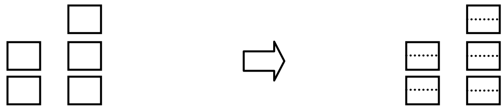


3. Repeat this for the problem *Cordial to water is 1:4, how much cordial for 20 litres of water?*

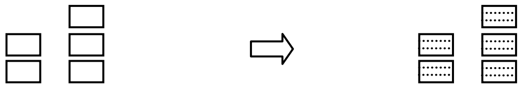
3.2 Ratio and proportion

There are four ways of dealing with proportion.

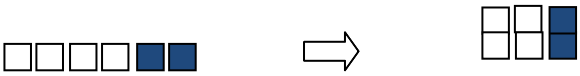
1. **Cutting squares**. Take diagram of 2:3 and cut squares in half. The ratio is now 4:6.



Cut squares into 3 parts, then 2:3 = 6:9.



2. **Joining unifix or squares** to shows that 4:2 = 2:1.



3. **Proportion sticks**. These are like fraction sticks but vertical not horizontal. There are 10 sticks which start 1, 2, 3, going down to 10; 2, 4, 6, going down to 20, and so on; until you get to 10, 20, 30, going down to 100. A full set of such sticks is shown below.

   These sticks are called the 1 stick, 2 stick, up to the 10 stick. To obtain proportion, put two sticks side by side, e.g. the 5 and the 2 stick as below (the shaded sticks). This will show that 5:2 = 10:4 = 15:6 and so on. They act like horizontal fraction sticks and can be used find equivalent ratios, the rule for equivalent ratios or proportion (i.e. equivalent ratios cancel down to same proportion) or to compare proportions.

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

5	2
10	4
15	6
20	8
25	10
30	12
35	14
40	16
45	18
50	20

4. **Stretching and squishing**. Now lines can be contracted and extended and can show relationships. This means that lines can show multiplicative relationship, e.g. length of A is 3 × length of B, so A to B is 3:1. This leads to many activities where length can be modified, such as in the activity Computer Madness.

A ________________

B _____

Activities

A. Showing proportion

1. Cut squares to show that 3:5 is 6:10, and 2:7 is 6 to 21. Join squares to show that 12:8 = 3:2.
2. Construct a set of proportion sticks and use them to find a set of proportions for 2:3 and 7:4.
3. Proportions for 2:3 can show the pattern that numbers in the 2 column increase by 2 and numbers in the 3 column increase by 3 as you go down, while the pattern for 7:4 is the 7 column increases by 7 and the 4 column by 4. Is there a pattern that is common to both these two proportions? What is it? Does it work for all proportions? [See Activity D]
4. Construct four pairs of complicated ratios which are in proportion.

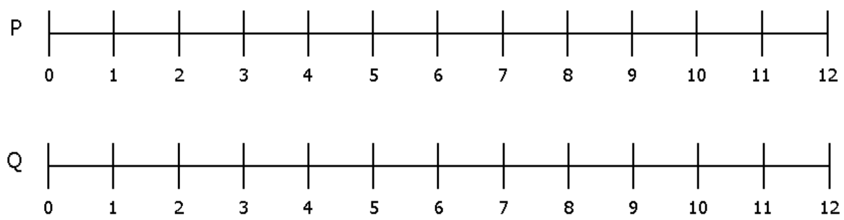
B. Computer madness

1. Use PowerPoint to draw two identical people.
   - Change A so that height A:B is 2:1
   - Change B so that width A:B is 2:3
   - Change A so height A:B is 1:3 and width A:B is 1:2
   - Play with the two people, stretching and squishing. Can you make size A:B as 2:1?

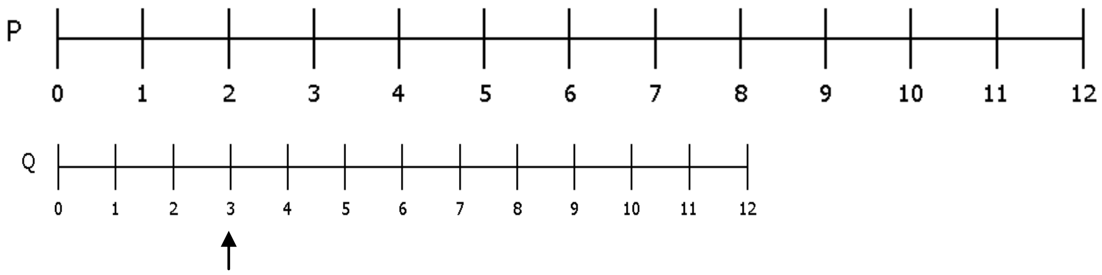
2. Construct two identical candles X and Y on PowerPoint. Change X and Y so that:
   - height X:Y is 1:4
   - height X:Y is 3:4
   - width X:Y is 9:1
   - height X:Y is 3:2 and width X:Y is 2:1

C. Developing the double number line

1. Two lines can be stretched and squished differently on a computer. **This can also be done with elastic**. To see this, construct two identical rulers P and Q on PowerPoint.



2. Stretch P so P:Q is **2:3** – that is, the 2 of P aligns with the 3 of Q.



3. Then, proportions can be seen in the diagram. For example, If P = 6, then Q is 9 (as is evident from the diagrams). This means that 2:3 = 6:9. List other proportions that can be seen in the diagram.
4. What if we put the 4 against the 3 – what proportions are available?

D. *Patterns in multiples*

1. Looking again at P and Q, we see that 2:3 = 4:6 = 6:9 = 8:12, and if we went on we would get 14:21 = 24:36 and so on. **Explore multiples** below and **notice patterns** – that multiples are the same if ratios are in proportion.

×2
2 : 3 = 4 : 6
×2

×7
2 : 3 = 14 : 21
×7

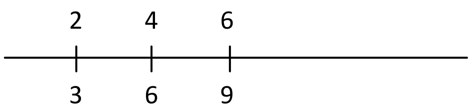
Note: We can reverse and work out proportions by keeping multiples the same.

×3
3 : 7 = 9 : ?
×3

×8
? : 7 = 24 : 56
×8

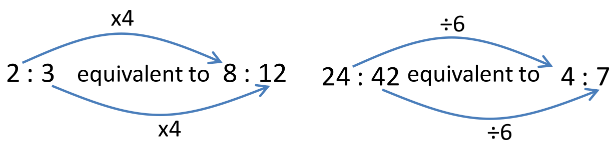
2. What does this mean for the pattern that shows proportion? That is, 2:3 = 4:6 = 6:9 and so on? How are all the proportions in the list relating to 2:3 if we use multiples? How can we use this to develop a rule for proportion around multiples?

3. If we take a line and put 2 on top and 3 under – what does this say for the relation to 4 and 6, 6 and 9?

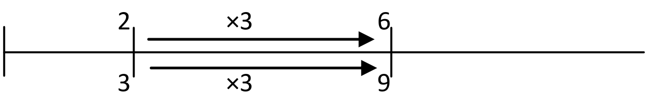


E. *Equivalence principle and double number line*

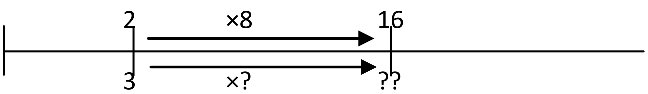
1. Equivalence is based, as for fractions, on multiplying by 1 (identity) or multiplying and dividing by the same number (compensation). However, for ratios this means that the two numbers are multiplied by the same number, that is, 2:3 is equivalent to 8:12 because both the 2 and the 3 have been multiplied by 4. This leads to two ratios being equivalent or in proportion if they can be *cancelled down to the same starting ratio* (e.g. 8:14 = 24:42 because 8:14 = 4:7 divided by 2 and 24:42 = 4:7 divided by 6). These ideas are diagrammatically shown as on right.



2. We can transfer this to both sides of a line as follows, then the change on both sides is the same.



3. This can be reversed to find answers to multiplicative comparison problems. That is, if cement to sand is 2:3, how much sand for 16 tonne of cement? We set up the double number line as below and we have to find ? and ??. First we see that ? = 8 and so ?? = 24 (or 24 tonne of cement). **This is the double number line method** (see Appendix B)**.**



4. Use the method to find the answer to problem: *If I have to spray for termites by mixing 25 mL of chemical with 2 L of water, how much chemical do I need for 15 L of water?*

3.3 Ratio applications–problems

Problem types

A ratio like 3:2 has three components – 2 the second amount, 3 the first amount, and 3:2 the ratio. The three problem types are:

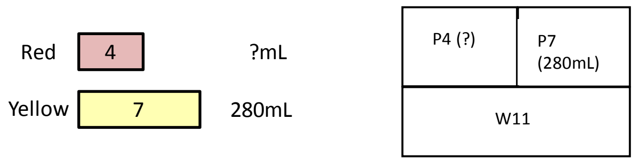
Type 1: The second amount is unknown (e.g. red and yellow colouring are mixed in ratio 4:7, how many mL of yellow for 280 mL of red?)

