YuMi Deadly Maths

AIM Module O2
Year A, Term 3

Operations:
Multiplication and Division for Whole Numbers

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The YuMi Deadly Centre acknowledges the traditional owners and custodians of the lands in which the mathematics ideas for this resource were developed, refined and presented in professional development sessions.

YUMI DEADLY CENTRE
The YuMi Deadly Centre is a research centre within the Faculty of Education at the Queensland University of Technology which is dedicated to enhancing the learning of Indigenous and non-Indigenous children, young people and adults to improve their opportunities for further education, training and employment, and to equip them for lifelong learning.

“YuMi” is a Torres Strait Islander Creole word meaning “you and me” but is used here with permission from the Torres Strait Islanders’ Regional Education Council to mean working together as a community for the betterment of education for all. “Deadly” is an Aboriginal word used widely across Australia to mean smart in terms of being the best one can be in learning and life.

YuMi Deadly Centre’s motif was developed by Blacklines to depict learning, empowerment, and growth within country/community. The three key elements are the individual (represented by the inner seed), the community (represented by the leaf), and the journey/pathway of learning (represented by the curved line which winds around and up through the leaf). As such, the motif illustrates the YuMi Deadly Centre’s vision: Growing community through education.

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DEVELOPMENT OF THE AIM MODULES
The Accelerated Inclusive Mathematics (AIM) modules were originally developed from 2010 to 2013 as part of the Accelerated Indigenous Mathematics project funded under the Commonwealth Government’s Closing the Gap: Expansion of Intensive Literacy and Numeracy program for Indigenous students. The project aimed to assist secondary schools with beginning junior-secondary Indigenous students who were at Year 2/3 level in mathematics by developing a junior-secondary mathematics program that accelerates the students’ learning to enable access to mathematics subjects in the senior-secondary years and therefore enhance employment and life chances. The project developed three years of modules (Years A to C) that are vertical sequences of learning to take students from their ability level to their age level in mathematics. The YuMi Deadly Centre acknowledges the role of the Department of Education, Employment and Workplace Relations in the development of the AIM project and these modules.

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Module Overview

This is the second of the Operations modules. It covers whole number multiplication and division from early childhood to early secondary years — moving from meanings and models for one-digit multiplication and division through to algebraic representations. It will be followed by other modules on operations, covering common and decimal fractions (Module O3); arithmetic and algebra principles, including estimation (Module O4) and financial mathematics (Module O5). Operations for percent, rate and ratio, and directed number and indices are covered in Number Modules N4 and N5 respectively.

Background information for teaching multiplication and division

This section describes the two foci of operations (meaning and computation), major connections between multiplication and division and other topics and within multiplication and division, and lists the big ideas for operations.

Two foci of operations

As we stated in Module O1, there are two components for operations. The first of these deals with meaning or operating — what do multiplication and division mean (what are their concepts), where are they used in their world, and what are their properties. The second deals with computation or calculating — what is the answer to the operation, what ways can we work it out, and how accurate does this have to be.

As further stated in Module O1, problem solving is based on meaning, not computation, as is understanding algebraic sentences with multiplication and division. Obviously algorithms are computation along with basic facts and estimation. As well, most of the operations work is based on strategies and properties of the operations (which this module will call principles). Multiplication and division operate in the world in two ways — on individual objects, and on measures. The major components of operations are given below, followed by a description of how they interact.

Meaning

1. Concepts — these are meanings that define the operation in terms of the everyday world; they also cover the different models in which the meanings can be encapsulated (for multiplication and division, there are three models – set, number line or length, and array or area); they relate story, action, drawing, language and symbols (these are called representations).

2. Principles — these are properties that hold no matter what the numbers are (i.e. they hold for whole numbers, decimals, fractions, and so on); for example, the commutative law, that first × second always equals second × first.

3. Problem-solving strategies — these are general rules of thumb that give direction towards the answer; for example, the major ones in this module are “identify given and wanted”, “act it out”, “make a drawing”, “restate the problem”, “solve a simpler problem”, “check the answer”, and “learn from the solution”. The strategies will also include Polya’s plan of attack (See, Plan, Do, Check).

4. Word problems — being able to interpret problems given in words and determine the operation to use, and construct problems from equations.

5. Extension to algebra — being able to repeat concepts for when there are variables (i.e. relate algebra sentences to actions and stories).
Computation

1. **Basic fact strategies** – ways to find answers to multiplication and division facts (albeit slowly); the major ones in this module are “patterns”, “connections”, “turnarounds”, “families”, and “think multiplication”.

2. **Basic facts** – automated answers to multiplication and division facts (one-digit numbers) from $0 \times 0$ to $9 \times 9$, from $81 \div 9$ to $0 \div 0$, and for multiples of 10 (e.g. $40 \times 500$, $18 \, 000 \div 60$).

3. **Computation strategies** – ways of finding answers to multiplication and division algorithms; the major ones in this module are “separation”, “sequencing”, and “compensation”.

4. **Algorithms** – calculating answers to multiplication and division algorithms (more than one digit numbers – mental, written and calculator).

5. **Estimation strategies** – ways to find approximate answers to large-number computations; the major ones are “front end”, “rounding”, “straddling”, and “getting closer” (covered in Module O4).

6. **Estimation** – calculating approximate answers to large-number computations; uses principles to be accurate (covered in Module O4).

Connections

The major connections between operations and the other topics are to topics that use number and/or operations as the basis of their mathematics (e.g. operations, algebra, measurement, statistics and probability). Major connections are as follows.

1. **Operations and number** – an obvious connection as operations need numbers to act on. In particular, the strategies for computation relate to the numeration concepts, that is: (a) separation strategy relies on a place-value understanding of 2- to 4-digit numeration; and (b) sequencing and compensation strategies rely on a rank understanding of numeration.

1. **Operations and algebra** – again an obvious relationship as algebra is generalisation of arithmetic activities. In particular, $3x$ relates to an example like $3 \times 5$. The difference is that 5 is an actual number while $x$ is a variable.

2. **Operations and measurement** – measurement involves a lot of operations particularly with respect to formulae (e.g. area, volume).

3. **Operations and statistics and probability** – both of these involve operations (e.g. in calculating mean and chance).

As well as between operations and other topics, there are connections between topics within the two foci of addition and subtraction and multiplication and division; and between topics within operations. The major connections within operations are as follows.

1. **Addition with multiplication** – one meaning of multiplication is repeated addition.

2. **Subtraction with division** – one meaning of division is repeated subtraction.

3. **Multiplication with division** – division is the inverse of multiplication and vice versa.

4. **Concepts and problem solving** – the meanings of the operations are the basis of solving problems as they determine which operations relate to which situations.

5. **Calculation and estimation** – estimation requires calculation but they also have strategies in common (the calculation strategies help to estimate).
Big ideas

The big ideas for operations are global and come from the principles of a field and equivalence class (or extensions of these principles) – a field is a mathematical structure that is followed by operations on numbers, while an equivalence class is a mathematical structure that is followed by equals. These principles/properties are taken up in more detail in Module O4 Arithmetic and Algebra Principles. The major big ideas are as follows.

Global/teaching principles

1. Symbols tell stories. The symbols of mathematics enable the world to be described succinctly and in a generalised way (e.g. $2 \times 3 = 6$ means 2 people each catch 3 fish, or you bought two $3 chocolates, or you joined two 3 m lengths of wood, and so on).

2. Relationship vs change. Mathematics has three components – objects, relationships between objects, and changes/transformations between objects. All relationships can be perceived as changes and vice versa. This is particularly applicable to operations; multiplication of 2 and 3 can be perceived as relationship $2 \times 3 = 6$ or change $\frac{2x3}{5} \rightarrow 5$.

