YuMi Deadly Maths

AIM Module G2
Year B, Term 2

Geometry:
Euclidean Transformations
(Flips, Slides and Turns)

Prepared by the YuMi Deadly Centre
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ACKNOWLEDGEMENT

The YuMi Deadly Centre acknowledges the traditional owners and custodians of the lands in which the mathematics ideas for this resource were developed, refined and presented in professional development sessions.

YUMI DEADLY CENTRE

The YuMi Deadly Centre is a research centre within the Faculty of Education at the Queensland University of Technology which is dedicated to enhancing the learning of Indigenous and non-Indigenous children, young people and adults to improve their opportunities for further education, training and employment, and to equip them for lifelong learning.

“YuMi” is a Torres Strait Islander Creole word meaning “you and me” but is used here with permission from the Torres Strait Islanders’ Regional Education Council to mean working together as a community for the betterment of education for all. “Deadly” is an Aboriginal word used widely across Australia to mean smart in terms of being the best one can be in learning and life.

YuMi Deadly Centre’s motif was developed by Blacklines to depict learning, empowerment, and growth within country/community. The three key elements are the individual (represented by the inner seed), the community (represented by the leaf), and the journey/pathway of learning (represented by the curved line which winds around and up through the leaf). As such, the motif illustrates the YuMi Deadly Centre’s vision: Growing community through education.

The YuMi Deadly Centre (YDC) can be contacted at ydc@qut.edu.au. Its website is http://ydc.qut.edu.au.

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DEVELOPMENT OF THE AIM MODULES

The Accelerated Inclusive Mathematics (AIM) modules were originally developed from 2010 to 2013 as part of the Accelerated Indigenous Mathematics project funded under the Commonwealth Government’s Closing the Gap: Expansion of Intensive Literacy and Numeracy program for Indigenous students. The project aimed to assist secondary schools with beginning junior-secondary Indigenous students who were at Year 2/3 level in mathematics by developing a junior-secondary mathematics program that accelerates the students’ learning to enable access to mathematics subjects in the senior-secondary years and therefore enhance employment and life chances. The project developed three years of modules (Years A to C) that are vertical sequences of learning to take students from their ability level to their age level in mathematics. The YuMi Deadly Centre acknowledges the role of the Department of Education, Employment and Workplace Relations in the development of the AIM project and these modules.

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Module Overview

As discussed in Module G1, human thinking has two aspects: verbal logical and visual spatial. Our senses and the world around us have enabled both these forms of thinking to evolve and develop. Because it is a product of human thinking that has emerged from solving problems in the world around us, mathematics has, historically and presently, two aspects at the basis of its structure: number and geometry. This resource looks at the teaching of the second of these bases, geometry. It is the second module on this strand and follows on from Module G1 Shape.

Background information for teaching Euclidean transformations

This section looks at background information – general framework, relationship vs change, and big ideas.

General framework for geometry

Geometry can be one of the most exciting and interesting sections of mathematics. It provides an opportunity for motivating learners that should not be missed. It can be colourful and attractive. Pattern and shape can be created and admired. Success can be enjoyed by the majority. However, some approaches make geometry teaching more effective:

1. The focus of geometry should be from and to the everyday world of the learner, and there should be a balance between experiences which enable learners to interpret their geometric world, and processes where problems are solved with visual imagery.

2. Learners’ activities should be multisensory (using actual physical materials and moving and transforming them) and structuring (recording results on paper in words and pictures) – the “typical” geometry classroom would have groups, physical materials and pens and books ready to record, and there should also be opportunities for learners to display what they have made.

2. Teaching activities should move through three levels of development (based on van Hiele levels):

   - the **experiential** level, at which learners learn through their own interaction with their environment (shapes are identified and named – e.g. this is a triangle);
   - the **informal/analysis** level, in which certain shapes and concepts are singled out for investigation at an active, non-theoretical level (e.g. triangles have three sides and three angles); and
   - the **formal/synthesis** level, where a systematic study is undertaken and relationships identified (e.g. interior angle sum of triangles is 180 degrees).

At the **experiential level** learners should be allowed to learn through experience with materials, not the teacher’s words. Shape can be labelled and described but not broken into its component parts. Learners should not be expected to be accurate in their statements. At the **informal level**, experiences can include analysing shapes and constructing shapes from their properties. The sub-concept approach discussed later would be appropriate here. At the **formal level** properties such as congruence and similarity can be investigated, and formulae discovered. There should be no attempt at deductive proof and posing abstract systems.

It is also important to **reverse** activities. We tend to make shapes and then explore the diagonals. Why not make diagonals and then see what shapes emerge as we move them, change their length, and their point of intersection? Similarly, give a shape and ask for lines of symmetry – then ask students to draw a shape with a given number, say three, lines of symmetry. Try to allow students to explore change in the two scenarios. It is also important to be **flexible** and to show shapes in non-prototypic ways (e.g. a rectangle at angle to
horizontal) and to look for generalities (e.g. all squares are rectangles, all rectangles are parallelograms, all parallelograms are quadrilaterals).

Note: School geometry appears to be the strand which is not so confronting to non-Western cultures, and to be an area in which nearly all cultures have excelled, particularly with respect to the geometric aspects of art.

Relationship vs change

Similar to other strands, geometry can be seen from the perspective of relationship and the perspective of change or transformation (where every relationship can be reinterpreted as a change and vice versa).

Types of geometries

Change or transformational geometries that can be used in Years P to 9 are based on three types of changes:

(a) topological: change of living things – length and straightness change;
(b) projective: how our eyes see the world – length changes, straightness does not change; and
(c) Euclidean: how the human-made world changes – length and straightness both do not change.

The Australian Curriculum: Mathematics gives precedence to Euclidean over projective and topology. Euclidean transformation geometry is built around three changes, flips (reflections), slides (translations) and turns (rotations). These changes relate to line and rotational symmetry and lead to tessellations and dissections. They also underpin congruence. These ideas are described in more detail in the units in this module and are summarised in Appendix A.

Importantly, because they deal with change, Euclidean transformations are actions like operations and therefore have some of their properties such as inverse and identity. Slides, flips and turns can all be undone (have an inverse) and the action that does not change anything (identity) is “do nothing” or turn 360 degrees. Finally, Euclidean change is obviously related to mental rotation (ability to rotate an object in your mind and to imagine yourself rotating around an object) and this is a powerful process in life, and many jobs and vocations.

This module, Module G2 Euclidean Transformations, comes from the perspective of change. Module G1 Shape has a relationship perspective, as does Module G3 Coordinates and Graphing to follow this module. The fourth geometry module, Module G4 Projective and Topology, also comes from a change perspective.

Teaching materials and techniques

Studying Euclidean transformations is best done actively – by experiencing them. The materials for doing this are first of all the students’ bodies – they can act out flips, slides, turns and symmetries. Then there is tracing paper; different techniques are needed for turns and slides than for flips. Computers and the manipulation of virtual shapes are also very effective methods, particularly for the art applications. Finally, there is a specific material for this topic, the Mira mirror (or simple Mira). Appendix B has activities to familiarise students with the Mira. It is effective with flips and line symmetry and good with art.

Big ideas for Euclidean transformations

1. **Change vs relationship.** Mathematics has three components – objects, relationships between objects, and changes/transformations between objects. Everything can be seen as a change (e.g. similar shapes are formed by “blowing one up” using a projector) or as a relationship (similar shapes have angles the same and sides in proportion or equivalent ratio). Geometry can be studied as change or relationship.

2. **Interpretation vs construction.** Things can either be interpreted (e.g. what are the line and angle properties for this shape) or constructed (construct a shape of 4 sides with 2 sides parallel). This is particularly true of geometry – shapes can be interpreted or constructed.
3. **Parts vs wholes.** Parts can be combined to make wholes and wholes can be partitioned to form parts (e.g. making or dividing a shape from or into a collection of smaller shapes). In geometry, this big idea is particularly applicable to dissections and tessellations.

4. **Identity and inverse.** Identity refers to actions that leave things unchanged, for example, 0 and 1 do not change things for operations +/− and ×/÷ respectively (e.g. 4+0=4, 26÷1=26). In geometry, a 360 degree turn does not change things for flips, slides, turns (as does “do nothing”). Inverse refers to actions that undo other actions, for example, + is inverse of −, × is inverse of ÷ for operations (e.g. +2 and −2; ×3 and +3). For geometry, 90 degree turns are inverse of 270 degree turns, and a flip is the inverse of itself.

5. **Angle properties.** This is not as important in Euclidean transformations as it is in general shape. But interior angles are important for determining whether shapes tessellate.

6. **Transformational invariance.** Topological transformations change straightness and length, projective transformations change length but not straightness, and Euclidean transformations change neither. *(Note: affine projections leave parallelness unchanged, while similarity projections leave parallelness unchanged and sides in ratio.)* This gives implications for shape, angle and what does not change (invariance) for each of the types of transformations.