Type 2: The first amount is unknown (e.g. red and yellow colouring are mixed in a ratio 4:7, how many mL of red for 280 mL of yellow?)

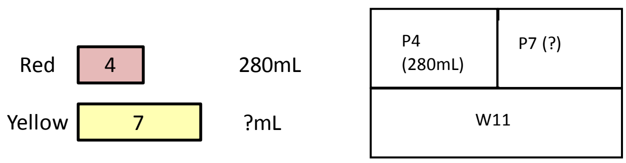
Type 3: The ratio is unknown (e.g. red and yellow colours are mixed, 280 mL of red and 490 mL of yellow, what is the ratio?)

To prevent errors, we again use a model to ensure that we use the right method for each problem type. The models chosen for this booklet are size/P-P-W models. The unitary method is again the main calculation strategy but proportion has to be used for Type 3 problems.

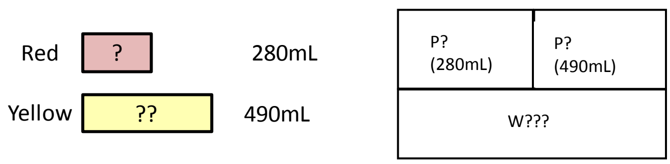
1. **Problem Type 1 – If the ratio of red to yellow is 4:7, how much red for 280 mL of yellow?** The steps are: draw and fill in information on diagram, and use unitary method to calculate answer. Here 7 Yellow is 280, so 1 is 280 ÷ 7 = 40. This makes Red = 4 × 40 = 160 ml.



2. **Problem Type 2 – If the ratio of red to yellow is 4:7, how much yellow for 280 mL of red?** The steps are: draw and fill in information on diagram, and use unitary method to calculate answer. Here 4 Red is 280, so 1 is 280 ÷ 4 = 70. This makes Yellow = 7 × 70 = 490 ml.



3. **Problem Type 3 – What is the ratio if 280 mL red is mixed with 490 mL of yellow?** The steps are: draw and fill in information on diagram, and use proportion to calculate answer. Here Red:Yellow = 280: 490 = 28: 49 (dividing both numbers by 10) = 4:7 (dividing both numbers by 7).



To do the above, students need to be able to go from problem to answer on their own. Thus, they need to be able to do two things **without the assistance** of the teacher. *Note*: Also need to **reverse** these steps – to go from solution to a problem.

Step A Take a problem and turn it into a diagram based on size with numbers and unknown correctly placed on diagram; and

Step B Use the diagram to solve the problem (one way to do this is the unitary method).

Activity

Use one of the models above to solve the following problems:

(a) Sand and cement is 7:3. How much sand for 38.4 kg of cement?

(b) Sand and cement is 7:3. How much cement for 115.5 kg of sand?

(c) 117.5 kg of sand is mixed with 47 kg of cement, what is the ratio?

(d) Chemical and water is mixed 35 mL to 3 L of water. How much chemical is in 10 L of water?

(e) 275 mL of mixture is needed for 2 m^2 of wall. How much mixture for a wall 2.5 m by 5.5 m?

Unit 4: Non-Financial Applications, Extensions to Estimation and Algebra, and Rich Tasks

Percent, rate and ratio have many applications in financial mathematics (e.g. interest, profit, loss) which will appear in Module O5. In this section, we look at an application in a non-financial situation, extensions into algebra plus investigations and rich tasks. Applications of percent, rate and ratio are excellent opportunities for longer investigations or rich tasks that explore situations in depth.

4.1 Ratio and chance applications

Many chance situations have odds. For example, *Winning on one game (game P) could be 3 chances in 7 (i.e. win:loss is 3:4) and in another game (game Q) 5 chances in 12 (i.e. win:loss is 5:7). Which of these gives better value?* There are two ways to do this, as follows:

Proportion

Similar to fractions, whether two ratios are equivalent, or in proportion, is determined by cancellation down to smallest possibility, and determining whether they are the same ratio. For example 24:40 and 33:55 both cancel down to 3:5. So they are equivalent or in proportion. This is because equivalence is a result of multiplying both numbers by the same amount (see example on right).

× 8
3:5 is equivalent to 24:40
× 8

To compare two ratios that are not in proportion, we do the same as we did with equivalent fractions – we look for a common second number. Similar to equivalent fractions, the common second number for ratios 3:4 and 5:7 is found by multiplying the existing two second numbers (e.g. 4 and 7) which gives 28. We now use the rule for proportion to find ratios equivalent to 3:4 and 5:7 with a second number of 28 (as below).

× 7 / × 4
3:4 is equivalent to ? :28 and 5:7 is equivalent to ? :28
× 7 / × 4

We see that 3:4 = 21:28 and 5:7 = 20:28, so game P (the first game) is better value.

Note: This way of relating ratios by using a common second term has been adopted (as we have said before) by race betting. Race betting odds are now always in relation to a fixed second number, 10. So 3:4 would be 7.5:10 and 5:7 would be 7.1:10, thus evaluating the first odds to be a better chance.

Percent

The ratios can be turned into a percent by dividing the second number by the first and changing the decimal to a %. Since the ratio is win:loss, the second number divided by the first gives percent loss while the first number divided by the second gives percent win:

2^{nd} by 1^{st} (% loss)	1^{st} by 2^{nd} (% win)
Game P: 4÷3 = 1.33 = 133%	Game P: 3÷4 = 0.75 = 75%
Game Q: = 7÷5 = 1.4 = 140%	Game Q: 5÷7 = 0.71 = 71%

Putting both these together, game P has the lowest loss % and the highest win %.

Question. What other applications of percent, rate and ratio can you find that are chance situations?

4.2 Extension to algebra

Algebra is the generalisation of arithmetic; thus, this extension, in particular, seeks to generalise the material so far in this module. This will be done in two subsections, the first on generalisation of models and procedures, and the second on formulae.

Generalisations

This section contains many options for generalisation. These often do not involve letters. They are a result of discussing examples and asking students for patterns – they are the final R in RAMR. **It is very important that this is done, and students are encouraged to develop their own "correct" generalisations even if they are informal and idiosyncratic**. Some useful ones are given below.

Percent ↔ Number

Percent is hundredths, therefore 27.8% becomes 0.278 as a decimal number. This change looks like as follows for place value chart:

Percent (%)

H	T	O	t	h	th
	2	7 •	8		

↔

Decimal Number

H	T	O	t	h	th
		0 •	2	7	8

Thus the **general rule** for percent–decimal conversions is (a) numerals shift two places to right (÷100) for percent to decimal number; and (b) numerals shift two places to left (×100) for decimal number to percent.

Note: The decimal number form is best for calculating percentage, for example, 27% of $60 = 0.27 × 60.

Rate

Rates are given as "number" "attribute"/"second attribute" (e.g. 24 km/hr, 6 Litres/km and $3/kg). Rate problems, therefore, deal with these attributes. Looking at a lot of problems will show that attributes "cancel" as well as numbers (e.g. 3 hr @ 40 km/hr = 120 km – the hr/hr seems to become 1). This means the following **general rules:**

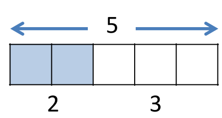
(a) if rate and first attribute is given, it is usually multiply; and

(b) if rate and second attribute is given, it is usually divide.

The generality (b) above comes from division, and its relation to multiplication by reciprocal. So, for example, 160 km @ 40 km/hr is 160 km ÷ 40 km/hr = 160 km × $\frac{\text{hr}}{40\text{km}}$ = 4 hrs (the km seem to divide away).

Ratio and proportion

Ratios are part to part and fractions are part to whole. For example, in diagram on right, the three components give a ratio of 2:3 (part to other part) and two fractions 2/5 and 3/5.



In general, this means that ratio A:B gives two fractions (A/A+B and B/A+B), while fraction C/D gives another fraction D−C/D and a ratio C:D−C.

Proportion divides ratios in sets equivalent to a starting ratio, that is, A:B = 2A:2B = 3A:3B and so on (e.g. 2:3, 4:6, 6:9, and so on). This means that the rule for proportion is that ratios are in proportion (equivalent) if they cancel down to the same starting ratio (e.g. 12:16 cancels down to 3:4 and 51:68 cancels down to 3:4, so 12:16 and 51:68 are in proportion).