3. Interpretation vs construction. Things can either be interpreted (e.g. what operation for this problem, what properties for this shape) or constructed (write a problem for $2 \times 3 = 6$; construct a shape of 4 sides with 2 sides parallel).

4. Accuracy vs exactness. Problems can be solved as accurately as required (e.g. find $27 \times 38$ to the nearest 100 by rounding and estimation) or exactly (e.g. $27 \times 38 = 1026$ – basic facts and algorithms).

5. Part-part-total/whole. Two parts make a total or whole, and a total or whole can be separated to form two parts – this is the basis of numbers and operations (e.g. fraction is part-whole, ratio is part to part; multiplication is knowing parts, wanting total).

Field properties for operations

1. Closure. Numbers and an operation always give another number (e.g. $2.17 + 4.34 = 6.51$ – for any numbers $a$ and $b$, $a + b = c$ which is another number; and $2.17 \times 4.3 = 9.331$ – for any numbers $a$ and $b$, $a \times b = c$, where $c$ is another number).

2. Identity. 0 and 1 do not change things (+/− and $\times$/÷ respectively). Adding/subtracting zero leaves numbers unchanged (e.g. $9 + 0 = 9$, where 0 can equal $+1$−1, $+6$−3−3, $+11$−14+$3$, and so on). Anything multiplied by 1 is itself (e.g. for any $a$, $a \times 1 = 1 \times a = a$). Anything multiplied by 0 = 0.

3. Inverse. A change that undoes another change. Addition is undone by subtraction and vice versa (e.g. $+5$−$5 = 0$, so $2 + 5$ = 7 means $7 – 5 = 2$). Multiplication’s inverse is division and vice versa (e.g. $\times 5 \div 5 = \frac{5}{5} = 1$, so $2 \times 5$ = 10 means $10 \div 5 = 2$). This principle holds for fractions and indices (e.g. for fractions, the inverse of $\frac{3}{2}$ is the reciprocal $\frac{2}{3}$ (or 1 over $\frac{2}{3}$) because $\frac{2}{3} \times \frac{3}{2} = \frac{3}{2} \times \frac{2}{3} = 1$; for indices, the inverse of $6^3$ is $6^{-3}$ and vice versa because $3 + (−3) = 0$ and $6^3 \times 6^{−3} = 6^{3−3} = 6^0 = 1$).

4. Commutativity. Order does not matter for addition but does for subtraction (e.g. $3 + 4 = 4 + 3$, but $7 − 5 ≠ 5 − 7$). Order does not matter for multiplication but does for division (e.g. $12 \times 4 = 4 \times 12$ but $12 \div 4 ≠ 4 \div 12$; for any $a$, $b$ and $c$, $(a \times b) \times c = a \times (b \times c)$. Also known as turnarounds.

5. Associativity. What is done first does not matter for addition and multiplication but does matter for subtraction and division (e.g. $(8 + 4) + 2 = 8 + (4 + 2)$, and $(8 \times 4) \times 2 = 32 \times 2 = 64$ and $8 \times (4 \times 2) = 8 \times 8 = 64$ but $(8 \div 4) + 2 ≠ 8 \div (4 + 2)$).

6. Distributivity. Multiplication and division are distributed across addition and subtraction and act on everything (e.g. $3 \times (4 + 5) = (3 \times 4) + (3 \times 5)$; $(21 + 12) \div 3 = 21 \div 3$). Distributivity does hold for all operations (e.g. $7 \times (8 − 3) = (7 \times 8) – (7 \times 3)$, $(56 + 21) \div 7 = (56 \div 7) + (21 \div 7)$ and $(56 – 21) \div 7 = (56 \div 7) – (21 \div 7)$).
**Extension of field properties**

1. **Compensation.** Ensuring that a change is compensated for so the answer remains the same – related to inverse (e.g. $5 + 5 = 7 + 3$; $48 + 25 = 50 + 23$; $61 – 29 = 62 – 30$).

2. **Equivalence.** Two expressions are equivalent if they relate by adding or subtracting 0 and multiplying or dividing by 1; also related to inverse (e.g. $48 + 25 = 48 + 2 + 25 – 2 = 73$; $50 + 23 = 73$, $2 \times 3 = \frac{2}{3} \times 2 \div \frac{4}{6}$).

3. **Inverse relation.** The higher the second number in subtraction and division, the smaller the result (e.g. $12 \div 2 = 6 > 12 \div 3 = 4; \frac{1}{2} > \frac{1}{3}$). For division, the more you divide by, the less you have (e.g. $24 \div 8$ is less than $24 \div 6$). This principle does not apply to addition or multiplication.

4. **Triadic relationships.** When three things are related there are three problem types where each of the parts are the unknowns. For example, $4 \times 3 = 12$ can have a problem for: $? \times 4 = 12$, $4 \times ? = 12$, and $4 \times 3 = ?$. This principle holds for all four operations.

**Sequencing in operations**

This section looks at sequencing in multiplication and division, and sequencing in this module.

**Sequencing in multiplication and division**

The diagram on the right shows how the various components of meaning and computation interact. There are two columns. Concepts, principles, problem solving, problem-solving strategies, and algebra are on the meaning side while basic facts, algorithms and estimation, along with their strategies, are on the computation side.

Previous versions of this module covered all the work on the diagram on right. However, this amount of material in one module caused problems. Covering all the ideas made the model too long for half a term. There were too many new ideas, particularly because the module also included a number of models associated with concepts, namely, number line, array, area, and tree diagrams.

The principles were the most difficult; these were more novel than concepts and strategies. This caused difficulty with estimation because it is based on knowing the principles, and because as a near-last idea, it tended to be rushed. Therefore, we restrict this version of the multiplication and division module for whole numbers to concepts and strategies, giving the module a focus on getting answers as well as understanding operations as a language that describes the world. To allow this to happen, a special module, Module O4 *Arithmetic and Algebra Principles*, was developed for Level C. It was designed to cover the operation principles and includes a section on estimation before application to algebra.

Therefore, this module will focus on concepts, word problems and algebra, and basic facts and algorithms, along with relevant strategies (see new diagram on next page).
Sequencing in this module

The sequence for the module is based on the figure on right. It starts with concepts and then moves to basic facts, and algorithms. It then returns to word problem solving and concludes with extension to algebra. Concepts, word problems and algebra are part of meaning, while basic facts and algorithms are computation. Basic facts, algorithms and word problems use a strategy approach.

The figure on the right provides the basis of the vertical sequencing for the units in this module. The module has seven sections including five units as below:

**Overview**: Background information, sequencing and relation to Australian Curriculum

**Unit 1**: Basic concepts – initial meanings, models, representations for multiplication and division of whole numbers, and relationship to stories and activity

**Unit 2**: Basic facts – basic and multiple-of-ten facts, and strategies to learn them

**Unit 3**: Algorithms – mental, pen-paper and calculator forms to multiply and divide larger whole numbers, and strategies of separation, sequencing and compensation

**Unit 4**: Word problems – more detailed concepts for more difficult word problem types, and using these to interpret and construct word problems

**Unit 5**: Extension to variable – meanings, models, representations for multiplication and division of variables, and relationship to stories and activity for variables

**Appendices**: RAMR cycle components and description; AIM scope and sequence showing all modules by year level and term.