7. **Line symmetry–reflection relationships.** Number of lines of symmetry equals number of rotations of symmetry when there are two or more lines of symmetry, with the angle between lines half the angle between rotations; two reflections through the centre equal one rotation where the angle between reflections is half the angle of rotation; two reflections equal one translation where the reflection lines are perpendicular to the translations and the distance between lines is half the distance of the translation.

### Sequencing for Euclidean transformations

This section looks at sequencing in geometry generally and then sequencing in this particular module.

**Sequencing in geometry**

By its very nature, geometry does not have the dominating sequential nature of arithmetic and much more teacher choice is available in determining appropriate teaching sequences. There are also many experiences in geometry not directly connected to the development of rules and general procedures but rather to the development of imagery and intuition and as such they may not be recognised as important by teachers.

Thus, YuMi Deadly Maths (YDM) has developed a sequence for geometry (see figure on next page) that is based on, but enriches, the geometry in the *Australian Curriculum: Mathematics*. YDM recommends that the study of geometry has:

(a) two overall approaches: relationship geometry and transformational (or change) geometry;

(b) two sections in relationship geometry: Shape (2D and 3D shape, line, angle and Pythagoras’s theorem) and Coordinates and Graphing (polar and Cartesian coordinates, line graphs, slope and y-intercept, and graphical solutions to unknowns); and

(c) two sections in transformational geometry: Euclidean Transformations (flips, slides, turns and congruence) and Projective and Topology (projections, similarity, trigonometry, perspective and networks).

This gives four sections to the sequence and these are the four geometry modules in AIM: G1 *Shape*, G2 *Euclidean Transformations*, G3 *Coordinates and Graphing*, and G4 *Projective and Topology*. These are listed in the AIM scope and sequence in **Appendix D**.
Sequencing in this module

Module G2 covers the changes that occur in the built environment when it operates without breakages, loss and other problems. That is, it covers the three changes that do not affect the size and shape of solid and plane shapes: flip (reflection), slide (translation) and turn (rotation).

With regard to pictures (and plane shapes), flips relate to and are the defining concept behind line symmetry while turns perform the same function for rotational symmetry. Lastly, flips, slides and turns have application in two other geometric ideas important to the built environment: (a) tessellations (e.g. tiling patterns) that provide the framework for much of the building in the built environment as they focus on how a shape can repeat itself to cover space without gaps and overlaps (like bricks in a wall); and (b) dissections (e.g. jigsaw puzzles) that provide the framework for placing much of the built environment together as they focus on how smaller shapes are joined to make a larger shape.

Flips, slides and turns, symmetry, tessellations and dissections are not the traditional geometry of the primary school, although parts are well known, but they are a collection of ideas that are full of interesting puzzles and activities. They also build visual thinking and visual imagery and relate to art. In terms of sequencing, the units at the start of the module (Units 1, 2, 3 and 4) focus on different topics each at the same sequential level but are sequenced in relation to the order in which the topics are considered in detail. The units at the end of the module (Units 5 and 6) cover higher level mathematics.

The sections and units for this module are therefore as follows:

**Overview:** Background information, sequencing and relation to Australian Curriculum

**Unit 1:** Flips, slides and turns – concepts, constructions and early work on properties and relationships

**Unit 2:** Line and rotational symmetry – definitions, constructions, interpretation and early work on modifications, applications and relationships

**Unit 3:** Tessellations – constructing tiling patterns, developing angle rules, using grids and early work on their application to solids, art and shape puzzles
Unit 4: Dissections – construction and solution to simple and complex dissections (jigsaws and puzzles) and their relation to visual imagery

Unit 5: Properties and puzzles – extension of Units 1 to 4 into properties and puzzles, exploring relationships, and visual and artistic applications

Unit 6: Congruence and relationships – developing congruence as a change through flips, slides and turns only; integration across units, exploration of relationships and consolidation of visual imagery and mental rotation

Test item types: Test items associated with the six units above which can be used for pre- and post-tests

Appendix A: Euclidean transformation ideas

Appendix B: Introductory activities for Mira

Appendix C: RAMR cycle components and description

Appendix D: AIM scope and sequence showing all modules by year level and term.

Like Module G1, this module focuses on providing rich experiences in the classroom by describing many teaching ideas. Very few are in the RAMR form, showing all parts of the cycle. Teachers are encouraged to translate the ideas into RAMR lessons. For ease of presentation, these ideas may be given under headings that refer to materials being used – body, tracing paper, Mira and computer. The RAMR cycle is in Appendix C.
## Relation to Australian Curriculum: Mathematics

<table>
<thead>
<tr>
<th>AIM G2 meets the Australian Curriculum: Mathematics (Foundation to Year 10)</th>
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<tbody>
<tr>
<td>Unit 1: Flips, slides and turns</td>
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<td>Unit 2: Line and rotational symmetry</td>
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<td>Unit 3: Tessellations</td>
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<td>Unit 4: Dissections</td>
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<td>Unit 5: Properties and puzzles</td>
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<td>Unit 6: Congruence and relationships</td>
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### Unit 1: Flips, slides and turns

#### Investigate the effect of one-step slides and flips with and without digital technologies

**Content Description:**

Investigate the effect of one-step slides and flips with and without digital technologies (ACMMG045)

<table>
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#### Identify symmetry in the environment

**Content Description:**

Identify symmetry in the environment (ACMMG066)

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### Unit 2: Line and rotational symmetry

#### Create symmetrical patterns, pictures and shapes with and without digital technologies

**Content Description:**

Create symmetrical patterns, pictures and shapes with and without digital technologies (ACMMG091)

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#### Describe translations, reflections and rotations of two-dimensional shapes. Identify line and rotational symmetries

**Content Description:**

Describe translations, reflections and rotations of two-dimensional shapes. Identify line and rotational symmetries (ACMMG114)

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#### Apply the enlargement transformation to familiar two-dimensional shapes and explore the properties of the resulting image compared with the original

**Content Description:**

Apply the enlargement transformation to familiar two-dimensional shapes and explore the properties of the resulting image compared with the original (ACMMG115)

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### Unit 3: Tessellations

#### Investigate combinations of translations, reflections and rotations, with and without the use of digital technologies

**Content Description:**

Investigate combinations of translations, reflections and rotations, with and without the use of digital technologies (ACMMG142)

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### Unit 4: Dissections

#### Draw different views of prisms and solids formed from combinations of prisms

**Content Description:**

Draw different views of prisms and solids formed from combinations of prisms (ACMMG161)

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### Unit 5: Properties and puzzles

#### Describe translations, reflections in an axis, and rotations of multiples of 90° on the Cartesian plane using coordinates. Identify line and rotational symmetries

**Content Description:**

Describe translations, reflections in an axis, and rotations of multiples of 90° on the Cartesian plane using coordinates. Identify line and rotational symmetries (ACMMG181)

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<th>Year</th>
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### Unit 6: Congruence and relationships

#### Define congruence of plane shapes using transformations

**Content Description:**

Define congruence of plane shapes using transformations (ACMMG200)

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YuMi Deadly Maths AIM
Unit 1: Flips, Slides and Turns

Flip, slide and turn are the everyday names for three mathematical changes in the diagrams below: (a) FLIP – reflection about a line, (b) SLIDE – translation in a direction for a certain distance, and (c) TURN – rotation around a centre point.

These basic ideas are explored in this unit by using body and hand (tracing paper) activities, and at the end, using the Mira. As well, early work on art, properties, and relationships is explored.

*Note:* One of the names, “flip”, can lead to misunderstanding in that the everyday meaning of the name (e.g. “flipping a car”) is not really a reflection but a turn or rotation in the vertical plane. This has to be clarified during teaching.

### 1.1 Concepts and constructions

This section looks at the meanings of flip, slide and turn, and how to construct and experience them.

**Meanings of flips, slides and turns (body)**

**Slides** are relatively straightforward. Get the students to strike a pose (should not be symmetrical), then walk in a direction (should not be directly forward – walk diagonally or backward or sideways) without turning their body or changing the pose. Students should then repeat this with other objects. Finally, draw an arrow and have objects moved along the direction and length of this arrow without changing orientation.

**Turns** are not quite as easy. At the beginning, they are straightforward – strike a non-symmetric pose and rotate body, and then rotate objects and toys similarly. The problem comes with rotating about a centre, when you are not exactly on top of the centre point.

One way to get this across is find the centre point to be rotated around and attach some rope. Have a student strike a pose about a metre away and attach the other end of the rope to the student. They should then move around the centre point, keeping the same distance away from the point. The trick is that orientation must be kept toward the centre so that the pose turns as the students move about the centre point. This concept of orientation toward the centre can be hard to grasp so this can be built by having a student pose with one hand pointing at the centre (along the rope) and they walk around the point, keeping their body fixed in the pose and their arm pointing at the centre, taking note of how their body turns as they turn about the circle.