Triad and three options

All percent, rate and ratio examples have three options as follows:

Percent: Amount, percent and percentage
Rate: Amount of second attribute, rate, and amount of first attribute
Ratio: Amount of first component, ratio, and amount of second component

Formulae

There are many formulae for financial situations, for example, simple and compound interest. This is left to Module O5 *Financial Mathematics*.

4.3 Percent rich tasks and investigations

The percent investigations usually involve interest rates and profit, loss and discount, but others are possible. Some investigations are below, starting with an early one.

Early investigation – estimating percentages

Need a 1 L measuring cylinder (or tall thin jug that you can calibrate), coloured water.

Teacher pours coloured water into cylinder – students guess percentage full, and record guesses and accuracy (how far out) – after each guess, give answer so that students can record errors. Begin with multiples of 10% (100 mL) in cylinder, then multiples of 5% (50 mL) and finally examples with multiples of 1% (10 mL).

Students record and graph error and find average for five trials. Students can also consider is it possible and appropriate to find error in terms of percentage of amount being estimated (and to average this).

Getting a feel for percent

There is a simple game to learn what percent means. It involves two players and one calculator (or two calculators exchanged).

(a) The first player takes the calculator and chooses a number between 9 and 100. They hide it from their opponent, then press [number] [÷] [number] [%] without the opponent seeing what they are pressing. They do not press the [=] key. The calculator should read 100%.

(b) The first player gives the calculator to their opponent who has to guess the number. They do not press clear! They press [guess] [%]. This should give a percent to help with the next guess (below 100% is too low, above 100% is too high). The opponent keeps using the [%] button to make guesses until they get the number (calculator will show 100%).

(c) Opponent's score is the number of guesses it takes to reach the correct number.

(d) Opponent sets up the calculator for the first player who now has to guess.

(e) Five trials each and the winner is the one with the lowest score.

Reversing the procedure

In the applications above, we start with a percent problem and end with an answer. Now we are going to start with an answer and end with a problem. The problem must have a percent between 10 and 90 and cannot be a multiple of 10. The number for amounts must be 50 or over.

An example is: The answer is $68 for a Type 2 problem. So what if $51 was the percent paid and then you had to find the total. Now $51 ÷ $68 = 0.75. Then the problem was, "Jack paid 75% of the cost, which was $51. What was the total cost?"

Find a problem for: (a) Type 1 problem, answer $124; (b) Type 2 problem, answer $245; and (c) Type 3 problem, answer 26%.

Interest rates

Gather information from banks and get students to work out how much they would buy something for if they had a bank loan or put the cost on a credit card and took a year to pay off the loan /credit card.

Compare ways to buy a phone or a car. Work out how much interest is paid during a normal home loan.

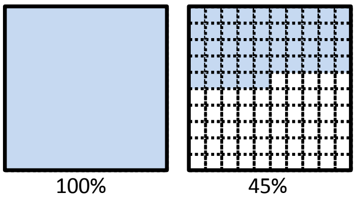
Compare simple and compound interest. For example, borrow $1000 at 12% interest for a year. How much more than simple interest do you pay back if compound every month? Every week? Every day?

Profit, loss and discount

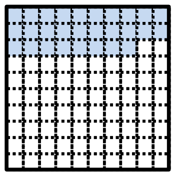
Investigations do require knowledge of what happens in percentages when you have profit or loss/discount. Students need to understand how to shade and represent percentages in these situations.

(a) Profit must be added to 100%, so 45% profit becomes a whole plus an extra 45% (as below left). The money after profit is 145%, that is, 1.45 times the original amount.

(b) Meanwhile loss and discount must be subtracted. It must also be understood that the amount after a loss or a discount is 100 − discount as a percent. That is, 28% discount gives a sale price of 100 – 28 = 72% (as on right).



Profit 45% is 145%



Discount 28% is sale price 72%

Examples:

(a) A dress is discounted 25%. What is the sale price if the original price was $60?

(b) A coat was also discounted. What is the original price if the sale price is $60?

(c) Jan made a profit of 45% on the sale of a painting. What did she sell the painting for if the painting cost her $800?

(d) Jan made a profit of 45% on the sale of a painting. What did she buy the painting for if she sold the painting for $800?

4.4 Rate investigations

The investigations in rate have to do with holding a variety of rates and seeing the way things could be. The following are some examples.

Truck company

Look at average speeds (km/hr) for truck drivers and average distances driven per hour. Consider how long a driver can drive until they have to rest.

(a) How long can a driver travel on average before they have to rest?

(b) How long would it take for a truck to travel to Melbourne and return with one driver?

(c) If you have to have four trucks per week travelling from Brisbane to Melbourne, how many drivers do you need?

Cheap houses and look out for baby

This investigation is into why square houses are cheap and why you have to worry about your children and babies when the temperature is hot or cold. The reasons for this are as follows:

(a) The area/perimeter rate for houses – you get the most area for the least perimeter in a square house. This makes it cheaper to build.

(b) The volume/surface area rate for babies – small objects have a much smaller volume to surface area and so babies have more skin area to lose/gain heat and a smaller volume to ensure there are no large changes in temperature.

These two investigations are in the YuMi Deadly Centre's PEMO MITI booklets titled *Cheap Houses* and *Look Out for the Baby!*

Recovering

This investigation is to look at recovering from giving or losing blood.

(a) There are around 7 500 white blood cells per mL3 of blood and 700 times as many red blood cells. What is the rate for red blood cells?

(b) In the blood bank they take around 450 mL of blood. How many red blood cells are lost?

(c) The replacement rate is 3 000 000 red blood cells per second. How long before they return to normal?

Rating our world

Task 1. Using the table headings below as a guide, make a list of all the electrical appliances in your home. For each appliance record the power rating. Estimate the number of hours each appliance is used in your home each week. Use this information to calculate the total energy consumed by electrical appliances in your home each week.

Table headings (guide only):

Appliance	Power Rating(W)	Hours/week	Energy used (kWh)
Jug	1500	2.5	3.75

Hint: You may find it easier to list all the appliances by placing them into categories.

Heating and cooling devices: Air conditioner reverse cycle, 3 bar heater etc.

Cooking: Microwave, fan-forced oven, etc.

General household use: Refrigerator, water heater, pool pump etc.

Entertainment: Radio, DVD, computer, etc.

Task 2. Use the information from Task 1 to calculate the quarterly cost of electricity for your household. Compare this with your usual household quarterly bill. What are the likely reasons for any differences in the amounts? List ways your family could reduce electricity use. Calculate the reduced energy use as a percentage of original total energy use.

Task 3. South-east Queensland sometimes suffers from a lack of rain. In the past, this has caused the government to implement water restrictions so we use less water. In the coming years, we will certainly have to pay more for our water. The table below gives the amount of water used for a typical household. All values are approximate.

Source http://www.melbourne.vic.gov.au/rsrc/PDFs/Water/CalculatorWaterMark.pdf

Water Use	Volume (L)
Dual flush toilet Single flush	5
Dual flush toilet Double flush	11
Shower (7.5 L/minute) for low water flow shower head	30 per 4 min shower
Shower (12 L/minute) regular shower head	48 per 4 min shower
Bath	96
Washing water efficient AAAA front loading per load	40
Regular washing machine per load	130
Meal preparation	5
Washing up	10
Jug	2
Dishwasher	40
Bucket for garden watering	9
Swimming pool top up for 50000L pool	720
Teeth cleaning with tap running	5
Teeth cleaning with set amount	1
Other water uses	

Use the table above to calculate the volume of water used in your household every week. Remember to take account of all the people who live in the house in a typical week.

Task 4. If the cost of water is \$2.00 per kilolitre, how much would you expect your household to pay for a quarter (3 months)? List ways your household could save water. Calculate the reduced amount of water as a percentage of your current use.

Many people have installed rainwater tanks. How much rain can we collect in Ipswich in a typical year? For example: If 5 mm of rain fell on an area 100 cm × 100 cm what volume of rain has fallen? [Volume = 0.005 m × 1 m × 1 m = 0.005 m³ = 0.005 m³ × 1000 L/m³ = 5 L] The average rainfall in Ipswich per year is 863.1 mm.

Task 5. Use a suitable method to calculate the area of your roof. Your answer should include a diagram. Use this answer and the information above to calculate the average volume of water which could be collected in Ipswich in a year.

