Accelerated Inclusive Mathematics Project

YuMi Deadly Maths

AIM Module O3

Year B, Term 2

Operations:

Common and Decimal Fractions

Prepared by the YuMi Deadly Centre
Queensland University of Technology
Kelvin Grove, Queensland, 4059

ACKNOWLEDGEMENT

The YuMi Deadly Centre acknowledges the traditional owners and custodians of the lands in which the mathematics ideas for this resource were developed, refined and presented in professional development sessions.

YUMI DEADLY CENTRE

The YuMi Deadly Centre is a research centre within the Faculty of Education at the Queensland University of Technology which is dedicated to enhancing the learning of Indigenous and non-Indigenous children, young people and adults to improve their opportunities for further education, training and employment, and to equip them for lifelong learning.

"YuMi" is a Torres Strait Islander Creole word meaning "you and me" but is used here with permission from the Torres Strait Islanders' Regional Education Council to mean working together as a community for the betterment of education for all. "Deadly" is an Aboriginal word used widely across Australia to mean smart in terms of being the best one can be in learning and life.

YuMi Deadly Centre's motif was developed by Blacklines to depict learning, empowerment, and growth within country/community. The three key elements are the individual (represented by the inner seed), the community (represented by the leaf), and the journey/pathway of learning (represented by the curved line which winds around and up through the leaf). As such, the motif illustrates the YuMi Deadly Centre's vision: *Growing community through education*.

The YuMi Deadly Centre (YDC) can be contacted at ydc@qut.edu.au. Its website is http://ydc.qut.edu.au.

DEVELOPMENT OF THE AIM MODULES

The Accelerated Inclusive Mathematics (AIM) modules were originally developed from 2010 to 2013 as part of the Accelerated Indigenous Mathematics project funded under the Commonwealth Government's *Closing the Gap: Expansion of Intensive Literacy and Numeracy* program for Indigenous students. The project aimed to assist secondary schools with beginning junior-secondary Indigenous students who were at Year 2/3 level in mathematics by developing a junior-secondary mathematics program that accelerates the students' learning to enable access to mathematics subjects in the senior-secondary years and therefore enhance employment and life chances. The project developed three years of modules (Years A to C) that are vertical sequences of learning to take students from their ability level to their age level in mathematics. The YuMi Deadly Centre acknowledges the role of the Department of Education, Employment and Workplace Relations in the development of the AIM project and these modules.

Contents

Module Overview

In Year A, modules were presented that covered whole and decimal number (N1 *Whole Number Numeration* and N2 *Decimal Number Numeration*) and operations with whole numbers (O1 *Addition and Subtraction* and O2 *Multiplication and Division*). In Year B, a number module (N3 *Common Fractions*) was presented that extended the previous number ideas to all of fractions. This module, O3 *Common and Decimal Fraction Operations*, extends the previous operation modules to fractions. It covers operations for common fractions (like and unlike denominator) and operations for decimal fractions.

This module is, therefore, based on the ideas covered in N1, N2, N3, O1 and O2. Thus, the module begins by summarising information from these previous five modules on which this module is based, that is, concepts and processes of common and decimal numbers and concepts, principles, strategies and models that hold for whole-number operations. Some of these are covered in this overview, and the remainder in the first unit.

As with O1 and O2, this module covers operation activities that can be easily completed with calculators. Where this is a legitimate alternative, it will be noted.

Background information for teaching operations with fractions

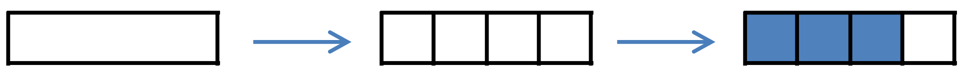
This section covers ideas from N1, N2, N3, O1 and O2 that apply to Module O3 – concept big ideas for fractions and operations, models for operations, principle big ideas for operations, information on language and symbols, and other big ideas.

Concepts for fractions and operations

The concepts for fractions and operations remain as they were developed in Modules N1, N2, N3, O1 and O2.

Number

Common fractions are where the denominator gives the number of parts into which the whole has been partitioned and the numerator the number of these parts that are being considered, as in the diagram below.



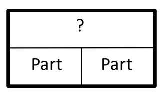
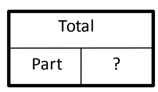
fourths 3/4

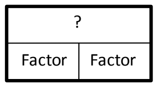
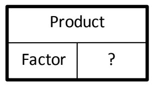
Decimal fractions are where numbers are presented in terms of fractional place values based around partitions of ten (tenths, hundredths, thousandths, and so on), plus the whole number place values, as below.

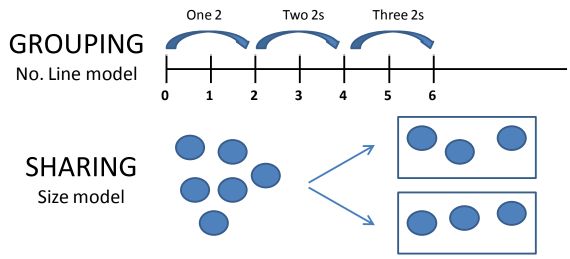
TH	H	T	O	t	h	th
			0 •	7	5	

75 hundredths

Operations

OPERATION	CONCEPTS	INTEGRATING CONCEPT
Addition	Joining, inverse of take-away, change/comparison (end unknown), inaction (where have to find superset)	Part-part-total (total is unknown) 
Subtraction	Take-away, inverse of joining, change/comparison (start or change unknown), inaction (where have to find subset)	Part-part-total (a part is unknown) 

Multiplication	Combining equal amounts, inverse of partitioning, change/comparison (end unknown), combinations	Factor-factor-product (product is unknown) 
Division	Partitioning into equal amounts (sharing and grouping), inverse of combining, change/comparison (start or multiplier unknown), combinations	Factor-factor-product (a factor is unknown) 

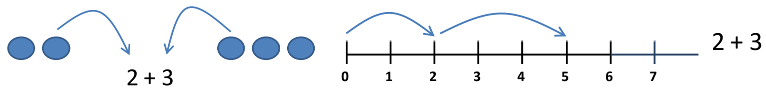
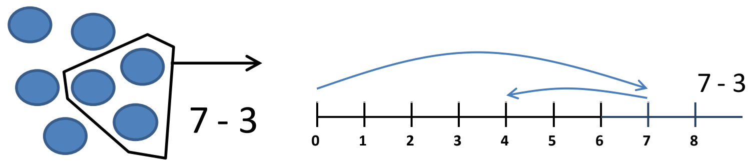
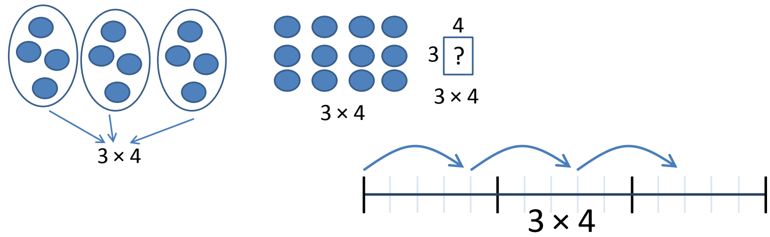
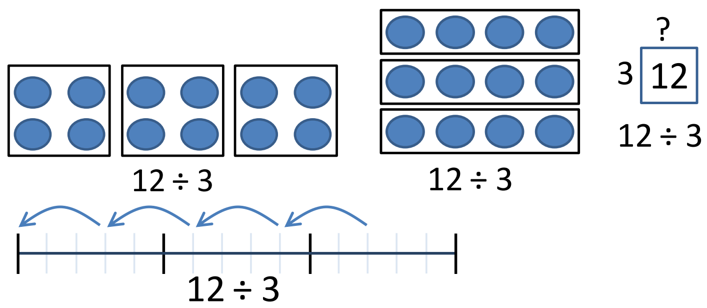


For example 6÷2, grouping is how many 2s in 6 (there are 3 2s in 6), while sharing is how much does each person get if 6 is shared equally amongst 2 (each person gets 3), as below.

Note: The two methods of division, grouping and sharing, will not work with examples like 2/3÷3/4 (we cannot share amongst 3/4 and cannot group as 3/4 is larger than 2/3). We have to revert to the true mathematical meaning of division as multiplication by inverse (reciprocal). In doing this, we must rebuild the idea of division for examples like 6÷2 as 6×1/2.

Operation models

The models for the operations are as below. We use the term **size** to replace the set model because with fraction you will not have sets of objects, you will have $^2/_3$ of an object – the model has to show the difference between $^2/_3$ and $^3/_4$ so it does this by showing size.

OPERATION	MODELS	INTEGRATING CONCEPT
Addition	Set/size, number line	
Subtraction	Set/size, number line	
Multiplication	Set/size, number line, arrays, area	
Division	Set/size, number line, arrays, area	

Principle big ideas for operations

Number-size principles

All the number-size principles/big ideas continue to hold for fraction operations. This means the following:

OPERATIONS	MAJOR NUMBER-SIZE PROPERTIES
Addition and Multiplication	• If either number becomes larger/smaller, the answer goes the **same** way for either the first or second number – 3+4=7 → 3+5=8, 5>4 and 8>7. • There is no situation in which something becoming larger/smaller makes the answer go the opposite way. • If either number becomes larger/smaller and the answer stays the same, the other number has to go the opposite way – 3+4=7 → 2+5=7, 2<3 and 5>4.
Subtraction and Division	• If the first number becomes larger/smaller, the answer goes the same way – 8-3=5 → 10-3=7, 10>8 and 7>5. • If the second number becomes larger/smaller, the answer goes the opposite way – 24÷3=8 → 24÷4=6, 4>3 and 6<8 (this is the inverse relation principle). • If either number becomes larger/smaller and the answer stays the same, the other number has to go the same way – 11-4=7 → 9-2=7, 9<11 and 2<4.

Operation principles

All the field principles hold for addition and multiplication as the following shows:

FIELD PRINCIPLES	APPLICATION/EXAMPLES
Closure	Fraction + fraction = fraction; fraction × fraction = fraction.
Identity	0 for addition and 1 for multiplication – $^2/_3 + 0 = {}^2/_3$; $^2/_3 \times 1 = {}^2/_3$.
Inverse	A fraction has additive inverse of fraction (e.g. $-^3/_4$ is the inverse of $^3/_4$); A fraction has multiplicative inverse of reciprocal of fraction (e.g. $^4/_3$ is the inverse of $^3/_4$).
Associativity	$(^2/_{11} + {}^3/_{11}) + {}^4/_{11} = {}^2/_{11} + ({}^3/_{11} + {}^4/_{11})$; $(^2/_3 \times {}^3/_5) \times {}^4/_7 = {}^2/_3 \times ({}^3/_5 \times {}^4/_7)$.
Commutativity	$^2/_{11} + {}^5/_{11} = {}^5/_{11} + {}^2/_{11}$; $^2/_3 \times {}^3/_5 = {}^3/_5 \times {}^2/_3$.
Distributivity	$^2/_3 \times (^3/_{11} + {}^5/_{11}) = (^2/_3 \times {}^3/_{11}) + (^2/_3 \times {}^5/_{11})$.

The equals/equivalence and order principles also hold for fractions as they do for whole numbers.

Language and symbols

Representations for fractions mean the language, models (physical, virtual and pictorial) and symbols for these fractions. We have covered models, so this leave language and symbols.

There are two symbol methods for fractions – decimals and common fractions (e.g. $^3/_8$ and 0.375 – both of these represent three eighths). The decimal fraction symbols are an extension of whole number symbols but common fraction symbols are very different. Overall the decimal symbols predominate in a metric society. However, the language for both decimal and common fractions is the same and is the common fraction language, but related to only some denominators – the place-value positions of tenths, hundredths, thousandths and so on. Thus, $^3/_8$ is stated as "three eighths" and 0.375 is stated as "three-hundred and seventy-five thousandths".