**Flips** could be the most difficult. Reflection can be practised by having two students sitting and/or standing facing each other. In turn, one of the students strikes poses (non-symmetric) or moves around and the other student acts as a mirror. However, getting a student to experience the act of reflection on their own requires them to strike a non-symmetric pose, walk towards an imaginary mirror (e.g. a line on the ground), and to imagine passing through the mirror to become the reflection. This would require the student turning around and facing the other way, and changing the pose so left becomes right and right becomes left. However, it is the important way for learning because, like the second turn way, it **requires the student to be active in achieving the change**. Again, students should repeat these activities with objects and toys.
Any of these could be done by having students lie on the ground in a pose, with other students looking over the top and have a second student perform a flip, slide or turn in relation to the first student. As a game you could have different actions in a hat and they select one to perform.

![Diagram](image)

**Construction of flips, slides and turns (hand – tracing paper)**

Small-sized copies of possible worksheets and instructions are below.

**FLIP**

To construct a flip:

(a) Draw a shape onto a page, with the reflection line as shown at the left.

(b) Line the edge of a piece of tracing paper along the edge of the reflection line and copy the shape.

(c) Flip the edge of the tracing paper over the line.

(d) Trace the shape (now upside down) onto the page.

**SLIDE**

To construct a slide:

(a) Draw a shape on a page with an arrow, as shown at left.

(b) Copy the shape onto tracing paper and place a dot at start of arrow.

(c) Holding original page fixed, move tracing paper so that dot slides along arrow to end without turning the tracing paper.

(d) Press hard and copy shape back onto original page where it is placed right now.

**TURN**

To construct a turn:

(a) Draw a shape on a page, with a “centre dot” close by.

(b) Copy shape and centre dot onto tracing paper.

(c) Put a pencil on the dot, and rotate the tracing paper the required amount of turn.

(d) Copy the shape on to the page where the tracing paper is.
1.2 Exploring early art, properties and relationships

This early work is experiential and does not lead to more formal analysis of properties and synthesis of properties and principles.

Exploring art (hand – tracing paper)

A design can be copied onto tracing paper and then the tracings used to reflect the design and turn the design to make a larger design, as below, which is better than the original. The reflections/rotations shown in the tables are symmetry designs and on the right are frieze patterns.

The “design” is the starting design and the “reflect” or “rotate” are in reference to this original design.

<table>
<thead>
<tr>
<th>design</th>
<th>reflect</th>
<th>design</th>
<th>rotate</th>
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</thead>
<tbody>
<tr>
<td>reflect</td>
<td>reflect</td>
<td>rotate</td>
<td>rotate</td>
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</table>

Exploring properties (body)

Determining the properties of the three changes can be helped by body movements or moving objects and toys. For instance, as they are acting out the changes students could be directed towards seeing that:

- for a slide, all parts of the body/object/toy move parallel the same distance and direction;
- for a turn, all parts of the body/object/toy turn but that parts close to the centre stay close to the centre and vice versa;
- for a flip, left and right interchange, with all parts near the flip line staying near the flip line and vice versa.

Note: A good way to experience turns is to put students in a cart and wheel them around in a circle (constructed by a rope to a peg).

Exploring relationships (body)

It is possible to act out the relationships that two flips make a slide or two flips make a turn. To do this, for the slide, get two students to strike the same pose (one in front of the other) and the front one to complete a slide (resulting in the slide being depicted by a start student and a finish student). Then get a third student to copy the pose, stand in front of the start and do two flips in the direction of the slide. The second flip will end with this student in exactly the same pose as the finish student for the slide so, if flip lines are the right distance apart (half the slide distance), it shows that two flips is the same as one slide.

This process can be repeated for the turn but here the flip lines will go through the centre and be half the angle of the turn. Both processes can be repeated with objects/toys.
Again, you could play this as a game where they pull out a card from a hat with either flip slide or turn that they have to do in sequential order; students who are not part of the ‘game’ can be analysing what is happening.

1.3 Using Mira mirrors for flips

The Mira is excellent for flips and reflections – it will not do slides and turns.

1. Complete the introductory activity with the Mira in Appendix B.

2. Complete the activities with the Mira below. The Mira is placed on the flip line (with bevelled edge down and towards the student) and looking from the shape through the Mira, the student can draw the flipped shape in dotted line form as a copy of the original shape on the other side of the Mira. Small copies of worksheets are below.
Unit 2: Line and Rotational Symmetry

Line symmetry is where a shape can be folded in half and have both sides match (see top figure on right) and rotational symmetry is where a copy of a shape can be turned on top of itself and match in a part turn (see bottom figure on right).

This unit looks at line and rotational symmetry in terms of activities, concepts, constructions and modifications. It also discusses properties and relationships but work on these should be left to Units 5 and 6.

2.1 Activities with line and rotational symmetry

The following is a list of things possible with symmetry. Most of these are covered within this module but it is useful to know the extent of what is possible.

1. **Concepts.** Defining line and rotational symmetries, determining if shapes have these symmetries and the number of them they have. *(Note: If a shape only matches at 360° it has no rotational symmetry but, if it matches at a part turn, the 360° is counted – thus rotational symmetries go from 0 to 2; it is not possible to have one rotational symmetry)*.

2. **Constructions.** These are the reverse of the above – instead of determining the number of line and rotational symmetries, students are required to construct shapes with given symmetries.

3. **Modifications.** Modifying symmetries is halfway between 1 and 2. Modification is the knowledge to change the number of lines and rotations of symmetry of a shape by simple changes to the shape. Activities investigate what changes in shapes cause changes in symmetry.

4. **Art.** This is using symmetries for art; however, this is the same as the art generated by flips, slides and turns so it will not be repeated here (the symmetric designs and frieze patterns).

5. **Classification properties.** This is using numbers of lines and rotations of symmetry to define shapes; for example, with regard to symmetry, a square has four lines and four rotations, a rectangle has two lines (not diagonals) and two rotations, a rhombus has two lines (diagonals) and two rotations, a parallelogram has no lines and two rotations, an isosceles trapezium has one line and no rotations, an equilateral triangle has three lines and three rotations, and an isosceles triangle has one line and no rotations.

6. **Relationships.** If a shape has more than two lines of symmetry, its number of line symmetries equal its number of rotations of symmetry, and the angle between consecutive lines is half the angle between consecutive turns.

2.2 Concepts

There are two ways to introduce symmetry, with body and with hand (tracing paper).

**Acting out symmetry (body)**

Line symmetric poses are easier. Students stand and adopt poses – other students indicate if line symmetrical. Students then deliberately strike line symmetrical poses. Look at pictures and sort by whether line symmetrical. Using a mirror to see that it looks the same for both directions and photographs that can be printed and folded is also effective.
Rotational symmetrical poses are more difficult – a group of students on the floor with the same pose radiating out from a centre could be the easiest way to use body to construct one.

**Determining symmetry (hand – tracing paper)**

Using tracing paper is a powerful way to test for line and rotational symmetry as the traced shape can be folded in half to test for line symmetry and can be rotated on top of the original shape to test for rotational symmetry easily.

The four symmetry cards at the end of this module are work cards that show the progression of the concept from: identifying symmetries → counting symmetries → investigating symmetry change → relating symmetries.

(a) Complete the **first card** – it gives experience of determining line and rotational symmetry.

(b) Complete the **second card** – it gives experience of determining the number of lines and rotations of symmetry.

**Determining and completing line symmetry (Mira)**

*Drawing in lines of symmetry*

Student places Mira on shapes/designs below and finds all lines of symmetry. If bevelled edge is facing student and is down, line can be drawn along this edge (and below the reflection plane of the Mira) to give line of symmetry.

*Completing line symmetric figures*

Student places Mira along the dotted lines, then draws the other half of the figures. If the result makes sense, the resulting figure is line symmetric. Does the result always make sense? Why or why not?

**2.3 Constructing symmetries**

We will look at how to construct figures with required lines and rotations of symmetry. This is a reversing of the shape → symmetries teaching process to symmetries → shape activities. It is important and enhances learning. Some examples of how to do this are as follows.
1. **One line, two rotations.** Draw a dotted line and a design on one side of the line as shown on right.

   For line symmetry, copy the design and the dotted line onto tracing paper, fold paper and retrace, and the result is a shape with one line of symmetry on the tracing paper.

   For rotational symmetry, copy the design and the line onto tracing paper as before but, this time, turn the tracing paper on top of the design and dotted line so that the dotted lines exactly match/cover each other, retrace the shape, and the result is a shape with two rotations of symmetry.

2. **Three rotations.** Construct three dotted lines at 120° to each other and draw a design in one of the three sections as on the right. Copy dotted lines and design on tracing paper, rotate 120° until dotted lines are aligned exactly and recopy design, and rotate a further 120° until dotted lines are aligned exactly again and copy design a third time. The result is a shape with three rotations of symmetry.

