Accelerated Inclusive Mathematics Project

YuMi Deadly Maths

AIM Module A1

Year B, Term 3

Algebra:

Equivalence and Equations

Prepared by the YuMi Deadly Centre
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ACKNOWLEDGEMENT

The YuMi Deadly Centre acknowledges the traditional owners and custodians of the lands in which the mathematics ideas for this resource were developed, refined and presented in professional development sessions.

YUMI DEADLY CENTRE

The YuMi Deadly Centre is a research centre within the Faculty of Education at the Queensland University of Technology which is dedicated to enhancing the learning of Indigenous and non-Indigenous children, young people and adults to improve their opportunities for further education, training and employment, and to equip them for lifelong learning.

"YuMi" is a Torres Strait Islander Creole word meaning "you and me" but is used here with permission from the Torres Strait Islanders' Regional Education Council to mean working together as a community for the betterment of education for all. "Deadly" is an Aboriginal word used widely across Australia to mean smart in terms of being the best one can be in learning and life.

YuMi Deadly Centre's motif was developed by Blacklines to depict learning, empowerment, and growth within country/community. The three key elements are the individual (represented by the inner seed), the community (represented by the leaf), and the journey/pathway of learning (represented by the curved line which winds around and up through the leaf). As such, the motif illustrates the YuMi Deadly Centre's vision: *Growing community through education*.

The YuMi Deadly Centre (YDC) can be contacted at ydc@qut.edu.au. Its website is http://ydc.qut.edu.au.

DEVELOPMENT OF THE AIM MODULES

The Accelerated Inclusive Mathematics (AIM) modules were originally developed from 2010 to 2013 as part of the Accelerated Indigenous Mathematics project funded under the Commonwealth Government's *Closing the Gap: Expansion of Intensive Literacy and Numeracy* program for Indigenous students. The project aimed to assist secondary schools with beginning junior-secondary Indigenous students who were at Year 2/3 level in mathematics by developing a junior-secondary mathematics program that accelerates the students' learning to enable access to mathematics subjects in the senior-secondary years and therefore enhance employment and life chances. The project developed three years of modules (Years A to C) that are vertical sequences of learning to take students from their ability level to their age level in mathematics. The YuMi Deadly Centre acknowledges the role of the Department of Education, Employment and Workplace Relations in the development of the AIM project and these modules.

Contents

Module Overview

Accelerated Indigenous Mathematics (AIM) mathematics-education modules are based around: (a) big ideas, connections and sequencing; and (b) the RAMR framework (Appendix B). The modules endeavour to do three things: (a) reveal the **structure** of mathematics; (b) show how the mathematics **symbols tell stories** about our everyday world; and (c) provide students with knowledge that they can access in real-world situations to help **solve problems** and to improve employment and life chances. **Algebra is the generalisation of arithmetic** – it is the parts of arithmetic that hold for any number. **Equivalence and equations** is the part of algebra that deals with understanding equations and solving equations for unknowns. It is based on the balance rule, that equations stay true if the same is done to each side (e.g. equation $2 \times ? + 3 = 11$ is the same as $2 \times ? = 8$, when 3 is subtracted from both sides, and is the same as ? = 4, when both sides are divided by 2); and the corollary to this, if one side cannot change then the other side's changes must be compensated for on that side (e.g. for the equation: money for living + money for entertainment = pay cheque, money for living must decrease if money for entertainment increases).

Thus, for AIM, this module on equations and equivalence is important for three reasons. First, it is necessary for mathematics that leads to **employment**, for example, the manipulation of formulae and the ability to do budgets and prepare quotes. To give a particular example, a welder building a tank of 3.5 m diameter to a height to hold 40,000 litres, has to know how to change the subject of a formula, that is, $V = \pi R^2 \times H$ becomes $H = \frac{V}{\pi R^2}$. Second, it represents the **importance of algebra** in ensuring that understandings of number and operations grow into structures that are powerful and portable. However, it should be noted that the algebra of this booklet is not the algebra of x's and y's (although we do get to these symbols); it is interesting and motivating arithmetic that enables the development of big ideas and prepares for the x's and y's. Third, as shall be argued at the end of this module, it represents teaching that, through its focus on big ideas, is in harmony with Aboriginal, Torres Strait Islander and low-SES students' **learning styles**.

Background information for teaching equivalence and equations

This section overviews the role algebra plays in the structure of mathematics, describing how it is connected to the other strands within the structure of mathematics, how it is based on a series of big ideas that recur across Years P to 9, and the particular models used in this module.

Mathematics as a structure

YuMi Deadly Maths believes that mathematics should be taught so that it is accessible as well as available, that is, learnt as a rich schema containing knowledge of when as well as how (see *AIM Overview* booklet). Rich schema has knowledge as connected nodes which facilitates recall (it is easier to remember a structure than a collection of individual pieces of information) and problem solving (content that solves problems is usually peripheral, along a connection from the content on which the problem is based). Through generalising, rich schema also provides a framework into which new learning can be placed. As is evident from its description, the RAMR cycle has many situations where it emphasises connections and generalisation.

AIM argues that knowledge of the structure of mathematics, particularly of connections and big ideas, can assist teachers to be effective and efficient in teaching mathematics. This is because it enables teachers to do the following: (a) **determine what mathematics is important to teach** (mathematics with many connections or based on big ideas is more important than mathematics with few connections or little use beyond the present); (b) **link new mathematics ideas to existing known mathematics** (mathematics ideas connected to other mathematics ideas or based on the one big idea are easier to recall and provide options in problem solving);

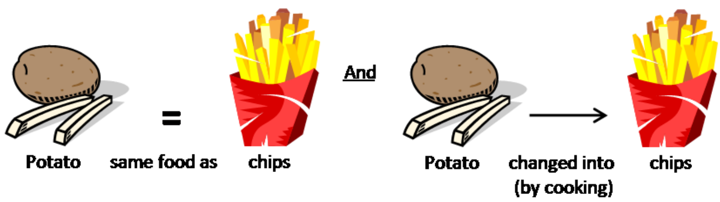
(c) **choose effective instructional materials, models and strategies** (mathematics that is connected to other mathematics or based around a big idea commonly can be taught with similar materials, models and strategies); and (d) **teach mathematics in a manner that makes it easier for later teachers to teach more advanced mathematics** (by preparing the linkages to other ideas and the foundations for the big ideas the later teacher will use).

The structural aspect of mathematics is discussed in the *AIM Overview* booklet which discusses: (a) the connections between number, operations and algebra; (b) the connections between number-operations-algebra, geometry, and measurement-statistics-probability; and (c) the nature, role and types of big ideas. However, the important things to note about big ideas are that, firstly, they are **generalisations** of particular ideas that applied initially to the particular numbers at which we were looking. Secondly, because they encompass many numbers and topics, they are **holistic** in their focus. Thus algebra and big ideas are almost synonymous. Both can be seen to reoccur across many topics and strands, to be generalisations and to be holistic in focus. As a consequence of what they are, algebraic thinking and ideas, along with big ideas and generalities, are very important parts of mathematics because they have the following important attributes: (a) they **last learners many years**; (b) they **connect many mathematical ideas**; (c) they **cover many mathematical situations**; (d) they **reduce the amount of mathematics that has to be learnt**; and (e) they **are holistic**.

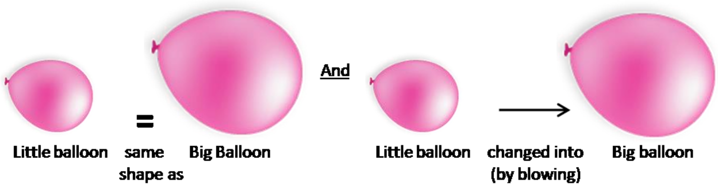
Relationship vs transformation (change) big idea

This is an **important global big idea** –it applies very widely (in fact it applies to all mathematics). Let us consider three examples.

1. The first is **potatoes being cooked** into chips. We can consider this as a relationship, the potatoes and chips are the same food; we can consider this as a change, the potatoes have been changed to chips by cooking. This gives rise to two different ways of thinking about, and two different symbol organisations for, one mathematical idea – *same as* and *equals*, and *change* and *arrow*, as below.



2. The second is **a balloon being blown up**. The half-filled balloon and the fully-filled balloon can be considered to be related because they are the same shape although different in size. However, from another perspective, the little balloon could be considered to have been changed into the large by being blown into. Again we have two different symbols and two different ways of thinking (see below) for one mathematical idea.



3. The third is **addition**. Consider 2 and 3. The joining of 3 to 2 could be considered as a relationship, *2 and 3 relate to 5 by addition*. However, it can also be considered as a change, *2 can be changed to 5 by the action of +3*. Again, one mathematical idea but two ways of thinking and two different forms of symbols, as seen below.

$$2 + 3 = 5 \qquad\qquad 2 \xrightarrow{+3} = 5$$

So, the everyday activities such as addition can be thought of in two ways and result in two symbol systems even though they only represent one mathematical idea. One way uses equations to express relationships such as that between 2, 3, 5 and addition; the other way uses arrows to show change such as how 2 can be changed to 5 by addition of 3. The first way leads to algebraic equations; the second to algebraic functions.

(*Note*: Equations use an equals sign to relate two sides but the module **uses equivalence instead of equals** because this is what the "=" sign actually means in mathematics. Equals is used in schools because it is more straightforward.)

Thus, algebra can also be seen as **both relationship and transformation**. For example, when we are algebraically describing or modelling a shopping situation such as *the cost of a coat is $1 more than double the cost of the pants*, we can: (a) think of this and express it as an **equation**, *the value of the coat* (y) *is equal to the value of the pants* (x) *multiplied by 2 plus 1*, that is, $y = 2x + 1$ (in formal equation notation); and (b) think of it as **transformation or change** and express it as a **function**, *the value of the pants* (y) *is found by doubling the price of the pants* (x) *and adding a dollar*, that is, $x \xrightarrow{\times 2} \xrightarrow{+1} y$ (in arrowmath notation). The first of these, the relationship perspective written in equations, is the perspective of this module, Module A1 *Equivalence and Equations*; the second, the transformation perspective using arrowmath, is the perspective of Module A3 *Change and Functions*.

Within each of these perspectives, relationship and transformation, there is an important big idea: (a) within **relationship, the big idea is the balance principle** – that to keep an equation equal, whatever is done to one side has to be done to the other; and (b) within **transformation, the big idea is the backtracking principle** (based on inverse) – that change can be undone by reversing the operations (and doing so in reverse order).

Equivalence and equations big ideas

The principles for algebra and for equivalence and equations are as follows (the placing of the principles into three sets is the decision of this module). Note that these big ideas are those for operations and arithmetic.

The equivalence or equals big ideas

1. **Reflexivity principle.** Anything equals itself (e.g. 2 = 2).
1. **Symmetry principle.** Equals can be turned around (e.g. if 2+3 = 5 then 5 = 2+3).
2. **Transitivity principle.** Equals continues across equal relationships; the first in a sequence, equals the last (e.g. if 2+3 = 5 and 5 = 6−1 then 2+3 = 6−1).
3. **Balance principle.** Equations stay true if whatever is done to one side is done to the other (e.g. 2+3 = 5 means that 2+3+5=5+5, that is, 2+8 = 10).

The operation structure big ideas *(the first five are the field properties)*

1. **Identity principle.** There are two identities that leave everything unchanged, namely, 0 for addition and 1 for multiplication (e.g. 27+0 = 27 and 1×¾ = ¾).
2. **Inverse principle.** If a change is to be made it can be undone by an inverse change, namely, −3 for +3 and ÷7 for ×7. (*Note:* This means that − is the inverse of + and ÷ is the inverse of ×).
3. **Commutative principle.** Operations with + and × can be "turned around" without error (e.g. 6+7 = 7+6 and 23×4 = 4×23). However, this is not true for − and ÷ (e.g. 6−2 ≠ 2−6 and 12÷3 ≠ 3÷12).
4. **Associative principle.** Operations with + and × which have more than two expressions can be completed with any association in any order (e.g. 6+3+5 = 9+5 or 6+8 or 11+3).
5. **Distributive principle.** Addition means adding "like things" but multiplication means multiplying everything, that is, × distributes across + (e.g. 3×(4+5) = (3×4) + (3×5); 23×2 = (20×2) + (3×2)). This means that 23+3 = 26 but 23×3 ≠ 29, and 23×3 = 69 but 23+3 ≠ 56.

6. **Compensation principle**. If there is a change in one number in + or ×, this is compensated by the inverse change in the other number (e.g. 8+5 = 10+3 (8+2 = 10, 5−2 = 3); 12×3 = 4×9 (12÷3 = 4, 3×3 = 9)). Because of their inverse nature, − and ÷ also compensate but not by using inverses; they compensate by doing the same to both numbers (e.g. 8−5 = 12−9 = 3 (8+4 = 12, 5+4 = 9); 12÷3 = 24÷6 = 4 (12×2 = 24, 3×2 = 6)).

7. **Inverse relation principle.** For + and ×, any increase in a number increases the total (e.g. 8×4 = 32, 8×6 = 48). However, for − and ÷, increasing the second number decreases the total (e.g. 8−3 = 5, 8−6 = 2 (3 increases so 5 decreases); 12÷3 = 4, 12÷6 = 2 (3 increases so 4 decreases)).

8. **Equivalence principle.** Since +0 and ×1 do not change anything, then two things are equivalent if one is +0 or ×1 of the other (e.g. 2/3×1 = 2/3×2/2 = 4/6, so 2/3 is equivalent to 4/6 because 2/2 is the same as 1; 404−186 = 404−186+0 = 404+14−186−14 = 418−200, so 404−186 can be solved by 418−200 because +14−14 is the same as +0).