Other big ideas

Because of what operations and numbers are, the following big ideas also continue to hold.

Global/teaching principles

1. **Symbols tell stories**. The symbols of mathematics enable the world to be described succinctly and in a generalised way (e.g. 2 + 3 = 5 means caught 2 fish and then caught another 3 fish, or bought a $2 chocolate and $3 drink, or joined a 2m length of wood to a 3m length … and so on).

1. **Relationship vs change.** Mathematics has three components – objects, relationships between objects, and changes/transformations between objects. All relationships can be perceived as changes and vice versa. This is particularly applicable to operations; 2 plus 3 can be perceived as relationship 2 + 3 = 5 or change $2 \xrightarrow{+3} 5$.

2. **Interpretation vs construction**. Things can either be interpreted (e.g. what operation for this problem, what properties for this shape) or constructed (write a problem for 2 + 3 = 5; construct a shape of 4 sides with 2 sides parallel).

3. **Accuracy vs exactness**. Problems can be solved accurately (e.g. find 5 275 + 3 873 to the nearest 100 – rounding and estimation) or exactly (e.g. 5 275 + 3 873 = 9 148 – basic facts and algorithms).

4. **Part-part-total/whole**. Two parts make a total or whole, and a total or whole can be separated to form two parts – this is the basis of numbers and operations (e.g. fraction is part-whole, ratio is part to part; addition is knowing parts, wanting total).

Extension of field properties

1. **Compensation**. Ensuring that a change is compensated for so answer remains the same – related to inverse (e.g. 5+5=7+3; 48+25=50+23; 61−29=62−30).

2. **Equivalence**. Two expressions are equivalent if they relate by adding or subtracting 0 and multiplying or dividing by 1; also related to inverse (e.g. 48+25=48+2+25−2=73; 50+23=73; $^2/_3={}^2/_3\times{}^2/_2={}^4/_6$).

3. **Inverse relation principle**. This is part of the number-size principles described above.

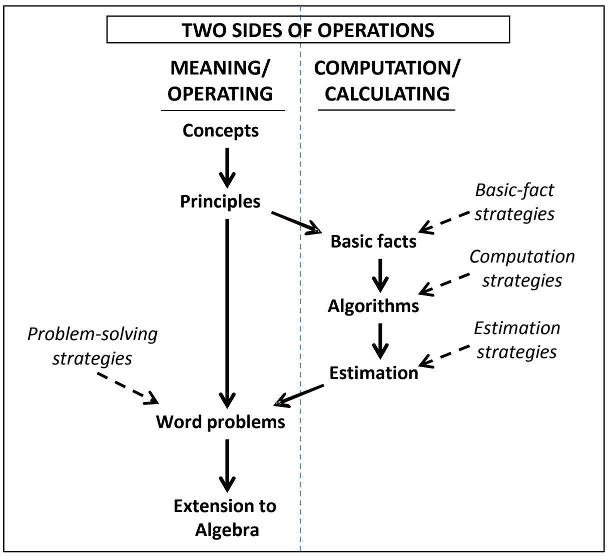
4. **Triadic relationships**. When three things are related there are three problem types where each of the parts are the unknowns. For example, 2+3=5 can have a problem for: ?+3=5, 2+?=5, 2+3=?.

Sequencing for operations with fractions

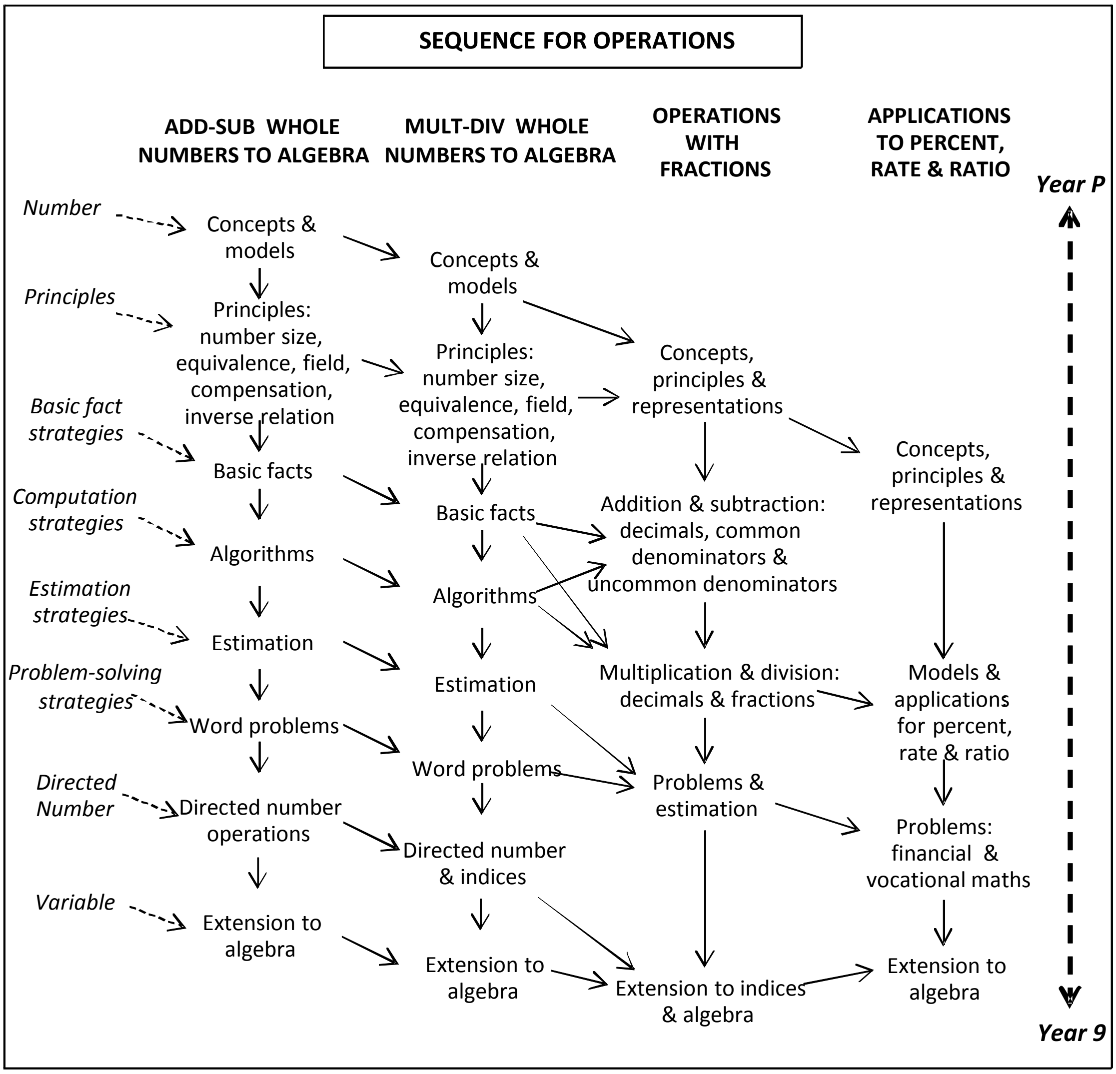
This section briefly looks at the role of sequencing in acceleration of learning and its importance in this project, then looks at the role of sequencing in this particular module.

Sequencing in operations

The sequencing that was the basis of Modules O1 and O2 remains, and is represented in the figure on the right. It shows that there are two components to operations, meaning and computation, and problem solving and algebra relate to meaning (concepts and principles). It also shows the strong role strategies now play in operations as calculators and other computational devices and software take over the actual calculation. This, of course, diminishes the role of pen-and-paper algorithms in fractions and decimals, but not the role of mental calculation and estimation. Thus the focus of this module will be on meaning, and understanding how computation can be achieved.



The total sequence for all operations is given below. This module means that we are covering enough to be able to see how all the parts fit together.



Sequencing in this module

The particular sequence followed by this module is highlighted in the figure on right. This module is covered in four units. All the sections in the module are listed below.

Overview: Background information, sequencing, and relation to Australian Curriculum

Unit 1: Concepts, models and representations for operations

Unit 2: Addition and subtraction – common and decimal fractions

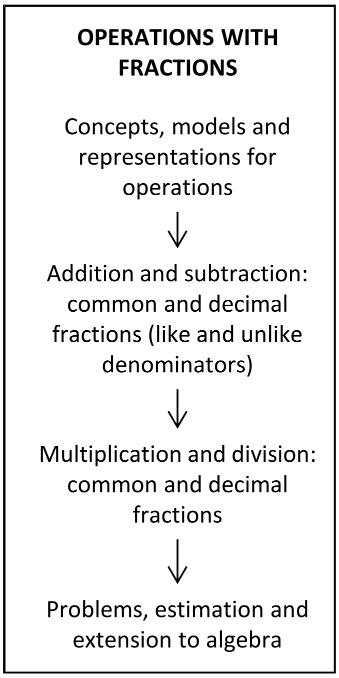
Unit 3: Multiplication and division – common and decimal fractions

Unit 4: Problems, estimation and extension to algebra

Test item types: Test items associated with the four units above which can be used for pre- and post-tests

Appendix A: RAMR cycle components and description

Appendix B: AIM scope and sequence showing all modules by year level and term.



The units above provide ideas for teaching this module with diagrams. They are not in the form of RAMR lessons with information under headings. It is recommended that the ideas are implemented in the classroom following the RAMR cycle included in Appendix A.

Relation to Australian Curriculum: Mathematics

AIM O3 meets the Australian Curriculum: Mathematics (Foundation to Year 10)

Unit 1: Concepts, models and representations
Unit 2: Addition and subtraction
Unit 3: Multiplication and division
Unit 4: Problems, estimation and extension to algebra

Content Description	Year	O3 Unit			
		1	2	3	4
Recognise and describe one-half as one of two equal parts of a whole. (ACMNA016)	2	✓			
Recognise and interpret common uses of halves, quarters and eighths of shapes and collections (ACMNA033)		✓			
Model and represent unit fractions including 1/2, 1/4, 1/3, 1/5 and their multiples to a complete whole (ACMNA058)	3	✓			
Investigate equivalent fractions used in contexts (ACMNA077)	4	✓	✓	✓	
Count by quarters halves and thirds, including with mixed numerals. Locate and represent these fractions on a number line (ACMNA078)		✓	✓	✓	
Compare and order common unit fractions and locate and represent them on a number line (ACMNA102)	5	✓	✓	✓	✓
Investigate strategies to solve problems involving addition and subtraction of fractions with the same denominator (ACMNA103)			✓	✓	✓
Compare fractions with related denominators and locate and represent them on a number line (ACMNA125)	6	✓	✓	✓	✓
Solve problems involving addition and subtraction of fractions with the same or related denominators (ACMNA126)			✓	✓	✓
Find a simple fraction of a quantity where the result is a whole number, with and without digital technologies (ACMNA127)				✓	✓
Add and subtract decimals, with and without digital technologies, and use estimation and rounding to check the reasonableness of answers (ACMNA128)			✓		
Multiply decimals by whole numbers and perform divisions by non-zero whole numbers where the results are terminating decimals, with and without digital technologies (ACMNA129)				✓	
Multiply and divide decimals by powers of 10 (ACMNA130)				✓	
Make connections between equivalent fractions, decimals and percentages (ACMNA131)		✓			
Solve problems involving addition and subtraction of fractions, including those with unrelated denominators (ACMNA153)	7		✓	✓	✓
Multiply and divide fractions and decimals using efficient written strategies and digital technologies (ACMNA154)				✓	✓
Express one quantity as a fraction of another, with and without the use of digital technologies (ACMNA155)		✓	✓	✓	✓
Connect fractions, decimals and percentages and carry out simple conversions (ACMNA157)			✓	✓	✓

Unit 1: Concepts, Models and Representations for Operations

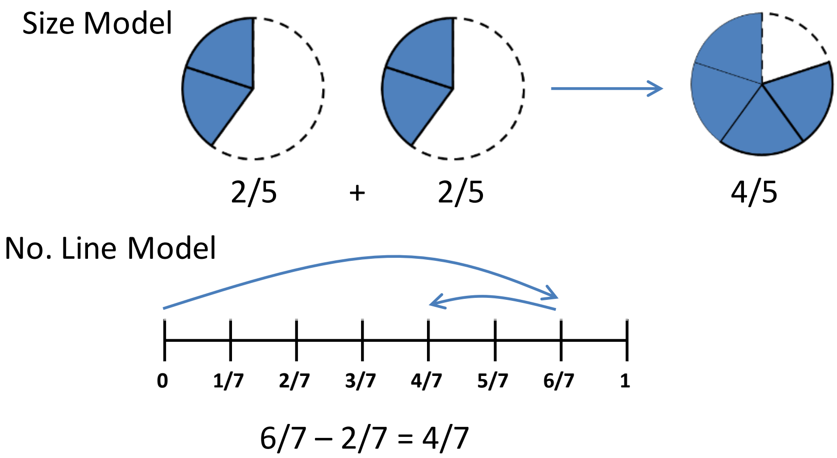
To teach operations with fractions, you must begin with meaning, revisiting the concepts, principles and representations, moving on to addition and subtraction, first as an extension of whole numbers for decimal numbers and separated into two stages (like and unlike denominators) for common fractions and mixed numbers. Next, look at multiplication and division, beginning with the area model and moving on to algorithms, and finally looking at problem solving, estimation and algebra. This unit looks at the concepts and models that will be used in this sequence.