3. **Two lines, three rotations.** Repeat step 1 above but for the situation where there are two dotted lines crossing each other. For two lines of symmetry, copy design and dotted lines and then reflect four times. For four rotations, copy design and dotted lines and rotate four times. *(Note: If for rotation the lines of the designs do not meet, join them along the dotted lines.)*

4. **More lines, rotations.** Students can continue in this way, adding lines, where students should generalise that:

   (a) dotted lines which go from a centre outward in 2, 4, 6, 8, and so on, directions give shapes with half the number of lines of symmetry and equal the number of rotations of symmetry, and

   (b) dotted lines that go from a centre outward 3, 5, 7, and so on, directions give shapes with equal number of rotations of symmetry, but no lines of symmetry.

### 2.4 Modifying symmetries

As well as constructing symmetries, learning how to change symmetries of shapes/designs with simple additions or deletions enhances understanding of symmetries. Some examples are as follows:

   (a) Complete the third card of the four symmetry cards at the end of the unit. What do you notice about shape A and C and G and I?

   (b) Find the symmetries of the shape on right [3 lines 3 rotations] and change to a design with 1 line 0 rotations by adding one line. How would you do this with a deletion of a line segment or part of a line segment?

   (c) Find the symmetries of the design on right [4 lines 4 rotations] and change to a design with 1 line 0 rotations by deleting one line segment, then change to a design with 2 lines 2 rotations with the deletion of a further line segment. How would you do this by adding lines?

*Note:* The final card in the sequence of symmetry cards (card 4) at the end of this unit is one to find the relation between number of lines and number of rotations of symmetry. This is left to Unit 5. However, the card remains in the unit because the sequence of four cards is so excellent in terms of coverage of symmetry ideas.
SYMMETRY CARDS (Card 1)

Which designs match themselves by folding along a line?

Which designs match by turning a tracing part of a full turn?

Which designs do not match in either way?

Note: The language is important here and needs to be developed – particularly the words “match”, “not match”, “folding along a line”, “part turn”, “full turn” and “either way”.
SYMMETRY CARDS (Card 2)

Which designs have line symmetry?

Which designs have rotational symmetry?

How many lines of symmetry do they have?

How many rotations of symmetry do they have?

Note: The lines of symmetry are the number of different ways a shape can be folded in half. The number of rotations is the number of different part-turn matches plus the 360° turn. The 360° turn is counted if there is a match on a part turn but not counted when there is no part-turn match – so the number of rotations goes from 0 to 2 – there is never 1 rotation of symmetry.
Which shapes can be turned part-way around and look just as they did at the start?

Which shapes can be folded so their parts match?

Do any of the shapes do both?

What is similar about the shapes A and C?

What is different about the shapes A and C?

What do you notice about shapes G and I?

Note: The key questions here are “what is alike about shapes A and C?” and “what is different?” These point towards an important symmetry activity which is understanding how small changes in a shape can affect symmetries.
SYMMETRY CARDS (Card 4)

On each design, find all the lines of symmetry and, without testing it, make your best judgement whether the design has ‘turning symmetry’.

Now test the turning symmetry for each design.

What do you notice about the number of lines of symmetry a design has and the number of part-turns it can make?

Note: This activity here focuses on the relationship between number of lines and number of rotations of symmetry and the angles between rotations and lines of symmetry.
Unit 3: Tessellations

We exist in a society that puts shapes together to build and cover and that packs shapes together to carry them around. With this in mind, it is important to investigate shapes that fit and pack well. Shapes that fit together without gaps or overlaps are called **tessellations**. Squares and rectangles tessellate, as shown in the figures on right.

Circles do not tessellate, as shown in the figures on right. When attempting to pack circles, there will always be either a gap or an overlap.

However, circles can be the starting point for shapes that do tessellate. For example, the shapes tessellate in the pattern at right. Here circles have had curved pieces taken out of opposite sides so that they fit together. As well, special shapes can be made that fit together with the circles so that both shapes together tessellate.

This unit looks at concepts and activities for tessellations, applications of tessellation to artwork and design, and explores how grids of tessellating shapes can develop larger tessellations and puzzles.

3.1 Concept and activities with tessellations

Tessellations are useful in developing **spatial visualisation** – the ability to mentally manipulate (flip, slide and turn) shapes. Tessellations also help develop an informal knowledge of the interior angles of polygons.

**Tessellation activities**

The following is a list of tessellation activities. All are somewhere in this module – this unit or Units 5 and 6. In this module, tessellations cover the following:

(a) **concepts and constructions** – what is a tessellation and how to construct them;

(b) **tessellations with more than one shape** – what sets of shapes naturally tessellate and how can we make something to add to other shapes that makes a tessellation;

(c) **art** – how tessellations can be used for art;

(d) **grids-puzzles** – how tessellating shapes can form grids and puzzles (e.g. pentominoes);

(e) **properties** – these range from general properties such as a tessellation pattern has to show evidence/convince the observer that it is a pattern that is ongoing in all directions, to more formal properties such as single shapes and groups of shapes only tessellate if their interior angles are factors of a complete turn (360 degrees) either on their own, in combination or in addition; and

(f) **solid tessellations** – 3D shapes that tessellate.

The idea is to build a concept of tessellations through constructing them. The sequence is as follows:

(a) one-shape tessellations – which shapes tessellate and which do not and why;

(b) two or more shape tessellations – which combinations tessellate and why and what non-tessellating shapes (as single shapes) can now tessellate with another shape; and

(c) building shapes from combining tessellating shapes and seeing which of these tessellate.
**Examples and activities**

The following shapes tessellate by themselves:

Some shapes do not tessellate by themselves, but some tessellate with another shape. Even for the circles, a shape can be found to tessellate with it. And circular pieces can be taken from and added to other shapes so they tessellate.

For example:

(a) **construct a tessellation for shape A**;
(b) **construct a second tessellation for shape B**; and
(c) **make another tessellating shape out of semicircles and rectangles**.

Some other examples:

(a) **investigate and find a shape that will tessellate with a circle and draw the tessellation**;
(b) **draw a double tessellation when one shape is a semi-circle**; and
(c) **can you think of any more double tessellations that include a circle or part of a circle**?

### 3.2 Art and design

Tessellating shapes are the bases of three art and design forms – Frieze patterns, Escher-type art and fabric design.

**Frieze patterns**

A simple design is constructed then flipped and turned to make an edging to a window or a strip of fabric as below:
Escher-type art

This is based on doing something to one side of a tessellating shape and undoing it to the other. This can be done by cutting a tessellating shape out of cardboard and then cutting a piece from one side and then adding it to the other with sticky tape. After one or more attempts at this, the resulting changed shape is copied on to two colours of paper and turned into a pattern of shapes as in examples below. Note the use of flipping, sliding and turning. Note also that these lead to the actual art of Escher—look this up on the Internet and learn how it is done.

(a) changing shapes by sliding (i.e. translating):

![Diagram showing the process of changing shapes by sliding](image)

Start with a square → Make a change in one side → Translate the change → The finished shape has the same area as the original shape. Why?

(b) changing shapes by rotation – note that if you use an isosceles triangle, then you will have to rotate and flip every alternate shape in order to make the shape tessellate:

![Diagram showing the process of changing shapes by rotation](image)

Start with an equilateral triangle. → Make a change in one side. → Rotate the change in AB to AC. → The finished shape has the same area as the original shape. Discuss why.

(c) changing shapes by reflection – this tessellation requires another shape where the pieces are added on, not removed:

![Diagram showing the process of changing shapes by reflection](image)

(d) Escher style art:

![Diagram showing the process of creating Escher style art](image)

Students can make their own tessellations by starting with a tessellating shape like a square, rectangle, triangle and so on. Then remove from one side and add to the other until you have something interesting which should tessellate. Cut out copies of the shape using two colours of paper and put together to make a pattern.

Note: Computers do this really well.
Fabric design

In this method, a tessellating shape is chosen and a design drawn in it. If the design goes outside the shape, this outside part is redrawn inside the shape on the opposite side. Then the shape with design is repeated to make the fabric (the repeat can be flipped, slid or turned) as follows. Again students can make their own designs.

3.3  Grids–puzzles

Tessellation puzzles

When a shape tessellates, it can form “graph” paper or grids of the shape. When it is in the form of a grid, then larger shapes are made up of 2, 3, 4, 5, and so on, of the repeated shapes. Investigations grow for this:

(a) Do the larger shapes themselves tessellate?

(b) How many different larger shapes can we make from a given number (say, 5) of the repeating shapes that form the grid?

(c) Do these different larger shapes make puzzles?

The answer to all three is pretty much a yes. Certainly we know of sets of larger shapes formed for this grid process that have become the basis of puzzles, for example: (a) the 12 pentominoes that are made from 5 squares; (b) the 12 hexiamonds that are made from 12 equilateral triangles; and (c) the 24 McMahon four-coloured triangles that form when the triangle in a grid is divided into three smaller triangles.