Algebra big ideas

1. **The act of generalisation.** This is skill in generalising – being able to develop a rule that holds for all numbers; and justification of generalisation.

2. **The meaning of variable.** This is being able to state a rule with a letter, e.g. the n^{th} position has 2n+1 objects.

3. **Unnumbered → numbered.** Start with unnumbered activities before moving onto numbered activities– unnumbered activities enable big ideas to be more easily seen.

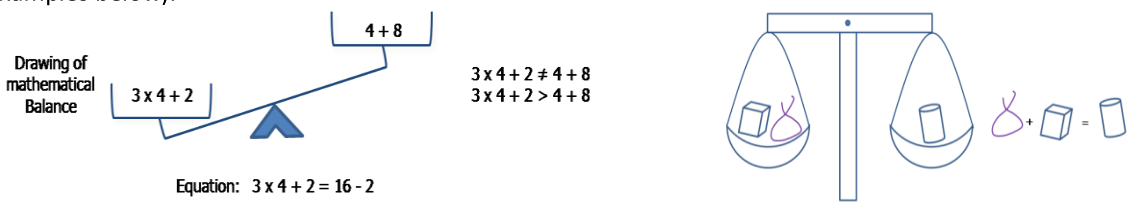
4. **Exploration/reality.** Act out real-world situations, enquire, discuss and allow students to come up with their rules – that is, think of real situations in which the patterns could exist (even if only evaluations).

Main models

The main models to be used are mass (the balance beam) and length (strips of paper and the double number line). Each of these will now be described. It should be noted that each of these models has to move from **physical model to virtual or pictorial model and then to abstract maths model**. Virtual models are also available.

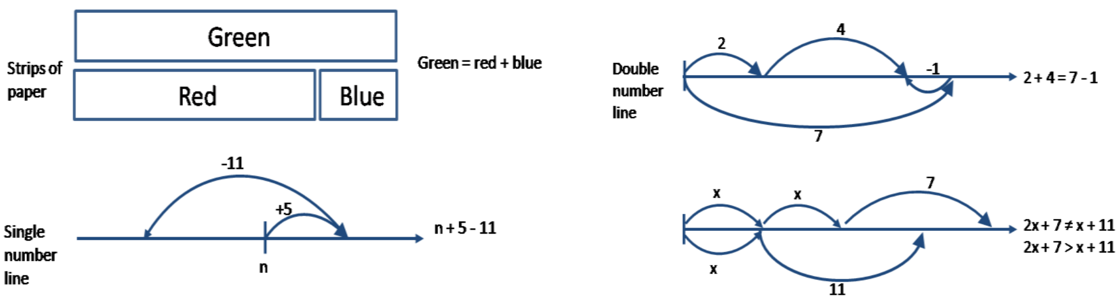
Mass or balance model

Equals is shown by the balance being "in balance", and not equals by the balance being "out of balance" (see examples below).



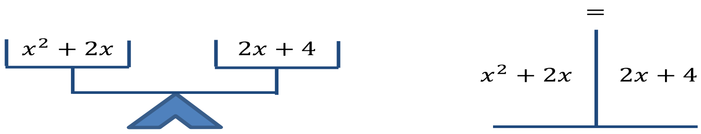
Length model

The materials for this model are strips of paper and single and double number lines (see below for diagrams).



Advanced models

As we go up the years, the equations become more complicated – we need to move on to imaginary extensions of balances and lines, where anything is possible. One can subtract, divide, multiply, and go into negatives and so on. Anything mathematical is allowed.



Maths balance **Maths ruler**

Once we have unknowns or variables, we can introduce models for unknowns. The following are useful: (a) a bag covered in question marks into which weights can be placed to act out ? + 3 = 11; (b) boxes of various shapes into which counters could be placed to act out Δ + O + 4 = Δ + 7; and (c) cups □ and counters O to act out equations with variables (cup acts as variable and counters as ones) to act out □ + □ + OOO = □ + OOOOO.

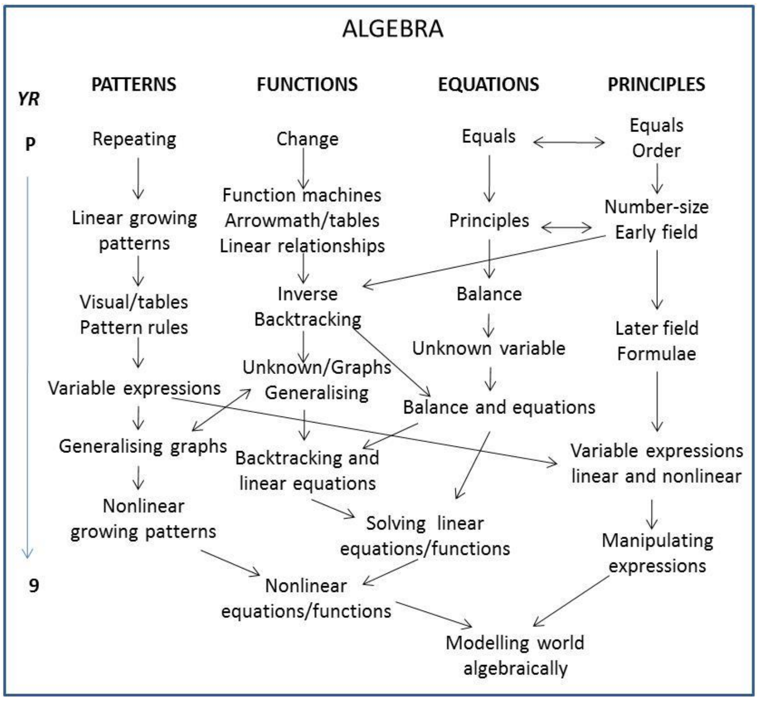
Sequencing for equivalence and equations

This section briefly looks at the role of sequencing in algebra and then sequencing in this module.

Sequencing in algebra

Overall sequence

The overall sequence for algebra is given in the figure below. It has four sections which are covered by the following AIM modules: Module A1 *Equivalence and Equations*, Module A2 *Patterns and Linear Relationships*, Module A3 *Change and Functions*, and Module O4 *Arithmetic and Algebra Principles*, with Module A4 *Algebraic Computation* covering some of the later activities.



It begins with **patterns** as training in the act of generalisation by finding pattern rules and relating to graphs. It then moves onto **functions**, starting from change rules in transformations, using real situations, tables and arrowmath notation before equations and graphs, solving for unknowns by the use of the balance rule. After this it moves to relationships that in arithmetic and algebra are represented predominantly by **equations**, solving them by the use of the balance rule. The sequence is completed by focusing on arithmetic and algebraic **principles** and extending these to methods such as substitution, expansion and factorisation.

Special features

There are three important aspects of sequencing for algebra because of its generalised nature and reliance on big ideas, particularly principle big ideas. These principles hold generally across mathematics in the building of big ideas.

1. **Unnumbered work before numbered**. Within each module, the sequencing will begin with unnumbered activities as these enable the big ideas to develop, move on to numbers and arithmetic situations and then move to generalised situations. YDM follows the view of the Russian mathematics educator Davydov that to build big ideas like the *balance rule*, requires **initially working in unnumbered situations** as numbers tend to result in students looking for answers, not generalisations of concepts, processes (relationships/changes) and strategies.

2. **Processes not answers**. The basis of algebra is things that hold for all numbers not particular answers. For 2+3 the processing is 2+3 (i.e. joining 2 things and 3 things) which gives answer 5. For x+3, the process and the answer are the same, that is, x+3. Thus, algebra has to be built around big ideas not computation. It should be noted that this has a consequence, that arithmetic does not teach two operation processes well as the students simply do each process as it happens, for example, 2×5+3 becomes 2×5=10 and 10+3=13, two single steps not one double step. This means that time needs to be spent on teaching the processes of two-step operations (e.g. the inverse of 2×5+3 is -3 and ÷2 not the other way around).

3. **Separation to integration**. The sequence begins simply, in a separated manner, but by the time junior secondary is reached, the components are more integrated and connected to allow patterns, functions and equivalence all to be expressed in the same way (by equations), and for results to cover nonlinear as well as linear relationships and changes.

4. **Modelling as end point**. The overall end point of algebra is modelling as well as manipulation of symbolics. Computers and special calculators can do the manipulations to simplify and solve for unknowns – what is important, like in arithmetic, is to apply the knowledge to the world and solve problems – to model the world algebraically. This is important because most students cannot see the relevance of, say, $x + y = 7$ to their everyday world. Yet, with understanding it is very relevant. It could mean that you bought two things at a shop for \$7. Then the cost of the first thing (x) plus the cost of the second thing (y) is equal to \$7. This gives parameters in which thinking can be used. Suppose we were working in whole dollars. Then the first thing could cost \$1 and the second cost \$6, or \$2 and \$5, or \$3 and \$4, and so on.

Sequencing in this module

Equivalence and equations explores how to represent everyday life in terms of relationships. Thus, it studies the symbols, notation and rules for equations (number sentences with equal signs). In the long run, this is equations with numbers, operations and letters. To get to this point requires studying: (a) equations in unnumbered form and in arithmetic; (b) equations with unknowns for which calculation is in arithmetic form (called pre-algebra by some curricula); and (c) equations with variables where calculation is in algebraic form (considered to be full algebra). It is imperative that students learn symbols as ways of telling stories about everyday life. The sequence involves changes in models as well as content as in the figure below. This means that the physical materials have to change to abstract in a way that can accommodate any operation (including division) and negative numbers.

Equals, equation and inequation (relation to reality)

↓

Principles (equivalence and order)

↓

Balance rule (unnumbered → numbered)

↓

Unknown/Solving linear equations

↓

Nonlinear equations/Formulae

↓

Modelling algebraically

<u>Materials</u>

Physical balance/Lengths

↓

Pictures and counters/ Number lines

↓

Abstract 'Math' balance/ Drawings/Equations

The major ideas to be covered in equivalence and equations include the following in sequence:

(a) introducing the notion of same and different and relating this to introduce equal, unequal, greater than and less than in length and mass (balance) situations, and using mass and length in unnumbered situations to build understanding of equals and equations and to develop the equivalence and order properties;

(b) using mass and length in numbered situations to build understanding of arithmetic equations and reinforce the equivalence and order properties in numbered situations, and relating arithmetic equations to real-life situations and vice versa (e.g. telling stories about the world);

(c) using mass and length models to introduce the balance principle that equations stay equal if the same thing is done to both sides of the equation;

(d) using mass and length models to introduce unknowns, and relate equations with unknowns to real-world situations and vice versa, and extending mass and length models to mathematical versions (in picture form) where all operations are possible; and

(e) using the balance principle to find solutions to equations with unknowns and to change the subject of a formula.

These points make up the five units. Thus, the module consists of the following:

Overview: Background information, sequencing, and relation to Australian Curriculum

Unit 1: Meaning of equation

Unit 2: Equivalence principles

Unit 3: Balance rule

Unit 4: Unknowns and inversing change

Unit 5: Solving equations and changing subject

Test item types: Test items associated with the five units above which can be used for pre- and post-tests

Appendix A: Cultural implications for teaching algebra

Appendix B: RAMR cycle components and description

Appendix C: AIM scope and sequence showing all modules by year level and term.

The modules are designed to provide resources – ideas to teach the mathematics. Some important points about this provision are:

(a) the activities are given as a sequence of activities – they are not in RAMR lesson form – so before using them with students, teachers should transform them into RAMR form (see **Appendix B**);

(b) the overall sequence moves from numbered to unnumbered activities;

(c) mass is the major model used with length being less mentioned;

(d) algebra has important cultural implications – read **Appendix A**; and

(e) although this module assists with solving linear equations by non-graphical means, coordinate work is left to Modules G3, A2, A3 and A4.

Relation to Australian Curriculum: Mathematics

AIM A1 meets the Australian Curriculum: Mathematics (Foundation to Year 10)

Unit 1: Meaning of equation
Unit 2: Equivalence principles
Unit 3: Balance rule
Unit 4: Unknowns and inversing change
Unit 5: Solving equations and changing subject

Content Description	**Year**	**A1 Unit**				
		1	**2**	**3**	**4**	**5**
Use equivalent number sentences involving addition and subtraction to find unknown quantities (ACMNA083)	**4**	✓				
Continue and create sequences involving whole numbers, fractions and decimals. Describe the rule used to create the sequence (ACMNA133)	**6**	✓	✓	✓		
Extend and apply the laws and properties of arithmetic to algebraic terms and expressions (ACMNA177)	**7**		✓	✓		
Solve simple linear equations (ACMNA179)				✓		✓
Apply the four operations to simple algebraic fractions with numerical denominators (ACMNA232)	**10**		✓	✓		
Substitute values into formulae to determine an unknown (ACMNA234)				✓	✓	

Unit 1: Meaning of Equation

In this unit we introduce that equations are number sentences with an equals sign where both sides of the equals sign are in balance like both sides of a beam balance when measuring mass.

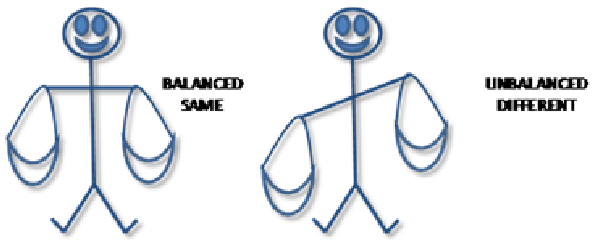
1.1 Same, difference, equals and unequals

This section looks at building balance as a model for equals.