1.1 Concepts and representations

With regard to fractions, some of these concepts and models in the overview section (background information) of this module become difficult to apply as discussed below.

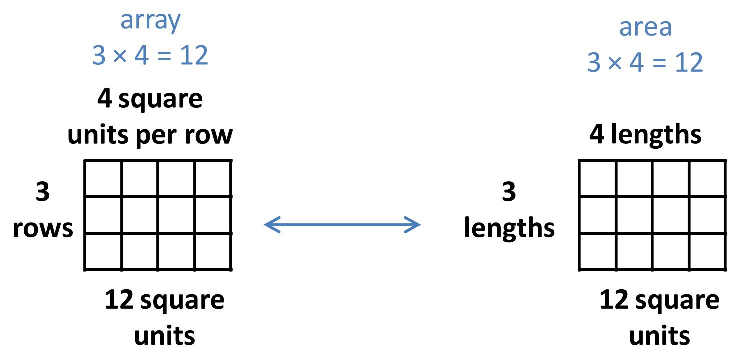
Addition and subtraction

The nature of addition and subtraction mean that you have to **"add-subtract like things"** (e.g. ones to ones and tens to tens). This can be extended to fractional place value in decimal numbers (e.g. tenths added-subtracted to tenths) and for like denominators (e.g. 2 fifths + 2 fifths = 4 fifths; 6 sevenths – 4 sevenths = 2 sevenths) – see below. However, if denominators are unlike, then something must be done before addition and subtraction (e.g. 2 fifths + 3 sevenths cannot be done unless we can change to the same thing, i.e., thirty-fifths).



Multiplication

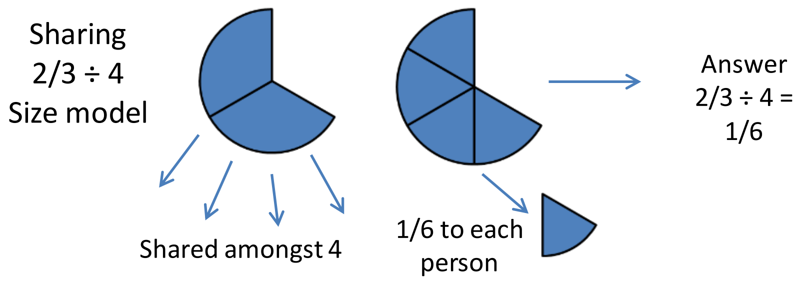
Multiplication has been built around combining equal amounts. However this assumes a whole number of these amounts. This means that we can multiply a fraction by a whole number using the set and number line models, but to multiply a fraction by a fraction we need to use an extension of the array model – the area model. It is possible for 3 rows of 4 square units to be seen as a 3×4 rectangle. In both examples (see below), the answer is 12 square units. However, in arrays, the 3 is the number of rows and the 4 is the number of square units in each row and the 12 is square units (i.e. number times rate, 3 rows × 4 square units/row); but in area, the 3 is lengths of side of the square unit, the 4 is the same while the 12 is square units (i.e. number × number, 3 lengths × 4 lengths = 12 square units). The extension of arrays to area must be taken with care as it is a change from combining concept to combinations concept.

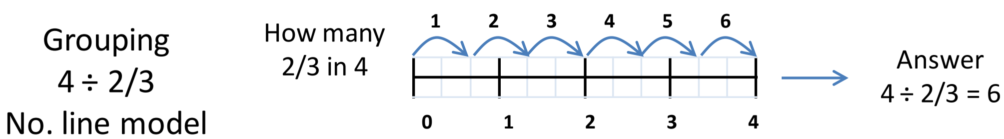


As well as this, multiplication of fractions makes things smaller (i.e. is very different to multiplication of whole numbers), for example, multiplying by $^{1}/_{3}$ changes 6 to 2.

Division

Division also is difficult with fractions – sharing requires a whole number of sharers, while grouping requires a whole number of groups (see below). To understand fraction divide by fraction requires using the proper mathematical meaning of division, multiplication by reciprocal (e.g. $^{2}/_{3} \div {}^{3}/_{4} = {}^{2}/_{3} \times {}^{4}/_{3}$).





As well, division by fractions makes things larger (i.e. is very different to division by whole numbers which makes things smaller), for example, dividing by $^{1}/_{3}$ changes 6 to 18.

Activities

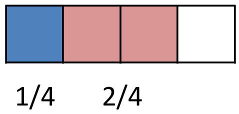
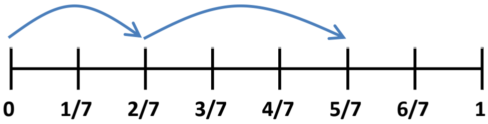
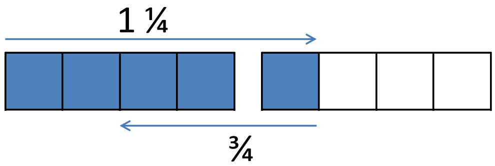
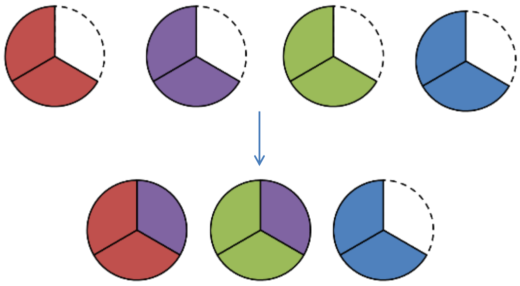
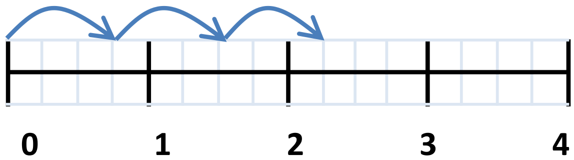
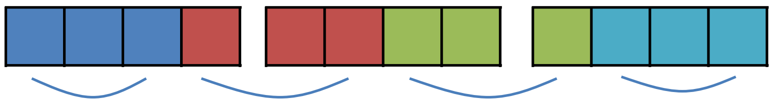
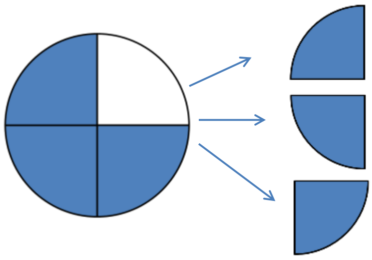
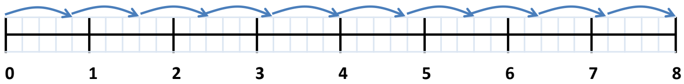
Use representations above to show how to do the following:

1. $^{3}/_{8} + {}^{7}/_{8}$ and $2\,^{1}/_{6} - {}^{5}/_{6}$ by set/size model and number line model.
2. $4 \times 2\,^{1}/_{3}$ by set model and array model.
3. $2\,^{2}/_{3} \div 4$ by sharing (set/size and number line models) and $2\,^{1}/_{4} \div {}^{3}/_{4}$ by grouping (set/size and number line models).

1.2 Using models

It is important at the start of fraction work to transfer the concepts, principles and representations (including models) from whole numbers to decimal and common fractions. In particular, it is important to be able to use the set/size, number line and array/area models for operating with fractions. However, it is also important to modify the models for the needs of fractions. The modifications will be covered in the next section. Here we look at examples where the whole number ideas can be easily transferred.

Use the models beside the operations to work out the answer to the operations for the models given.

OPERATION	MODEL	WORKING	ANSWER
$^{1}/_{4} + ^{2}/_{4}$	Set/Size		$^{3}/_{4}$
$^{2}/_{7} + ^{3}/_{7}$	Number line		$^{5}/_{7}$
$1^{1}/_{4} - ^{3}/_{4}$	Set/Size		$^{2}/_{4}$
$4 \times ^{2}/_{3}$	Set/Size		$^{8}/_{3}$ or $2^{2}/_{3}$
$3 \times ^{3}/_{4}$	Number line		$^{9}/_{4}$ or $2^{1}/_{4}$
$3 \div ^{3}/_{4}$	Set/Size		4
$^{3}/_{4} \div 3$	Set/Size		$^{1}/_{4}$
$8 \div ^{4}/_{5}$	Number line		10

Activities

For each of the above examples, use another model to solve the example.

Unit 2: Addition and Subtraction – Common and Decimal Fractions

Operations with fractions require students to have a good understanding of the many concepts of addition, subtraction, multiplication and division, and of what a fraction is. Addition and subtraction require like things to be added or subtracted. This will not be a problem for like-denominator fractions where numerators can be added or subtracted in an extension of whole-number addition and subtraction. However, for unlike denominators, it is necessary to rename fractions so that the operations are acting on like things, that is, **like denominators**. Students need a well-developed understanding of the equivalence ideas relating to fractions to be able to complete computations involving fractions.

2.1 Addition/subtraction for like-denominator common fractions

When fractions have the same denominators, the numerators can be added using any whole-number computation strategy that would work for the numbers involved. So the Separation, Sequencing and Compensation strategies described for whole numbers would each apply to the addition and subtraction of fractions with the same denominators. An example is provided below for both addition and subtraction.

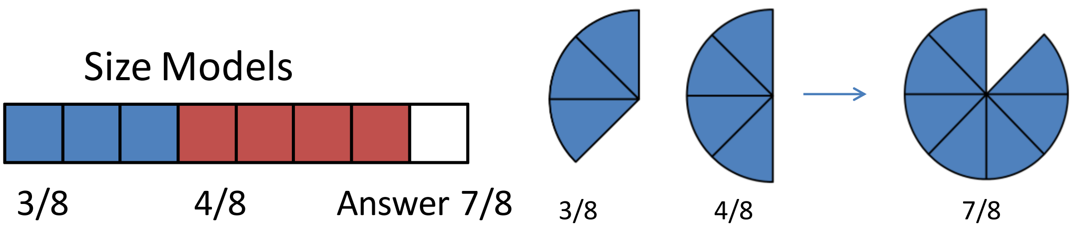
Common fractions

There are two options here – size and number-line models. Students need to understand addition as parts combined to make a total. The models represent the parts and the total and allow the computation strategies to be used. The models act similarly to whole numbers. There are three steps in the teaching process.