All these different sets of shapes form the basis of shape puzzles – look these up on the Internet or other sources of ideas. You will find many books of puzzles. Some ideas for pentominoes are given below.

Pentomino activities

There are 12 different shapes that can be made with 5 squares and they are called pentominoes. They are shapes which look like these letters: N, Z, P, T, U and C, L, Y, X, F, W, I, V – some of the pentominoes are not exactly like their letters – they look a little strange.

These 12 shapes can form other shapes as puzzles – some examples are below:

(a) C, N and P; C, Y and P; and C, P and V can combine to form a 3×5 rectangle;

(b) Y, L, W and P; P, N, Y and C; and V, T, W and P can form a 4×5 rectangle; and

(c) all 12 pentominoes can form 4×15, 5×12, and 6×10 rectangles, and a large W, as on right.

Try the other tessellation puzzles – hexiamonds and McMahon four-colour triangles – use the Internet.
Unit 4: Dissections

Dissections are an extension of jigsaw puzzles. There are two types: (a) \textit{simple} – pieces of the puzzle are given and have to be joined to form the shape – an example is the circle below on left where the pieces are cut out and mixed up and the student has to reassemble it to make a circle; and (b) \textit{complex} – two shapes are given and the first shape has to be cut up so that its pieces can form the second shape – an example of a complex dissection is given below on right.

![Circle dissection example](image)

One cut and form

This unit will cover concepts and constructions for dissections, look at one dissection activity, and discuss how dissections lead to visual imagery.

4.1 Concepts and constructions

Dissections cover the following:

(a) \textit{concepts and constructions} – activities which introduce students to dissections (the two types) and enable students to construct dissections;

(b) \textit{puzzles} – activities that allow students to experience puzzles where parts are put together to form a whole, or more difficult, one shape is cut into pieces to form another shape (e.g. tangrams, egg puzzles, soma cubes); and

(c) \textit{visual imagery} – activities to practise finding shapes that match other shapes or fit into spaces left in a design.

Complete examples of simple dissections and complex dissections. Examples are:

(a) jigsaw puzzles – these are the beginning of dissections;

(b) any construction kit (making things out of components – including furniture, toys, and so on);

(c) Lego and Duplo constructions; and

(d) any 2D and 3D object that can be pulled apart and put together in one or a variety of ways.

Construct dissections as follows:

(a) \textit{Simple} – hand out grid paper (e.g. squares, triangles, rectangles), cut out the biggest shape that you can by following the grid lines (e.g. a rectangle, a diamond, a hexagon), cut this into 5 or 6 pieces along the grid lines, use the original grid paper to make a template, and provide the pieces to a student to reform into the shape (note that students can make up these puzzles for each other).

(b) \textit{Complex} – take a simple shape (e.g. square, circle, triangle, t-shape, etc.) and cut it into 2 or 3 (or more) pieces using straight cuts, reform these pieces into a different shape (stick together with tape), and trace around this – the final shape becomes the starting shape and the first shape the final shape; a copy of both is given to a student with the instruction to cut the first shape into 2 or 3 or more parts by straight cuts and form the second shape.
### 4.2 The dissection puzzle called tangrams

A jigsaw has parts and is difficult to put together but it only makes one thing. Many dissection puzzles are made up of components that can be formed and reformed into many things. The tangram is one of the most common 2D dissection puzzles that can do this – an example is shown on right.

It consists of a square that is cut into five different-sized triangles, a small square and a rhombus. (The diagram on right shows how the pieces are produced.) These pieces can be arranged to form over 300 different figures. Tangrams are Chinese in origin and appear to be about 4000 years old. The name was probably derived from the Chinese word “t’an” which means “extend” or the Cantonese word “t’ang” which means “Chinese” and then the European term “gram” was added. The word *tangram* seems to have been coined between 1847 and 1864.

To make the tangram pieces, a square (8 cm × 8 cm) is cut up as on right above. To make it: draw the diagonal BD; find the midpoints of BC and DC; label these X and Y respectively and draw the line XY; find the midpoint of the diagonal BD and label it Z and draw the line AZ and extend it to meet XY at E; from E, draw a line parallel to BC, meeting BD at F; and from Y, draw a line perpendicular to BD, meeting it at G.

A brief sequence of tangram activities is given below (many more can be found on the Internet or in books):

(a) try these four activities; and

(b) determine what other letters can be made with all the pieces.

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**Note:** Particularly good materials with similar chances to make many shapes are the following:

(a) Egg puzzles – an egg shape cut into pieces that can be re-formed for many shapes;
(b) Hexiamonds and Pentominoes – each has 12 puzzle pieces made from looking at tessellating equilateral triangles and squares respectively; and

(c) McMahon four-colour triangles and squares – triangles and squares coloured four different colours that form 24 pieces and can be reformed to make puzzles where the final puzzle has the same colour on edge all around and two shapes are only fitted together if they join at the same colour.

4.3 Visual imagery

Tessellation and dissection activities are important in testing and in real life. Then build mental rotation – rotating a shape in the mind and imagining yourself rotating around the shape and what it would look like as you did. These activities are important in many jobs, e.g. trades such as carpentry, design, art, architecture, engineering, medicine, dentistry, and so on.

The examples of activity include:

(a) determining which shape is the same as another or identifying the shape that will fit into a space;

(b) working out the 12 pentominoes on square graph paper;

(c) giving students a shape (e.g. a key) and then asking which of four shapes is the same as it except for orientation and position (Note: Allow younger or less experienced students to cut out a copy of the shape and try to fit it with the other shapes by actually flipping, sliding and turning the copy); and

(d) asking students to fill in a gap in a puzzle by choosing the appropriate shape (in fact end games of puzzles are good for many things – give the half-finished puzzle and ask for the piece that will complete the puzzle).
This unit extends the ideas in the first four units. They allowed students to experience flips-slide-turns, symmetry, tessellation and dissection, and to try some puzzles and art. Now we move on to analysing for properties (see the Van Hiele levels in the Overview section) and looking more deeply at art.

5.1 Properties

Flips, slides and turns (tracing paper)

Get the students to revisit flips, slides and turns. However, this time use tracing paper on the shape as accurately as possible (using the process in 1.1) and use most of a sheet of A4 paper and large shapes to show the flips, slides and turns. This can allow more formal ways of identifying properties. Do the following:

1. **Flips.** Draw lines from corresponding points on the start picture to the end picture. Look at the direction of these lines. Look at how they cut the dotted flip line. Encourage students to look for parallelness and perpendicularity, and distances either side of the flip line. Also look for other properties.

   You should be able to get students to determine these properties: (a) change is along parallel lines; (b) change is at right angles to flip line; (c) change is different length (if a point is near to flip line, it will stay near to flip line after change and vice versa); (d) flip line bisects change line (perpendicular bisector); and (e) facing the flip line, what is on the left becomes on the right and vice versa.

2. **Slides.** Also draw lines from corresponding points and look again at lines as they are and in relation to the slide line. You should be able to get students to determine these properties: (a) the change is along parallel lines; (b) these lines are parallel to the slide line; (c) these lines are the same length as the slide line; and (d) right and left do not change.

3. **Turns.** Start by drawing straight lines between corresponding points in start and finish. These will show no pattern. Ask students to use a compass to draw curves from the centre that join corresponding points. You should be able to get students to determine that, for a turn, change is through concentric circles from the centre or turning the same angle.

Line and rotational symmetries

Properties of symmetries are straightforward – can the shape be folded in half so both sides are the same; can a copy be rotated on top of it and match in a part turn.

If you wanted to dig a little more deeply, you could get students to investigate:

(a) angles between lines of symmetry and angles between rotations of symmetry; and

(b) how lines of symmetry divide the shape in terms of fractions.

In this case, one would find that:

(a) angles between lines of symmetry and angles between rotations of symmetry are divisors of 360 degrees;

(b) the divisor for lines is double the number of symmetries (e.g. 3 lines of symmetry means an angle of $360\div6 = 60$ degrees) and the divisor for rotations is the number of symmetries (e.g. 3 rotations means an angle of $360\div3 = 120$ degrees); and
(c) the lines of symmetry divide a shape into equal parts double the number of lines (e.g. 3 lines of symmetry means that the lines divide the shape into sixths).

Note: For rotational symmetry, one match in a part turn means two rotations of symmetry because where there is at least one rotation, you can count the full turn.

Tessellations

There are two properties for tessellations. Once again they can be found by students studying a variety of tessellations and discussing properties. The two properties are:

(a) that the pattern in the tessellation must convince anyone looking at it that it is going in all directions with things repeating forever and is not just a pretty confined design; and

(b) shapes that tessellate have angles so that wherever the shapes touch at a point, the interior angles at the point must total 360° (see examples below).