Task 6. What percentage of your typical household use (answer to Task 3) could be accommodated by a rainwater tank? If the Ipswich average rainfall fell to 60% of its current level what percentage of your household use could be accommodated by a rainwater tank?

Construct a table to show the percentage of household water a tank would hold if the average Ipswich rainfall is 10%, 20%, 30%, 40%, 50%, 60%, 70%, 80%, 90% and 100% of the current average.

4.5 Ratio investigations

Ratio has many applications through proportion, particularly in the trades. It is also useful to analyse relationships such as those relating to size and money. Some examples are as follows.

Drawing faces

(a) Get students to draw their face.

(b) Get students to measure the position of their hair, eyes, mouth and nose to the sides of their face. For example, they will find the eyes are halfway down the face.

(c) Use this data to place the hair, nose, eyes, and mouth in a drawing of their head.

(d) Is this a more accurate drawing than the face they drew in step (a)?

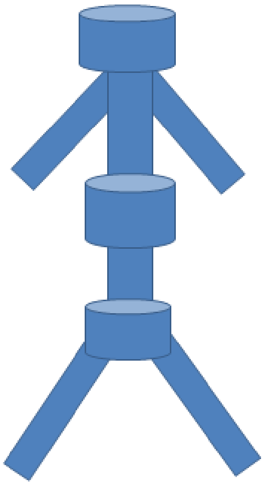
Making skeletons

(a) A skeleton can be made with strips of paper for backbone, arms and legs, and circular strips for head, chest, hips etc. (as shown on the right).

(b) Pick someone in your group and make a 1/2 and a 1/3 size skeleton.

(c) Compare the 1/2 size one with the actual measurements of a 1 to 2 year old baby.

(d) Are there differences between the proportions between babies and students? [Yes there are, the head is larger proportionally in a baby.]



Is Barbie a monster?

(a) Get a Barbie doll (preferably an older version) and measure her proportions.

(b) Make an adult size version of the doll (think of a way to do it). How do its proportions relate to ordinary people?

(c) Is it a monster? Discuss.

How tall is the thief?

(a) A house got robbed. The only clue is the footprint of a shoe in the ground beside the window used to break in. The shoe print is 34 cm long. Prepare a plan to determine the height of the thief.

(b) Make sure you have many trials to find a relation between height and foot length.

(c) This is in a YDC PEMO MITI booklet titled *How Tall is the Criminal?*

Best buy in weed killer

There were two products for sale, with the characteristics shown in the table on right.

Work out which is the cheaper/better value.

Product	Cost	Contents	Strength
A	$24.95	250 mL	80 mL/L
B	$26.45	100 mL	30 mL/L

Money exchange

The exchange rate for money is given in terms of overseas money per Australian dollar (a rate). However, the exchange rate between two overseas countries is a ratio. Can you work out the exchange rate between Europe and USA as a rate in terms of (a) 1 Euro, and (b) 1 US Dollar?

World poverty / Australian poverty

(a) Look up gross national product/person for poor and rich countries (and wages).

(b) Look up how much food per person in poor and rich countries (in terms of types of food and minimum daily requirements).

(c) Look up health in rich and poor countries (access to doctors, life expectancies, infant mortality).

(d) Compare these things in tables and graphs.

(e) Repeat this, comparing Australian Indigenous with non-Indigenous people.

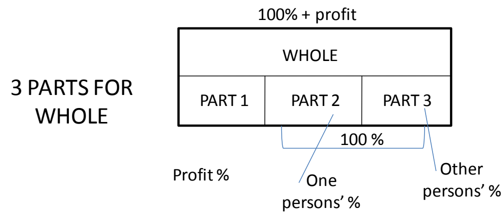
Final notes

Rich tasks should involve the students in using the Internet to gather information to make decisions about something that is motivating to them. For example: (a) buying a smart phone and working out the best plan/agreement to be on; (b) borrowing money to buy a car; and (c) the difficulties with using money on a credit card.

The YuMi Deadly Centre has a set of 12 Prevocational Mathematics books built around rich tasks. These are available for download free from the YuMi Deadly Centre Website <ydc.qut.edu.au>. Many of the rich tasks in these booklets are built around rate and ratio.

Finally, it should be noted that, as problems become more complex, the P-P-W model can extend as follows:

Problem: *A painting was sold for a profit of 22%. The profit was kept in an account. The rest of the money was split 35% to Fred and the remainder to John. If the sale price was $2,350, how much did John get?*



We will now need to have a P-P-P-W model as on right.

Test Item Types

This section presents instructions and the test item types for the subtests associated with the units. These will form the bases of the pre-test and post-test for this module.

Instructions

Selecting the items and administering the pre-post tests

This section provides an item bank of test item types, constructed around the units in the module. From this bank, items should be selected for the pre-test and post-test; these selected items need to suit the students and may need to be modified, particularly to make post-test items different to pre-test items. The purpose of the tests is to measure students' performance before and after the module is taught. The questions should be selected so that the level of difficulty progresses from easier items to more difficult items. In some modules this will follow the order of the units and subtests, and in other modules it will not, depending on the sequencing across the module and within units. The pre-test items need to allow for students' existing knowledge to be shown but without continual failure, and the post-test items need to cover all the sections in a manner that maximises students' effort to show what they can do.

In administering the pre-test, the students should be told that the test is not related to grades, but is to find out what they know before the topic is taught. They should be told that they are not expected to know the work as they have not been taught it. They should show what they know and, if they cannot do a question, they should skip it, or put "not known" beside questions. They will be taught the work in the next few weeks and will then be able to show what they know. Stress to students that **any pre-test is a series of questions to find out what they know** before the knowledge is taught. They should do their best but the important questions come at the end of the module. For the **post-test**, the students should be told that **this is their opportunity to show how they have improved**.

For all tests, **teachers should continually check to see how the students are going.** Items in later subtests, or more difficult items within a particular subtest, should not be attempted if previous similar items in earlier subtests show strong weaknesses. Students should be allowed to skip that part of the test, or the test should be finished. Students can be marked zero for these parts.

Information on the percent, rate and ratio item types

The percent, rate and ratio test items have been divided into four subtests to match the four units in the module. Like the measurement units in Modules M1, M2 and M3, Units 1, 2 and 3 of this N4 module each cover a different topic in sequence from simple to advanced. Unit 4 is an advanced unit. This means that the pre-test should include the simpler item types from the each of Subtests 1, 2 and 3, while the post-test should include all the test item types from Subtests 1, 2 and 3, plus a rich task for Subtest 4.

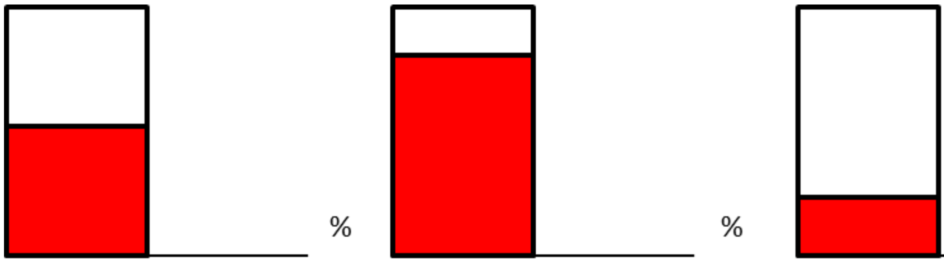
The grade for the rich task should be A completely done [4], B reasonably done [3], C borderline [2], D fail but some information [1] and E nothing correct/did not attempt [0].

Note: Always read the questions to the students and explain any contextual information as long as this does not direct to answer.