The module covers:

(a) five meanings of multiplication/division – (i) combining equal groups/partitioning equal groups, (ii) comparison, (iii) factor-factor-product, (iv) inverse of combining/partitioning (backward problem), and (v) combinations;

(b) four models – set, number line, array, and area; and

(c) three major strategies – separation, sequencing and compensation.

The module has a very large amount of work to cover, so we have selected only the crucial aspects. It will be assumed that all activities will relate to the real world of the students as far as possible and that teaching will be active, involving the students acting out situations kinaesthetically.

The modules are designed to provide resources – ideas to teach the mathematics. Within each unit, there is a sequence for the teaching, and the stages are designed to be followed in sequence. Although it is expected that teaching this module will use the RAMR framework or cycle (see Appendix A and the AIM Overview booklet for more detail), many of the ideas in this module are not given in RAMR form.
## Relation to Australian Curriculum: Mathematics

| AIM O2 meets the Australian Curriculum: Mathematics (Foundation to Year 10) |
|---|---|---|---|---|---|
| Unit 1: Basic concepts | Unit 2: Basic facts | Unit 3: Algorithms | Unit 4: Word problems | Unit 5: Extension to variable |
| Content Description | Year | O2 Unit |
|---|---|---|---|---|---|
| Recognise and represent multiplication as repeated addition, groups and arrays (ACMNA031) | P to 2 | ✓ | ✓ | | |
| Recognise and represent division as grouping into equal sets and solve simple problems using these representations (ACMNA032) | | ✓ | ✓ | | |
| Recall multiplication facts of two, three, five and ten and related division facts (ACMNA056) | 3 | | ✓ | | |
| Represent and solve problems involving multiplication using efficient mental and written strategies and appropriate digital technologies (ACMNA057) | | | ✓ | | |
| Investigate number sequences involving multiples of 3, 4, 6, 7, 8, and 9 (ACMNA074) | | ✓ | ✓ | | |
| Recall multiplication facts up to 10 × 10 and related division facts (ACMNA075) | | ✓ | ✓ | | |
| Develop efficient mental and written strategies and use appropriate digital technologies for multiplication and for division where there is no remainder (ACMNA076) | 4 | | ✓ | ✓ | |
| Explore and describe number patterns resulting from performing multiplication (ACMNA081) | | ✓ | ✓ | | |
| Solve word problems by using number sentences involving multiplication or division where there is no remainder (ACMNA082) | | ✓ | ✓ | ✓ | |
| Identify and describe factors and multiples of whole numbers and use them to solve problems (ACMNA098) | | ✓ | ✓ | ✓ | |
| Use estimation and rounding to check the reasonableness of answers to calculations (ACMNA099) | 5 | | ✓ | ✓ | ✓ | |
| Solve problems involving division by a one digit number, including those that result in a remainder (ACMNA101) | | | ✓ | ✓ | ✓ | |
| Use efficient mental and written strategies and apply appropriate digital technologies to solve problems (ACMNA291) | | | ✓ | ✓ | ✓ | |
| Use equivalent number sentences involving multiplication and division to find unknown quantities (ACMNA121) | | | ✓ | ✓ | ✓ | ✓ | |
| Identify and describe properties of prime, composite, square and triangular numbers (ACMNA122) | 6 | | ✓ | ✓ | ✓ | ✓ | |
| Select and apply efficient mental and written strategies and appropriate digital technologies to solve problems involving all four operations with whole numbers (ACMNA123) | | | ✓ | ✓ | ✓ | ✓ | |
| Explore the use of brackets and order of operations to write number sentences (ACMNA134) | | | ✓ | ✓ | ✓ | ✓ | |
| Apply the associative, commutative and distributive laws to aid mental and written computation (ACMNA151) | 7 | | ✓ | ✓ | ✓ | |
| Introduce the concept of variables as a way of representing numbers using letters (ACMNA175) | | | ✓ | ✓ | ✓ | |
| Carry out the four operations with rational numbers and integers, using efficient mental and written strategies and appropriate digital technologies (ACMNA183) | 8 | | ✓ | ✓ | ✓ | ✓ |
This unit looks at the meanings of multiplication and division. It covers the different representations (stories, actions, drawings, language, and symbols), and the initial real-world situations that are multiplication and division (combining/partitioning, comparison/change and factor-factor-product). The more complex situations and meanings of inverse and combinations are left to Unit 4, Word Problems. More detailed work on comparison and change is left to Module A3 Change and Function.

The unit covers the first three models for multiplication and division (set, number line, array). It begins by looking at the initial and most basic meaning (combining/partitioning) and using these to reinforce models and representations, and briefly overviews the other basic meanings, comparison, and factor-factor-product. It concludes by looking in more detail at symbolic representations and the role of equals. Finally, as an added complexity, two types of division are overviewed – grouping where the number of groups is unknown, and sharing where the number in each group is unknown.

### 1.1 Multiplication and division as combining and partitioning

The initial and most basic meanings are actions:

- **multiplication** is combining equal groups (e.g. combine 3 lots of 5); and
- **division** is partitioning into equal groups (e.g. partitioning 15 into lots of 3 – grouping, and partitioning 15 into 3 equal groups – sharing).

Teaching these incorporates different representations from stories to symbols.

#### Reality

**Prior experience**

It is important to differentiate multiplication and division from addition and subtraction at the start. Compare the operations with those previously introduced as you start on an operation. For example, when introducing multiplication for the first time, compare it with addition. Use examples: *How did we see 3 + 4 in our surroundings and how do we see 3 × 4? What is the same about them, what is different?*

Encourage students to recognise that: (a) addition and multiplication are both joining groups, and subtraction and division are both separating a whole into groups; (b) addition and subtraction can have unequal groups but multiplication and division must have equal groups; and (c) addition and subtraction have numbers that all refer to the same objects but multiplication and division have one number referring to groups of objects not the objects themselves.

Make the difference with counters or blocks, for example:

- $3 + 4$  
  ![3+4 diagram](image)
  a group of 3 joining a group of 4
- $3 \times 4$  
  ![3x4 diagram](image)
  3 rows each row with 4 objects

**Local knowledge**

Discuss local examples of multiplication and division.

**Kinaesthetic**

Pose problems from these local situations and act them out with the students.
Abstraction

Set model \((15 \div 5 = 3 \rightarrow \text{sharing})\)

Start with acting out a set problem (and finding the answer, if important) and getting students to model the acting out with counters. We shall look at a sharing example: **15 apples were shared equally among 5 bags, how many apples in each bag?**

Then get students to draw a picture; **15 apples were shared equally among 5 bags, how many apples in each bag?**

Number-line model \((15 \div 5 = 3 \rightarrow \text{sharing})\)

Start with acting out a number-line problem (and finding the answer if important) and getting students to model the acting out with a number line: **The relay race was 15 km, if there are 5 runners to a relay team, how far does each runner have to run?**

Array model \((15 \div 5 = 3 \rightarrow \text{sharing})\)

Start with acting out an array problem (and finding the answer if important) and getting students to model the acting out with counters in rows and columns: **15 students lined up in 5 rows, how many students in each row?**

Mathematics

**Language/Symbols**

Introduce language, for example, *Fifteen divided by five is three*. Introduce symbols and notation – *How many apples were there? [15] How many bags? [5] and How many apples in each bag? [3]*

Practice

Begin with any of the representations and complete the others, that is, relate all five representations: stories ↔ act out ↔ pictures ↔ language ↔ symbols as on right.

Use the diagram/mat on the right, fill in one area (any of the areas) and ask for the other areas to be completed. It is important to ensure that:

1. all meanings and models are covered, that is, multiplication, division (grouping and sharing) and set, number line and array models;

2. all connections are both ways, that is, students can write a story for language or symbols, and can interpret a drawing in a story or symbols; and

3. stories are used for a variety of situations – shopping, sporting, fishing, driving, TV stories, and so on, and related to grouping and sharing at home, school, and in the community.