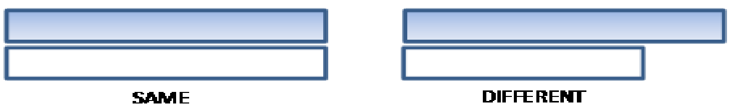
Activity sequence

1. Students identify objects which are the same and which are different. They learn to describe what is the same and what is different about two objects. They sort objects into those that are the same and notice that different groups are different.

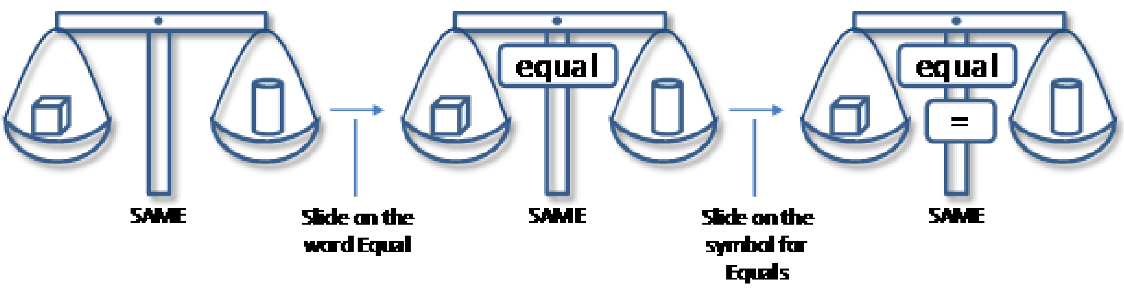
2. The understanding of same and different is then considered in terms of, particularly, mass and length. For mass, teacher takes two plastic bags and places on students' arms, puts things in the bags and allows students to feel when things are the same and when they are different. *Note*: This is really effective if the students with the bag are blindfold.



   For length, look at objects (e.g. paper strips) and see if same or different lengths.

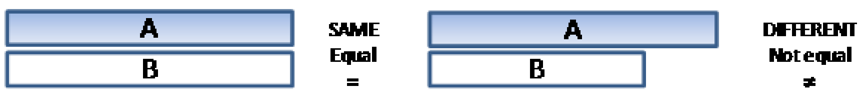


3. Once same and different are understood, the material can be used to introduce the formal language for same and different, namely, equals, not equals, less than and greater than, and symbols, namely =, ≠, < and >.

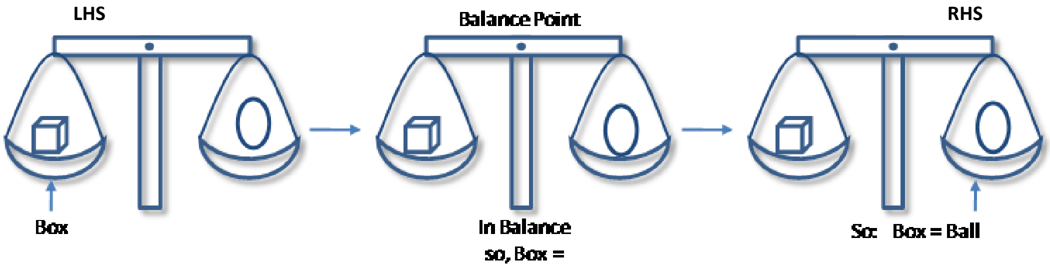


   The technique is to discuss what is happening with respect to the balance [it is balanced] and introduce words and symbols by sticking the language and symbols on the balance. Relate the notion of balance to "equals" and imbalance to "not equals".

   Note: Length can also be used in this way but maybe not as strongly. For example:



4. Move hands along the balance to introduce equations for objects; for example below, point to left-hand side (LHS) and say “box”, then move hand to centre of balance and say “equals”, then finally to RHS and say “ball”. Get students to repeat this, saying “box equals ball”. Reinforce that balance is equals.

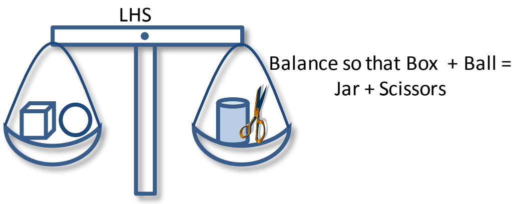


1.2 Developing equations

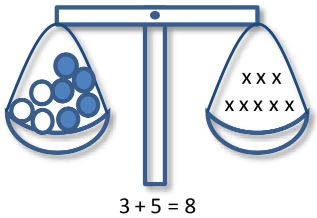
This section looks at building balance (and length) as a model for equations.

Activity sequence

1. The first formal type activities should not use number; just different objects and different lengths. These are explored for equals and not equals and, later, greater than and less than. Students find different things to balance and not balance and record these as “equations” (not in the strictest sense); see example at right.

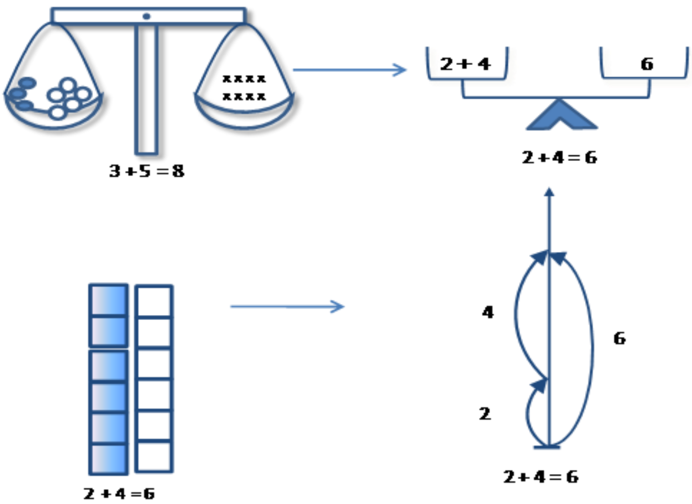


2. Once non-numerical situations have been explored, numbers can be introduced by using same-size weights (recommend small baked beans cans or similar). Then we can use models to represent equations with numbers, as on right. It is important to read the equations from the materials, e.g. 3 cans plus 5 cans balances 8 cans so 3+5=8.

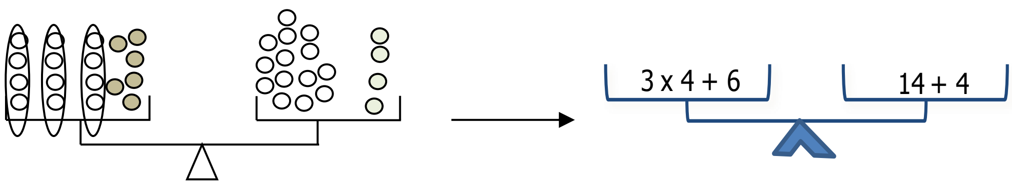


3. Then we move to a picture of a balance, placing materials on the picture and finally numbers on the picture as at right.

   Note: This can also be done for length, using: (a) unnumbered – strips of coloured paper (e.g. where red strip equals a blue strip beside a green strip, gives red = blue + green); (b) numbered – cubes; and (c) pictorial – the double number line. Also note that for this example, we have turned the cubes and the line vertical because this way it shows LHS and RHS. This does not have to be done but it makes the equation easier to relate to the picture.



   More complicated equations can also be represented as below:



1.3 Relating to real-world situations

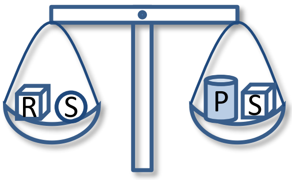
1. It is important to relate real-world situations to equations. To do this, relate stories to actual components of the equation, for example, as on right.

Story	Symbols
Two boys join three others	2 + 3
Two boys join three others, how many in all?	2 + 3 =
Two boys join three others to make 5 boys	2 + 3 = 5
Two boys join three others and this is the same number as 6 boys and one leaves	2 + 3 = 6 - 1

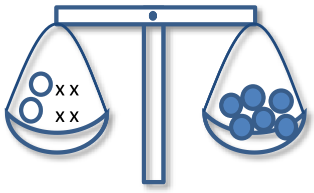
2. To do this, students have to build that two or more objects (or numbers of objects) joining each other is addition. This means that 1 soap and 1 pasta put in the same side of the balance is "soap + pasta". It is also important to note that if students have only had "sums" to do, they will not believe that 5=2+3 and 2+3=6−1 are allowed. There will be a reluctance to write them.

3. It must be reinforced that = means "same value as" (not "where to write the answer"). So 2+3=5 because 2+3 has the same value as 5 or 5 has same vale as 2+3.

4. A good way to teach the relation of symbols and stories is to use a material context. Two examples are, "The rice and the soap are the same as the pasta and the sugar", and "the two weights and the four weights are the same as six weights".



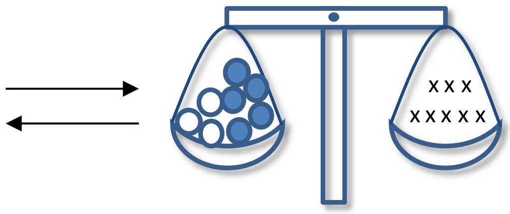
Rice + Soap = Pasta + sugar



2 + 4 = 6

5. It is important to reverse this process, for example, by asking students "what is a shopping story for 3+5=8?"

"I bought a chocolate for $3 and a hamburger for $5, and spent $8"

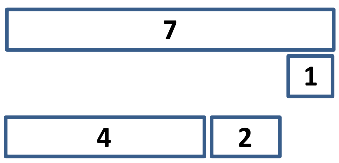


3 + 5 = 8

Once again, the focus has to be on addition being understood as 2 or more things joining, where the things can be different.

Note: One can also use length processes, for example:

"There were 7 boys and 1 left, which came to the same number as the girls, where 4 joined 2."



Unit 2: Equivalence Principles

This unit introduces the properties/principles that equals follows. These are often not taught and so often not known by students. They emerge from the "same value as" meaning of equals. The *equivalence* principles are as follows. The one that many students have difficulty with is symmetry.

- **Reflexivity principle** – anything equals itself (e.g. 2 = 2);
- **Symmetry principle** – equals can be turned around (e.g. if 2+3 = 5 then 5 = 2+3); and
- **Transitivity principle** – equals continues across equal relationships; the first in a sequence, equals the last (e.g. if 2+3 = 5 and 5 = 6−1 then 2+3 = 6−1).

2.1 Unnumbered activities

This section attempts to illustrate the power of unnumbered activities in teaching properties and principles.

Activity sequence

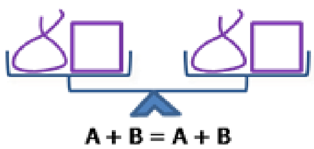
Where possible, find real-life contexts to embed the activities in; for example, using relevant objects or situations.

1. The principles are easily shown physically with a balance. Get the students to experience the principles with plastic bags and beam balances and groceries. (*Note*: It is always best to start with (a) *human body*, e.g. hang plastic bags on arms and become a beam balance and walk different distances; and, as stated before, (b) in *unnumbered* situations (e.g. compare groceries, use unmeasured strips of paper).

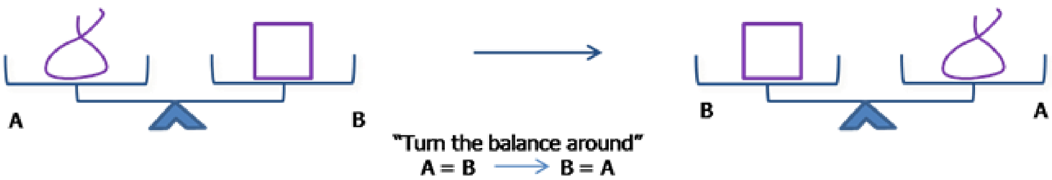
2. Direct the students to state out aloud the equation as you move your hand from their left -hand side of the balance to their right-hand side – "equals" is said as the hand goes past the balance point (centre) of the balance (it can be useful to stick "=" on the centre of the balance).

3. Direct the students to record the balanced groceries as informal equations, e.g. "salt equals soap plus pasta". Discuss any generalities they find; encourage them to see the generalities below.

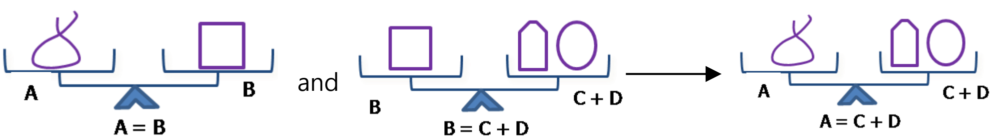
   Reflexivity: This is fairly obvious (that two things that are the same will be equal) but it needs to be made explicit with equations. Students can investigate if same things always balance.



   Symmetry: This can be seen by turning the balance 180°.

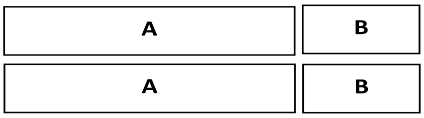


   Transitivity: Compare three things in three different ways to show A=B and B=C means A=C.



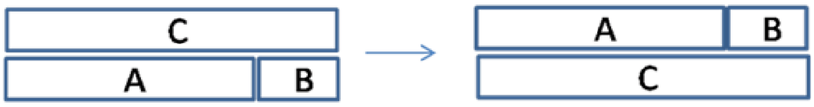
Note: This can also be done with length (the example below is strips but cubes (e.g. Unifix) would be just as effective).

Reflexivity:



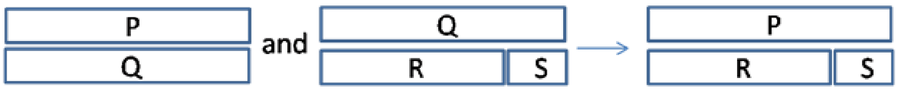
A + B = A + B

Symmetry:



C = A + B → **A + B = C**

Transitivity:



P = Q **and** **Q = R + S** → **P = R + S**

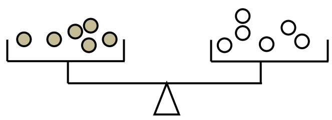
2.2 Numbered activities

This section attempts to illustrate how the equals principles can be extended to numbered situations. This can be difficult as the results clash with what students have picked up from doing sums like 6+8=___ .