Subtest item types

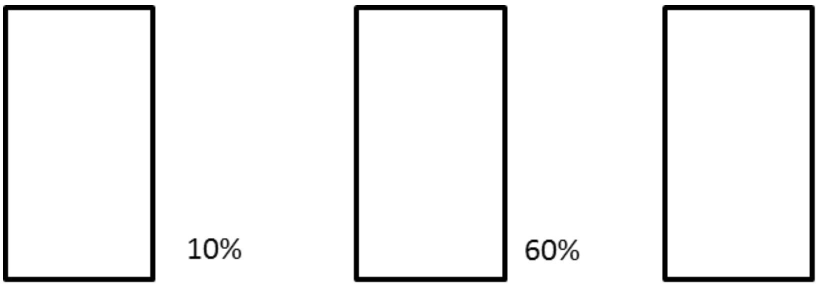
Subtest 1 items (Unit 1: Percent)

1. State the amount of cordial as a percent of the glass.



__________ % __________ % __________ %

2. Draw the amount of cordial in the glass.



10% 60% 90%

3. (a) Write 15% as a decimal ____________________________________

   (b) Write 7.45 as a percent ____________________________________

   (c) Write 3.5% as a decimal ____________________________________

   (d) Write 0.58 as a % ____

4. Use the P-P-W diagrams to solve these % problems. The first one has been done for you.

<table>
<tr><th></th><th>Working</th><th>P-P-W Diagram</th></tr>
<tr><td></td><td>Find 60% of $300
100% = $300
10% = $30
60% = 6 × $30 = $180</td><td><table><tr><td>P</td><td>P</td></tr><tr><td colspan="2">W</td></tr></table> <table><tr><td>60% (?)</td><td>40%</td></tr><tr><td colspan="2">100% ($300)</td></tr></table></td></tr>
<tr><td>(a)</td><td>Find 40% of $150</td><td><table><tr><td>P</td><td>P</td></tr><tr><td colspan="2">W</td></tr></table> <table><tr><td>40% (?)</td><td>60%</td></tr><tr><td colspan="2">W 100% ($150)</td></tr></table></td></tr>
<tr><td>(b)</td><td>30% is $75. How much is 100%?</td><td><table><tr><td>P</td><td>P</td></tr><tr><td colspan="2">W</td></tr></table> <table><tr><td>30% ($75)</td><td>70%</td></tr><tr><td colspan="2">100% (?)</td></tr></table></td></tr>
</table>

5. (a) Find 35% of $175.

(b) 35% is $175, what is 100%?

Subtest 2 items (Unit 2: Rate)

1. (a) There are 7 apples per bag, how many apples in 22 bags? ______________

   (b) There are 7 apples per bag, how many bags for 119 apples? _____________

2. Use the rate-to-unit diagrams to solve these rate problems. The first one is done for you.

<table>
<tr><th></th><th>Working</th><th colspan="2">Diagram</th></tr>
<tr><td></td><td>Potatoes were $2.50/kg. How many kg of potatoes could you buy with $10?
2.50 = $10
1 = 10 ÷ 2.50 = 4
Answer = 4 kg</td><td><table><tr><td colspan="2">Amount</td></tr><tr><td>1 kg</td><td>$2.50</td></tr></table></td><td><table><tr><td colspan="2">$10</td></tr><tr><td>? kg</td><td>$2.50</td></tr></table></td></tr>
<tr><td>(a)</td><td>Potatoes were $2.50/kg. How much would 7.5kg cost?</td><td><table><tr><td colspan="2">Amount</td></tr><tr><td>1 kg</td><td>$2.50</td></tr></table></td><td><table><tr><td colspan="2">?</td></tr><tr><td>1 kg</td><td>$2.50</td></tr></table></td></tr>
<tr><td>(b)</td><td>Petrol was $1.60/L. How many litres could you buy for $64?</td><td><table><tr><td colspan="2">Amount</td></tr><tr><td>1 L</td><td>$1.60</td></tr></table></td><td><table><tr><td colspan="2">$64</td></tr><tr><td>? L</td><td>$1.60</td></tr></table></td></tr>
</table>

3. (a) The rate for the phone call was $1.70 per minute.

   How much money for a call of 19.5 minutes? _________________________

   (b) The rate for the phone call was $1.70 per minute. The phone call cost $36.55.

   How long did the call take? _____________________________

   (c) On another phone, a call of 45.5 minutes cost $86.45.

   What is the rate per minute on this phone? __________________________

Subtest 3 items (Unit 3: Ratio)

1. (a) Flavouring and water are mixed to make cordial drink. The fraction of flavouring is $^{3}/_{10}$.

   What is the ratio of flavouring to water? ___________________

   (b) Sand and cement are mixed to make concrete. If the ratio of sand to cement is 7:2, what is the fraction of cement in the concrete? _______________

2. Circle the ratios that are equivalent to 3:5.

   (a) 12:25 (b) 21:35 (c) 24:40 (d) 32:52

3. Use the P-P-W diagrams to solve these ratio problems. The first one is done for you.

<table>
<tr><th></th><th>Working</th><th>P-P-W Diagram</th></tr>
<tr><td></td><td>Cordial to water is 2:5. How much cordial would you need for 10L of water?
5 = 10L
1 = 10 ÷ 5 = 2L
2 = 4L of water</td><td><table><tr><td>P</td><td>P</td></tr><tr><td colspan="2">W</td></tr></table><table><tr><td>2 (?L)</td><td>5 (10L)</td></tr><tr><td colspan="2">7</td></tr></table></td></tr>
<tr><td>(a)</td><td>Cordial to water is 2:5. How much water would you need for 15.5L of cordial?</td><td><table><tr><td>P</td><td>P</td></tr><tr><td colspan="2">W</td></tr></table><table><tr><td>2 (15.5L)</td><td>5 (?L)</td></tr><tr><td colspan="2">7</td></tr></table></td></tr>
<tr><td>(b)</td><td>To make the concrete, sand is mixed with cement in ratio 7:3. How much sand would you need for 12 tonnes of cement? How much concrete would this make?</td><td><table><tr><td>P</td><td>P</td></tr><tr><td colspan="2">W</td></tr></table><table><tr><td>7 (? sand)</td><td>3 (12t)</td></tr><tr><td colspan="2">10 (? concrete)</td></tr></table></td></tr>
</table>

4. (a) The ratio of red to yellow is 4:9.

How much yellow for 44 litres of red?

(b) The ratio of red to yellow is 4:9.

How much red for 54 litres of yellow?

(c) In a new mixture, 56 litres of red was mixed with 96 litres of yellow.

What is the ratio of red to yellow? ________________________

(d) During one day (24 hours), the local radio station played 9 hours of rock and roll and the rest of the time was country music. What is the ratio of rock and roll to country music on the station? _______________________

Subtest 4 items (Unit 4: Non-financial applications, extensions and rich tasks)

Use one of the rich tasks from section 4.5 in the module (pp. 30–31).

Appendix A: Other Methods for Teaching Percent, Rate and Ratio Applications

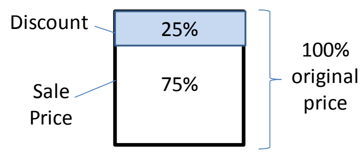
In this module, we have focused on teaching percent, rate and ratio applications by the size/part-part-whole (PPW) model. There are three models that could be used to solve percent, rate and ratio applications. Only one model was provided in the units of this module for three reasons: (a) to simplify the module (with all three present it was confusing to teach and too long); (b) to overcome a scope and sequence problem (the perquisites for one of the other models had not been covered); and (c) to meet the needs of schools to have a method that was easy to teach and easy to relate to real world situations. The size/PPW model is also the most straightforward of the three models and was continually chosen by teachers as the method they would adopt when all three were being taught.

This Appendix provides information on all three models and models, provides a process for using the models and discusses other ways of teaching applications for percent, rate and ratio that would have to be considered if there was a variety of strategies/models.

The three models – size, double number line, and change

This subsection describes the three model types for percent, rate and ratio

Three models for percent

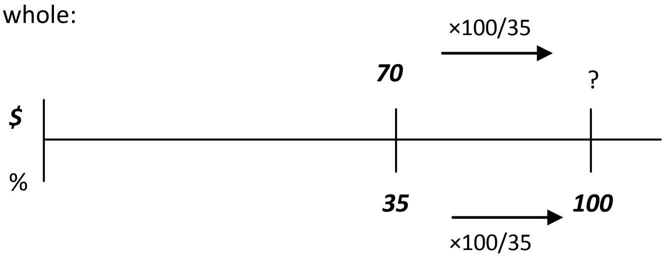


1. **Size**. Using a picture (e.g. as on right) to represent the percentage as area in problem such as *How much for an $80 coat when discount is 25%*? Can see that, if the 100% is $80, the discount is 1/4 or $20, so the coat costs $60.

   This picture can be a diagram with all 100 squares shown (and 25 shaded), a picture like the above, or a part-part-whole (P-P-W) picture as on right. For the problem *if 35% is $70, what is the whole amount?*, 35% being $70, means 2% for each 1%, which means that the whole is $200.

Part 35%, $70	Part 65%
Whole 100% ?	

2. **Double Number Line**. Using both sides of the line (see below) to show problem *if 35% is $70, what is the whole amount?* Then can find whole by seeing that change from 35% to 100% is same as from $70 to whole:



Thus, ? = 100/35 × 70 = $200

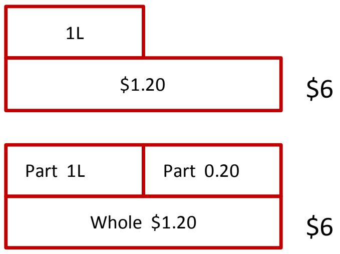
3. ***Change***. Using an arrow to represent the problem *if 35% is $70, what is the whole?* as change from the whole to the part (the $70) ×0.35, as below. Then the change has to be reversed.