This means that any tessellating shape must:

(a) have angles that are divisor of 360 degrees (like equilateral triangles);

(b) have angles adding to a divisor of 360 degrees (any scalene triangle’s angles add to 180 so they can be fitted so any point adds to a divisor of 360 degrees); or

(c) be with another shape whose angles when combined with the first shape are a divisor of 360 degrees.

5.2 Three-dimensional Euclidean transformations

The flips, slides and turns of plane shapes are less complicated than the flips, slides and turns of solid shapes, and represent what happens in most vocations. Thus, we need to spend time on 3D activities.

Solid tessellations

These are solid objects that can be piled or packed together and they can fill up a truck or room – they are 3D shapes that pack together without gaps or overlaps. Such solids that tessellate tend to have tessellating faces and flat surfaces. Thus, square, rectangular, triangular, and hexagonal prisms tessellate while pentagonal and octagonal prisms do not (see on right).

Square, rectangular, triangular and hexagonal prisms tessellate while other polygon based pyramids do not. This holds as long as the vertex is above the base (e.g. on right, (i) tessellates and (ii) does not). It is also possible to mix a second (or third) solid shape to produce a tessellation. For example, triangular and square prisms will tessellate. Solid shapes which are the combination of tessellating solids will also tessellate (e.g. the solids on right tessellate).

To investigate these tessellations do the following:

(a) collect the following: prisms (e.g. erasers, various shapes of dowelling cut in sections, various types of small packets), pyramids (e.g. tetrapaks), cylinders (e.g. circular dowelling cut into sections, cans and...
bottle tops), spheres (balls), and other solids (e.g. L-shaped dowelling sections, small wood and plastic houses);

(b) stack examples of the following solids to determine whether they tessellate: cube, rectangular prism, triangular prism, square pyramid, tetrahedron, sphere, cone and cylinder; and

(c) visit a supermarket and consider articles you find on the shelves and how they relate to solid tessellations.

Packaging in our society is a compromise between strength, cost, appearance and packing (tessellation). The best shape to pack (or tessellate) is a rectangular prism (the common box). It is also cheap to make because it can be folded from a net. However, it is weak as corners and edges are points of weakness. The strongest surface is a curve like a sphere or cylinder with semi-spherical ends but this does not tessellate well. Also the sphere is the package where the least material (surface area) encloses the most contents inside (volume). Square prisms are the best in this regard when we consider prisms.

3D puzzles

Three-dimensional puzzles are very important as these act as preparation for our 3D world – architecture, brain surgery, and so on. Two examples are:

(a) Soma cubes – 7 pieces that make a 3×3×3 cube and many other shapes; and

(b) wooden cube pentominoes – pentomino pieces that can be used for 3D as well as 2D puzzles.

Of course, there are always real-world examples such as taking apart and repairing, or building from flat packed materials, and so on.

5.3 Puzzles and sequencing

As we have seen in Units 3 and 4, there are many puzzles that emerge from flips, slides and turns and can help visual imagery. However, students’ success with the puzzles will be enhanced if we follow these sequencing properties. The components of puzzles that have to be taken into account and their sequencing is as follows.

<table>
<thead>
<tr>
<th>COMPONENT</th>
<th>SEQUENCE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of pieces</td>
<td>Small number → larger number → all the pieces</td>
</tr>
<tr>
<td>Shape outline</td>
<td>Full size template of shape → small picture → no outline at all</td>
</tr>
<tr>
<td>Complexity of shape</td>
<td>Partially completed → clues → no clues</td>
</tr>
<tr>
<td>Type of movement</td>
<td>Only turning required → turning and flipping required</td>
</tr>
<tr>
<td>Presentation</td>
<td>Simpler subtasks → full tasks only</td>
</tr>
<tr>
<td>Organisation of tasks</td>
<td>Similar tasks → dissimilar</td>
</tr>
</tbody>
</table>

5.4 Computer-based activities

Computers are excellent at doing slides. A shape or picture moved by click and drag will always slide and never change orientation. They will also easily do turns and flips. Students should experience these early.

However, to flip around a line or to turn around a centre needs more work: (a) to flip around a line, group the line and shape, copy it, flip the copy, drag the flipped shape and line until the original line and flipped line are directly over each other; and (b) to turn around a centre, group the shape and centre, copy it, turn the copy the required amount, and drag the turned shape until the two centres overlap.

The computer is effective here because its accurate depiction of flips, slides and turns enables properties and relationships to be easily and accurately explored. And, in art, it is very effective because complicated designs can be grouped and then turned and flipped with ease.
Unit 6: Congruence and Relationships

This is the final unit. It covers congruence, relationships between components principles and integration with other topics.

6.1 Congruence and congruence activities

This section looks at congruence and begins the process of looking at how its parts relate.

Concept

The best way to teach and think of congruent shapes is that congruent shapes are when one shape changes to the other by flips, slides and turns only. Properties can then be discovered. A good sequence of activities is as follows.

1. Constructing congruence. Draw a shape and copy it from cardboard – starting from the original shape, flip, slide and turn the copy and draw what you produce – repeat this many times trying to get something that looks different – everything that you draw will be congruent to the original shape.

2. Determining congruence. You have two shapes – copy one of them – flip, slide and turn the copy to see if can make the other shape – if yes, congruent and, if not, not congruent.


The property of congruent shapes is that: (a) corresponding sides are the same length; and (b) corresponding angles are the same angle. This means that congruent shapes are the same size and shape – only their position and orientation change.

Relationship

Slides obviously don’t change anything but position – however, what is the difference between flips and turns?

One way to start this is to look at their effect on the same starting shape. An example of this is the construction of the “shield” on right. Students are given a shield shape and two dotted lines crossing. They put a design in one of the quadrants (as on right) and use the Mira to flip it three times into the other quadrants and tracing paper to rotate it three times.

This leaves us with a flip design and a rotation design. Both can look effective as a shield if the artistry is good. But which one looks better?

Explore this different design and this will show the difference in flips and turns (Note: Flips tend to provide a line symmetrical static design while rotation gives a turning effect). The above design causes debate whenever it is used.
6.2 Relationships and principles

Relationships between flips, slides and turns

If we use tracing paper, we can draw fairly accurate slides and turns. If we use a Mira, we can see if we can duplicate the effect of slides and turns with flips (and the answer is yes). We use the Mira to flip corresponding points in the slide and turn process and the following happens:

*Slide and two flips*

![Diagram of slide and two flips]

*Turn and two flips*

![Diagram of turn and two flips]

**Relationships**

The relationships are as follows:

(a) **two flips equal one slide** – with flip lines perpendicular to the direction, and half the distance of the length of the slide apart; and

(b) **two flips equal one turn** – with the two flip lines meeting at and running through the centre and the angle where the flip lines meet being half the angle for the turn.

**Relationship between line and rotational symmetry**

Card 4 of the symmetry cards at the end of Unit 2 begins the exploration of this relationship.

1. Complete card 4 from Unit 2 and, until you discover the pattern, record the details on a table such as this.

2. Reinforce and extend pattern by drawing a variety of shapes, determining the number of line symmetries and rotations of symmetry for each one; determine the angle between lines and angle between rotations for each one, and putting data on table as below.

<table>
<thead>
<tr>
<th>Shape</th>
<th>Number of lines of symmetry</th>
<th>Number of rotations of symmetry</th>
<th>Angle between lines</th>
<th>Angle between rotations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square</td>
<td>4</td>
<td>4</td>
<td>45°</td>
<td>90°</td>
</tr>
<tr>
<td>Isosceles triangle</td>
<td>1</td>
<td>0</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>Equilateral triangle</td>
<td>3</td>
<td>3</td>
<td>60°</td>
<td>120°</td>
</tr>
</tbody>
</table>

3. Use the above data to obtain the following relationships:

- a shape can have no lines of symmetry but any number of rotations,
- a shape can have one line of symmetry and no rotations; and
- if the shape has two or more lines, the number of rotations equals the number of lines (and the angle between the lines is half the angle between rotations).

**Note:** This is very similar to the flips, slides and turns relationship.

### 6.3 Extensions

The relationships above can lead to extension. Two ways are discussed.

**Finding centre by reversing flips and turns**

If students are given a turn as on the right, the relationship referred to in flips, slides and turns can be used to find the angle and centre of the turn.

To do this, find two pairs of corresponding points on the figure and the rotation. Use a Mira to, in turn, mark in the Mira line that takes a point to its corresponding point.

Where the two Mira lines meet will be the centre and the angle of turn will be double the angle between the Mira lines.

**Coordinates and flips, slides and turns**

The final activity is to relate flips, slides and turns to coordinate points.

1. Use students’ bodies on a grid (or mat) – use a rope or band to divide the grid into two. Have 3-4 students stand at intersections on one side (acting as corners of a 3- or 4-sided shape) and have 3-4 other students (with help from class) stand on the other side for three different changes – the 3-4 people having been slid to the other side (so shape is same distance away from line on each side), reflected about line, and rotated 180°.