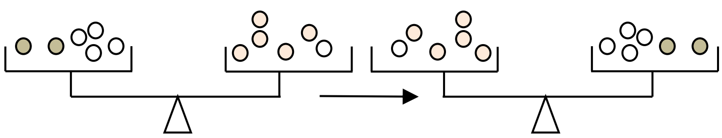
Activity sequence

Where possible, find real-life contexts to embed the activities in; for example, using relevant objects or situations.

1. Show reflexivity by examples where 6 = 6, 11 = 11, and so on, e.g. 6 = 6 as on right. Elicit from students that same numbers are always equal.

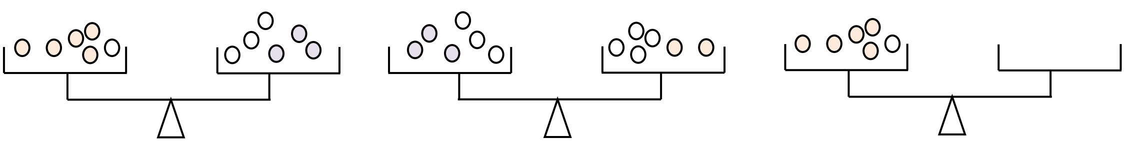


2. Show symmetry by "turning around" the balance – reading each equation as this is done – point at LHS then to centre of balance while saying "equals" and then to RHS – get students to say equation and write equation as you do this. Elicit from students that this is true for all equations of this type.



2+4 = 5+1 turn 180^0 5+1 = 2+4

3. Show transitivity as a three-step process as below: 5+1 = 3+3 → 3+3 = 4+2 → 5+1 = what?. Students can discuss, for example above, what is the relation between 5+1 and 4+2? Once they have seen that 5+1 = 4+2, then have to realise that this always happens (e.g. if first sum equals second, and second sum equals third, then it is always true that first sum equals third).



Unit 3: Balance Rule

In this unit, we introduce the crucial principle for equations and equivalence which is the balance rule or principle (e.g. if what is done to LHS of a true equation is also done to RHS, the equation stays true). The unit also relates the rule to the real world.

3.1 Introducing the balance rule

This section looks at the beginning of balance rule.

Activity sequence

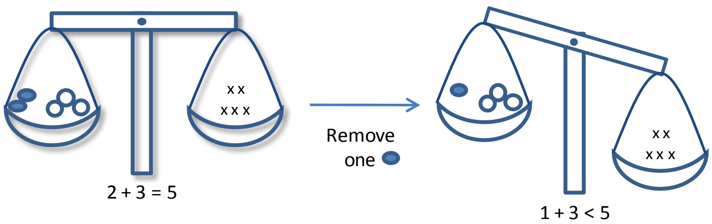
Find real-life contexts to embed the activities in; for example, using relevant objects or situations.

1. The best way to begin is to return to unnumbered activities. For example: (a) use the plastic bags and the blindfold– get a balance, add a book to one side, then add the same book to the other; and (b) place, for example, soap and rice on one side and balance tuna and pasta on the other (e.g. soap+rice = tuna+pasta), then add, say, sugar to RHS, and balance becomes unbalanced and can be rebalanced again by adding sugar to other side. Involve the students in discussion. Unbalance the bags or the beam balance and ask student for how we can balance again. Students tend to say to remove what you have added. Say that this is excellent way to rebalance, but ask if they can find another way. Most students see quickly that we need to add the same to the other side.

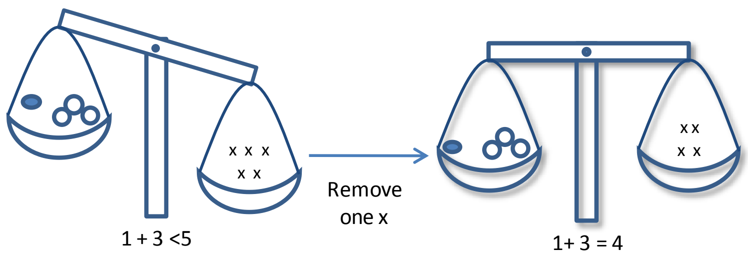
2. Do examples where you remove from one side. Do this by ensuring that one of the same thing is on both sides, for example, soap+rice = soap+pasta. Then remove soap from LHS and balance become unbalanced. Once again, students can easily see that the way to rebalance (other than to put back soap on LHS) is to remove soap from RHS.

3. Once this has been done, need to elicit from students that anything done to LHS must be done the same to RHS. One way to do this is to say something like the following – "I know this is not possible with this balance, but what if I added/subtracted 27 soaps to/from LHS, what has to be done to rebalance?". Students pick this up quickly and can be led to say, whatever you add/remove to/from RHS, add/remove to/from LHS.

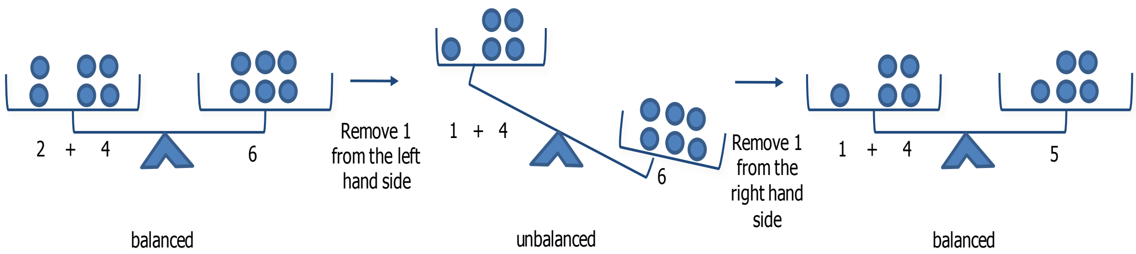
4. With numbered activities, the balance can also be used to explore what happens when extra weights (e.g. one weight) are added or removed from a balanced equation.



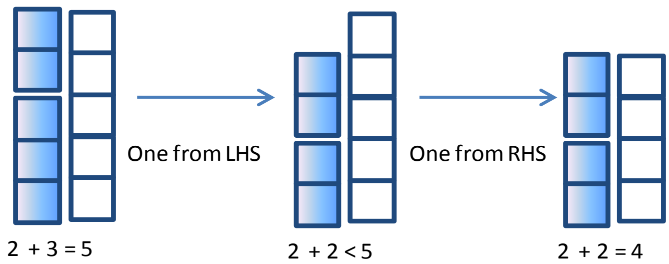
   Students can be asked how to balance the equation again. There are three possibilities for the example above: (1) put the weight back again (this returns the equation to 2+3−5); (2) add another weight to the 3 on the LHS (this makes equation 1+4=5); and (3) remove a weight from RHS (this makes the equation 1+3=4). The third possibility is the beginning of the balance principle and should be the focus of questioning.



5. Direct the students to add and remove different weights and to rebalance. For example:



Note: The balance principle can also be introduced and demonstrated with length models (as on right).



6. With questioning, try to get students to generalise this process (e.g. "whatever you do to the one side you do to the other") to the full balance principle.

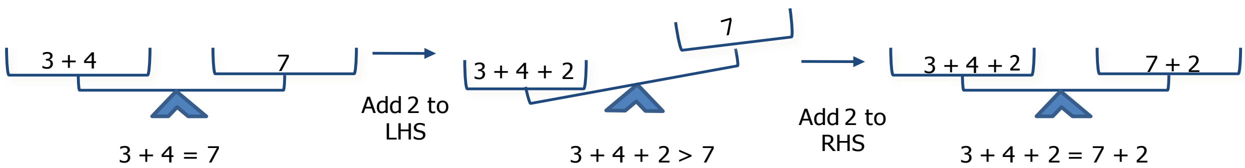
3.2 Extending the balance rule and relating to real world

This section looks at extending the balance rule to pictures and equations; and finally to images of equations where all operations are possible. This means moving towards a changed idea of balance (or line) from what a "real" balance (line) will be able to do to what a **"mathematical balance"** (or a "mathematical line" or **"mathematical ruler"**) will be able to do. A real balance cannot solve x+7=4 because after removing the 4 from both sides, we are left with x+3=0 and a real balance can do no more – but a mathematical balance would subtract a further 3 from both sides to get x=−3.

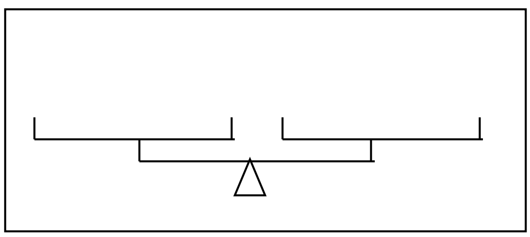
Activity sequence

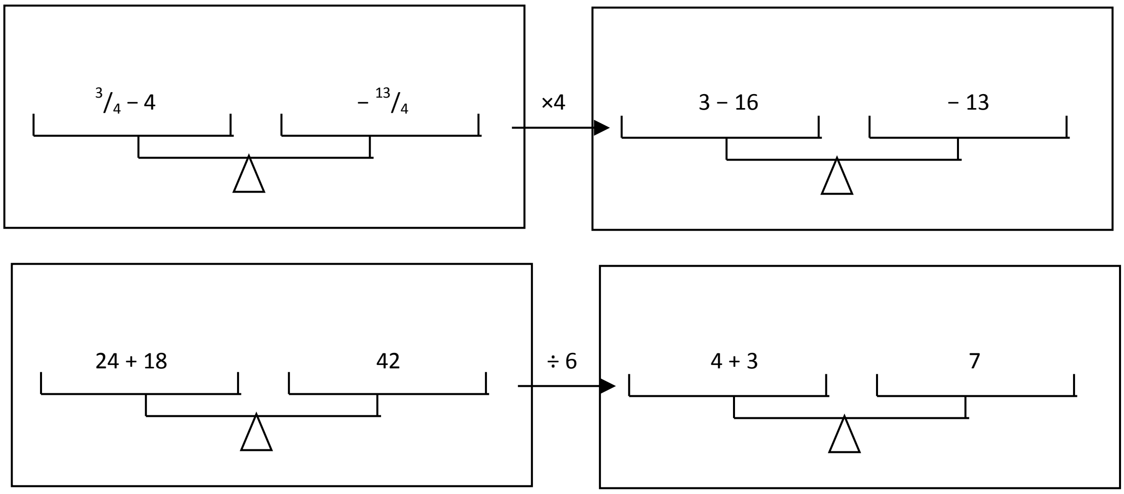
Find real-life contexts to embed the activities in; for example, using relevant objects or situations.

1. Extend the work with masses and numbers from section 3.1 to pictures by putting counters on the pictures and adding and subtracting counters and by using equations on a "mathematical balance" as follows. This reinforces the work in section 3.1.

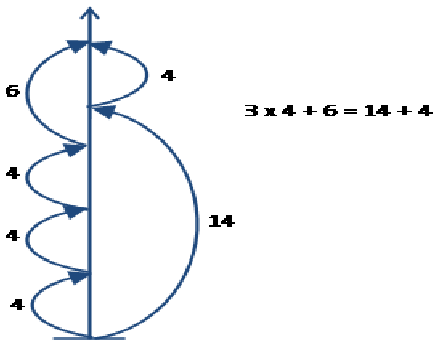


2. Make up a "mathematical balance" from an A4 sheet with drawing on it (see right) and use this for writing on (and counters). Spend time making changes that are not possible to make on a real balance, such as dividing and subtracting (see the examples on the next page).

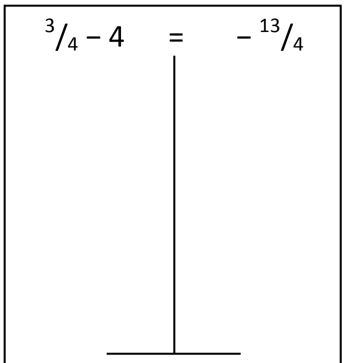




Note: The length model with Unifix can be extended to a more abstract form by using the double number line and this can be extended to a "mathematical ruler" with a drawing as below:



Double number line

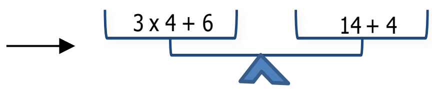


"Mathematical ruler"

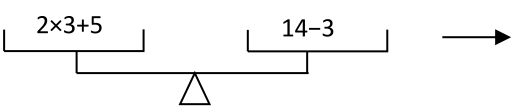
3. It is important in this to continue to reinforce the relationship between real-world situations and equations. The relationship is best taught by experience in which equations are deconstructed into parts and related to stories and vice versa (i.e. stories are deconstructed and related to equations). Some examples are as follows.

Story to equation

"I bought 3 chocolates for $4 each and a pie for $6 from the store. I spent the same as June, who bought a meal for $14 and a drink for $4 from the deli."



Equation to story



"The 2 brothers each caught 3 fish. A friend gave them 5 more. Their father caught 14 fish and threw back 3. So the brothers and the father caught the same number of fish."

It is important to ensure that students understand that subtraction is separating or taking away, multiplication is combining equal groups and division is partitioning whole into equal groups. In this way, the stories can make sense.

Note: The story → equation and equation → story can be done with length model. Example: the chocolate story above can be shown on a double number line and as an equation as above.

4. Now, as the equations are changed on one side, the story can be changed and this helps students see that the same change must be done to the other side. For example, if an extra $2 is spent in store after the chocolates and pie, then the only way to make it equal again is to spend more for the meal.