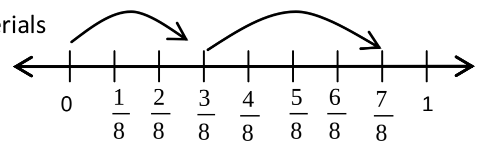
Step 1: Simple operations

Example: ${}^{3}/_{8} + {}^{4}/_{8}$

For the **size model**, shading areas will work as will joining eighth pieces (see below).

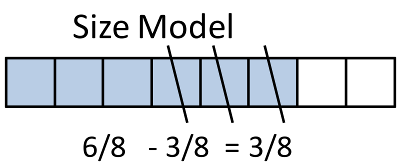


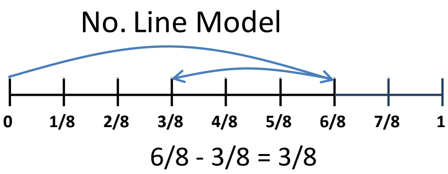
For the **number-line model**, we replace the shading or the materials with hops along a line as on right. Eighths are shown on a line with 0–1 divided into eight equal parts. We can do this example by 3 hops and then 4 hops, or by a hop of ${}^{3}/_{8}$ and a hop of ${}^{4}/_{8}$.



Example: ${}^{6}/_{8} – {}^{3}/_{8}$

Subtraction is done the same way, as shown below.





Step 2: Pattern

Generalise to complete examples as in Step 1, record what is done in symbols, look at symbols and discover pattern:

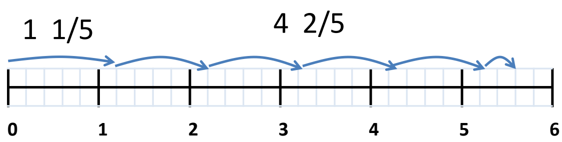
$$^{m}/_{p} + {^{n}/_{p}} = {^{(m+n)}/_{p}} \quad \text{and} \quad ^{m}/_{p} - {^{n}/_{p}} = {^{(m-n)}/_{p}}$$

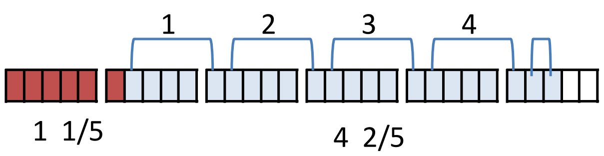
Step 3: Mixed numbers

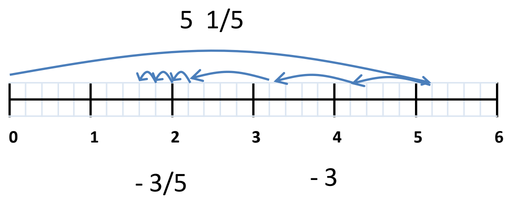
Example: $1^{1}/_{5} + 4^{2}/_{5}$ *and* $5^{1}/_{5} - 3^{3}/_{5}$

Mixed-number addition and subtraction can be done in three ways:

Way 1: Use **models** as in Step 1 on previous page (see below)







Way 2: Change to improper fractions and use **pattern**

$1^{1}/_{5} + 4^{2}/_{5} = {^{6}/_{5}} + {^{22}/_{5}} = {^{28}/_{5}} = 5^{3}/_{5}$ and $5^{1}/_{5} - 3^{3}/_{5} = {^{26}/_{5}} - {^{18}/_{5}} = {^{8}/_{5}} = 1^{3}/_{5}$

For subtraction, can also do **additive sequencing** as follows:

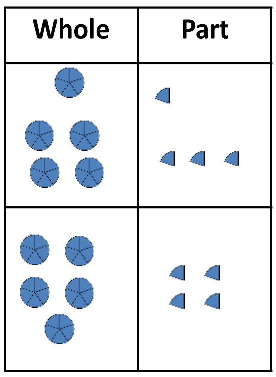
$3^{3}/_{5} \rightarrow {^{2}/_{5}}$ to $4 \rightarrow 1^{1}/_{5}$ to $5^{1}/_{5}$, so difference is $1^{3}/_{5}$

or **compensation** as follows:

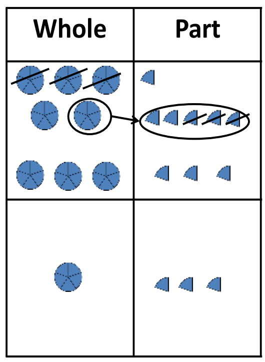
$5^{1}/_{5} - 3^{3}/_{5} = 5^{3}/_{5} - 4$, by adding $^{2}/_{5}$ to each number

Way 3: Use **whole-part chart** (see below for addition $1^{1}/_{5} + 4^{3}/_{5}$ and subtraction $5^{1}/_{5} - 3^{3}/_{5}$)

Whole	Part
1	1/5
+ 4	+ 3/5
5	4/5



Whole	Part
4 ~~5~~	6/5 ~~1/5~~
- 3	- 3/5
1	3/5



However, for these mixed-number examples, we recommend using Way 3 as this follows on the whole-number work. This means that it is seen in terms of mathematics the students may have already.

2.2 Addition/subtraction for decimal numbers

Addition and subtraction of decimal numbers can be done as an extension of whole-number operations. In fact, where possible, YDM recommends doing decimal addition and subtraction **as the reflection stage** of RAMR lessons of whole-number addition and subtraction. The sections below look at the three strategies for algorithms and how they are applied to decimals.

Separation

The **separation strategy** for addition works by adding like place values and then combining. Students need to understand decimal place values to partition numbers into parts and manage the value of these parts. An example of this strategy being used with decimals is provided on right. Using this strategy also has potential to assist student understanding of place value as they need to consider the value of the digits in the computation not just the digits themselves as can be the case when using the traditional written algorithm.

23.6 + 45.58	
23 + 45 = 68	(whole numbers)
0.6 + 0.5 = 1.1	(tenths)
0 + 0.08 = 0.08	(hundredths)
68 + 1.1 + 0.08 = 69.18	

Three examples of the separation strategy for subtraction involving decimals are provided below. The first example manages the decimal component of the number as hundredths. The second example manages the decimal component one place value at a time. The third example shows regrouping – ways of separating numbers to manage subtraction so as to manage computations requiring bridging.

34.78 – 23.55	**34.78 – 23.55**	**29.315 – 18.27**
Separate 34.78 into 34 + 0.78	Separate 34.78 into 34 + 0.7 + 0.08	Separate 29.315 into 29 + 0.31 + 0.005
Separate 23.55 into 23 + 0.55	Separate 23.55 into 23 + 0.5 + 0.05	Separate 18.27 into 18 + 0.27
34 – 23 = 11 (whole numbers)	34 – 23 = 11 (whole numbers)	29 – 18 = 11 (whole numbers)
0.78 – 0.55 = 0.23 (hundredths)	0.7 – 0.5 = 0.2 (tenths)	0.31 – 0.27 = 0.04 (hundredths)
11 + 0.23 = 11.23 (sum of parts)	0.08 – 0.05 = 0.03 (hundredths)	0.005 – 0 = 0.005 (thousandths)
	11 + 0.2 + 0.03 = 11.23 (sum of parts)	11 + 0.04 + 0.005 = 11.045 (sum of parts)

Sequencing

This strategy will also work with decimals. For simpler examples, it is possible to develop a number board that involves decimals and to use it in the same way as a number board with whole numbers (the jumps will change so that to go down a row will not be +10 but will be +0.1 or 0.01 depending on the numbers used). It is also fairly straightforward to use a number line similar to whole numbers to do the operations. An example is given on right.

26.75 + 43.88	
Start with 26.75	
Separate 43.88 and add progressively	
+ 40	= 66.75
+ 3	= 69.75
+ 0.25	= 70.00
+ 0.25	= 70.25
+ 0.30	= 70.55
+ 0.05	= 70.60
+ 0.03	= **70.63**
43.88	

Compensation

All of these representations can be used to assist with addition and subtraction computations involving decimals (see two examples below). The adjustments and compensations will require sound understandings of decimal place value. Introducing these strategies and practising them could work to improve student understanding of decimals as well.

23.86 + 18.75	
Change to 24 + 18.61	(+0.14 and –0.14)
Change to 30 + 12.61	(+6 and – 6)
= 42.61	

123.32 – 58.75	
Change to 123.57 – 59	(+0.25 and +0.25)
Change to 124.57 – 60	(+1 and + 1)
= 64.57	

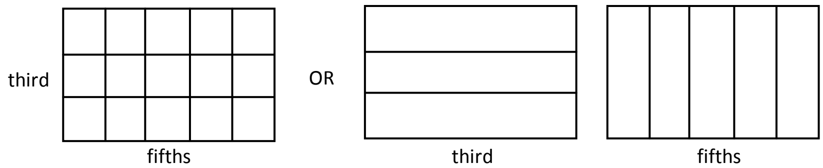
2.3 Addition/subtraction for unlike-denominator common fractions

To complete operations with fractions that have different denominators students need to use understandings of equivalent fractions. There are three ways to do this.

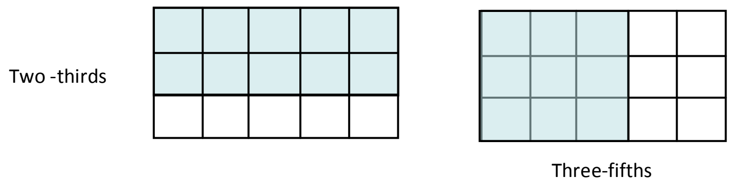
Way 1: Array/Area model

Representing fractions using an area model was advocated as a representation of one of the concepts of a fraction – a fraction is equal parts of a whole. When fractions with different denominators are to be added there needs to be consideration of equivalent fraction versions of the fractions being added. This can be helped using an area model.

Example $^{2}/_{3} + {}^{3}/_{5}$



To represent these fractions the whole needs to be the same. Using one rectangle to show both fractions assists with the maintenance of the whole and in seeing the equivalent fractions. One dimension of the rectangle is used to show each fraction. For example, in the figure above, one dimension is divided into thirds and the other into fifths. If this is difficult for students to visualise they can start with two congruent rectangles and divide one into thirds and the other into fifths and overlay them for the same effect.

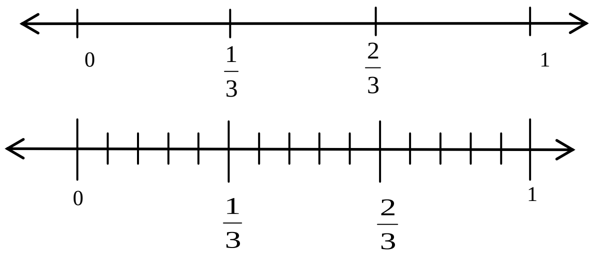


Using the combined diagram on right, the fractions in the computation can be represented. It can now be seen that two-thirds is the same as $^{10}/_{15}$ and that three-fifths is the same as $^{9}/_{15}$. The two fractions now have the same denominator and can be added using the computation strategy that works for the whole numbers that are the numerators, 10+9 in this case. The answer will be fifteenths – $^{19}/_{15}$. This can be converted to a mixed number or other equivalent fraction.

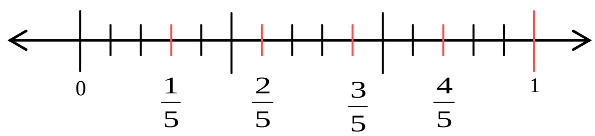
Use the area method to find the answer to $^{3}/_{4} – {}^{1}/_{3}$.

Way 2: Number lines

Example $^{2}/_{3} + {}^{3}/_{5}$



The representation of equivalent fractions can be done on a number line in a similar way to the area model. The same example as above is shown here using a number line. First, one of the fractions is used to divide a line into equal parts. Then each segment is divided into the number of equal parts represented by the other fraction (fifths).



Two-thirds is represented as $^{10}/_{15}$ so $^{2}/_{3}$ is equivalent to $^{10}/_{15}$. This number line can now also be shown divided into fifths (which will be every third mark), so $^{3}/_{5}$ is seen as $^{9}/_{15}$. So, as with the area model, the two equivalent fractions with the same denominator can be added by focusing on the numerators and adding these as whole numbers.