2. Repeat this for another 3-4 students but this time label the lines with numbers and students have to say where students go by stating coordinates.
3. Repeat the above but use elastic bands to make a shape. Coordinates of corners will be stated to show flips-slides-turns.

4. Repeat the above but for the mat/grid divided into four sections using $x$ and $y$ axes (the coordinates can now be negative). Make shapes in one quadrant (top right is normal). Use coordinates to slide the shape across axes, flip the shape about axes and rotate the shape 90°.

5. Place a grid on a computer, draw simple shapes, use the mouse and the software to slide shape along a line that starts and ends on coordinates, flip shape about a line that starts and ends on coordinates, and rotate shape 90°, 180° and 270° about a centre which is a coordinate. Write down the starting and finishing coordinates, and compare differences in these coordinates with the extent of the flip, slide and turn.

6. Use this comparison to suggest coordinate rules for these slides, flips and turns.

7. **Investigation.** Use coordinates and grids to: (a) slide a simple shape along an arrow (arrow gives direction and distance – start and end of arrow have to be a coordinate position); (b) flip a shape about a line (line starts and ends on coordinates); and (c) rotate a shape about a centre (which is a coordinate) angles of 90°, 180° and 270°.

8. Compare the start and finish coordinates – use these to propose coordinate rules for slides, flips and turns.
Test Item Types

This section presents instructions and the test item types for the subtests associated with the units. These will form the bases of the pre-test and post-test for this module.

Instructions

Selecting the items and administering the pre-post tests

This section provides an item bank of test item types, constructed around the units in the module. From this bank, items should be selected for the pre-test and post-test; these selected items need to suit the students and may need to be modified, particularly to make post-test items different to pre-test items. The purpose of the tests is to measure students’ performance before and after the module is taught. The questions should be selected so that the level of difficulty progresses from easier items to more difficult items. In some modules this will follow the order of the units and subtests, and in other modules it will not, depending on the sequencing across the module and within units. The pre-test items need to allow for students’ existing knowledge to be shown but without continual failure, and the post-test items need to cover all the sections in a manner that maximises students’ effort to show what they can do.

In administering the pre-test, the students should be told that the test is not related to grades, but is to find out what they know before the topic is taught. They should be told that they are not expected to know the work as they have not been taught it. They should show what they know and, if they cannot do a question, they should skip it, or put “not known” beside questions. They will be taught the work in the next few weeks and will then be able to show what they know. Stress to students that any pre-test is a series of questions to find out what they know before the knowledge is taught. They should do their best but the important questions come at the end of the module. For the post-test, the students should be told that this is their opportunity to show how they have improved.

For all tests, teachers should continually check to see how the students are going. Items in later subtests, or more difficult items within a particular subtest, should not be attempted if previous similar items in earlier subtests show strong weaknesses. Students should be allowed to skip that part of the test, or the test should be finished. Students can be marked zero for these parts.

Information on the Euclidean transformation item types

There are six subtests to match the six units in this module. Subtests 1 and 2 focus on the foundational ideas of flips-slides-turns and line and rotational symmetry; Subtests 3 and 4 look at the ideas of tessellations and dissections that arise from these; and Subtests 5 and 6 put all the ideas together to look at puzzles and congruence. It would be reasonable to restrict the pre-test to Subtests 1 and 2 and the easier parts of Subtests 3 and 4. However, teachers need to ensure that the harder parts of Subtests 3 and 4 and all of Subtests 5 and 6 are a major part of the post-test.

This is a similar structure to that for the subtests for Module N1 Whole Number Numeration.

Note: The item types of this module are not something that can be rote learnt, so it is important to continue with puzzles like this as students progress through all other modules.
Subtest item types

Subtest 1 items (Unit 1: Flips, slides and turns)

1. (a) Draw a flip of the shape, over the line.

(b) Draw the shape flipped over the line.

2. Turn the shapes 90° clockwise about the dot.

(a) 

(b)

3. Show the arrow after a slide.
4. (a) If I flip the shape over the first line and then over the second, the final result would be the same as if the triangle has:

slid or turned

(circle the correct response)

(b) If I flip the triangle over the first line and then over the second, the final result would be the same as if the triangle has:

slid or turned

(circle the correct response)
Subtest 2 items (Unit 2: Line and rotational symmetries)

1. Draw a line of symmetry for each of the following shapes.
   
   (a) ![Shape A]
   
   (b) ![Shape B]

2. Look at this shape:
   
   ![Shape C]
   
   (a) How many lines of symmetry does this shape have? _________
   
   (b) How many ways can this shape be rotated onto itself? _________
   
   (c) Draw a shape with only one line of symmetry.

3. Look at this shape:
   
   ![Shape D]
   
   (a) How many lines of symmetry does this shape have? _________
   
   (b) How many ways can this shape be rotated onto itself? _________
Subtest 3 items (Unit 3: Tessellations)

1. (a) Circle the shapes which will tessellate.

(b) Circle the shapes which will NOT tessellate.

2. This is a tessellating pattern:

(a) How many degrees is each interior angle of a regular hexagon? _________________

(b) What is the total number of degrees around the point where the shapes meet together? _________________

(c) What other regular shapes have angles that can do this (tessellate)?

__________________________________________________________________________

(d) What is the rule for regular shapes tessellating? __________________________________________

__________________________________________________________________________

3. (a) Draw a tessellating pattern (tiling pattern) using this shape:

Your pattern must go horizontally as well as vertically.

(b) What is the total number of degrees around the point where the shapes meet together? _________________

(c) What is the general rule for tessellating shapes and their angles? ________________________

____________________________________________________________________________________
Subtest 4 items (Unit 4: Dissections)

1. For each shape below, draw a dotted line for where you would cut to form two pieces that could be put together to make a square.

(a) ![Shape A]

(b) ![Shape B]

(c) ![Shape C]

2. Draw your own shape and a dotted line where you would cut the shape so that it could be put together to make a square.

3. Here are some shapes:

![Shape D]

![Shape E]

![Shape F]

Draw a picture to show how you would put these shapes together to make a shape like this:

![Shape G]
Subtest 5 items (Unit 5: Properties and puzzles)

1. Challenge question

Here are two Pentominoes. They are made up of five squares joined at their edges:

![Pentominoes](image)

How many different Pentominoes can you draw?

*(Note: Pentominoes are the same if they can equal each other by flips, slides and turns, e.g. \[\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array}\] and \[\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array}\] are the same but \[\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array}\] and \[\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array}\] are different.)*
Subtest 6 items (Unit 6: Congruence and relationships)

1. Circle the shape(s) below that are congruent to the shape on right:

   ![Shapes](image1)

2. Circle the shape below that is congruent to the shape on right:

   ![Shapes](image2)

3. What is the relationship between:
   
   (a) two flips and one slide? 
   
   (b) two flips and one rotation?
Appendix A: Euclidean Transformation Ideas

Flips, slides and turns

A flip is a reflection (a line of reflection can be given around which the shape is flipped), a slide is a translation (a movement in one direction without change of orientation), and a turn is a rotation (a centre point can be given about which the shape is turned):

![flip][slide][turn]

Flips, slides and turns are the only changes allowed in Euclidean transformations. Thus Euclidean transformations are the changes of our human-made world. They are changes which will not affect the object being moved.

Slides and turns are related to flips – a slide can be achieved by two flips where the lines of reflection are perpendicular to the slide and the distance between lines of reflection is half the length of the slide, and a turn can be achieved by two flips where the lines of reflection pass through the centre of the turn and the angle between lines of reflection is half the angle of the turn.

Symmetry

Shapes can be studied for both line and rotational symmetry and for the relationship between these two symmetries (for more than 2 lines of symmetry, the number of lines equals the number of rotations of symmetry and the angle between lines of symmetry is half the angle between consecutive rotations). Line symmetry means that the shape can be folded in half along a line; rotational symmetry means that the shape looks the same after a part turn (see below). Shapes can be classified by symmetry (e.g. square has 4 lines/4 rotations of symmetry, while rectangle only 2 lines/2 rotations). Line symmetry can be related to reflection (flip); while rotational symmetry is related to rotation (turn).

![line symmetry][rotational symmetry]

Tessellations

When a shape is such that many copies of it can be joined like tiles to cover a space without overlapping or leaving gaps, the shape is said to tessellate (see below). Tessellating shapes have angles that combine to give 360 degrees or divide into 360 degrees. Tessellating shapes give rise to different “graph” papers. Fabric designs result from tessellating patterns, as do Escher-type drawings. Shape puzzles come from tessellating shapes (e.g. pentominoes come from squares).