5. One way to reinforce these relationships between stories and equations is with worksheets with headings as below in which teachers fill in one space in each row and the students fill in the other spaces. (*Note*: Students tend to have the most difficulty with creating their own stories. The pictures can be balance pictures, double number line pictures, "mathematical balance" pictures, or "mathematical ruler" pictures.)

STORY	PICTURE	EQUATION

Unit 4: Unknowns and Inversing Change

This unit introduces unknown and discusses how such equations can be solved, leading to the need to develop the notion of inverse. The inverse idea is developed with number line.

4.1 Introducing unknown

This section introduces the idea of an unknown that later is represented by a letter. We then realise the need for inverse.

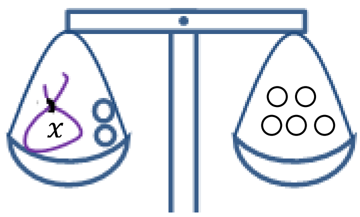
Activity sequence

Find real-life contexts to embed the activities in; for example, using relevant objects or situations.

1. Discuss with students what they see an unknown as (older students tend to choose "x" but others could choose a box or "?"). Talk about it being **any number**.

2. Make a bag with the chosen symbol on it (e.g. x), put in 3 weights, and then balance it and 2 weights with 5 weights.

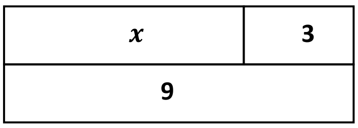
3. Discuss that the bag is unknown. Ask how we could find what is in the bag without opening it. Most students will know it is 3 because 3+2=5, but ask students to find a way to make it without this knowledge – state that the numbers could be large.



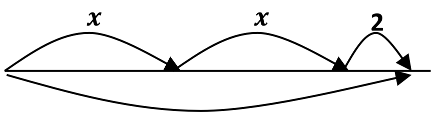
$x + 2 = 5$

4. Talk about how we can **get the unknown on its own**. Say we need to learn how to undo what has been added, and so on, to the unknown.

   Note: Unknown can also be introduced using the length model by a paper strip (labelled x or ?) which relates to other strips with numbers on them as below.



$x + 3 = 9$



$2x + 2 = 14$

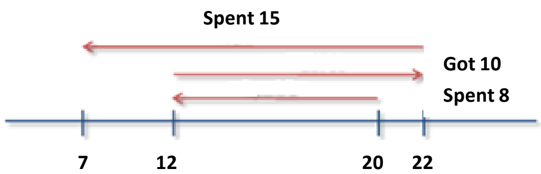
4.2 Introducing inverse

This section introduces the idea of inverse using a number line.

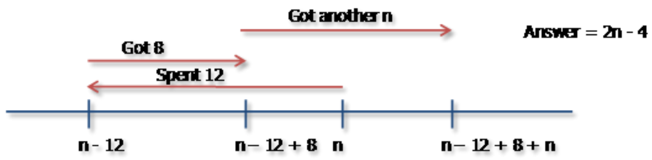
Activity sequence

Find real-life contexts to embed the activities in; for example, using relevant objects or situations.

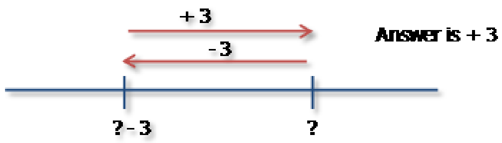
1. At this point, it is useful to introduce single number line activities. In these activities, the line is used to act out real-world situations. For example, "I had \$20, I spent \$8, got an extra \$10 and then spent \$15, how much do I have?" Moving along the line will give the answer.



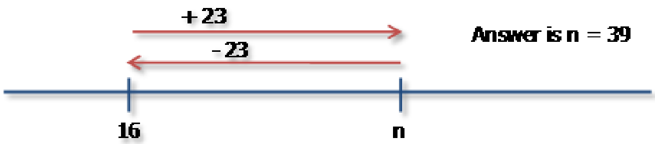
2. It is possible to use this number line model when there is an unknown or variable. For example, "My dad gave me some money, I spent $12, I received $8, then my Mum gave me the same money as my Dad, how much do I have?" Need to come up with a symbol for the unknown – could be *n* or ?. Then the line helps.



3. The line is useful for teaching inverse. This is particularly so for the inverse that makes an unknown on its own when solving an equation such as ?−3=11. For example, how do we change ?−3 so that we get back to ? on its own? The line on the right can help students understand, and the line is a good model to explain or act out the process.



4. Thus, the number line can be used to find the answer to unknowns. For example, "I went out and bought a CD for $23, this left me with $16, how much did I start with?" To solve this, make the start *n* and use the line as below; *n* goes to *n*-23, this is $16, so +23 to get back to *n* which is $39.



Unit 5: Solving Equations and Changing Subject

In this unit, we look at how we use the balance rule to solve equations for unknowns and change subject in a formula. This is based on the balance rule and inverse as described in the previous two units. It also relates to real-world situations.

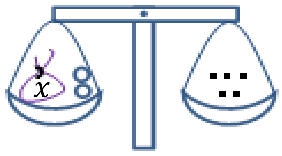
5.1 Solving equations based on real balances (lines)

This section looks at continuing sections 3.1 and 4.1 to solution for situations and equations where pictures of real balances and real number lines are possible (e.g. no negatives and simple multiplication and division).

Activity sequence

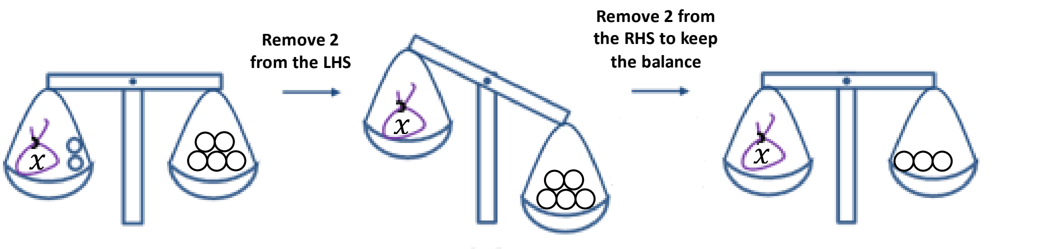
Find real-life contexts to embed the activities in; for example, using relevant objects or situations.

1. Continue discussion from section 4.1. Set up the equation ?+2=5 using materials. We know the answer is 3 masses, but what can we do to understand a process that always works? Elicit that we have to get the unknown on its own. Focus on the fact that LHS is unknown +2. How do we get rid of the +2?



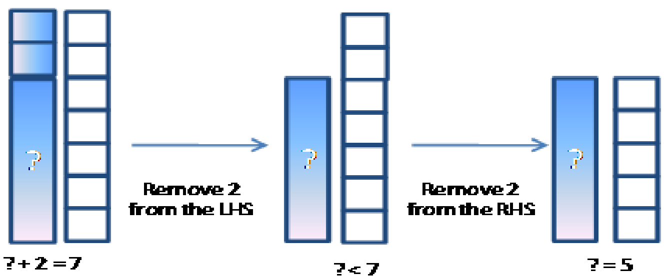
$$x + 2 = 5$$

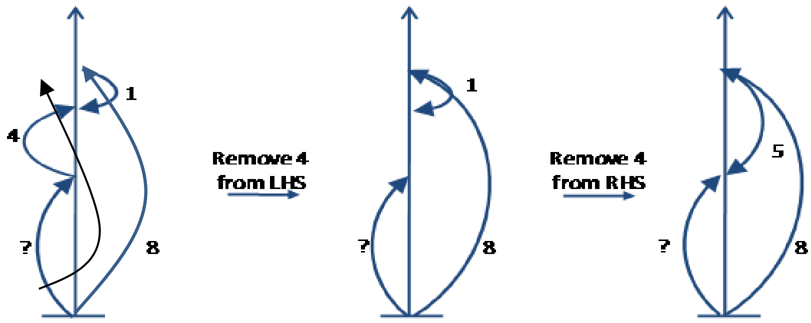
2. Students can usually see that we can get unknown on its own by removing 2 weights. Then remind of balance rule – students are usually quick to say that this means removing 2 weights from RHS as below.



? + 2 = 5   ? < 5 after removing 2 from LHS   ? = 3 after removing 2 from RHS

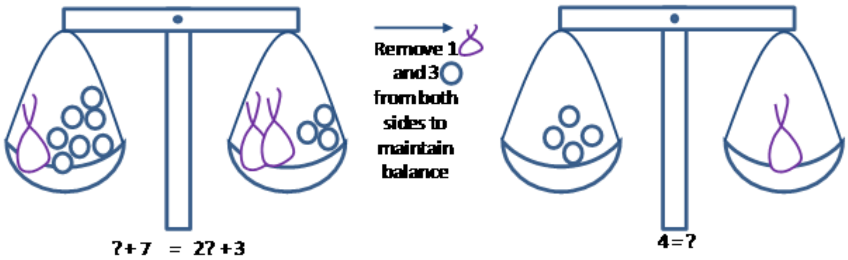
Note: This can also be done with length as below.



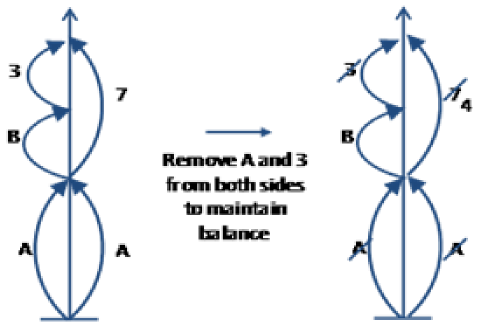


? + 4 = 8 – 1　　? < 8 – 1　　? = 8 – 5 = 3

3. It is quite easy to extend this to more than one unknown on each side (for example, see below) and even different types of unknown on each side:

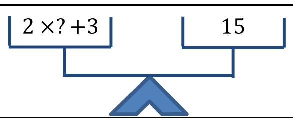
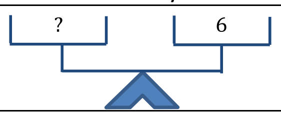


Note: This can also be acted out with the length model – an example that actually has two types of unknown is given on right.



A + B + 3 = A + 7　　B = 4

4. At this point, the process can be extended to symbolic equations. The balance drawings are the best way to make the transition because they can do all operations. Continue to stress that the drawings show a "mathematical balance" that can do all operations.

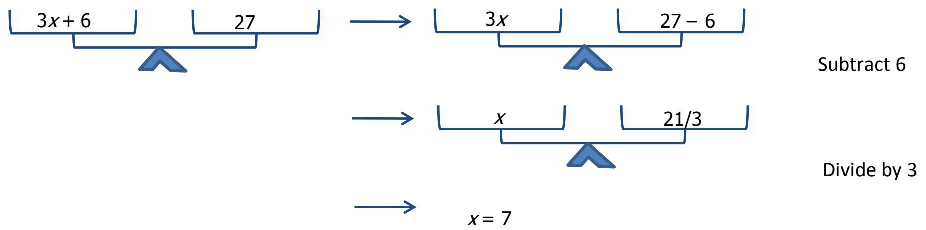
Picture		? notation		Variable notation
		$2\times?+3 = 15$		$2x + 3 = 15$
remove 3 and divide by 2	-3 $\div 2$	$2\times? = 12$	-3	$2x = 12$
		$? = 6$	$\div 2$	$x = 6$

5. However, we must never forget that one of the crucial things to develop is the ability to **translate real-world situations into equations and back again, so that we can translate real-world problems with unknowns into equations with unknowns, solve the equations for the unknowns and translate the**

answer back into the real-world problems. This is done by teaching the students what the equation means as a story followed by activities interpreting the story in terms of symbols and back again. Spend time on **translating symbols → stories as well as stories → symbols** (an example is below)**.** Constructing real-world problems is an excellent way to teach interpreting problems.

Equation	Story
$3x + 6 = 27$	$3x$ means 3 × any number; +6 means to add 6 So we need 3 lots of the same thing plus 6 What about maxi taxis? "Three maxi taxi loads of children were brought to the game. 6 children were already there. This made 27 children."

We can use the balance principle on these equations to work out the unknowns, e.g.



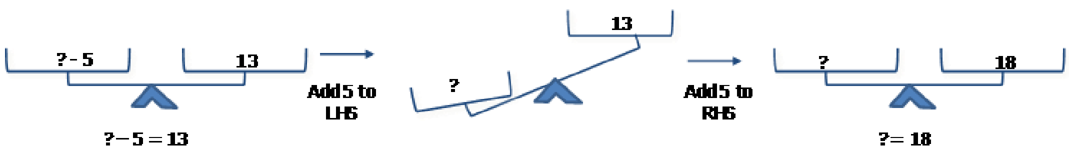
5.2 Solving equations with any symbols

This section looks at continuing sections 3.2 and 4.2 by using "mathematical balances" and "mathematical rulers" to undertake equations and changes that it is not possible to complete on a balance or number line that reflects reality.