Use the number-line method to find the answer to $^{3}/_{4} – {}^{1}/_{3}$.

Way 3: Fraction sticks

Fraction sticks can give equivalence. The sticks are named by their left-hand digit (e.g. 2 stick, 5 stick and so on). A fraction is made by putting two rows together, that is, $^2/_5$, with the 2 stick above the 5 stick as below (note that the 2 stick is the multiple of 2s and the 5 stick is the multiple of 5s):

2	4	6	8	10	12	14	16	18	20
5	10	15	20	25	30	35	40	45	50

The sticks then give the sequence for equivalent fractions. This can be used to show the pattern for equivalent fractions, and the sticks can be used to **compare, add or subtract** fractions.

The equivalent fraction pattern was shown in Module N3 *Common Fractions* and is:

- that the fractions equivalent to $^2/_5$ are found by multiplying $^2/_5$ by 1, that is, multiplying $^2/_5$ by $^2/_2$, $^3/_3$, $^4/_4$ and so on (e.g. $^2/_5 \times {}^2/_2 = {}^4/_{10}$, $^2/_5 \times {}^3/_3 = {}^6/_{15}$, and so on); and that, reversing, fractions are equivalent to $^2/_5$ if they cancel down to $^2/_5$ (e.g. $^{14}/_{35} = {}^2/_5$, cancelling the 7)
- that two fractions are equivalent to each other if they cancel down to the same starting fraction; and
- if we have a fraction, then any fraction with a denominator which is a multiple of the first fraction's denominator can be equivalent to it if its numerator is the same multiple of the first fraction's numerator (e.g. $^7/_{12} = {}^{21}/_{36}$ because 36 = 3 × 12 and 21 = 3 × 7).

Examples $^3/_7 + {}^2/_5$ and $^3/_7 - {}^2/_5$

There are four steps as follows:

Step 1: Use sticks

Put out the 3 stick and the 7 stick to make $^3/_7$ and the 2 stick and the 5 stick to make $^2/_5$. Look along the denominators for a common denominator, in this case 35ths. Align the 35s and look at the two fractions aligned: $^{15}/_{35}$ and $^{14}/_{35}$. These are equivalent to $^3/_7$ and $^2/_5$. See fraction sticks below.

3	6	9	12	15	18	21	24	27	30
7	14	21	28	35	42	49	56	63	70

2	4	6	8	10	12	14	16	18	20
5	10	15	20	25	30	35	40	45	50

This means that:

$^3/_7 + {}^2/_5 = {}^{15}/_{35} + {}^{14}/_{35} = {}^{29}/_{35}$, and $^3/_7 - {}^2/_5 = {}^{15}/_{35} - {}^{14}/_{35} = {}^1/_{35}$

Step 2: Use sticks to reveal pattern

The fraction sticks are a **patterning** material in that their actions reveal the pattern behind unlike-denominator addition and subtraction. We see that 35 = 7 × 5 is the common denominator because it is a multiple of both fractions' denominators; and so $^3/_7 = {}^{15}/_{35}$ because both top and bottom are multiplied by 5 and $^2/_5 = {}^{14}/_{35}$ because both top and bottom are multiplied by 7, thus:

$$\frac{3}{7} + \frac{2}{5} = \frac{3 \times 5 + 7 \times 2}{7 \times 5}$$

Generalisation. This means in general that $\frac{a}{b} + \frac{c}{d}$ can be added by common denominator b × d and that $\frac{a}{b} = \frac{(a \times d)}{(b \times d)}$ (multiplying top and bottom by d) and that $\frac{c}{d} = \frac{(b \times c)}{(b \times d)}$ (multiplying top and bottom by b), meaning that:

$$\frac{a}{b} + \frac{c}{d} = \frac{a \times d + b \times c}{b \times d}$$

A similar pattern exists for subtraction.

Step 3: Give up sticks

When this pattern is known, the fraction sticks can be removed.

Step 4: Practise pattern

Use the pattern to add $^1/_6$ and $^3/_8$ and to subtract $^1/_2$ from $^7/_9$.

Unit 3: Multiplication and Division – Common and Decimal Fractions

In addition and subtraction of decimals, tenths are added to tenths and hundredths to hundredths in an extension of whole-number addition and subtraction. In multiplication and division of decimals, we multiply as if whole numbers and then determine place values by rules for number of digits in decimal place values.

For multiplication of common fractions, we use the area model. For division of common fractions, we have to rebuild a new understanding of division, that as multiplication by reciprocal.

Note: Of course, we need not develop any of these pen-and-paper methods. We can use a calculator to simply multiply and divide the decimal numbers, focusing on problem solving rather than computation.

3.1 Multiplication of decimal numbers

Multiplication of decimal numbers follows the pattern of whole numbers but determining the ones position is the problem. The following steps are useful.

Step 1: Complete whole × decimal examples

These follow the pattern for multiplication of whole numbers.

Step 2: Use area models for simple examples

Look at 0.6 × 0.7 and 0.1 × 0.1 using area model as in fractions ($^6/_{10}$ x $^7/_{10}$) – can also do this for examples like 3.2 × 4.7 if students need to see this. This gives rise to 6 tenths x 7 tenths being 42 hundredths and this is part of the pattern that:

tens × tens = hundreds	→	tenths × tenths = hundredths
tens by hundreds = thousands	→	tenths by hundredths = thousandths
and so on.		

Step 3: Relate decimals to whole numbers by naming in last place-value position

3.2 × 0.7 is 32 tenths × 7 tenths = 32 × 7 hundredths = 224 hundredths = 2.24

Step 4: Use calculator patterns

Use calculators to identify the pattern for the ones position.

Use calculators:	38 × 6 = _______	→ 3.8 × 0.6 = __________
	74 × 5 = _______	→ 0.74 × 0.5 = __________
	and so on	
Don't use calculators:	27 × 3 = 81	→ 2.7 × 0.3 = __________
	and so on.	

Step 5: State the pattern

The pattern/rule is that we have to add the decimal place-value position to find the ones position. That is, for example 0.36 × 0.6, there is a total of three decimal place-value positions which makes the 6 a thousandth. This in turn makes the ones position fourth from the right. Thus, if 36 × 6 by whole numbers is 216, then the answer is 0.216.

Note: Refrain from putting a rule forward that describes the pattern in terms of decimal points. Say that the two decimal place-value positions in 0.36 and the one decimal place-value position in 0.6 means three decimal place-value positions in answer, which means the ones are the fourth position from right. However, note that for an example such as 4 × 5 = 20, 0.4 × 0.5 = 0.2 but this really is 0.20 – this can upset the pattern if not taken into account.

Step 6: Use pattern and whole-number methods together

Complete any decimal multiplication using whole-number algorithm methods and then use the pattern to determine the ones. That is, 2.4 × 4.5 is 24 × 45 = 1080 in whole numbers, there are a total of two decimal place-value positions, so the ones position is third from right, so the answer is 10.80 or 10.8.

3.2 Division of decimal numbers

As stated earlier, the problem for division is that sharing requires the number of sharers to be a whole number and grouping requires the number of groups to be a whole number. In this subsection we will look at this situation but spend most time looking at the situation where both factors are decimals.

One factor is a whole number

Division can be an extension of whole-number division if the divisor is a whole number. For instance, algorithms like 7.32 ÷ 3.

Similarly, division can be an extension of whole-number division when the divisor can fit into the dividend, for example 43.2 ÷ 0.9.

Both factors are decimal numbers

Here we look at examples like 0.378 ÷ 0.7. To be able to do this using extension of whole-number techniques, we have to change the example, without changing the answer, to where 0.7 is a whole number. There are several methods for this.

1. **Using reality**. Consider 0.378 ÷ 0.7 as representing a reality situation, say m and mm. Then 0.378 ÷ 0.7 is in m and we change to mm and it becomes 378 ÷ 700. This can be done by whole-number methods. (*Note*: this method works best when translation to reality is simple. For example 0.21 ÷ 0.07, changing it from m to cm changes the example to 21 ÷ 7 – easy to solve.)

2. **Using the number-size compensation principle**. In this method, 0.7 changes to 7 by ×10, so to compensate change 0.378 the same way, that is, 0.378 ÷ 0.7 = 3.78 ÷ 7, and this can be solved by whole-number methods.

3. **Using fraction equivalence**. The example 0.378 ÷ 0.7 is the same as fraction 0.378/0.7, multiplying top and bottom by 10; this is equivalent to fraction 3.78/7 which is the same as 3.78 ÷ 7.

4. **Using calculator patterns**. We can do the same four steps for decimal division as we did for multiplication (use calculators to relate 0.345 ÷ 0.5 to 345 ÷ 5 and find the pattern, then don't use calculators to practise the pattern, describe the pattern, and use the pattern). This will give the pattern that for decimal division 0.345 ÷ 0.5, we do the whole-number division and subtract the decimal place-value positions to find the ones in the whole-number answer. That is, 345 ÷ 5 is 69, it is three decimal places subtract one decimal place = two decimal places, so ones are three from right – so decimal division answer is 0.69.

3.3 Multiplication of common fractions

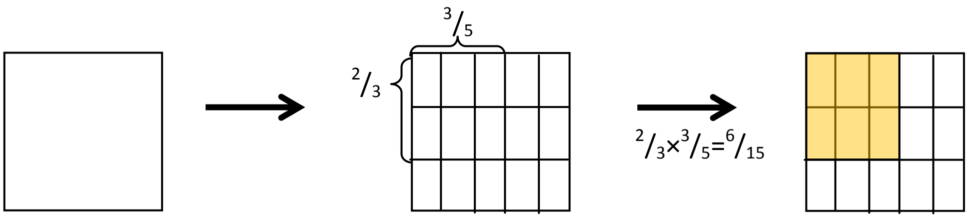
Multiplication of fractions is fairly straightforward if using the area model.

Multiplication of normal fractions

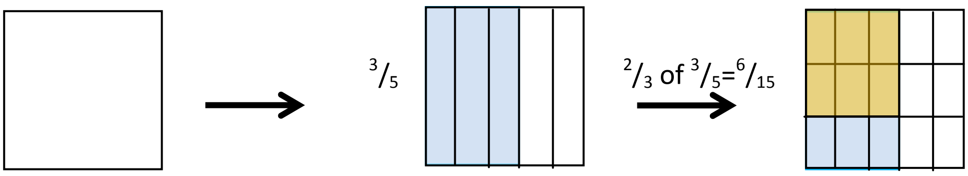
Constructing the multiple

There are two ways of doing this – similar to each other.

Way 1: Area in a square unit. For this method, we think of $^{2}/_{3} \times {}^{3}/_{5}$ as the area of $^{2}/_{3}$ unit × $^{3}/_{5}$ unit. We begin by drawing the square unit (must be square) and then constructing $^{2}/_{3}$ on one side and $^{3}/_{5}$ on the other. The rectangle $^{2}/_{3}$ by $^{3}/_{5}$ can be calculated as a fraction of one square unit. Similarly for $^{3}/_{4} \times {}^{2}/_{7}$.



Way 2: Part of part. Multiplying fractions involves finding a part of a part. In this understanding, it is helpful to think of multiplication in relation to fractions using "of" instead of "by". Thus, for the example $^{2}/_{3} \times {}^{3}/_{5}$, we first construct $^{3}/_{5}$ from a rectangle, then we find $^{2}/_{3}$ of this $^{3}/_{5}$, and calculate this as a fraction. Similarly for $^{3}/_{4} \times {}^{2}/_{7}$.