Whole Start $\xrightarrow{\times\%}$ ***Part End***   ***?*** $\xrightarrow{\times 0.35}$ ***$70***   ? = $70 ÷ 0.35 = $200

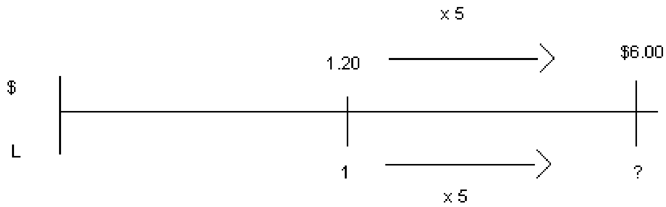
The three models for rate

Rate is also represented by the three types of models – size, double number line, and change. However, rate is like fraction in that it is something that multiplies – it acts like a multiplicative operator or multiplier. Actually, multiplication is best understood as number by rate, for example, 4 bags of lollies at 3 lollies/bag = 12 lollies.

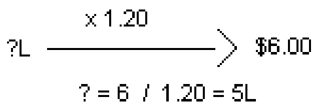
1. **Size.** Rate is represented with the "per" attribute as one. That is, $1.20/litre is as on right – in the first diagram, the 1 litre is compared to the $1.20, in the second diagram (P-P-W) the 1L is one part and there is the other part and the $1.20 is the whole. Thus, if we have the problem *How many litres can I buy for $6 if petrol costs $1.20/litre?*, the $1.20 represents $6 so is × 5; and this means that the 1 litre is x5 as well. Thus, the answer is 5 litres.



2. **Double Number Line.** The top of the line is one attribute while the bottom of the line is the other. So, if we have $1.20/litre and the problem *How many litres can I buy for $6 if costs $1.20/litre?*, the double number line works as on right. We can see that the $1.20 to $6 is ×5, so the answer is 5 × 1L = 5 litres.



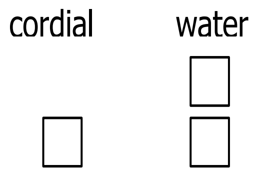
3. **Change (Multiplier).** Rate can be considered as a multiplication change from second to first attribute. Thus, for the problem *How many litres can I buy for $6 if costs $1.20/litre?*, litres change to $ by ×1.20, and so unknown litres changes to $6 by ×1.2. To solve, reverse/inverse the change and we have 6 ÷ 1.2 = 5, that is 5 litres.



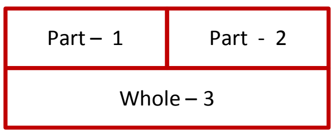
The three models for ratio

Again there are the three models as before.

1. **Size.** The first of these is to look at ratio in terms of size (this can be in terms of area or sets). For the problem, *If cordial to water is 1:2, and we have 16 litres of water, how much cordial do we need?,* the diagrams are on the right.

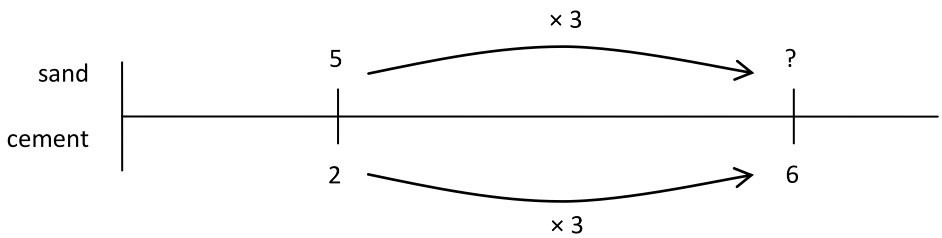


   For the sets, we have 1 cordial as unknown and 2 waters as 16 litres, so 8 for each square means cordial is 8 litres. For the P-P-W diagram, ratio is part to part (P-P) and we have 1:2, the 2 is 16 litres, so again the cordial is 8 litres.



   Note: Ratio normally differs from fraction in that fraction is part to whole while ratio is part to part.

2. **Double number line.** This is a good model for problem solving with ratio. Consider the problem *Sand to cement is 5:2, how much sand for 6 kg of cement?* The diagram is as below:



The diagram places the 5 for sand opposite to the 2 for cement and the unknown next to the 6. As the two multiples are the same, the 2 being multiplied by 3 to get 6 means that the 5 has to be multiplied by 3 also. Thus, the amount of sand is 5 × 3 = 15 kg of sand.

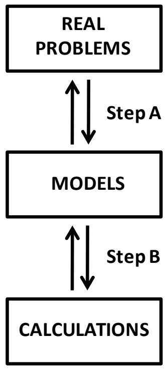
3. **Change.** This is a difficult method for ratio unless you realise that 5:2, for example, means the same as ×2÷5, that is ×0.4. Again we consider the problem *Sand to cement is 5:2, how much sand for 6 kg of cement?* The diagram is as on the right. We consider ratio of 5:2 as transforming the sand to cement by ×0.4. Since we know the cement is 6 kg, then we have to reverse/inverse the change to get the amount of sand. That is, the sand is 6 ÷ 0.4 = 15 kg of sand.

$$\text{Sand} \xrightarrow{\times 0.4} \text{Cement}$$

Processes and problem types

This section looks at a two-step process for using all three models and how it takes account of problem types (a consequence of the triadic big idea).

Two-step process

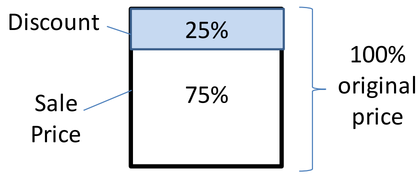


It is important to remember that the teaching objective for this section is to have mental models of percent, rate, and ratio we can use to get answers to problems and applications. This means we need to be able to do both of the following: (a) translate real-world problems to mental models; and (b) use models to calculate answers. Thus, we focus our teaching on two steps:

- Step A: lessons and worksheets that **relate real problems to models as diagrams**, and the reverse; and
- Step B: to have lessons and worksheets that **use the models to determine the calculations** and the reverse (the actual calculations can be done with calculators).

This is a two-step process, namely, **real problems ←→models** and **models ←→ calculations**, as on right.

It is our aim is for students to be able to **do the two steps in their minds** (using no drawings, or just quick doodlings). So there is a need to ensure that students move away from formal drawn models to their own quick way of drawing things or to visualisation in the mind. Body → Hand → Mind is a good way to do this: (a) act out problems so students see what is involved; (b) use drawings of models on which to place information so students can work out what calculations to use (and do not make incorrect suppositions); and (c) get students to visualise the models and put problems into these imagined models to determine the calculations that have to be done.



The reason for models is that there are more than one problem type for percent, rate and ratio and procedures for one type cause errors in the other. For example, for problem *to find 45% of $80*, students can be told to put the 45 over a 100 and multiply by the amount (i.e. 45/100 × 80/1). This gives the correct answer. But when given the problem *to find the total when 45% is $80*, the students are told to put 80/45 and multiply by 100, an entirely different use of 100. This can cause errors. Models assist the students to know what to do. For example, many students when given the problem, *There was a 25% discount, dress sold for $60, how much was the pre-sale price?* will find 25% of $60 = $15 and add this to $60 to get $75. However, a model such as that on right be will show that 75% is sale price and 100% is original price so $60 has to be divided by 75 and multiplied by 100 (which gives $80 using a calculator).

Teaching all problem types

Problem types. Multiplicative comparison, percent, rate and ratio problems involve three components as below:

1. for **percent**, it is the starting amount, the percent, and the finished percentage (e.g. in problem, *25% of $48 is $12*, the $48 is the starting amount, the 25% is the percent, and the $12 is the finishing percentage);

2. for **rate**, it is the amount of the second attribute, the rate and the amount of first attribute (e.g. in problem, *30L at $1.50/L is $45*, the 30 is the starting litres, the 1.50 is the rate $/L, and the 45 is the finishing dollars); and

3. for **ratio**, it is the first amount, the ratio, and the second amount (e.g. in problem, *20kg of sand at ratio sand:cement is 5:2 needs 8kg of cement*, the 20 for sand is the starting amount, the 5:2 is the ratio, and the 8 for cement is the finishing amount).