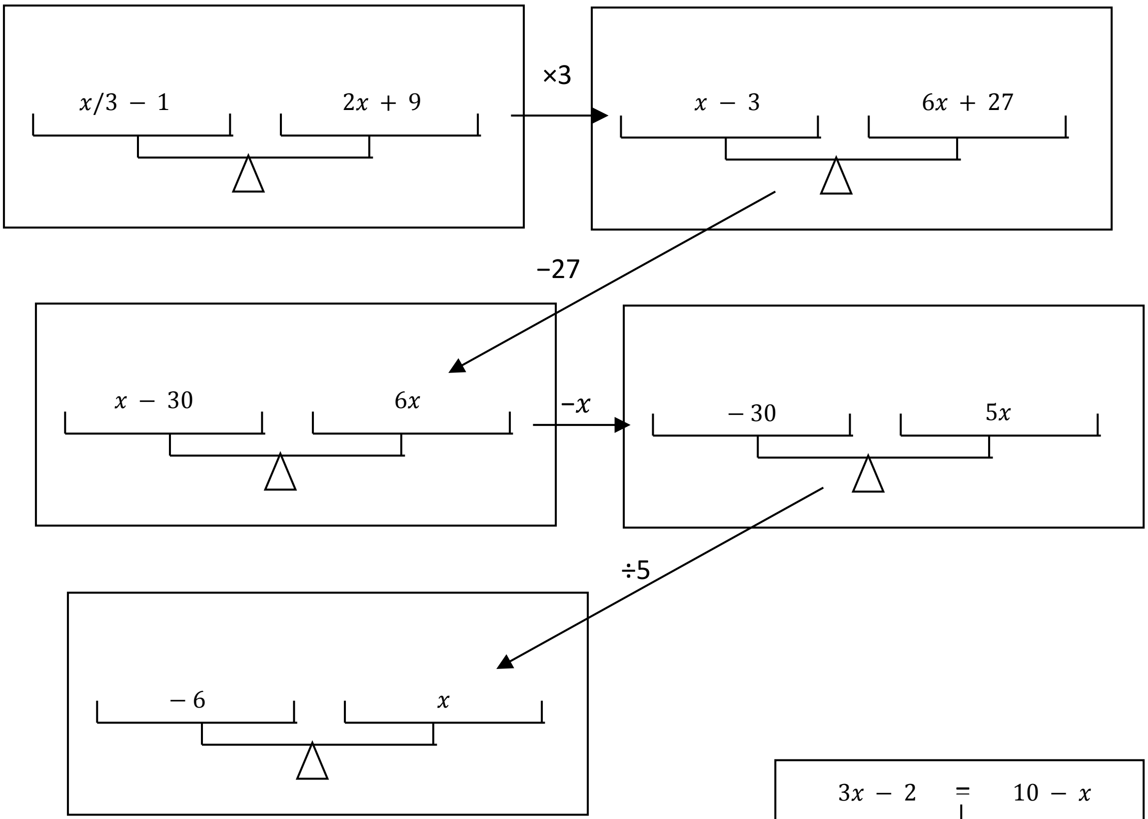
Activity sequence

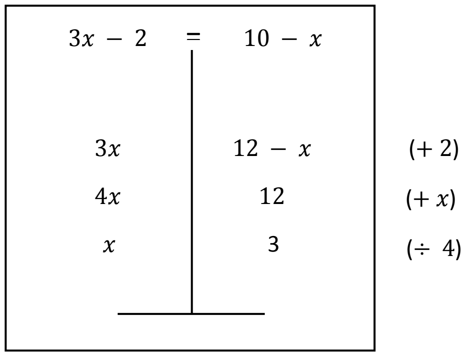
Try to find real-life contexts to embed the activities in; for example, using relevant objects or situations. This will be harder for these examples.

1. Say to students that we are now moving to "mathematical balances" where anything is possible (negatives, division, and so on). Discuss the balance rule and generalise to doing anything to each side of the equation as long as it is the same.
2. Translate simpler problems to the mathematical balance and solve them without reference to weights or counters – just use the balance rule. For example:



3. Repeat this for more complex problems as on next page. **Do not move onto simple rules** like "change sign when move to other side" – focus on the **method being solving a problem** – "how do I get the unknown on its own"? Students may discover that they can follow short cuts like "change sign" but let this evolve from seeing everything as a balance. If there are problems, make the students put out their hands and turn themselves in a balance in their mind. Say, for example, you have $3x - 1$ on this side and $x/2 + 1$ on the other side – "how do we keep things in balance and get the unknown on its own – just as x?"





Note: This can be also done with the "mathematical ruler" as on right. The problem is that John has \$2 less than 3 times the amount and Jenny has \$10 less the amount; what is the amount? The equation is $3x - 2 = 10 - x$. To solve it, 2 is added to both sides (to remove the –2), x is added to both sides to remove the $-x$, and both sides are divided by 4 to give x = 3.

4. Finally, you can remove the diagrams and directly solve for unknowns from the equations as follows:

$3x - 1 = x + 5$
$3x = x + 6$ (add 1 to both sides)
$2x = 6$ (subtract x from both sides)
$x = 3$ (divide both sides by 2)

$11 - x = 4$
$11 = 4 + x$ (add x to both sides)
$x + 4 = 11$ (turn around equation for ease of working)
$x = 7$ (subtract 4)

This is the point to practise many examples.

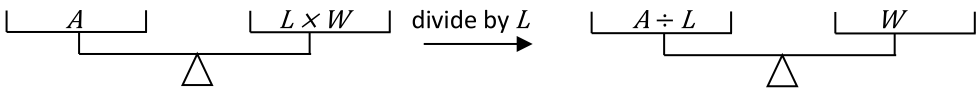
5.3 Changing subject of a formula

This section looks at continuing 5.1 and applying it to formulae.

Activity sequence

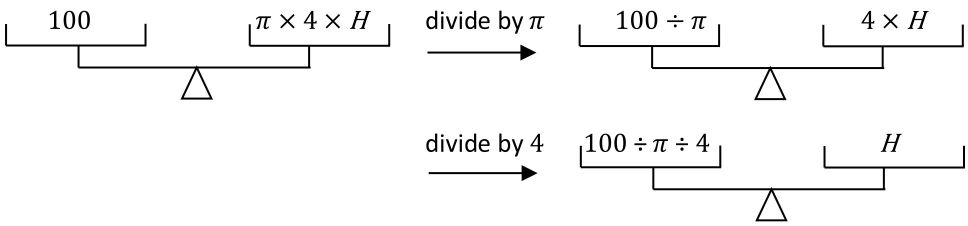
Where possible, find real-life contexts to embed the activities in; for example, using relevant objects or situations. Relate what the subject has to be to the unknown in the previous work – how do we get that alone.

1. Problem – "We have to make a rectangular garden of a certain area and a certain length; how wide does it have to be?" So we need $W =$ but the formula is $A = L \times W$. What is W equal to? We have to get it on its own. As can be seen below, this means $W = A \div L$ or A/L.

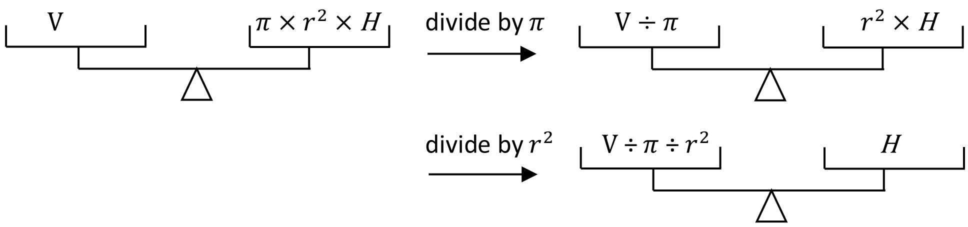


2. Problem – "We have to make a tank to hold grain. It has to be 4 m wide or have a radius of 2 m. It has to hold 100 m^3 of grain or 100,000 litres. How high does it have to be?" This means setting up a more complicated formula, e.g. $V = \pi r^2 \times H$. We can solve it two ways:

   (a) We take the amounts – radius of 2 m, volume of 100 m^3 or 100,000 litres. Placing this in the formula, and keeping everything in metres, we have $100 = \pi \times 2 \times 2 \times H$. What is H? As can be seen below $H = 100 \div \pi \div 4$.



   (b) We take the formula, $V = \pi r^2 \times H$, and find H for this.



   We now have $= V \div \pi \div r^2$, we substitute 2 for r and 100 for V and we have the answer.

   Use the idea that we have to keep in balance – so we do things to both sides that lead to what we want being on its own – becoming the subject of the formula.

Test Item Types

This section presents instructions and the test item types for the subtests associated with the units. These will form the bases of the pre-test and post-test for this module.

Instructions

Selecting the items and administering the pre-post tests

This section provides an item bank of test item types, constructed around the units in the module. From this bank, items should be selected for the pre-test and post-test; these selected items need to suit the students and may need to be modified, particularly to make post-test items different to pre-test items. The purpose of the tests is to measure students' performance before and after the module is taught. The questions should be selected so that the level of difficulty progresses from easier items to more difficult items. In some modules this will follow the order of the units and subtests, and in other modules it will not, depending on the sequencing across the module and within units. The pre-test items need to allow for students' existing knowledge to be shown but without continual failure, and the post-test items need to cover all the sections in a manner that maximises students' effort to show what they can do.

In administering the pre-test, the students should be told that the test is not related to grades, but is to find out what they know before the topic is taught. They should be told that they are not expected to know the work as they have not been taught it. They should show what they know and, if they cannot do a question, they should skip it, or put "not known" beside questions. They will be taught the work in the next few weeks and will then be able to show what they know. Stress to students that **any pre-test is a series of questions to find out what they know** before the knowledge is taught. They should do their best but the important questions come at the end of the module. For the **post-test**, the students should be told that **this is their opportunity to show how they have improved**.

For all tests, **teachers should continually check to see how the students are going.** Items in later subtests, or more difficult items within a particular subtest, should not be attempted if previous similar items in earlier subtests show strong weaknesses. Students should be allowed to skip that part of the test, or the test should be finished. Students can be marked zero for these parts.

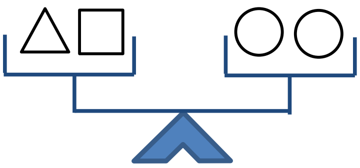
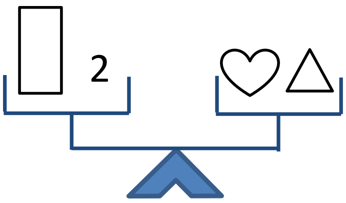
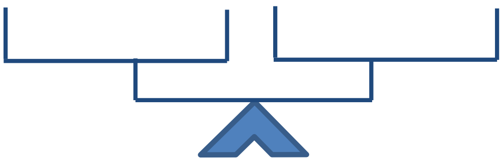
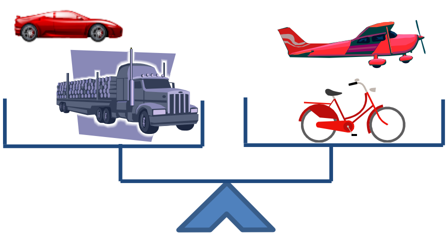
Information on the equivalence and equations item types

The algebra equivalence and equations item types are divided into five subtests, one for each of the five units in this module. The five units are in sequence. Therefore, the pre-test should cover the early subtests/units in order until teachers feel that students cannot answer (from here on their mark is zero), while the post-test should cover all the subtests/units, including all the later ones.

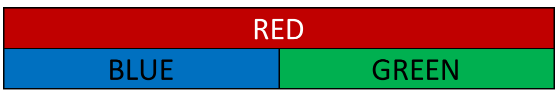
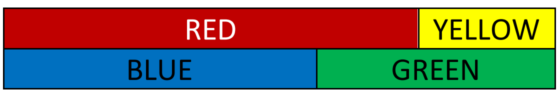
Subtest item types

Subtest 1 items (Unit 1: Meaning of Equation)

1. Fill in the empty boxes in the table. The first one has been done for you.

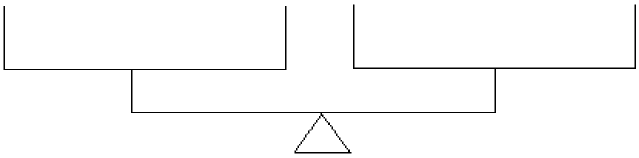
	Balance	Equation
		△ + □ = ○ + ○
(a)		
(b)		circle + square = rectangle + rectangle + rectangle
(c)		

2. Fill in the empty boxes in the table. The first one has been done for you.

	Drawing	Equation
		Red = Blue + Green
(a)		
(b)		Pink + White = Red + Red

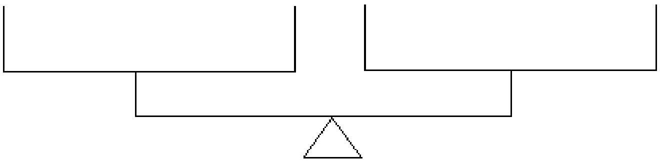
3. Use the stories below to fill in the trays on the scale and to write an equation.

(a) I bought 5 bottles of water for \$2 each and a packet of biscuits for \$4. I spent the same amount of money as Dave who bought 4 packets of chips for \$3 each and an orange for \$2.



Equation:

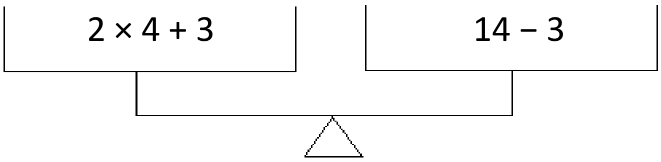
(b) Sally bought 3 drinks for \$2 each and a banana for \$1.50. She spent the same amount of money as George who bought 5 donuts for \$1 each and a bag of chips for \$2.50.



Equation:

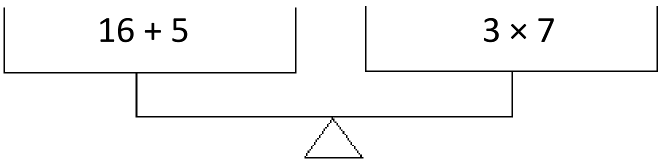
4. Use the balancing amounts to write an equation and a story.