Connections

Relate multiplication to repeated addition, and relate division to repeated subtraction.
Reflection

Validation

In reflection, begin by getting students to apply their knowledge to their world so as to validate it – to find multiplication and division in their world. A poster could be made of different examples.

Application

Set problems and applications using the operations – add in multiplication and division for two-step problems.

Extension

Flexibility. Ensure that students understand all words associated with the operations and different ways to express them (e.g. three multiplied by four, two fives, three times as many, shared among, divided by, and so on) and explore a variety of ways of doing multiplication and division.

Reversing. Make sure that students can reverse ideas – that is, can start with an equation and write stories, act out the equation and draw the equation. Can give students materials and get them to make up a play with the materials and show it to the class.

Generalising. Focus on seeing operations as generic – that 5 × 3 = 15 means that 5 bags of 3 fish makes 15 fish, 5 bottles at $3 each costs $15, 5 women run 3 km each to run the 15 km, and so on. Thus, 5 × 3 = 15 holds for every set of objects and every measure in the world.

Any examples used for student practice should include a variety of problems (contexts) and models (set, line and array).

Note: In contrast to the action meanings above, there is an inaction meaning of multiplication in terms of the superset: for example, there were 5 Holdens, 5 Fords, and 5 Toyotas, this made 15 cars. The inaction meaning of division is in terms of the subset: there were 15 cars made up of equal numbers of Fords, Holdens, Toyotas, Mazdas and Hyundais, there were 3 Holdens.

1.2 Multiplication and division as comparison

The other meanings that are introduced early are: comparison and factor-factor-product. These meanings should be taught similarly to the action meanings above – using RAMR structure in the same manner, that is:

(a) covering all representations – telling stories ↔ acting out situations and modelling them with materials ↔ drawing the situations ↔ language ↔ symbols; and

(b) using both set, number-line and array models and ensuring stories are in a variety of situations for all these models.

There are three forms of comparison: (a) numerical – greater and less than, for example, 24 > 8; (b) additive – difference, for example, 24 is 16 more than 8; and (c) multiplicative – times as much, for example, 24 is 3 times as much as 8. The last of these, multiplicative comparison, is one of the most important meanings of multiplication.

Thus, multiplicative comparison is a comparison where there is x times more than the starting number: John caught 3 fish, Jack caught 5 times as many fish as John, Jack caught 15 fish; division is the opposite: Jack caught 5 times as many fish as John, Jack caught 15 fish, John caught 3 fish.

Interestingly, comparison in multiplication is very close to looking at multiplication as a change or transformation. Relationship is the traditional way of looking at operations where three numbers are related, e.g. 3 × 5 = 15 or 15 ÷ 5 = 3. However, transformation thinks of operations as movements, e.g. 3 goes to 15 by ×5 and 15 goes to 3 by ÷5. They are best represented by arrowmath as on right.
Change is multiplication in terms of increase: John’s pay was $3, it increased 5-fold to $15 \(3 \times 5 = 15\); division meaning is in terms of decrease: the price had decreased by a multiple of 5 from $15 and was now $3 \(15 \div 5 = 3\). Thus it is important to think of multiplication as a change or transformation as well as combining and partitioning, and to use the arroowmath notation as an alternative to equations if this notation makes more sense.

1.3 Multiplication and division as factor-factor-product

Factor \(\times\) factor = product is the most important meaning, as it can include all other meanings and prevent difficulties in interpreting word problems. It is an overarching method to determine whether operations are multiplication or division. The rules are:

- multiplication is knowing the factors (F, F) and wanting the product (P); and
- division is knowing the product (P) and one factor (F) and wanting the other factor (F).

It can be used to determine the operation for more complex word problems. It requires being able to determine what are the parts and what is the total in the situation and what is known and unknown.

The idea is to use a storyboard or thinkboard to represent different problems that have an element missing, as in the summary below, and to identify part, part and total and which is unknown.

<table>
<thead>
<tr>
<th>OPERATION</th>
<th>MEANING</th>
<th>PROBLEM</th>
<th>THINKING</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiplication</td>
<td>Know factors – want product</td>
<td>Straightforward problem: I bought 15 radios at $165 per radio, how much did I pay?</td>
<td>“The 15 and $165 are factors. The wanted amount is the product. So, the operation is multiplication.”</td>
</tr>
<tr>
<td>Division</td>
<td>Know product – want a factor</td>
<td>Complex problem: Each of the 17 people received the same amount of money. The total given out was $3400. How much did each person get?</td>
<td>“The $3400 is the product. The 17 is a factor. The wanted amount is a factor. So, the operation is division.”</td>
</tr>
<tr>
<td>Division</td>
<td>Know product – want a factor</td>
<td>Complex problem: I divided the winnings among the 15 in the group, each member got $165, how much did we win?</td>
<td>“The $3400 is the product. The 17 is a factor. The wanted amount is a factor. So, the operation is division.”</td>
</tr>
</tbody>
</table>

However, it is important for real-life understanding to go further than this – to know the difference between grouping and sharing for all models, as the following shows.

1. The set model drawing on the right is 15 ÷ 5 = 3 (5 groups – how many in each group?) in terms of sharing, and 15 ÷ 3 = 5 in terms of grouping (3 in each group – how many groups?).

2. The array drawing on right is 15 ÷ 5 = 3 in terms of sharing since there are 5 rows, and 15 ÷ 3 = 5 in terms of grouping since there are 3 objects in each row.
3. The number line drawing below is $15 \div 5 = 3$ in terms of sharing since there are 5 jumps, and $15 \div 3 = 5$ in terms of grouping since there are 3 spaces in each jump.

![Number line]

For sharing to be $15 \div 3$ and for grouping to be $15 \div 5$, there has to be
- 3 bags of 5 for set model,
- 3 rows of 5 for array, and
- 3 jumps of 5 for number line.

### 1.4 Role of equals

The real meaning of the equals sign (=) is “same value as”, not a symbol for “put the answer here” or “do something” which it has become for many students.

Thus although $7 \times 4$ is 28 is represented with symbols as $7 \times 4 = 28$, it must be seen as $7 \times 4$ is the same value as 28. This means that it is possible and equally correct to show $7 \times 4 = 28$ as $28 = 7 \times 4$.

Thus many forms of equations are possible and all relate to stories, as the following shows.

<table>
<thead>
<tr>
<th>STORY</th>
<th>SYMBOLS</th>
</tr>
</thead>
<tbody>
<tr>
<td>There were 7 bags of 4 lollies.</td>
<td>$7 \times 4$</td>
</tr>
<tr>
<td>There were 7 bags of 4 lollies, how many lollies altogether?</td>
<td>$7 \times 4 =$</td>
</tr>
<tr>
<td>There were 7 bags of 4 lollies, this made 28 lollies altogether; OR</td>
<td>$7 \times 4 = 28$</td>
</tr>
<tr>
<td>There were 7 bags of 4 lollies on the table, this was the same number</td>
<td>28 = $7 \times 4$</td>
</tr>
<tr>
<td>of lollies that were on the bench which was 28.</td>
<td></td>
</tr>
<tr>
<td>There were 28 lollies on the plate, this was the same number of</td>
<td></td>
</tr>
<tr>
<td>lollies as on the bench where there were 7 bags of 4 lollies; OR</td>
<td></td>
</tr>
<tr>
<td>28 lollies is the same number of lollies as in 7 bags of 4 lollies.</td>
<td></td>
</tr>
<tr>
<td>Amy had 7 bags of 4 lollies, James had 30 lollies but he ate 2, both</td>
<td>$7 \times 4 = 30 - 2$, or</td>
</tr>
<tr>
<td>Amy and James had the same number of lollies.</td>
<td>$30 - 2 = 7 \times 4$</td>
</tr>
</tbody>
</table>

The important point here is that the following teaching components must focus on the line in the notation on the right and the equals sign in $7 \times 4 = 28$ as meaning “same as value as”:

(a) the sequence, stories $\rightarrow$ acting out/modelling $\rightarrow$ pictures $\rightarrow$ language $\rightarrow$ symbols;

(b) the reverse of this sequence; and

(c) the RAMR cycle, reality $\rightarrow$ abstraction $\rightarrow$ mathematics $\rightarrow$ reflection.