![rectangles tessellate][circles do not]
Tessellations can be introduced through tiling and shapes that tile without gaps or overlaps identified. Tessellations of more than one type of shape can be explored as can 3-dimensional or solid tessellations (i.e., containers that pack well). Finally, tessellations can be used to identify shapes which form the basis of shape puzzles (e.g. pentominoes).

**Dissections**

Dissections emerge from jigsaw puzzles – they are puzzles where pieces (normally all different shapes) have to be put together to make a larger shape. There are two types of dissection: (a) *simple* – where pieces (starting shapes) are given and have to be put together to form a final shape; and (b) *complex* – where a first shape is given which has to be cut into pieces and reassembled to form a second shape. Dissections also include families of puzzles where one set of pieces is used to make many other shapes (e.g. Tangrams, Soma cubes).

Simple dissection:

![Simple dissection diagram](image)

Complex dissection:

![Complex dissection diagram](image)

**Congruence**

Shapes are congruent if one can be changed to the second by flips, slides and turns. This means that congruent shapes are the same size and shape (they have the same angles and length of sides), they differ only by orientation.

![Congruence diagram](image)
Appendix B: Introductory Activities for Mira

Introducing the Mira

A Mira is a red plastic mirror as shown on right. Its strength is that the red plastic allows you to look through the Mira to what is behind and, at the same time, to see the reflection of what is in front of the Mira. This means that a Mira can be used to superimpose and to draw reflections.

The diagram below on right shows how to use the Mira to draw reflections and superimpose a shape. Its purpose is to develop spatial visualisation (the ability to mentally manipulate, twist, rotate, reflect, slide or invert shapes), an understanding of the role of angles in reflection, and an understanding of the relationship between reflections and symmetry.

The Mira will only do flips and line symmetry but it does them well. Here is an activity sequence for the Mira.

Learning how to use a Mira

Which hat fits best on the man?

For each hat in turn, student places Mira between hat and man (with Mira perpendicular to an imaginary line between hat and man and bevelled edge down and facing student) and adjusts Mira until hat is on the man (or man is under the hat – depends on what way student are looking into the Mira).

After trying each hat, student judges which is best.

Use Mira to read the reflected writing on right.

Use Mira to determine what is wrong with the reflection of the clock face.
Drawing flips

Put Mira on dotted lines and copy – result is a flip.

Drawing in lines of symmetry

Student places Mira on shapes/designs below and finds all lines of symmetry. If bevelled edge is facing student and is down, line can be drawn along this edge and be below the reflection plane of the Mira.

Completing line symmetric figures

Student places Mira along the lines, then draws the other half of the figures. If the result makes sense, the resulting figure is line symmetric. Does the result always make sense? Why or why not?
Appendix C: RAMR Cycle

AIM advocates using the four components in the figure on right, reality—abstraction—mathematics—reflection (RAMR), as a cycle for planning and teaching mathematics. RAMR proposes: (a) working from reality and local culture (prior experience and everyday kinaesthetic activities); (b) abstracting mathematics ideas from everyday instances to mathematical forms through an active pedagogy (kinaesthetic, physical, virtual, pictorial, language and symbolic representations, i.e. body → hand → mind); (c) consolidating the new ideas as mathematics through practice and connections; and (d) reflecting these ideas back to reality through a focus on applications, problem solving, flexibility, reversing and generalising (see figure on right).

The innovative aspect of RAMR is that the right half develops the mathematics idea while the left half reconnects it to the world and extends it. For example, whole-number place value built around the pattern of threes where hundreds-tens-ones applies to ones, thousands, millions, and so on, can be easily extended to metrics by considering the ones to be millimetres, the thousands to be metres and the millions to be kilometres.

Planning the teaching of mathematics is based around the RAMR cycle if it is deconstructed into components that are applied to a mathematical idea. By breaking instruction down into the four parts and taking account of the pedagogical approaches described above, the cycle can lead to a structured instructional sequence for teaching the idea. The table below briefly outlines how this can be done. Prerequisite mathematical ideas are considered in the Reality and Mathematics components of the cycle, while extensions and follow-up ideas are considered in the Reflection component.

<table>
<thead>
<tr>
<th>REALITY</th>
<th>ABSTRACTION</th>
<th>MATHEMATICS</th>
<th>REFLECTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Local knowledge: Identify local student cultural-environmental knowledge and interests that can be used to introduce the idea.</td>
<td>• Representation: Develop a sequence of representational activities (physical to virtual to pictorial materials to language to symbols) that develop meaning for the mathematical idea.</td>
<td>• Language/symbols: Enable students to appropriate and understand the formal language and symbols for the mathematical idea.</td>
<td>• Validation: Facilitate reflection of the new idea in terms of reality to enable students to validate and justify their new knowledge.</td>
</tr>
<tr>
<td>• Prior experience: Ensure existing knowledge and experience prerequisite to the idea is known.</td>
<td>• Body-hand-mind: Develop two-way connections between reality, representational activities, and mental models through body → hand → mind activities.</td>
<td>• Practice: Facilitate students’ practice to become familiar with all aspects of the idea.</td>
<td>• Applications/problems: Set problems that apply the idea back to reality.</td>
</tr>
<tr>
<td>• Kinaesthetic: Construct kinaesthetic activities, based on local context, that introduce the idea.</td>
<td>• Creativity: Allow opportunities to create own representations, including language and symbols.</td>
<td>• Connections: Construct activities to connect the idea to other mathematical ideas.</td>
<td>• Extension: Organise activities so that students can extend the idea (use reflective strategies — flexibility, reversing, generalising, and changing parameters).</td>
</tr>
</tbody>
</table>
## Appendix D: AIM Scope and Sequence

<table>
<thead>
<tr>
<th>Yr</th>
<th>Term 1</th>
<th>Term 2</th>
<th>Term 3</th>
<th>Term 4</th>
</tr>
</thead>
</table>
| A  | N1: Whole Number Numeration
   Early grouping, big ideas for H-T-O; pattern of threes; extension to large numbers and number system | O1: Addition and Subtraction for Whole Numbers
   Concepts; strategies; basic facts; computation; problem solving; extension to algebra | O2: Multiplication and Division for Whole Numbers
   Concepts; strategies; basic facts; computation; problem solving; extension to algebra | G1: Shape (3D, 2D, Line and Angle)
   3D and 2D shapes; lines, angles, diagonals, rigidity and properties; Pythagoras; teaching approaches |
|  | N2: Decimal Number Numeration
   Fraction to decimal; whole number to decimal; big ideas for decimals; tenths, hundredths and thousandths; extension to decimal number system | M1: Basic Measurement (Length, Mass and Capacity)
   Attribute; direct and indirect comparison; non-standard units; standard units; applications | M2: Relationship Measurement (Perimeter, Area and Volume)
   Attribute; direct and indirect comparison; non-standard units; standard units; applications and formulae | SP1: Tables and Graphs
   Different tables and charts; bar, line, circle, stem and leaf, and scatter graphs; use and construction |
|  | N3: Common Fractions
   Concepts and models of common fractions; mixed numbers; equivalent fractions; relationship to percent, ratio and probability | G2: Euclidean Transformations (Flips, Slides and Turns)
   Line-rotation symmetry; flip-slides-turns; tessellations; dissections; congruence; properties and relationships | A1: Equivalence and Equations
   Definition of equals; equivalence principles; equations; balance rule; solutions for unknowns; changing subject | SP2: Probability
   Definition and language; listing outcomes; likely outcomes; desired outcomes; calculating (fractions); experiments; relation to inference |
|  | O3: Common and Decimal Fraction Operations
   Addition, subtraction, multiplication and division of common and decimal fractions; models, concepts and computation | O4: Arithmetic and Algebraic Principles
   Number-size, field and equivalence principles for arithmetic; application to estimation; extension to algebra; simplification, expansion and factorisation | G3: Coordinates and Graphing
   Polar and Cartesian coordinates; line graphs; slope and y-intercept; distance and midpoints; graphical solutions; nonlinear graphs |
| B  | A2: Patterns and Linear Relationships
   Repeating and growing patterns; position rules, visual and table methods; application to linear and nonlinear relations and graphs | A3: Change and Functions
   Function machine; input-output tables; arrowmath notation, inverse and backtracking; solutions for unknowns; model for applications to percent, rate and ratio | O5: Financial Mathematics
   Applications of percent, rate and ratio to money; simple and compound interest; best buys; budgeting and planning activities |
|  | N5: Directed Number, Indices and Systems
   Concept and operations for negative numbers; concept, patterns and operations for indices; scientific notation and number systems | G4: Projective and Topology
   Visualisation; divergent and affine projections; perspective; similarity and trigonometry; topology and networks | SP3: Statistical Inference
   Gathering and analysing data; mean, mode, median, range and deviation; box and whisker graphs; large data sets, investigations and inferences |  |

**Key:** N = Number; O = Operations; M = Measurement; G = Geometry; SP = Statistics and Probability; A = Algebra.