(a)



Equation: Story:

(b)



Equation: Story:

Subtest 2 items (Unit 2: Equivalence principles)

1. Circle which of the following equations are correct.

$4 + 3 = 7$ $\quad$ $5 + 1 = 2 + 4$

$6 - 2 = 4 - 1$ $\quad$ $7 = 4 + 3$

$6 + 2 = 9 + 1$

2. Complete the following:

   (a) $6 + 2 = 9 - 1$ is the same as $9 - 1 = 6 +$ ____

   (b) $4 + 3 = 6 + 1$ and $6 + 1 = 5 + 2$, so $4 + 3 =$ _____ + _____

   (c) $7 - 3 = 6 - 2$ and $7 - 3 = 2 + 2$, so $6 - 2 =$ _____ + _____

3. Define the following:

   (a) symmetry rule for equivalence

   (b) transitivity rule for equivalence

   (c) reflexivity rule for equivalence

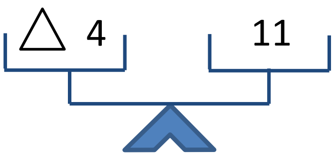
Subtest 3 items (Unit 3: Balance rule)

1. Use the balance rule to make the equations true again (the first one has been done for you):

	TRUE	NOT TRUE	TRUE AGAIN
	$4 + 5 = 7 + 2$	$4 + 5 + 11 \neq 7 + 2$	$4 + 5 + 11 = 7 + 2 + 11$
(a)	$8 - 3 = 4 + 1$	$8 - 3 - 25 \neq 4 + 1$	
(b)	$5 \times 12 = 3 \times 20$	$5 \times 12 + 66 \neq 3 \times 20$	
(c)	$81 \div 3 = 3 \times 3 \times 3$	$81 \div 3 \times 54 \neq 3 \times 3 \times 3$	
(d)	$6 + 3 = 15 - 6$	$6 + 3 \times 34 \div 2.3 \neq 15 - 6$	

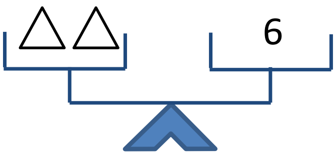
2. Work out what △ equals by using the balance rule.

(a)



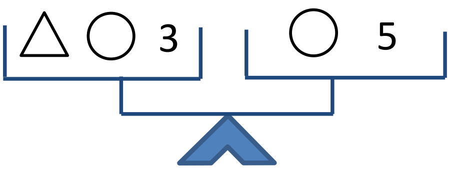
△ = ______________

(b)



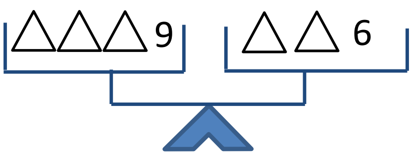
△ = ______________

(c)



△ = ______________

(d)



△ = ______________

Subtest 4 items (Unit 4: Unknowns and inversing change)

1. Fill in the empty boxes in the table by either writing the story or writing an equation (the equation has to represent the story). We do not want the answer. The first one has been done for you.

	Story	Equation with unknown
	I bought a pizza slice for \$3 and a drink. I spent \$5. (Cost of drink is d)	$3 + d = 5$
(a)	Mum gave me some money and then I spent \$3. I have \$7 left. (Amount of money Mum gave me is m)	
(b)		$4 \times a = 12$ (a can be any number of people, or amount paid – whatever you need to fit the story)
(c)	I had 24 lollies – they were in four bags with the same number of lollies in each bag (there were n lollies in each bag).	
(d)		$6 = 42 \div q$ (q can be anything as long as it fits your story)

2. Fill in the empty boxes in the table. The three columns are: expression, changes made to x in the expression, and reversing the changes to get back to the start which is x. The first one is done for you.

Note: (f) is a challenge

	Expression	**Changes to x**	**Reversing to get back to x**
	$3x + 2$	$\times 3$ and $+2$	-2 and $\div 3$
(a)	$5x - 4$		
(b)	$\frac{x}{4} + 4$		
(c)		$\times 7$ and -6	
(d)	$7x + 15 - 4 + 6$		
(e)			$+4$ and $\div 3$
(f)	$3x + 7 - x + 11$		

Subtest 5 items (Unit 5: Solving equations and changing subject)

1. What is the value of 'x'? Remember to show your working.

   (a) $x + 6 = 10$

   (b) $x - 7 = 12$

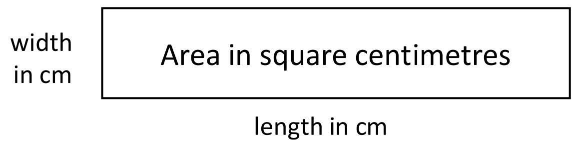
   (c) $2x = 14$

   (d) $12 - x = 4$

   (e) $2x - 5 = x + 1$

   (f) $2x + 7 = 4x - 19$

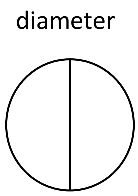
2. (a) Change the formula to find the length in cm of the rectangle in terms of Area and width



   Formula: Area = length × width $(A = l \times w)$

   Changed formula: Length = ____

   (b) Change the formula to find the diameter of the circle



   Formula: Circumference = π × diameter ($C = \pi d$)

   Changed formula: Diameter = ______________________

(c) Find the base of a triangle which has an area of 10 square centimetres and is 2 cm high.

3. **Challenge question**

Find the formula for the height (h) of a cylinder in terms of volume (V). Use the formula $V = \pi r^2 h$ as a starting point.

Appendix A: Cultural Implications for Teaching Algebra

This appendix discusses the cultural implications of algebra teaching. More emphasis than normal is placed on this aspect because algebra is holistic and best taught whole to part and this is similar to the cultural learning style of Aboriginal and Torres Strait Islander people (and low SES students). This has two implications: first, algebra understanding is best facilitated through big ideas; and second, this approach to teaching facilitates Aboriginal, Torres Strait Islander and low SES learning of algebra, and is a vehicle for mathematics understanding.

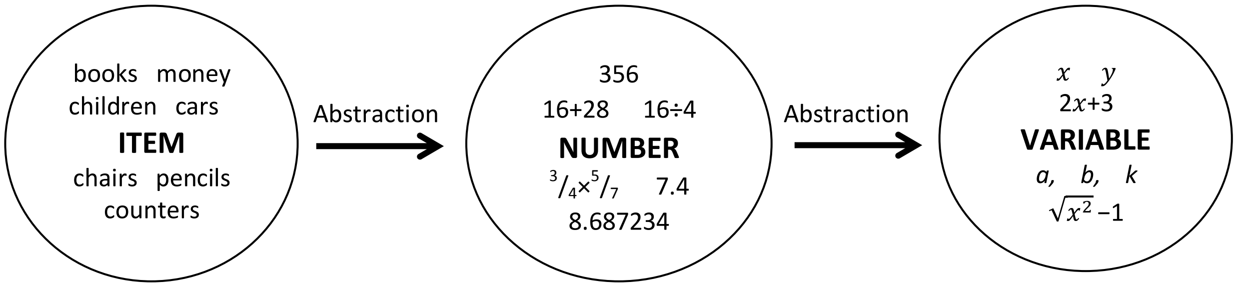
Algebra as abstraction of abstraction

Abstraction is a process by which a **generality** is determined from particular examples. In Western mathematics, an important abstraction is number. By experiencing, for example, many examples of two items (e.g. 2 eyes, 2 hands, 2 chairs, 2 students, and so on), learners generalise the language "two" and the symbol "2" as representing the "twoness" that is common to the examples. In a similar way, learners gradually build understanding of the language and symbols of all numbers.

When numbers and their names and symbols are new to learners, meaning lies with the items. For example, 2+3 is thought of as 2 items and 3 items. Counting all the items gives the solution 5 items. Thus 2+3 = 5 is thought of as 2 items joining 3 items to make 5 items. The focus of thinking is on the items. However, over time as more and more experience is gained, it becomes less necessary to think of items when we use numbers. After a while, 2+3 can be considered as equal to 5 without having to think of 2, 3 and 5 as specific items. The thinking simply happens on the symbols 2, 3 and 5. That is, the numbers become the focus or "objects" of thought; not the items that underlie them. At this point, the activity with real-world items has been **abstracted to numbers and arithmetic.**

However, abstraction does not stop with number. After a further time, learners start to see that sometimes things are the same regardless of the size and type of the numbers. An example of this are "turnarounds" (what is mathematically called the commutative principle), that is, for any number addition is the same regardless of the order in which numbers are added (e.g. 2+3 = 3+2; 656+172 = 172+656; $3^1/_4+2^2/_5 = 2^2/_5+3^1/_4$, and so on). For this principle, letters such as x and y can be introduced as symbols for variables (i.e. to stand for "any number") and used to represent the principle, that is, $x + y = y + x$. (*Note*: The commutative principle can be extended to more than two numbers and to algebra, and it only holds for addition and multiplication.)

Similar to numbers, when variables and their names and symbols (letters) are new to learners, meaning lies with the numbers that the variables could represent. For example, $2x + 3$ is thought of as two multiplied by "any number" plus 3. Solving $2x + 3 = 11$ means thinking like "I have a number, I multiply it by two, add 3 and end up at 11; to solve it, I subtract the 3 from 11 (get 8) and divide the 8 by 2 (get 4), so $x = 4$". The focus of thinking is on the numbers. However, over time as more experience is gained, it becomes less necessary to think of variables as numbers. The thinking simply focuses on the letters (e.g. $2x + 3x = 5x$ without thinking of x as a number). Thus, the variables become the focus or the "object" of thought. At this point, the numbers and arithmetic have been **abstracted to variables and algebra**. Overall, what this means is that the development from the real-world items to variables and algebra involves two steps: (1) abstraction from items to numbers and arithmetic, and (2) abstraction from numbers and arithmetic to variables and algebra. That is, algebra is an **abstraction of an abstraction** (see figure following).



Step 1: Items **Step 2: Numbers and arithmetic** **Step 3: Variables and algebra**

Interestingly, the process of abstraction involves gain and loss. Power is gained – we end up with knowledge that is much more portable and applies to a wider set of situations (i.e. the knowledge can be used in many situations). However, meaning is lost – the knowledge is highly symbolic and relationship back to the items it initially came from becomes more difficult (i.e. the knowledge is in a form that appears disconnected from the real world). Therefore, it is important in both arithmetic and algebra to **continually connect to the real world** (as in the RAMR cycle), to **show the role of symbols in mathematics**, and to understand mathematics **as a language**.

Abstraction, structure and holistic teaching

Recapping, the abstraction from arithmetic to algebra is an abstraction from particular activities represented by numbers and operations (i.e. arithmetic) to generalised activities represented by variables and operations (i.e. algebra). These generalised activities are interesting in that, to hold for all numbers, they must reflect **structural things in arithmetic**. In fact, they reflect what is called the underlying structure of arithmetic. This means, at its most powerful level, algebra reflects the **"big ideas" in arithmetic** – ideas that hold for whole numbers, fractions, measures as well as variables.

These big ideas are always present in what we do in arithmetic but are often undeveloped. A particular example may help in discussing this.

Example

The new mental computation approaches to computation are recommending that addition tasks such as 25+48 should be done by a strategy called compensation; that is 25+48 is calculated by changing one of the numbers to something easy to add and then compensating for this change on the other number. Because 50 is easy to add, we could change 48 to 50 by adding 2 and compensate by changing 25 to 23 by subtracting 2. In this way, the addition can be easily calculated (i.e. 25+48 = 23+50 = 73).

Most teachers stop here; they teach the strategy then support students to use it on other examples. To build big ideas, they need to go further. The important question that should be followed up is, "why does this work?"

The reason is that 23=25−2 and 50=48+2, so we are adding and subtracting 2. Since −2 and +2 are opposites or **inverses**, this is the same as adding 0, the **identity** (that which does not change anything). This means that the value of 25+48 does not change when it becomes 23+50 because all we are doing is adding 0. Putting in all the steps, what we have done is:

Start $\longrightarrow$ Finish

25+48 = 25+48+0 = 25+48−2+2 = 25−2+48+2 = 23+50 = 73

However, the big idea behind compensation is more than the −2+2 in this example. The big idea is that a first thing always equals a second as long as all we do is add 0 or something equivalent to 0. Thus to work out something complicated, all we have to do is find something the same as zero which changes it to something simple. This is an idea that can help us right across all mathematics (that is why it is called a **big idea**). For example,

PROBLEM	ZERO	WORKING
238+387	−13+13	238+387 = 225+400 = 625
4.8+2.7+3.6	+0.2+0.3−0.5	4.8+2.7+3.6 = 5+3+3.1 = 11.1
3h48m+2h37m	+12m−12m	3h48m+2h37m = 4h+2h25m = 6h25m
$a^2 - b^2$	−ab+ab	$a^2 - ab + ab - b^2 = a(a-b) + b(a-b) = (a+b)(a-b)$

This example shows a general method for teaching difficult additions by changing them to simple additions by finding things equivalent to zero to add to them. It is an example of algebra in action because the method does not identify the actual numbers to be used to bring about the change to a simpler form addition. Basically, the method says that any numbers will do as long as they add and subtract to 0. Thus, the example provides evidence for the *power of big ideas, underlying mathematics structure, and teaching mathematics using algebra.* Algebra teaching that is based on the structure of mathematics can enable students to learn big ideas that they can apply to particular examples right across mathematics. (*Note:* The other way to change without changing is to ×1. Combining +0 and ×1 together gives a more generic way to show equals or equivalence (e.g. it includes equivalent fractions and proportion). This really big idea is called equivalence of expressions and is the basis of algebra.)