The triadic big idea (see above sub-section) tells us that, as there are three components, there are three possible problems as below:

PROBLEM OVERALL	TYPE 1	TYPE 2	TYPE 3
Percent: 25% of $48 is $12	What is 25% of $48?	25% is $12, what is the total?	$12 is what percent of $48?
Rate: 30L at $1.50/L is $45	How much does 30L cost at $1.50/L?	How many litres can you buy for $45 at $1.50/L?	What is the rate when buy 30L for $45?
Ratio: 20kg of sand at sand to cement ratio of 5:2 needs 8kg of cement	How much cement for 20kg of sand at sand to cement ratio of 5:2?	How much sand for 8kg of cement at sand to cement ratio of 5:2?	What is the ratio for 20kg of sand mixed with 8kg of cement?

Further information on the double number line model

The double number line is a mental model that can be used for percent, rate, and ratio to get answers to problems and applications. It is used in two steps: **Step A** – translating real-world problems to the double number line; and **Step B** – to use the double number line to calculate answers. Double number lines are actually single lines on which numbers appear above and below for different attributes and amounts and using different scales. The numbers above and below are equivalent when on the same side of lines that cross the number line. The application of the double number line to percent, rate and ratio is now illustrated with examples.

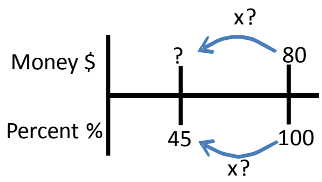
Percent

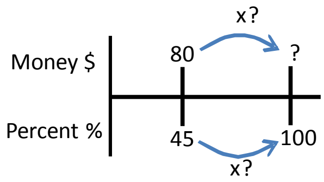
Problems: *What is 45% of $80?* And, *45% is $80, what is total?*

One side of the line is for percent and the other for what the percent is of. The 100% is highlighted and is the whole, and so are any other percents or numbers. Steps are as follows.

Step A – Draw the double number line with cross lines, mark in what above and below the line are called (here money and percent – place percent under the line); place little lines for 100% and any other percents or amounts and put numbers on either side; and mark in all information from the problem (including ? for unknowns) and place in arrows (arrows have to go in same direction and towards the ?).

Step B – Calculate the multiple that can be calculated for the arrow from number to number; and use the fact that the arrows have the same multiple to calculate answers.





What is 45% of $80?

45% is $80, what is total?

Answers

? on percent arrow is 45/100, so
? money is $80 × 45/100 = $36

? on percent arrow is 100/45, so
? money is $80 × 100/45 = $177.77

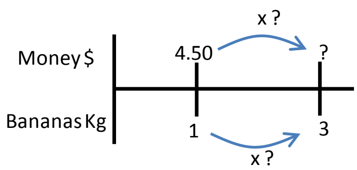
Rate

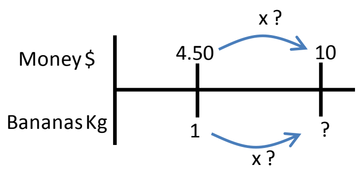
Problems: *Bananas are $4.50 a kg, what is the cost of 3kg of bananas?* and *Bananas are $4.50 a kg, how many kg of bananas for $10?*

Rate is difficult because one attribute always relates to number one. Of course, this is no different to percent where one number always relates to 100% which is one, but it seems to be different. The same steps are gone through. (*Note*: It is useful to always put the attribute that is 1 under the line.)

Step A – Draw the double number line model, mark in the information on the line, place in unknowns, mark in arrows (going the same way and towards the unknown amount).

Step B – Calculate the multiple for the arrow from number to number and use this to calculate answers.





Bananas are $4.50 a kg, what is the cost of 3 kg of bananas?

Bananas are $4.50 a kg, how many kg of bananas for $10?

Answers

? on bananas arrow is 3/1 =3, so
? money is $4.50 × 3 = $13.50

? on money arrow is 10/4.5, so
? bananas is 1 × 10/4.5 = 2.22kg

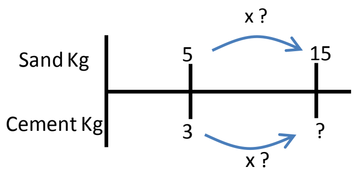
Ratio

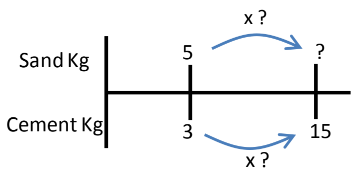
Problems: *Sand and cement are 5:3, how much cement for 15kg of sand?* and *Sand and cement are 5:3, how much sand for 15kg of cement?*

The double number line is excellent for ratio. There is no whole but it is normal to have the second attribute under the line. The two steps are as for percent.

Step A – Draw the double number line model, mark in the information on the line, place in unknowns, mark in arrows (going the same way and towards the unknown amount).

Step B – Calculate the multiple for the arrow from number to number and use this to calculate answers.





Sand and cement are 5:3, how much cement for 15 kg of sand?

Sand and cement are 5:3, how much sand for 15 kg of cement?

Answers

? on sand arrow is 15/5 = 3, so
? cement is 3 × 3 = 9 kg

? on cement arrow is 15/3 = 5, so
? sand is 5 × 5 = 25 kg

Short look at multiplicative comparison and the change model

Although we are not using the change model in this module and leaving its use to Module A3 *Change and Functions*, it is useful to briefly look at multiplicative comparison's/change's role in percent, rate and ratio.

In their applications, the three areas, percent, rate and ratio, are multiplicative, that is, they involve multiplication and division in their solutions. This is because **all three are a form of multiplicative comparison**:

- percent compares the original amount with the amount after the percent is calculated multiplicatively – for example, 200% of \$70 is 2×\$70 = \$140;
- rate compares one attribute with another different attribute multiplicatively – for example, \$3/kg means that \$4kg costs 3×4 = \$12; and
- ratio compares the second amount with the first (same attribute) multiplicatively – for example, butter to flour in ration 1:3 means that \$200g of butter is mixed with 3×200 = 600g of flour.

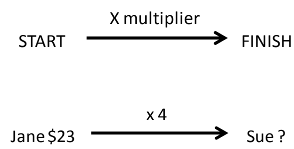
Multiplicative comparison is a way that compares two amounts through multiplication (and, backwards, through division) – it is best seen in relation to the other forms of comparison: (a) *numerical* – 24 is larger than 8; (b) *additive* – 24 is 16 more than 8; and (c) *multiplicative* – 24 is 3 times as large as 8.

Using multiplicative comparison with percent, rate and ratio should follow:

- using multiplicative comparison with whole numbers and fractions and decimals (e.g. John has 3 times the money Jack has, Sue's house was 3/4 the value of Jo's house, and so on); and
- developing the concepts of multiplicative comparison for multiplication of whole numbers.

Since percent, rate and ratio are similar in their mathematical structure, their applications and problems can be solved by using the same methods (we will show three models, all of which can solve all problem types). [*Note:* this is a **teaching big idea** – *mathematically similar ideas are taught using similar models and their problems are solved with similar methods.]*

Note: One way to think about multiplicative comparison situations is to think of them as change or transformation where a starting number is multiplied by a multiplier to get a finishing number (see diagram on the right). For the problem "Sue has 4 times the money of Jane, how much does Sue have, if Jane has \$23?" it is easy to see that the multiplier in the multiplicative comparison is ×4 as on right (answer \$92). We will look at this when we do the module on change and functions.



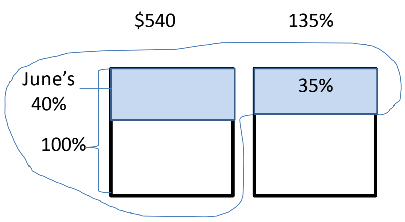
Effective teaching when using all models

Finally two points need to be made concerning effective teaching in this area.

Use a variety of methods with a variety of examples. First, it is important to ensure we practise both problem → models and models → calculation for a **variety of situations.** These would include the following.

(a) *Percent:* Ensure that students practise doing percent examples where there is profit, loss, and discount in all the three problem types.

Ensure that students practise more-than-one-step problems, where you move from a profit to a percent (e.g. John made a profit of 35% on the sale, selling an antique desk for \$540, how much did he pay back June who had given him 40% of the original price of the desk?). Note that our diagrams help here (see on right) – \$540 is 135% so June gets 40% which is 40/135 of the \$540.



(b) *Rate:* Give examples that relate a lot of measures, not just \$ and L or kg. Give scales that are large, 850kg/m^2, and small, 0.07g/m^3.

Get students used to seeing what is the one or the unit.