The wide variety of problems given to students should give precedence to equals as “the same value as” because it is the long-term meaning used in algebra.
Unit 2: Basic Facts

Once the concepts of the operations are introduced, it is time to teach ways to calculate the answers more quickly than representing the operation with counters and counting to get the answer. The first of the calculations to teach are those that form the basis of the later algorithms and estimation – the basic facts. While it is widely accepted that these facts have to be learnt off by heart, that is, automated by practice (drill), it is NOT something that, at this stage of these students’ schooling, should have an inordinate amount of time spent on memorising to the detriment of time for other concepts that must be accelerated. The reason for still automating facts is that automated facts are available in task situations without taking any thinking away from the task – automated facts have no cognitive load.

The basic facts are all the calculations with numbers less than 10 for multiplication and the inverse operations for division, that is:

- \(0 \times 0, 0 \times 1, 0 \times 2, ..., 0 \times 9\);
- \(1 \times 0, 1 \times 1, 1 \times 2, ..., 1 \times 9\);
- \(2 \times 0, 2 \times 1, ..., 2 \times 9; \ldots; 9 \times 0, 9 \times 1, 9 \times 2, ..., 9 \times 9\);
- \(1 \div 1, 2 \div 1, ..., 10 \div 1\);
- \(2 \div 2, 4 \div 2, ..., 18 \div 2\);
- \(3 \div 3, ..., 27 \div 3; \ldots; 9 \div 9, 18 \div 9, ..., 81 \div 9\).

2.1 Diagnosing and practising facts

Multiplication and division grids can be used to determine the strategies needed by students making errors. The errors show the × tables that are not known and the strategies to be used are determined from these errors.

- Multiplication has two strategies – “patterns” and “connections”.
- Division has only one, “think multiplication”. (“Turnarounds” and “families” can also be used.)

Diagnosis

Give students a list of random multiplication basic facts to complete. Keep all students together on the facts by reading each fact with a short time to write the answer. Mark and record the results on a multiplication grid.

![Multiplication Grid](image)

Thus the multiplication grid can be used to determine both the facts with which students make errors and the strategies needed to help students with their errors. If a student’s errors are placed on the grid, the position of the errors will determine which strategy or strategies are needed. For example, if students know one fact but not its turnaround, they need to be taught “turnarounds”.
Practice

Determine the facts that students do not know from the multiplication grid and set up practice. Use a student tracking worksheet to aid students with the process. Set up a regular daily practice program – 10 minutes per day with speed practice that uses different practice for each student depending on what students do not know (e.g. 4 minute mile, flash cards, bingo). Where possible, for each student, mark facts and graph the correct number of answers each day to compare with previous days. As well, record errors for special practice.

2.2 Teaching basic facts using strategies

There are five types of strategies for multiplication and division basic facts:

- turnarounds
- patterns
- connections
- think multiplication
- families

Strategies are used differently for multiplication and division than they are for addition and subtraction. In addition and subtraction, strategies covered a variety of facts and there was no need to focus on tables. In multiplication and division, there is more focus on tables (e.g. 4× and 7× tables).

The big ideas relevant for the facts are identity and inverse and the commutative, associative and distributive principles. These big ideas were covered in the Module Overview section.

Turnarounds (commutative principle)

This strategy is applied to all facts. It means that “larger × smaller” is the same as “smaller × larger”, for example, 7 × 3 = 3 × 7.

Body

Use the Maths Mat to make rectangles, for example, 3 squares by 4 squares, and discuss the rectangles as being representations of 3 × 4 AND 4 × 3. Include activities where the answer is 12 squares, what does the array look like? Is there only one answer?

Hand

Compare groups of counters by rotating the array model of multiplication: Put out 3 groups of 5 counters, and put out 5 groups of 3 counters. Which is larger? Are they the same? Construct an array for 4 × 7. Turn this array 90 degrees. Has the amount changed? What does this mean?

Mind

Using two dice, coloured pencils and some squared paper (an A4 sheet of 1 cm grid paper is ideal), this game reinforces the multiplication facts and the concept of area. Students can work in pairs or threes, each taking a turn.

A student rolls the two dice, and with coloured pencils, shades the rectangle formed by that number of rows and columns. For example, a dice roll of 3 and 6 means they shade a 3 × 6 rectangle. Write the numbers for the lengths in, and the total number of squares (3 × 6 is 18 squares).

The next student/s repeats the process and shades their rectangle a different colour. Continue until a student cannot fit their rectangle onto the grid. The winner is the student who last fitted their rectangle on the sheet.

For older students, vary the numbers on the dice (to develop multiplication of numbers greater than 6).
Patterns

This is one of the two major strategies for multiplication. This strategy applies to any of the tables for which there is a pattern that could help students remember the facts. The following tables have patterns:

<table>
<thead>
<tr>
<th>TABLE</th>
<th>PATTERN</th>
</tr>
</thead>
<tbody>
<tr>
<td>0×</td>
<td>Gives zero for all multiplication, e.g. $3 \times 0 = 0 \times 3 = 0$; $8 \times 0 = 0 \times 8 = 0$</td>
</tr>
<tr>
<td>1×</td>
<td>Gives the number (identity), e.g. $3 \times 1 = 1 \times 3 = 3$; $8 \times 1 = 1 \times 8 = 8$</td>
</tr>
<tr>
<td>2×</td>
<td>Doubles: 2, 4, 6, 8, 0, ... and so on</td>
</tr>
<tr>
<td>5×</td>
<td>Fives: 5, 0, 5, 0, 5, ... and so on; 5, 10, 15, ...; half the 10× tables; hands; clockface (minutes in one hour)</td>
</tr>
<tr>
<td>9×</td>
<td>Nines: tens are one less than number to be multiplied by 9, ones are such that tens and ones digits add to 9; 9, 18, 27, ... and so on</td>
</tr>
</tbody>
</table>

Teaching

These patterns can most easily be seen with a calculator, Unifix, and large and small 99 boards.

The table for the pattern is chosen (e.g. 4×). The number of the table is entered on the calculator and [+][=] pressed (e.g. [4][+] [=]). The result (4) is covered on the 99 board with a Unifix.

From there, [=] is continually pressed (adding 4) and the number shown is covered. Once sufficient numbers are covered to see the visual pattern on the 99 board, this pattern is transferred to the small 99 board by colouring squares.

The numbers coloured are discussed to arrive at the pattern.

If more reinforcement is needed, [number][+] [=] [=] [=] [=] [=] ... is pressed on the calculator and the ones or tens called out at each [=] press (see below). This enables students to verbally hear patterns. The numbers could also be written down for inspection for pattern.