In many schools, the teaching of arithmetic tends to focus on mathematics as disconnected parts, teaching the next activity as if it is a completely new thing, and relying on the weight of years for students to put together all the bits to make a whole. We will call this **part-to-whole** teaching. What the example above shows us is that algebra enables us to teach the more powerful mathematics where we learn a big idea and use it in particular situations. We shall call this **whole-to-part** teaching or **holistic teaching**.

Thus, algebra based on the structure of mathematics gives us a chance to teach holistically from the big picture down to the special case.

Indigenous culture, mathematics and holistic teaching

A danger in teaching Western mathematics (and science) to Aboriginal and Torres Strait Islander people is that teachers can make their teaching become a celebration of the growth and success of Western or European knowledge. It is particularly easy to represent Western knowledge as successful because it can be presented as continually advancing in terms of technology (e.g. cars, planes, rockets, computers) and as coming to dominate the planet. However, this same knowledge has been particularly unsuccessful in handling the intransigent and long-term problems of the planet such as destruction of the environment, poverty, war and violence, and climate change.

Teaching that presents mathematics as a celebration of this “linearly advancing technological process” can marginalise Aboriginal and Torres Strait Islander people, undermine the significance of their Indigenous identity and devalue Indigenous knowledges and cultures as simplistic societies (Matthews, 2003). Western mathematics places importance on number and arithmetic because this is where linear advancement in technological progress starts and what drives its progress. However, it can be argued that the invention of arithmetic was a consequence of a society in which material assets were considered more important than the individual. In Indigenous society, without the need to work out one’s assets in fine detail, number was not developed to the same level as in Western culture. However, this does not mean that Aboriginal and Torres Strait Islander cultures do not have their own mathematical knowledge.

Considerations of Indigenous knowledge of mathematics require recognition and respect of such knowledge, which should, in turn, be reflected in the teaching and learning of mathematics to students. For example, as argued by Matthews (2003), Yolngu children, from a young age, have a good understanding of their kinship system which governs the Yolngu way of life. This system is very complex and relies on cyclical and recursive patterns. Such patterns can be found within numbers themselves and other areas of mathematics (Divola & Wells, 1991; Jones, Kershaw, & Sparrow, 1996) and forms a good basis for Yolngu children to start their journey into Western mathematics. As most Aboriginal and Torres Strait Islander knowledge systems are based on interactions within the environment and groups of people, they can form algebraic systems because they can relate numbers in flexible ways.

Traditionally, in Australian schools, mathematics and its teaching both reflect Western culture. Therefore, differences in mathematics performance can stem from a different cultural view of what it means to be good at mathematics. Commonly, in most school environments, this is determined by gauging students' performance levels from test items that reflect non-Indigenous learning styles, namely solving meaningless problems by pen-and-paper means. In those problems, there are often marked differences in errors between Indigenous and non-Indigenous students. One case study of Indigenous students' errors found that underperformance tended to reflect mistakes in procedures rather than understanding (reflecting the position of Grant (1998) that Indigenous students see the whole rather than the parts).

Therefore, it is important to teach mathematics on an equitable basis with Western mathematics reflecting "both ways" approaches (Ezeife, 2002). Western teaching is traditionally compartmentalised, resulting in an education system in schools (whether oral or written) focusing on the details of the individual parts rather than the whole and relationships within the whole. By contrast, Indigenous students tend to be holistic learners, appreciating overviews of subjects and conscious linking of ideas (Grant, 1998). In fact, Indigenous people have been characterised as belonging to "high-context culture groups" (Ezeife, 2002) which are characterised by: a holistic (top-down) approach to information processing in which meaning is "extracted" from the environment and the situation. Low-context cultures use a linear, sequential building block (bottom-up) approach to information processing in which meaning is constructed (Ezeife, 2002).

What this means is that students who use holistic thought-processing are more likely to be disadvantaged in mainstream mathematics classrooms. This is because Westernised mathematics is largely presented as hierarchical and broken into parts with minimal connections made between concepts and with the students' culture and community. It potentially conflicts with how they learn. If this is to change, curriculum and assessment need to be made more culturally sensitive and community oriented (see Deadly Maths Consortium, Tagai09 Maths for Employment Interim Report, June 2009).

Thus, we have a confluence of results. Indigenous students are high context and learn best with holistic teaching (Grant, 1998; Ezeife, 2002). Mathematics in its most powerful form is based on structural understanding that is learnt best by holistic teaching. Algebra is the component of mathematics that is based on mathematical structure, and is capable of presenting mathematics holistically.

As a consequence, it seems that algebra is the form of mathematics that is most in harmony with Indigenous culture and learning style. Because of this, algebra understanding should be a **strength** of Indigenous students if it is taught through pattern and structure (rather than through sequential teaching of rules and algorithms). It seems likely that algebra is a subject in which Aboriginal and Torres Strait Islander students should excel. Finally, because of its relationship with arithmetic, this understanding of algebra should enable enhanced understanding of and proficiency with arithmetic.

Cultural implications for teaching algebra

There are two implications for algebra from the discussion above: (a) what is the best way to teach it, and (b) what is the best way to teach it to Aboriginal and Torres Strait Islander and low SES students?

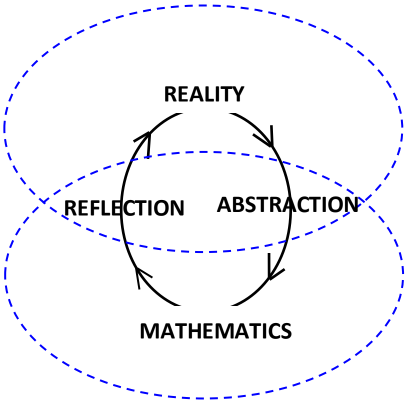
1. **Teaching algebra**. The power of mathematics lies in the structured way it relates to everyday life. Knowledge of these structures gives learners the ability to apply mathematics to a wide range of issues and problems. This is best achieved if the knowledge is in its most generalised form, which is algebraic form. Thus, the most effective way to present mathematical knowledge is through algebra. However, any topic of mathematics can be presented instrumentally (as a set of rules). Although algebra is the direction for power in mathematics, it has to be algebra that is presented structurally, showing the generalisations that can be used in many examples. Powerful algebra teaching focuses on extending arithmetic to generalisations that can apply across all arithmetic. That is, teaching that builds holistic understandings of

structure that can then be applied to particular instances (from the whole to the part). If students are fortunate enough to gain this structured understanding of mathematics, the subject becomes easy. This is because it is no longer seen as a never-ending collection of rules and procedures but rather the reapplication of a few big ideas.

2. **Teaching Indigenous students**. Aboriginal and Torres Strait Islander students tend to be high context. Their learning style is best met by teaching that presents mathematics structurally without the trappings of Western culture. Powerful Indigenous teaching is therefore holistic, from the whole to the part. As Ezeife (2002) and Grant (1998) argue, Indigenous students should flourish in situations where teaching is holistic (from the whole to the parts). Thus, holistic algebra teaching has two positive outcomes for Indigenous students: (a) it teaches a powerful form of mathematics; and (b) it teaches it in a way that is in harmony with Indigenous learning styles. Algebra taught structurally, then, is something in which Indigenous students should excel. However, this is just a general finding. What does this mean in practice for the teaching of algebra? First it means that we will not be teaching rules for manipulating letters. Letters and algebraic expressions and equations will be understood in terms of everyday life and algebraic ideas will be generalised from arithmetic. This will mean a lesser focus on algorithms and rules, and a greater focus on generalisations and applications to everyday life.

Thus for Indigenous and low SES students, for whom YDM and AIM were developed, algebra is the key for mathematics success – not x's and y's but the generalised holistic thinking that is the basis of it.

Appendix B: RAMR Cycle



AIM advocates using the four components in the figure on right, reality–abstraction–mathematics–reflection (RAMR), as a cycle for planning and teaching mathematics. RAMR proposes: (a) working from reality and local culture (prior experience and everyday kinaesthetic activities); (b) abstracting mathematics ideas from everyday instances to mathematical forms through an active pedagogy (kinaesthetic, physical, virtual, pictorial, language and symbolic representations, i.e. body → hand → mind); (c) consolidating the new ideas as mathematics through practice and connections; and (d) reflecting these ideas back to reality through a focus on applications, problem solving, flexibility, reversing and generalising (see figure on right).

The innovative aspect of RAMR is that the right half develops the mathematics idea while the left half reconnects it to the world and extends it. For example, whole-number place value built around the **pattern of threes** where hundreds-tens-ones applies to ones, thousands, millions, and so on, can be easily extended to metrics by considering the ones to be millimetres, the thousands to be metres and the millions to be kilometres.

Planning the teaching of mathematics is based around the RAMR cycle if it is deconstructed into components that are applied to a mathematical idea. By breaking instruction down into the four parts and taking account of the pedagogical approaches described above, the cycle can lead to a structured instructional sequence for teaching the idea. The table below briefly outlines how this can be done. Prerequisite mathematical ideas are considered in the Reality and Mathematics components of the cycle, while extensions and follow-up ideas are considered in the Reflection component.

REALITY • **Local knowledge**: Identify local student cultural-environmental knowledge and interests that can be used to introduce the idea. • **Prior experience**: Ensure existing knowledge and experience prerequisite to the idea is known. • **Kinaesthetic**: Construct kinaesthetic activities, based on local context, that introduce the idea.
ABSTRACTION • **Representation**: Develop a sequence of representational activities (physical to virtual to pictorial materials to language to symbols) that develop meaning for the mathematical idea. • **Body-hand-mind**: Develop two-way connections between reality, representational activities, and mental models through body → hand → mind activities. • **Creativity**: Allow opportunities to create own representations, including language and symbols.
MATHEMATICS • **Language/symbols**: Enable students to appropriate and understand the formal language and symbols for the mathematical idea. • **Practice**: Facilitate students' practice to become familiar with all aspects of the idea. • **Connections**: Construct activities to connect the idea to other mathematical ideas.
REFLECTION • **Validation**: Facilitate reflection of the new idea in terms of reality to enable students to validate and justify their new knowledge. • **Applications/problems**: Set problems that apply the idea back to reality. • **Extension**: Organise activities so that students can extend the idea (use reflective strategies – *flexibility*, *reversing*, *generalising*, and *changing parameters*).

Appendix C: AIM Scope and Sequence

Yr	Term 1	Term 2	Term 3	Term 4
A	**N1: Whole Number Numeration** Early grouping, big ideas for H-T-O; pattern of threes; extension to large numbers and number system	**O1: Addition and Subtraction for Whole Numbers** Concepts; strategies; basic facts; computation; problem solving; extension to algebra	**O2: Multiplication and Division for Whole Numbers** Concepts; strategies; basic facts; computation; problem solving; extension to algebra	**G1: Shape (3D, 2D, Line and Angle)** 3D and 2D shapes; lines, angles, diagonals, rigidity and properties; Pythagoras; teaching approaches
	N2: Decimal Number Numeration Fraction to decimal; whole number to decimal; big ideas for decimals; tenths, hundredths and thousandths; extension to decimal number system	**M1: Basic Measurement (Length, Mass and Capacity)** Attribute; direct and indirect comparison; non-standard units; standard units; applications	**M2: Relationship Measurement (Perimeter, Area and Volume)** Attribute; direct and indirect comparison; non-standard units; standard units; applications and formulae	**SP1: Tables and Graphs** Different tables and charts; bar, line, circle, stem and leaf, and scatter graphs; use and construction
B	**M3: Extension Measurement (Time, Money, Angle and Temperature)** Attribute; direct and indirect comparison; non-standard units; standard units; applications and formulae	**G2: Euclidean Transformations (Flips, Slides and Turns)** Line-rotation symmetry; flip-slides-turns; tessellations; dissections; congruence; properties and relationships	**A1: Equivalence and Equations** Definition of equals; equivalence principles; equations; balance rule; solutions for unknowns; changing subject	**SP2: Probability** Definition and language; listing outcomes; likely outcomes; desired outcomes; calculating (fractions); experiments; relation to inference
	N3: Common Fractions Concepts and models of common fractions; mixed numbers; equivalent fractions; relationship to percent, ratio and probability	**O3: Common and Decimal Fraction Operations** Addition, subtraction, multiplication and division of common and decimal fractions; models, concepts and computation	**N4: Percent, Rate and Ratio** Concepts and models for percent, rate and ratio; proportion; applications, models and problems	**G3: Coordinates and Graphing** Polar and Cartesian coordinates; line graphs; slope and y-intercept; distance and midpoints; graphical solutions; nonlinear graphs
C	**A2: Patterns and Linear Relationships** Repeating and growing patterns; position rules, visual and table methods; application to linear and nonlinear relations and graphs	**A3: Change and Functions** Function machine; input-output tables; arrowmath notation, inverse and backtracking; solutions for unknowns; model for applications to percent, rate and ratio	**O4: Arithmetic and Algebra Principles** Number-size, field and equivalence principles for arithmetic; application to estimation; extension to algebra; simplification, expansion and factorisation	**A4: Algebraic Computation** Arithmetic to algebra computation; modelling-solving for unknowns; simultaneous equations, quadratics
	N5: Directed Number, Indices and Systems Concept and operations for negative numbers; concept, patterns and operations for indices; scientific notation and number systems	**G4: Projective and Topology** Visualisation; divergent and affine projections; perspective; similarity and trigonometry; topology and networks	**SP3: Statistical Inference** Gathering and analysing data; mean, mode, median, range and deviation; box and whisker graphs; large data sets, investigations and inferences	**O5: Financial Mathematics** Applications of percent, rate and ratio to money; simple and compound interest; best buys; budgeting and planning activities

Key: N = Number; O = Operations; M = Measurement; G = Geometry; SP = Statistics and Probability; A = Algebra.

Accelerated Inclusive Mathematics Project

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