Finding the pattern

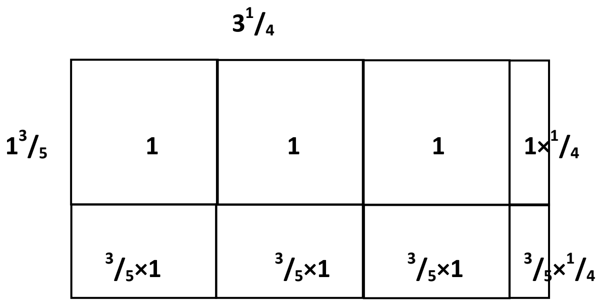
If we use either way above, we can look for a pattern. Using the example $\frac{3}{8} \times \frac{2}{7} = \frac{(3\times2)}{(8\times7)}$ it is fairly obvious that the numerators and denominators are multiplied. This pattern can then be used.

Multiplication of mixed numbers

Mixed numbers can be multiplied in two ways. For example, $1^{3}/_{5} \times 3^{1}/_{4}$:

Way 1: By changing mixed numbers to improper fractions and multiplying with pattern $1^{3}/_{5} \times 3^{1}/_{4} = {}^{8}/_{5} \times {}^{13}/_{4} = {}^{(8\times13)}/_{(5\times4)} = {}^{104}/_{20} = 5^{4}/_{20}$ or $5^{1}/_{5}$; or

Way 2: By constructing a rectangle of length numerator and width denominator (or vice versa) and calculating the area.



Notes: In these constructions, whether for mixed numbers or proper fractions, the following hold:

- a representation of the whole needs to be identified and then the fractions can be represented along each dimension and the multiplication can be modelled;
- using this model to represent the process works best with smaller fractions that are able to be shown this way;
- once the reasons why the methods work are established, then the generalised form (pattern) of the operation can be described and used with any examples; and
- it is always useful to place the problem in a real setting (e.g. how many square plaster boards are needed to cover an area 1.6 m by 3.25 m, that is $1^3/_5$ m by $3^1/_4$ m?).

Of course, we need not develop these pen-and-paper methods. We can use a calculator to simply divide numerators by denominators to change fractions to decimals and then multiply the decimals with the calculator.

3.4 Division of common fractions

It is in division of common fractions that the grouping and sharing meaning of fraction finally becomes unworkable. We can share $^3/_4$ amongst 6 and we can find how many groups of $^1/_4$ go into $2^1/_2$, but how can we share or group $^2/_3 \div {}^3/_4$? We have to return to the **true mathematical meaning of division which is multiplication by inverse which is the reciprocal**. This subsection looks at methods for doing fraction division and builds towards multiplication by reciprocal by using grouping and sharing methods where one factor is a whole number.

It should be noted that, for the division of fractions, students need a sound understanding of the operation of division as well as knowledge of fractions. Students who have been encouraged to write stories to model different operations should be able to think of real situations where the division of fractions makes sense.

Simpler divisions: Grouping and sharing

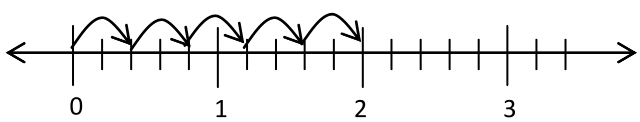
The following methods are available. Each is described and then, if it is applicable, how to use them to build towards reciprocal is discussed.

1. **Using reality**. Any work with fractions is best begun with reality. A division story for whole numbers could be to find how many boxes will be needed to put 100 pencils into boxes of 12 (100 ÷ 12). The same operation just needs a sensible context where fractions are used instead of whole numbers, for example, *How many times will I need to fill a bucket that holds $^1/_8$ of a litre to fill a tank that holds 35½ litres of water?* This is $35^1/_2 \div {}^1/_8$. Using a contextual problem like this helps students to see the connection to using the inverse when dividing fractions. It makes sense to realise that it will take 8 buckets to make 1 L and the number of buckets needed to fill 35½ L can be found by completing the computation 35½ × 8 (which is multiplication by reciprocal).

2. **Using sharing**. A problem could be to share $^3/_4$ cake amongst 6 students. This could be done by cutting a diagram of the cake portion into 6 pieces and seeing that each child receives $^1/_8$ of the whole cake.

   This process can be discussed as finding $^1/_6$ of an amount (here $^3/_4$), thus it is $^1/_6 \times {}^3/_4$. It only works if the divisor is a whole number.

3. **Using models**. A more complex story could be, *How many pieces of ribbon that are $^2/_5$ of a metre long can I cut from 2 m of ribbon?* ($2 \div {}^2/_5$). This problem could be solved logically using repeated subtraction of $^2/_5$ if students can manage this. A number line representation would be helpful – show a number line divided into fifths from 0 to 2 and count how many sections of $^2/_5$ can be made, as shown on number line on right.



If the problem is more complex, say $10^1/_2 \div {}^2/_5$, students can be encouraged to see that five ribbon lengths can be made from 2 m so for every 2 m you will be able to make 5 ribbons (inverse of ${}^2/_5$). So to find how many in $10^1/_2$ m they would divide $10^1/_2$ by 2 (to find how many 2 m lengths there were) and then multiply this by 5 because there are 5 ribbons able to be cut from each 2 m length. So the operation is $10^1/_2 \div 2 \times 5 = {}^{21}/_2 \times {}^5/_2$ which is multiplication by reciprocal.

Complex fractions: Multiplication by reciprocal

Examples like ${}^2/_3 \div {}^3/_4$ require us to **build a new meaning of division** that is the mathematically correct one, multiplication by reciprocal. To do this, go through these steps.

1. **Relook at simple divisions**. Start with $6 \div 2$ for which the answer is 3. Ask if there is a × that will give the same result, or 6 lots of what give 3. Encourage the students to see that $6 \times {}^1/_2 = 3$. Repeat this for other simple examples. Introduce ${}^1/_2$ as reciprocal of 2, and so on.

2. **Practise this new meaning of division with larger whole numbers**. Use a calculator to look at pairs of examples like $91 \div 7$, $91 \times {}^1/_7$; $812 \div 24$ and $812 \times {}^1/_{24}$; $23\,456 \div 128$ and $23\,456 \times {}^1/_{128}$; and so on. Reinforce that division by number is the same as multiplication by reciprocal.

3. **Play the change game for multiplication to show multiplication by fraction less than one gives a smaller answer**. Use the arithmetic excursions activity where students construct paths from one number to another by arrows with operations on top of them; extend these to multiplication only examples where students have to go from a large to a small number.

4. **Find reciprocal of fractions**. The inverse of ×2 is ÷2 because $2 \div 2 = 1$. We also know that $2 \times {}^1/_2 = 1$ and so ${}^1/_2$, the reciprocal of 2, is also the inverse of 2 for multiplication. Now ${}^3/_5 \div {}^3/_5 = 1$ so that $\div{}^3/_5$ is inverse of $\times{}^3/_5$. Similar to the argument above, the reciprocal of ${}^3/_5$ (which is $1/{}^3/_5$) is the inverse of ${}^3/_5$ (${}^3/_5 \times 1/{}^3/_5 = 1$). However, ${}^3/_5 \times {}^5/_3 = {}^{15}/_{15} = 1$. Thus, $1/{}^3/_5 = {}^5/_3$, and so the reciprocal of a fraction is the inversion of the fraction – numerator to denominator and vice versa. This means that the reciprocal of ${}^3/_5$ is ${}^5/_3$, of ${}^5/_7$ is ${}^7/_5$, of ${}^{11}/_{13} = {}^{13}/_{11}$, and so on.

5. **Extend reciprocal to fraction division**. Now we apply the above ideas. For fraction ${}^2/_3 \div {}^3/_4$, we use the proper mathematical form of division which is multiplication by reciprocal and we use the fact that reciprocal is the inversion of the fraction (i.e., the reciprocal of ${}^3/_4$ is ${}^4/_3$). This means ${}^2/_3 \div {}^3/_4 = {}^2/_3 \times {}^4/_3 = {}^8/_9$.

Note: Of course, we need not develop these pen-and-paper methods. We can use a calculator to simply divide numerators by denominators to change fractions to decimals and then divide the decimals with the calculator.

Unit 4: Problems, Estimation and Extension to Algebra

This unit covers some final extensions of fraction operations – to problem solving, estimation and algebra, and to a rich task.

4.1 Problems with fractions

Once again there are a few points to be made.

Fractions in problems

Once again the use of fractional amounts does not change normal word/operation problems. Whether we say "I took 5 cows from the paddock and this left 6 cows, how many in the paddock to start with?" or "I took $^5/_8$ of a tonne of sand from the pile, this left $^1/_4$ tonne, how much sand to start with?", the problem remains the same – what operation do we use, addition or subtraction? Here it is addition because the two parts are known (the part taken away and the part left) and the total (the start) is not known.

Thus the problem-solving aspects of word problems remain the same:

- we use the sameness of the parts and that one number is describing "lots of" to determine whether a problem is multiplication-division or addition-subtraction;
- we use part-part-total and factor-factor-product to determine whether addition-subtraction problems are addition or subtraction, and whether multiplication-division problems are multiplication or division; and
- we use break into parts, act it out/drawing, systematic/exhaustion, given-needed-wanted, and simplify/restate the problem strategies to do multi-step problems and break them down into components where the earlier ideas apply.

Fraction comparison problems

There is one type of problem that does need some extra ideas and that is the use of multiplicative comparison fraction problems; e.g. "if I paid $48 for $^3/_4$ of the package, how much did it cost in total?" These are best understood as size diagrams or change.

Equivalence problems

Like ratio and proportion, some problems rely on equivalence. For example, "Three-quarters of the material was to be taken away, there were 68 items, how many taken away?" This is done by equivalence which says top and bottom are multiplied by the same amount.

4.2 Estimation with fractions

Strategies

No matter the number, the estimation strategies of rounding, straddling, and getting closer still apply. However, front-end is not applicable.

Equivalence

The above strategies rely on being able to easily order fractions – so equivalence will be needed for unlike-denominator fractions. For example, $10\ ^2/_3 \times {}^5/_7 = {}^{32}/_3 \times {}^5/_7$ is easy to round to $^{32}/_3 \times {}^6/_8 = {}^{32}/_8 \times {}^6/_3 = 4 \times 2 = 8$. Is this too low or too high? Equivalent fractions show that $^6/_8 = {}^{42}/_{56}$ and $^5/_7 = {}^{40}/_{56}$, so $^6/_8$ is larger than $^5/_7$, so we are too high.

Benchmarking

For rounding, we need to know whether fractions are less than or greater than half. For example $7^3/_7$ is rounded to 7 for the nearest whole number because $^1/_2$ is $3^1/_2$ out of 7 whereas $^3/_7$ is 3 out of 7, therefore $^3/_7$ is smaller than $^1/_2$.

4.3 Extension of fractions to algebra

Here we will look at fractions as unknowns and algebraic understanding of fraction processes.

Fractions as unknowns

Operations can extend to where there are no numbers given, that is, to algebra. Fractions can be the unknown as well as whole numbers. For example, pizzas were divided up so Frank got a whole one and a quarter and the other 6 people got the same fractional amount of the remainder. There were 5 pizzas. How much did each person other than Frank get? The story can be written as 6PP + $1^1/_4$ = 5. This means that there were 6 part pizzas (PP) plus $1^1/_4$ to give 5. Backtracking, 6PP = 5 – $1^1/_4$ = $3^3/_4$ = $^{15}/_4$ and PP = $^{15}/_4 \div 6 = {}^{15}/_4 \times {}^1/_6 = {}^{15}/_{24} = {}^5/_8$ of a pizza.

Fraction processes and algebraic processes

We saw above that fraction processes can be generalised to algebra. For example:

- inverse or reciprocal: reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$ thus $\frac{b}{a} = \frac{1}{a/_b}$
- mixed number to improper fraction: $A\frac{b}{c} = \frac{(Ac+b)}{c}$
- addition: $\frac{a}{b} + \frac{c}{d} = \frac{(ad+bc)}{bd}$
- subtraction: $\frac{a}{b} - \frac{c}{d} = \frac{(ad-bc)}{bd}$
- multiplication: $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$
- division: $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$

4.4 Fractions and operations rich task

What equations can you write for:

$$6\frac{1}{2}$$

Think of lots of different kinds of equations for $6\frac{1}{2}$

Make lists of equations that are alike

What do you notice?