(c) *Ratio:* Give a variety of ratios – examples with the same units and examples with different units. Have large unit differences, e.g. 20mL:5L which is 20:5000. Explore examples of scale in maps as well as mixing things.

Practise using a variety of models, don't just practise calculations. Spend time practising **problem → model or diagram**. Also reverse everything, practise **model or diagram → problem**.

Make school decisions regarding variety taught. Second, each school has to make a decision on what to do regarding **variety of methods** with respect to percent, rate, and ratio. These three are the same thing mathematically but different in symbols and words used. There are three models (see Appendix A), so a school decision has to be made between three options as follows:

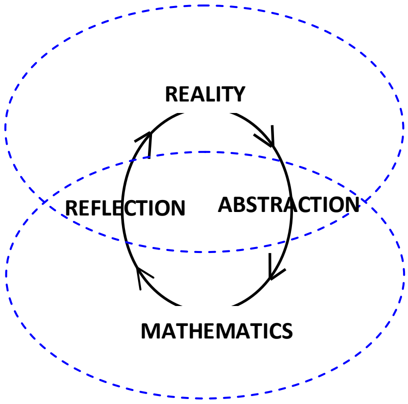
(a) *Teach all models in all situations.* Teach the full variety of ways to solve all percent, rate and ratio problems. This is based on believing that giving students a repertoire of strategies/models is the important outcome of mathematical teaching. As we have focused in size, with some activity on the double number line (see Appendix B), this module is insufficient for this. However, after the *Change and functions* model, this approach will be possible.

(b) *Teach one model for all situations.* Schools choose to use one method for all situations. The size model, particularly using the part-part-whole model for percent and ratio, is the easiest to teach – it is not powerful but quick to teach. The double number line has been successfully used in VET training in TAFEs for low performing students. Finally, the change model is very powerful and, if taught successfully, the quickest and most powerful to use. For this first foray into percent, rate and ratio we have focused on the easiest to teach model, size.

(c) *Teach one model, but a different model, for the three situations.* There is some argument that percent is best through size diagrams, rate through change, and ratio through double number line.

Appendix B: RAMR Cycle

AIM advocates using the four components in the figure on right, reality–abstraction–mathematics–reflection (RAMR), as a cycle for planning and teaching mathematics. RAMR proposes: (a) working from reality and local culture (prior experience and everyday kinaesthetic activities); (b) abstracting mathematics ideas from everyday instances to mathematical forms through an active pedagogy (kinaesthetic, physical, virtual, pictorial, language and symbolic representations, i.e. body → hand → mind); (c) consolidating the new ideas as mathematics through practice and connections; and (d) reflecting these ideas back to reality through a focus on applications, problem solving, flexibility, reversing and generalising (see figure on right).



The innovative aspect of RAMR is that the right half develops the mathematics idea while the left half reconnects it to the world and extends it. For example, whole-number place value built around the **pattern of threes** where hundreds-tens-ones applies to ones, thousands, millions, and so on, can be easily extended to metrics by considering the ones to be millimetres, the thousands to be metres and the millions to be kilometres.

Planning the teaching of mathematics is based around the RAMR cycle if it is deconstructed into components that are applied to a mathematical idea. By breaking instruction down into the four parts and taking account of the pedagogical approaches described above, the cycle can lead to a structured instructional sequence for teaching the idea. The table below briefly outlines how this can be done. Prerequisite mathematical ideas are considered in the Reality and Mathematics components of the cycle, while extensions and follow-up ideas are considered in the Reflection component.

REALITY • **Local knowledge**: Identify local student cultural-environmental knowledge and interests that can be used to introduce the idea. • **Prior experience**: Ensure existing knowledge and experience prerequisite to the idea is known. • **Kinaesthetic**: Construct kinaesthetic activities, based on local context, that introduce the idea.
ABSTRACTION • **Representation**: Develop a sequence of representational activities (physical to virtual to pictorial materials to language to symbols) that develop meaning for the mathematical idea. • **Body-hand-mind**: Develop two-way connections between reality, representational activities, and mental models through body → hand → mind activities. • **Creativity**: Allow opportunities to create own representations, including language and symbols.
MATHEMATICS • **Language/symbols**: Enable students to appropriate and understand the formal language and symbols for the mathematical idea. • **Practice**: Facilitate students' practice to become familiar with all aspects of the idea. • **Connections**: Construct activities to connect the idea to other mathematical ideas.
REFLECTION • **Validation**: Facilitate reflection of the new idea in terms of reality to enable students to validate and justify their new knowledge. • **Applications/problems**: Set problems that apply the idea back to reality. • **Extension**: Organise activities so that students can extend the idea (use reflective strategies – *flexibility*, *reversing*, *generalising*, and *changing parameters*).

Appendix C: AIM Scope and Sequence

Yr	Term 1	Term 2	Term 3	Term 4
A	**N1: Whole Number Numeration** Early grouping, big ideas for H-T-O; pattern of threes; extension to large numbers and number system	**O1: Addition and Subtraction for Whole Numbers** Concepts; strategies; basic facts; computation; problem solving; extension to algebra	**O2: Multiplication and Division for Whole Numbers** Concepts; strategies; basic facts; computation; problem solving; extension to algebra	**G1: Shape (3D, 2D, Line and Angle)** 3D and 2D shapes; lines, angles, diagonals, rigidity and properties; Pythagoras; teaching approaches
	N2: Decimal Number Numeration Fraction to decimal; whole number to decimal; big ideas for decimals; tenths, hundredths and thousandths; extension to decimal number system	**M1: Basic Measurement (Length, Mass and Capacity)** Attribute; direct and indirect comparison; non-standard units; standard units; applications	**M2: Relationship Measurement (Perimeter, Area and Volume)** Attribute; direct and indirect comparison; non-standard units; standard units; applications and formulae	**SP1: Tables and Graphs** Different tables and charts; bar, line, circle, stem and leaf, and scatter graphs; use and construction
B	**M3: Extension Measurement (Time, Money, Angle and Temperature)** Attribute; direct and indirect comparison; non-standard units; standard units; applications and formulae	**G2: Euclidean Transformations (Flips, Slides and Turns)** Line-rotation symmetry; flip-slides-turns; tessellations; dissections; congruence; properties and relationships	**A1: Equivalence and Equations** Definition of equals; equivalence principles; equations; balance rule; solutions for unknowns; changing subject	**SP2: Probability** Definition and language; listing outcomes; likely outcomes; desired outcomes; calculating (fractions); experiments; relation to inference
	N3: Common Fractions Concepts and models of common fractions; mixed numbers; equivalent fractions; relationship to percent, ratio and probability	**O3: Common and Decimal Fraction Operations** Addition, subtraction, multiplication and division of common and decimal fractions; models, concepts and computation	**N4: Percent, Rate and Ratio** Concepts and models for percent, rate and ratio; proportion; applications, models and problems	**G3: Coordinates and Graphing** Polar and Cartesian coordinates; line graphs; slope and y-intercept; distance and midpoints; graphical solutions; nonlinear graphs
C	**A2: Patterns and Linear Relationships** Repeating and growing patterns; position rules, visual and table methods; application to linear and nonlinear relations and graphs	**A3: Change and Functions** Function machine; input-output tables; arrowmath notation, inverse and backtracking; solutions for unknowns; model for applications to percent, rate and ratio	**O4: Arithmetic and Algebra Principles** Number-size, field and equivalence principles for arithmetic; application to estimation; extension to algebra; simplification, expansion and factorisation	**A4: Algebraic Computation** Arithmetic to algebra computation; modelling-solving for unknowns; simultaneous equations, quadratics
	N5: Directed Number, Indices and Systems Concept and operations for negative numbers; concept, patterns and operations for indices; scientific notation and number systems	**G4: Projective and Topology** Visualisation; divergent and affine projections; perspective; similarity and trigonometry; topology and networks	**SP3: Statistical Inference** Gathering and analysing data; mean, mode, median, range and deviation; box and whisker graphs; large data sets, investigations and inferences	**O5: Financial Mathematics** Applications of percent, rate and ratio to money; simple and compound interest; best buys; budgeting and planning activities

Key: N = Number; O = Operations; M = Measurement; G = Geometry; SP = Statistics and Probability; A = Algebra.

Accelerated Inclusive Mathematics Project

YuMiDeadly

Growing community through education

Faculty of Education
School of Curriculum
S Block, Room 404
Victoria Park Road
KELVIN GROVE QLD 4059

CRICOS No. 00213J

Phone: +61 7 3138 0035
Fax: +61 7 3138 3985
Email: ydc@qut.edu.au
Website: ydc.qut.edu.au