- Press [5][+] [=] [=] [=] ... stating the ones position
- Press [9][+] [=] [=] [=] ... stating the ones position, then repeat, stating the tens position
- Press [4][+] [=] [=] [=] ... stating the ones position

Some teachers also do 4× and 3× with patterns on a 99 board as below:

<table>
<thead>
<tr>
<th>4×</th>
<th>Fours: This is a pattern in the ones position, 0, 4, 8, 2, 6, 0, 4, ... and so on; odd tens is 2 and 6 for ones and even tens is 0, 4 and 8 for ones, that is,</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4 8</td>
</tr>
<tr>
<td>12</td>
<td>16</td>
</tr>
<tr>
<td>20</td>
<td>24 28</td>
</tr>
<tr>
<td>32</td>
<td>36</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>3×</th>
<th>Threes: This is a “one back” pattern, that is,</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3 6 9</td>
</tr>
<tr>
<td>12</td>
<td>15 18</td>
</tr>
<tr>
<td>21</td>
<td>24 27</td>
</tr>
</tbody>
</table>
Connections

The other major strategy is for all the tables not covered by patterns. Here, the unknown table is connected to a known table using the distributive principle.

We can use counters, Unifix, dot paper and graph paper for the models. The idea is that the answers to the unknown table are found from the known tables.

<table>
<thead>
<tr>
<th>UNKNOWN TABLE</th>
<th>KNOWN TABLE(S)</th>
<th>CONNECTION</th>
<th>EXAMPLE</th>
<th>DIAGRAM</th>
</tr>
</thead>
<tbody>
<tr>
<td>3×</td>
<td>2×</td>
<td>3× is 2× + 1×</td>
<td>3×7 is the same as 2×7 + 1×7 = 14+7 = 21</td>
<td>o o o o o o o o o o o o o o o o</td>
</tr>
<tr>
<td>4×</td>
<td>2×</td>
<td>4× is double 2× or 4× is double doubles</td>
<td>4×7 is the same as 2×7 + 2×7, i.e. double 2×7 is double 14 = 28</td>
<td>o o o o o o o o o o o o o o o o</td>
</tr>
<tr>
<td>6×</td>
<td>2×, 3×</td>
<td>6× is double 3× or 6× is 3× + 3×</td>
<td>6×7 is the same as 3×7 + 3×7, i.e. double 3×7 is double 21 = 42</td>
<td>o o o o o o o o o o o o o o o o</td>
</tr>
<tr>
<td>6×</td>
<td>5×</td>
<td>6× is 5× + 1×</td>
<td>6×7 is the same as 5×7 + 1×7 = 35+7 = 42</td>
<td>o o o o o o o o o o o o o o o o</td>
</tr>
<tr>
<td>7×</td>
<td>2×, 5×</td>
<td>7× is 5× + 2×</td>
<td>7×7 is the same as 5×7 + 2×7 = 35+14 = 49</td>
<td>o o o o o o o o o o o o o o o o</td>
</tr>
<tr>
<td>8×</td>
<td>2×, 4×</td>
<td>8× is double 4× or 8× is double doubles</td>
<td>8×7 is the same as 4×7 + 4×7 = 28+28 = 56</td>
<td>o o o o o o o o o o o o o o o o</td>
</tr>
</tbody>
</table>

An excellent teaching sequence, taking into account patterns and connections, is: 2×, 5×, 9×, 4×, 8×, 3×, 6×, and 7×. This, of course, is not the only correct or appropriate sequence. For example, 2×, 4×, 8×, 5×, 3×, 6×, 9×, 7× is also efficient.

Think multiplication

This strategy is for all division facts. The division facts are reversed in thinking to multiplication form; for example, 36 ÷ 9 is rethought as “what times 9 equals 36”. It can be taught by looking at combining and partitioning: Take 3 groups of 5 counters and combine. Partition 15 into groups of 5. Repeat. State “15 divided by 5 is the same as 5 multiplied by what is 15”. Do the same for 4×7 = 28.
Families

This strategy reinforces “think multiplication” and relates multiplication to division. For each multiplication/division fact, there are four members of the fact family, for example:

\[ 3 \times 5 = 15, \ 5 \times 3 = 15, \ 15 \div 5 = 3, \ \text{and} \ 15 \div 3 = 5. \]

Families for \( 4 \times 7 \) and \( 36 \div 9 \) are:

\[ 4 \times 7 = 28, \ 7 \times 4 = 28, \ 28 \div 7 = 4, \ 28 \div 4 = 7 \]
\[ 4 \times 9 = 36, \ 9 \times 4 = 36, \ 36 \div 9 = 4, \ 36 \div 4 = 9. \]

2.3 Teaching multiple-of-ten facts

Once basic facts are known, it is important to extend them to multiple-of-ten facts. Multiple-of-ten facts are the basic facts applied to tens, hundreds, thousands, and so on.

For example:

\[ 3 \times 5 = 15 \text{ can be extended to } 30 \times 50 = 1500 \]
\[ 3 \times 4 = 12 \text{ can be extended to } 300 \times 40 = 12000. \]

Materials/pictures that are best for this are arrays (area model) and calculators.

These facts are not as simple as addition and subtraction. They are based on the facts:

\[ 10 \times 10 = 100 \]
\[ 10 \times 100 = 1000 \]
\[ 100 \times 100 = 10000 \]

Multiple-of-ten multiplication facts

One way to see the pattern with respect to multiple-of-ten multiplication facts is to use calculators for examples like those below:

\[
\begin{align*}
2 \times 4 &= \quad 5 \times 3 &= \quad 8 \times 7 = \\
20 \times 4 &= \quad 50 \times 3 &= \quad 80 \times 7 = \\
2 \times 40 &= \quad 5 \times 30 &= \quad 8 \times 70 = \\
20 \times 40 &= \quad 50 \times 30 &= \quad 80 \times 70 = \\
200 \times 40 &= \quad 500 \times 30 &= \quad 800 \times 70 = \\
\text{and so on ...}
\end{align*}
\]

Care must be taken NOT to use “add a zero”. The language, when used correctly, will assist to develop the concept:

\[ 4 \times 2 \text{ is } 4 \text{ groups of } 2 \]
\[ 4 \times 20 \text{ is } 4 \text{ groups of } 2 \text{ tens} \]
\[ 4 \times 2 = 8 \]
\[ 4 \times 20 = 8 \text{ tens} \]
\[ 40 \times 20 \text{ is } 4 \text{ groups of } 10 \times 2 \text{ groups of } 10 = 8 \text{ groups of } 10 \text{ groups of } 10 = 800 \]

Create a worksheet or do on the board using examples similar to those below: USE A CALCULATOR

1. \[ 4 \times 2 \quad 40 \times 2 \quad 40 \times 20 \]
2. \[ 8 \times 4 \quad 80 \times 4 \quad 80 \times 40 \]
3. \[ 4 \times 6 \quad 400 \times 6 \quad 40 \times 600 \]
4. \[ 30 \times 8 \quad 30 \times 80 \quad 300 \times 80 \quad \text{and so on} \]

The students are asked to complete the examples and then to look for any patterns that emerge that enable the multiple-of-ten facts to be calculated from the basic facts.
The pattern that should emerge is that the “factors of ten” are combined, that is, $4 \times 6 = 24$ means that $40 \times 600$ is 24 with three “factors of ten”, i.e. 24 000. To check that the students understand the patterns, they should be asked to complete new exercises by using patterns and without a calculator, such as the following:

<table>
<thead>
<tr>
<th>$7 \times 6$</th>
<th>$70 \times 6$</th>
<th>$70 \times 600$</th>
<th>$7000 \times 60$</th>
</tr>
</thead>
</table>

Note: Must be careful with examples like $40 \times 50$ because $4 \times 5 = 20$, so $40 \times 50 = 2000$ (three “factors of ten” but one is from the $4 \times 5$).