In this task, students explore equations for $6^1/_2$. There are clearly many possible answers to this problem, for the number of possible equations is limitless. What is interesting is how students tackle the problem, how they organise their thinking, and what kinds of equations they are comfortable listing.

The results can be revealing in a number of ways. Students may not list equations you feel they have had lots of exposure to. Error patters can suggest misunderstandings you may never have suspected. Or students may list equations that are more sophisticated than you expected.

The mathematical ideas are:

- Understanding fractions
- Understanding fraction / decimal relationship
- Using patterns
- Making organised lists.

Presenting the problem

- You may need to discuss what an equation is. Write **_____ = 4** on the board. Each statement you and the students write should equal 4 (for example, 1 + 3 = 4; 6 – 2 = 4).

- Help students recognise that you have an addition and a subtraction equation so addition and subtraction lists can be made.

- Help students to recognise patterns in their lists.

Assessment criteria

The major emphasis of the task is how many different kinds of equations the student can write, and whether they can organise their equations in any way. The organisation may be as simple as four lists – one for each operation. However, within these lists students may reveal patterns (for example 3 + 1 = 4; 2 + 2 = 4; 1 + 3 = 4; 0 + 4 = 4, etc). To a lesser degree, look at the accuracy of the equations.

Ask yourself the following questions:

- Are any error patterns or misconceptions revealed in the equations?

- Were there any creative or unusual equations? Creativity may be using a context to create an equation, for example:

  There was a flower bed which was ½ a metre wide and 1 ½ metres long. Its perimeter was 4 m.

(**Rich Task modified from**: Westley, J. (1994). *Puddle questions: Assessing mathematical thinking (Grade 5).* Creative Publications, California, USA.)

Test Item Types

This section presents instructions and the test item types for the subtests associated with the units. These will form the bases of the pre-test and post-test for this module.

Instructions

Selecting the items and administering the pre-post tests

This section provides an item bank of test item types, constructed around the units in the module. From this bank, items should be selected for the pre-test and post-test; these selected items need to suit the students and may need to be modified, particularly to make post-test items different to pre-test items. The purpose of the tests is to measure students' performance before and after the module is taught. The questions should be selected so that the level of difficulty progresses from easier items to more difficult items. In some modules this will follow the order of the units and subtests, and in other modules it will not, depending on the sequencing across the module and within units. The pre-test items need to allow for students' existing knowledge to be shown but without continual failure, and the post-test items need to cover all the sections in a manner that maximises students' effort to show what they can do.

In administering the pre-test, the students should be told that the test is not related to grades, but is to find out what they know before the topic is taught. They should be told that they are not expected to know the work as they have not been taught it. They should show what they know and, if they cannot do a question, they should skip it, or put "not known" beside questions. They will be taught the work in the next few weeks and will then be able to show what they know. Stress to students that **any pre-test is a series of questions to find out what they know** before the knowledge is taught. They should do their best but the important questions come at the end of the module. For the **post-test**, the students should be told that **this is their opportunity to show how they have improved**.

For all tests, **teachers should continually check to see how the students are going.** Items in later subtests, or more difficult items within a particular subtest, should not be attempted if previous similar items in earlier subtests show strong weaknesses. Students should be allowed to skip that part of the test, or the test should be finished. Students can be marked zero for these parts.

Information on the common and decimal fraction operations item types

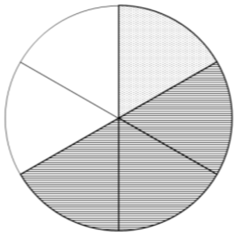
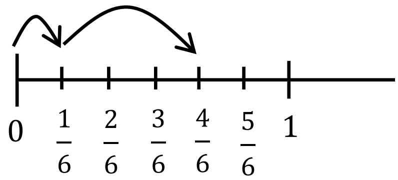
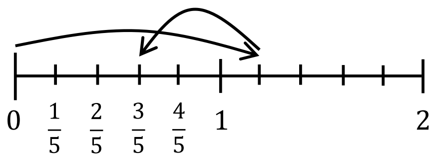
The common and decimal fraction operations item types are divided into four subtests, one for each of the four units in the module. The four units consist of a basic unit, Unit 1 – concepts, models and representations for operations, which underlies the other units; and a final unit, Unit 4 – problems, estimation and extension to algebra, which covers the highest level ideas. In the other two units, Units 3 – addition and subtraction of common and decimal fractions and Unit 4 – multiplication and division of common and decimal fractions, the parts of the units cover different ideas not necessarily higher ideas.

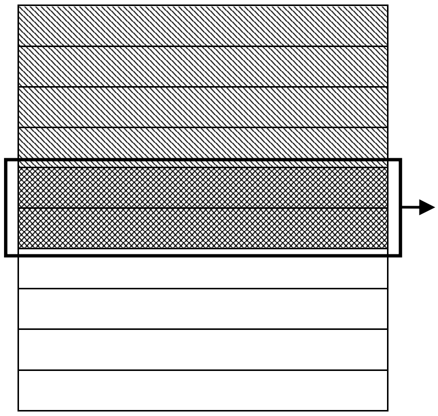
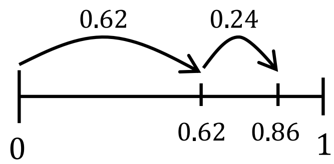
Therefore, the pre-test could be restricted to Subtest 1 and the easier parts of Subtests 2 and 3, while the post-test should contain all of Subtests 2, 3 and 4. It is important to ensure that there are sufficient common items in each of the pre-test and post-test to enable pre-post comparison, and teacher knowledge of their students is important in selecting the pre-test items to ensure the test gives an idea of what students can do but is not continuous failure for the students. Items that teachers know the students cannot do can be considered to score zero in the pre-test.

Subtest item types

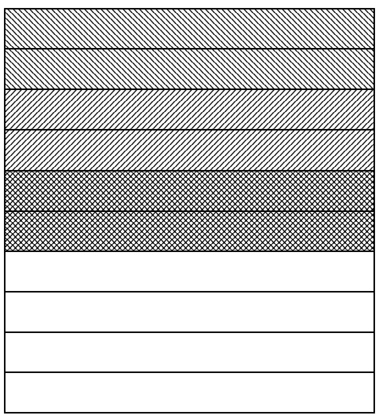
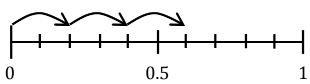
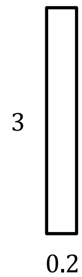
Subtest 1 items (Unit 1: Concepts, models and representations for operations)

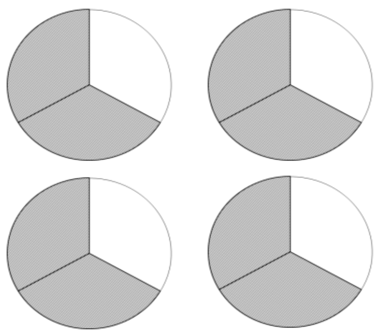
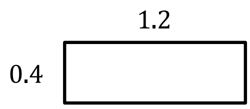
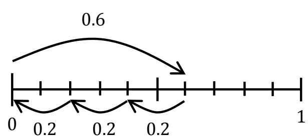
1. Complete the empty squares. The first one has been done for you.

	Operation	Size model	Number-line model
	$\frac{1}{6} + \frac{3}{6}$		
(a)	$\frac{3}{8} + \frac{2}{8}$		
(b)	$1\frac{1}{5} - \frac{2}{5}$		
(c)	$0.4 + 0.3$		
(d)	$0.84 - 0.22$		
(e)			
(f)			

	Operation	Size model	Number-line model
(g)			
(h)			

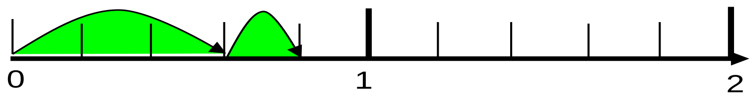
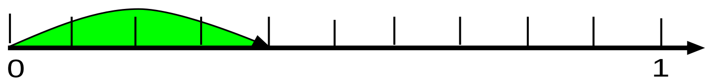
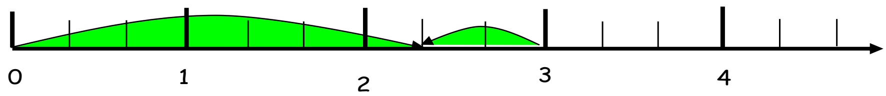
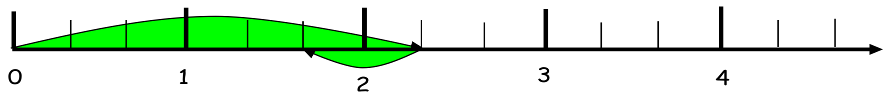
2. Complete the empty squares. The first one has been done for you.

	Operation	Size model	Number-line model	Array/area model
	3×0.2			
(a)	$0.8 \div 4$			

	Operation	Size model	Number-line model	Array/area model
(b)	$4 \times \frac{3}{5}$			
(c)	$\frac{6}{5} \div 3$			
(d)				
(e)		(Not applicable)	(Not applicable)	
(f)				

Subtest 2 items (Unit 2: Addition and subtraction – common and decimal fractions)

1. Circle the picture that matches the operation.

	Operation	Picture
(a)	$\frac{3}{5} + \frac{1}{5}$ means:	 OR 
(b)	$2\frac{1}{3} - \frac{2}{3}$ means:	 OR 

2. Complete the following operations:

(a) $\frac{3}{5} + \frac{1}{5}$ = ______________

(b) $\frac{13}{15} - \frac{11}{15}$ = ______________

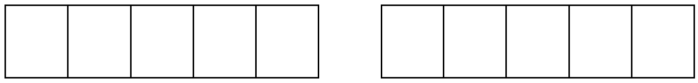
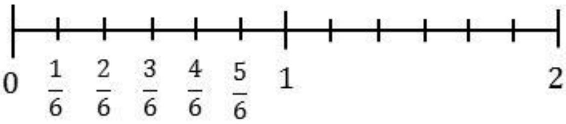
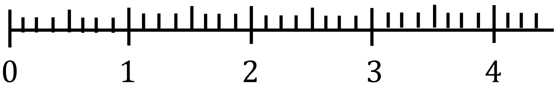
(c) $\frac{2}{3} + \frac{1}{6}$ = ______________

(d) $\frac{3}{4} + \frac{2}{4}$ = ______________

(e) $\frac{11}{16} - \frac{9}{16}$ = ______________

(f) $\frac{2}{5} + \frac{1}{10}$ = ______________

3. Use the models shown to solve the operations.

(a)	$\frac{2}{5} + \frac{4}{5}$ = ________________	
(b)	$\frac{7}{6} - \frac{3}{6}$ = ________________	
(c)	$2\frac{3}{8} + 1\frac{7}{8}$ = ________________	

4. Complete the following operations:

(a) 1.3 + 20.62 = ________________

(b) 3.75 + 0.2 = ________________

(c) 10.7 – 2.5 = ________________

(d) 4.1 – 1.95 = ________________

5. Complete the following using equivalent fractions:

(a) $\frac{1}{2} + \frac{3}{4}$ = ________________

(b) $1\frac{1}{2} - \frac{1}{3}$ = ________________

(c) $\frac{3}{5} + \frac{2}{7}$ = ________________

(d) $1\frac{1}{4} - \frac{5}{6}$ = ________________

Subtest 3 items (Unit 3: Multiplication and division – common and decimal fractions)