**Multiple-of-ten division facts**

In a similar manner, division multiple-of-ten facts could be done with examples such as those below and the “complete with calculator – look for a pattern – complete without calculator” method:

<table>
<thead>
<tr>
<th>$12 \div 6 = \phantom{0}$</th>
<th>$21 \div 7 = \phantom{0}$</th>
<th>$56 \div 8 = \phantom{0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$120 \div 6 = \phantom{0}$</td>
<td>$210 \div 7 = \phantom{0}$</td>
<td>$560 \div 8 = \phantom{0}$</td>
</tr>
<tr>
<td>$1200 \div 60 = \phantom{0}$</td>
<td>$2100 \div 70 = \phantom{0}$</td>
<td>$5600 \div 80 = \phantom{0}$</td>
</tr>
</tbody>
</table>

and so on ...

Studying these examples can lead to the pattern that, when numbers are divided, the “factors of ten” are subtracted, that is, $56000 \div 80$ is 700 because $56 \div 8 = 7$ and three “factors of ten” subtract one “factor of ten” = two “factors of ten”.

Once again, students’ understanding of the patterns can be checked by asking them to complete the following without a calculator:

$45 \div 9 \phantom{0} 450 \div 9 \phantom{0} 45000 \div 90 \phantom{0} 450000 \div 9000$

Similarly to multiplication, the $3000 \div 60$ examples are difficult as one of the “factors of ten” is from the $6 \times 5$ – so the answer is 50 not 500. Give special attention to these and always get students to check by multiplying so they are sure that they are not using a “factor of ten” twice. For example, $40000 \div 80$ has a 5 in it. Because $8 \times 5 = 40$, there are only three “factors of ten” in the 40 000 to be considered; thus, $40 \times 80$ is 500 not 5000.

**Using arrays**

The array or area model can also be used for the multiple-of-ten facts. For example, as an array, $40 \times 30$ is:

- 30	imes 10 = 300
- 30	imes 20 = 600
- 30	imes 30 = 900

Thus, $40 \times 30 = 1200$.
Unit 3: Algorithms

Multiplication and division algorithms for whole numbers are accurate (not approximate) multiplication and division for numbers of two or more digits. Today, they can be completed mentally, with pen and paper, and with calculators. As in Module O1, this module looks at them through the three strategies that can be used, which we call: separation, sequencing and compensation. We recommend that these strategies be reserved for computations of up to three digits – over this size we recommend estimation and calculators.

As also stated in Module O1, schools need to plan what they want from algorithms. The position of YDM is that there be opportunities for students to create their own ways of recording and that they should be able to show their working in some way. This requires students to examine their own thinking and, thus, develop metacognition. These creations need not be the same for each learner.

In general, the position of YDM is that all methods should be taught to all students as the methods are more important than getting answers which can be done with a calculator or, close enough, by estimation. However, in a remedial situation, as in AIM, only one method is needed. YDM’s recommendation would be to ask the students how they would do a sum – determine which method it is and, if they are happy with it, support the student to be accurate with that method. If students have no method, choose one which is common across your school and teach that.

With regard to recording, there are three ways it could be done:

(a) answer only – when full mental methods are used or when calculators are used;
(b) informal writing or doodling – numbers and drawings that assist the mental processes, mostly idiosyncratic to the learner; and
(c) pen-paper recordings that imitate the material manipulation – ways of recording that lead on from materials and can replace the material thinking.

3.1 Separation

The separation strategy involves separating the problem into parts, completing each part separately and then combining. This strategy is used in a variety of contexts – it is useful for whole numbers (e.g. $346 \times 8$), decimal numbers (e.g. $4.65 + 0.8$), measures (e.g. $3 \text{m} \ 342 \text{mm} \times 5$), mixed numbers (e.g. $\frac{3}{6} \times 4$), and algebra (e.g. $2a \times (3a + 2b)$).

It is best taught by set or array/area models. The set model is best taught with place value charts (PVCs) and size materials such as bundling sticks, MAB and money placed on top of these PVCs. The array model is best taught with pictures, although earlier work with counters and dot/graph paper for simpler examples can assist.

Set model (MAB): Multiplication ($37 \times 4 = 148$)

Reality

If it costs $37 for a meal, how much do we pay for 4 meals? Act this out with money, food and $37. For clothes, how much do we pay? Act this out with money or other materials.
**Abstraction**

Students have to recognise that 4 meals at $37 is reality for $37 \times 4$ and that this means 4 lots of 37.

**Step 1:** Put out 4 lots of 37 with tens and ones (here MAB but could be money) in tens and ones columns on PVC. Record as you go.

**Step 2:** Combine the 4 lots of 7 ones separately and trade to make tens, moving the tens to the tens PV position.

This means that 4 lots of 7 becomes 28 ones, and 2 tens and 8 ones after trading/carrying. Record as you go.

**Step 3:** Combine the tens and trade to make hundreds, moving the hundreds to the hundreds PV position. This means that 4 lots of 3 tens are 12 tens which is 1 hundred and 2 tens. Altogether this is 1 hundred, 4 tens and 8 ones. Record as you go and write the answer at the end (as in the diagrams).

**Step 4:** Get students to imagine the activity with sticks, MAB or money in their mind and to just record with numbers on pen-paper or in the mind.

**Mathematics**

*Language/Symbols*

The recording structure below is useful for students even though it is not as efficient as the traditional pen-and-paper algorithm but it is a close imitation of the process. For this method, recording (and MAB/money/sticks activities) can be largest place value first or smallest (ones) first – this means that there are two ways of recording (both of which imitate the physical material activity) as follows. At the start, it is a good idea to write what is being done on the side as shown:

<table>
<thead>
<tr>
<th>Smallest PV first (for 37 × 4, this is ones)</th>
<th>Largest PV first (for 37 × 4, this is tens)</th>
</tr>
</thead>
</table>
| $\begin{array}{c}
  \text{37} \\
  \times \text{4} \\
  \text{28} \\
  \text{120} \\
  \text{148} \\
\end{array}$ | $\begin{array}{c}
  \text{37} \\
  \times \text{4} \\
  \text{120} \\
  \text{28} \\
  \text{148} \\
\end{array}$ |
Set model (MAB): Division – sharing (92 ÷ 4 = 23)

Reality

Look at real-world situations for 92 ÷ 4, say, where you share $92 equally among 4 people.

Abstraction

Students have to recognise that $92 shared among 4 people is 92 ÷ 4 and that this means starting with 9 tens and 2 ones and having 4 groups to share among.

Step 1: Put the 92 with MAB on the PV chart and put out the 4 groups. Record as you go.

Step 2: Share the tens, trade left over tens for ones – record as you go. Ask: How many tens can each person get? Can we give one ten to each person, can we give two tens? and so on. How many tens did each person get? [2] – this is put above the line. How many tens were used? [8] – put this number below. How many tens left? [1] – do the subtraction. Trade left over tens for ones – How many ones left? [2].

Step 3: Share the ones – record as you go. Ask: How many ones can each person get? Can we give one one to each person, more than one one? How many ones did each person get? [3] – this is put above the line. How many ones were used? [12] – put this number below. How many ones left? [0] – do the subtraction. (If ones left over, these become remainder.)

Step 4: Students imagine materials in their mind and then complete algorithm without material – either pen-paper or in the mind.