1. Circle the language that **matches** the symbols. [There may be more than one answer.]

	Operation language		Symbols
(a)	7 times 3.5 equals 23.5 3.5 times 7 equals 23.5	7 times 35 equals 23.5 7 and 3.5 equals 23.5	$7 \times 3.5 = 23.5$
(b)	8 sixes = 4.8 4.8 divided by 8 equals 0.6	8 times 0.6 equals 4.8 4.8 minus 8 equals 0.6	$4.8 \div 8 = 0.6$
(c)	0.7 multiplied by 9 = 6.3 0.9 × 7 equals 6.3	9 times 7 equals 6.3 6.3 shared amongst 7 equals 0.9	$0.9 \times 7 = 6.3$
(d)	$^1/_2$ of 24 equals 12 24 × $^1/_2$ does not equal 12	$^1/_2$ of 12 equals 24 24 ÷ 12 = $^1/_2$	$^1/_2 \times 24 = 12$

2. Use the given operation to do the second operation.

(a) $7 \times 6 = 42$ 0.7×0.6 = ________________

(b) $3 \times 4 = 12$ 0.03×4 = ________________

(c) $11 \times 5 = 55$ 1.1×0.05 = ________________

(d) $18 \div 3 = 6$ $0.18 \div 3$ = ________________

(e) $36 \div 4 = 9$ $3.6 \div 0.4$ = ________________

(f) $15 \div 5 = 3$ $0.015 \div 0.03$ = ________________

3. Complete the following operations:

(a) 2×0.3 = _______________

(b) 0.2×0.75 = _______________

(c) $3 \div 0.2$ = _______________

(d) $24.6 \div 1.5$ = _______________

(e) $0.62 \div 3.1$ = ________________

(f) 2.8×0.4 = ________________

(g) 1.8×0.03 = ________________

(h) $24 \div 0.06$ = ________________

4. Use the models shown to solve the operations.

(a)	$\frac{3}{4} \times \frac{2}{5}$ = ________________	
(b)	$1\frac{1}{6} \times \frac{2}{3}$ = ________________	

5. Complete the following operations:

(a) $\frac{3}{4} \times \frac{1}{2}$ = ________________

(b) $\frac{5}{9} \div \frac{1}{2}$ = ________________

(c) $\frac{3}{5} \times \frac{1}{2}$ = ________________

(d) $\frac{3}{5} \div \frac{1}{2}$ = ________________

(e) $\frac{2}{3} \times \frac{4}{7}$ = ________________

(f) $\frac{2}{3} \div \frac{4}{7}$ = ________________

(g) $\frac{1}{5} \times \frac{3}{4}$ = ________________

(h) $1\frac{3}{4} \div \frac{4}{7}$ = ________________

Subtest 4 items (Unit 4: Problems, estimation and extension to algebra)

1. Tick (✓) the **operation** that matches the story.

	Story	Operation
(a)	After shopping, Denise had \$98.65 left. If she spent \$62.75, *how much did she have before shopping*?	☐ + \$98.65 = \$62.75 \$98.65 + \$62.75 = ☐ \$98.65 – \$62.75 = ☐
(b)	At Shop A, sneakers are cheaper than at Shop B. The difference in price is \$4.50. Shop B sells them for \$62.85; *what does Shop A sell them for*?	\$62.85 – \$4.50 = ☐ \$62.85 + \$4.50 = ☐
(c)	This week, Mary earned \$25 pocket money; this was \$3.25 less than the week before. *What did she earn last week*?	25 + 3.25 = ☐ 25 – 3.25 = ☐ 25 + ☐ = 3.25

2. Tick (✓) the **question** that makes sense for each story.

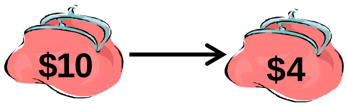
	Story	Question
(a)	John has $^3/_4$ of the marbles that June has. John has 120 marbles.	*Does June have ¼ of the marbles that John has?* *How many marbles does June have?*
(b)	Trevor walked $^3/_4$ of a distance; he walked 12 km.	*Is the whole distance more than 12 km?* *Does he still have to walk another 12 km?*
(c)	7 plates; $2^1/_2$ sandwiches per plate	*How many plates are needed?* *How many sandwiches altogether?*

3. Tick (✓) the **answer** that matches each story.

	Story	Answer
(a)	25 people shared \$200.	*Each share was $^1/_4$ of the \$200.* *Each share was less than $^1/_4$ of the \$200.*
(b)	$^1/_3$ of the 12 eggs were used to make the cake.	*There were more than 12 eggs left.* *There were 8 eggs left.* *There were 9 eggs left.*

4. Circle the story that matches the picture.

Picture



Story

Brigid spent $^4/_5$ of the money.

OR

Brigid spent $^3/_5$ of the money.

5. What is the rule for each of the following operations – write it in your own words.

(a) $\frac{a}{b} + \frac{c}{d}$ ____

____

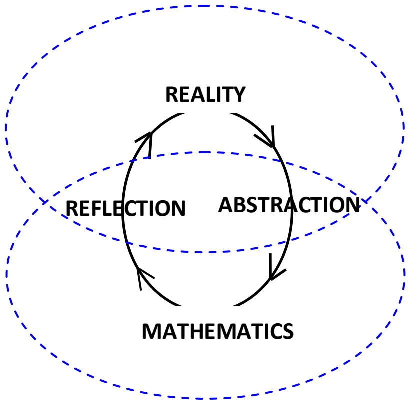
(b) $\frac{a}{b} \times \frac{c}{d}$ ____

____

(c) $\frac{a}{b} \div \frac{c}{d}$ ____

____

Appendix A: RAMR Cycle



AIM advocates using the four components in the figure on right, reality–abstraction–mathematics–reflection (RAMR), as a cycle for planning and teaching mathematics. RAMR proposes: (a) working from reality and local culture (prior experience and everyday kinaesthetic activities); (b) abstracting mathematics ideas from everyday instances to mathematical forms through an active pedagogy (kinaesthetic, physical, virtual, pictorial, language and symbolic representations, i.e. body → hand → mind); (c) consolidating the new ideas as mathematics through practice and connections; and (d) reflecting these ideas back to reality through a focus on applications, problem solving, flexibility, reversing and generalising (see figure on right).

The innovative aspect of RAMR is that the right half develops the mathematics idea while the left half reconnects it to the world and extends it. For example, whole-number place value built around the **pattern of threes** where hundreds-tens-ones applies to ones, thousands, millions, and so on, can be easily extended to metrics by considering the ones to be millimetres, the thousands to be metres and the millions to be kilometres.

Planning the teaching of mathematics is based around the RAMR cycle if it is deconstructed into components that are applied to a mathematical idea. By breaking instruction down into the four parts and taking account of the pedagogical approaches described above, the cycle can lead to a structured instructional sequence for teaching the idea. The table below briefly outlines how this can be done. Prerequisite mathematical ideas are considered in the Reality and Mathematics components of the cycle, while extensions and follow-up ideas are considered in the Reflection component.

REALITY • **Local knowledge**: Identify local student cultural-environmental knowledge and interests that can be used to introduce the idea. • **Prior experience**: Ensure existing knowledge and experience prerequisite to the idea is known. • **Kinaesthetic**: Construct kinaesthetic activities, based on local context, that introduce the idea.
ABSTRACTION • **Representation**: Develop a sequence of representational activities (physical to virtual to pictorial materials to language to symbols) that develop meaning for the mathematical idea. • **Body-hand-mind**: Develop two-way connections between reality, representational activities, and mental models through body → hand → mind activities. • **Creativity**: Allow opportunities to create own representations, including language and symbols.
MATHEMATICS • **Language/symbols**: Enable students to appropriate and understand the formal language and symbols for the mathematical idea. • **Practice**: Facilitate students' practice to become familiar with all aspects of the idea. • **Connections**: Construct activities to connect the idea to other mathematical ideas.
REFLECTION • **Validation**: Facilitate reflection of the new idea in terms of reality to enable students to validate and justify their new knowledge. • **Applications/problems**: Set problems that apply the idea back to reality. • **Extension**: Organise activities so that students can extend the idea (use reflective strategies – *flexibility*, *reversing*, *generalising*, and *changing parameters*).

Appendix B: AIM Scope and Sequence

Yr	Term 1	Term 2	Term 3	Term 4
A	**N1: Whole Number Numeration** Early grouping, big ideas for H-T-O; pattern of threes; extension to large numbers and number system **N2: Decimal Number Numeration** Fraction to decimal; whole number to decimal; big ideas for decimals; tenths, hundredths and thousandths; extension to decimal number system	**O1: Addition and Subtraction for Whole Numbers** Concepts; strategies; basic facts; computation; problem solving; extension to algebra **M1: Basic Measurement (Length, Mass and Capacity)** Attribute; direct and indirect comparison; non-standard units; standard units; applications	**O2: Multiplication and Division for Whole Numbers** Concepts; strategies; basic facts; computation; problem solving; extension to algebra **M2: Relationship Measurement (Perimeter, Area and Volume)** Attribute; direct and indirect comparison; non-standard units; standard units; applications and formulae	**G1: Shape (3D, 2D, Line and Angle)** 3D and 2D shapes; lines, angles, diagonals, rigidity and properties; Pythagoras; teaching approaches **SP1: Tables and Graphs** Different tables and charts; bar, line, circle, stem and leaf, and scatter graphs; use and construction
B	**M3: Extension Measurement (Time, Money, Angle and Temperature)** Attribute; direct and indirect comparison; non-standard units; standard units; applications and formulae **N3: Common Fractions** Concepts and models of common fractions; mixed numbers; equivalent fractions; relationship to percent, ratio and probability	**G2: Euclidean Transformations (Flips, Slides and Turns)** Line-rotation symmetry; flip-slides-turns; tessellations; dissections; congruence; properties and relationships **O3: Common and Decimal Fraction Operations** Addition, subtraction, multiplication and division of common and decimal fractions; models, concepts and computation	**A1: Equivalence and Equations** Definition of equals; equivalence principles; equations; balance rule; solutions for unknowns; changing subject **N4: Percent, Rate and Ratio** Concepts and models for percent, rate and ratio; proportion; applications, models and problems	**SP2: Probability** Definition and language; listing outcomes; likely outcomes; desired outcomes; calculating (fractions); experiments; relation to inference **G3: Coordinates and Graphing** Polar and Cartesian coordinates; line graphs; slope and y-intercept; distance and midpoints; graphical solutions; nonlinear graphs
C	**A2: Patterns and Linear Relationships** Repeating and growing patterns; position rules, visual and table methods; application to linear and nonlinear relations and graphs **N5: Directed Number, Indices and Systems** Concept and operations for negative numbers; concept, patterns and operations for indices; scientific notation and number systems	**A3: Change and Functions** Function machine; input-output tables; arrowmath notation, inverse and backtracking; solutions for unknowns; model for applications to percent, rate and ratio **G4: Projective and Topology** Visualisation; divergent and affine projections; perspective; similarity and trigonometry; topology and networks	**O4: Arithmetic and Algebra Principles** Number-size, field and equivalence principles for arithmetic; application to estimation; extension to algebra; simplification, expansion and factorisation **SP3: Statistical Inference** Gathering and analysing data; mean, mode, median, range and deviation; box and whisker graphs; large data sets, investigations and inferences	**A4: Algebraic Computation** Arithmetic to algebra computation; modelling-solving for unknowns; simultaneous equations, quadratics **O5: Financial Mathematics** Applications of percent, rate and ratio to money; simple and compound interest; best buys; budgeting and planning activities

Key: N = Number; O = Operations; M = Measurement; G = Geometry; SP = Statistics and Probability; A = Algebra.

Accelerated Inclusive Mathematics Project

YuMiDeadly
Growing community through education

Faculty of Education
School of Curriculum
S Block, Room 404
Victoria Park Road
KELVIN GROVE QLD 4059

CRICOS No. 00213J

Phone: +61 7 3138 0035
Fax: +61 7 3138 3985
Email: ydc@qut.edu.au
Website: ydc.qut.edu.au