Array/Area model: Multiplication (26 × 138 = 3588)

Step 1: Set up the problem/exercise as a diagram – best if problem is an area problem, e.g. how many tiles are needed for a wall 138 tiles wide and 26 tiles high?
Step 2: Break sides into hundreds, tens and ones and area into sub-areas – top is 100, 30 and 8; side is 20 and 6; and areas are as follows:

Step 3: Calculate each sub-area and then combine as follows:

Step 4: Imitate this with an algorithm showing all the sub-areas.

(Note: This algorithm can be done both ways: starting with the smaller PVs, i.e. 6 × 8 → 20 × 100, as well as starting with the larger PVs, i.e. 20 × 100 → 6 × 8)

It is best to use materials with smaller examples such as 34 × 6 and 276 ÷ 6 – get across the idea and then move on to harder exercises with thinking and symbols only.

3.2 Sequencing

This strategy is often referred to as a mental-computation strategy but it can be completed with recordings. It is a useful strategy with wide application and is best taught from the area model.

This strategy is different to the separation strategy as follows – separation breaks both components into parts (usually on place value), whereas the sequencing strategy breaks one number up (can be PV but need not be) and keeps the other number whole. The parts are done with the whole number in sequence. In these examples, we will just show the steps through abstraction to mathematics. However, it is always better to find a reality to start.

Area model: Multiplication (45 × 63 = 2835)

Step 1: Represent 45 × 63 with an area. Again a tiling problem would help.

Step 2: Keep 45 as is and break 63 into parts with which 45 can multiply easily. These would be 10s and 1s, and also 5s (1/2 a ten), 25s (1/4 a 100), and 50s (1/2 a 100). That is, the 63 would break into 50, 10 and 3.

Step 3: Calculate the components – 45 × 50 is 1/2 of 45 × 100 which is 1/2 of 4500 = 2250; 45 × 10 = 450; and 45 × 3 is 45 + 45 + 45 = 135. Altogether this is 2250 + 450 + 135 and this equals 2835.
Step 4: This is imitated with an algorithm as on right:

\[
\begin{array}{c}
 \times \\
2250 \\
450 \\
135 \\
2835
\end{array}
\]

Product 45×50

Product 45×10

Product 45×3

Product

Note: We could also have kept the 63 unseparated and separated the 45 as below, and imitate the sequencing as in the algorithm – 63×20 is double 63 tens = 1260 and 63×5 is \(\frac{1}{2}\) of 63×10.

Area model: Division – grouping (936 ÷ 4 = 234)

Step 1: Represent 936 ÷ 4 with an area. Again a tiling problem would help. It has to be thought of as a grouping – how many lots of four in 936? It is ? we have to find.

Step 2: Ask: Are there 100 fours in 936? – yes, there are 100 fours in 936, this is 400 and reduces 936 to 536, so adjust the area model. Then again ask: Are there 100 fours in 536? Yes, there are 100 fours in 536, so adjust the diagram again. This reduces the amount to 136 (as on right).

Step 3: Ask: Are there some 10 fours in the remaining 136? – well 10 fours is 40 and there are three 40s or 120 in 136. So let’s remove them and adjust our diagram (as on right).

Step 4: Look at what is left – the 16 is 4 fours, so this completes the top, and the diagram. It shows that 100 fours, 100 fours, 30 fours and 4 fours gives 936, so 100+100+30+4 = 234 is the number of fours in 936.

Step 5: Imitate this with the grouping algorithm on right.

\[
\begin{array}{c}
4 \big) 936 \\
- 400 \\
536 \\
- 400 \\
136 \\
- 120 \\
16 \\
- 4 \\
0
\end{array}
\]

0 234 lots of 4

100 lots of 4

100 lots of 4

30 lots of 4

4 lots of 4
Set model (money): Division – grouping \(2580 \div 12 = 215\)

The sequencing or grouping algorithm can be done with the set model – particularly using money.

For the example \(2580 \div 12 = 215\), the idea is to think – How many lots of $12 will make $2580? This idea can be translated into How many times can I subtract $12 from $2580?

\[
\begin{array}{cccc}
12 & ) & 2 & 5 & 8 & 0 \\
-1 & 2 & 0 & 0 & & \text{100 lots of 12} \\
-1 & 2 & 0 & 0 & & \text{100 lots of 12} \\
-1 & 2 & 0 & & \text{10 lots of 12} \\
-6 & 0 & & \text{5 lots of 12} \\
-6 & 0 & & \text{215 lots of 12} \\
\end{array}
\]

This algorithm is a good one for thinking how many – it is best to underestimate as you can always do an extra 100 or 10 – it does not have to have the “lots of” section – just put the numbers on the side, keep subtracting and then at the end, add up all the parts at the side.

Note: This “thinking” method is for the long term when you want to do the multiplication and division in the head with just the numbers. For the example \(58 \times 65\), you can leave 58 unchanged and 65 can be broken into 50, 10, and 5. So \(58 \times 65\) is \(58 \times 50 (\frac{1}{2}\) of \(58 \times 100) = 2900\), plus \(58 \times 10 = 580\), plus \(58 \times 5 (\frac{1}{2}\) of \(58 \times 10) = 290\).

This thinking can be done while doing the algorithm – adding new rows as the sequencing is followed as on right:

\[
\begin{array}{cccc}
5 & 8 \\
\times & 6 & 5 \\
\hline
2 & 9 & 0 & 0 & 58 \times 50 \\
5 & 8 & 0 & 58 \times 10 \\
2 & 9 & 0 & 58 \times 5 \\
3 & 7 & 7 & 0 & \text{Total} \\
\end{array}
\]

3.3 Compensation

This method leaves both numbers unseparated, but changes the example to an easy one and then compensates for the change → choosing numbers that are easier to multiply. Good number sense and thinking is required through the algorithm.

For the example \(64 \times 89\), you can think: This is easier to do if it was \(64 \times 100 = 6400\ and\ then,\ to\ compensate,\ I\ will\ subtract\ 64 \times 11\), that is:

\[
\begin{align*}
\text{6400} & \quad \text{64\times100} \\
-\text{640} & \quad \text{subtract 64\times10} \quad \text{subtract 64\times11} \\
-\text{64} & \quad \text{subtract 64\times1} \\
\text{5696} & \quad \text{which is 64\times89}
\end{align*}
\]

Often it is useful to use the sequencing strategy with the compensation strategy. The area model can help in some instances.

Some examples are given below.

1. \(23 \times 47\) (compensation – area model shown): To calculate this, change 23 \(\times\) 47 to 23 \(\times\) 50. Then, solve 23 \(\times\) 50 = \(\frac{1}{2}\) of 23 \(\times\) 100 = 1150, subtract 23 \(\times\) 3 or 69 from 1150, and get the answer 1150 – 69 = 1081. The area model shows this well: 23 \(\times\) 47 is the original area – a small section of 23 \(\times\) 3 is added to make 23 \(\times\) 50. 23 \(\times\) 50 is easily calculated as is 23 \(\times\) 3, so we subtract 23 \(\times\) 3 from 23 \(\times\) 50 and we have the answer to 23 \(\times\) 47.
2. \(339 \times 68\) (compensation and sequencing – area model shown): Thinking compensation, it easier to do \(339 \times 70\) and then subtract \(339 \times 2 = 678\). To do \(339 \times 70\), think of \(339\) as 300 and 30 and 9 (sequencing). Thus, the area diagrams (if used) and the algorithm are as follows:

---

3. \(136 \div 4\) (compensation – area model shown): To calculate, change 137 to something that is easily divisible by 4, say 120. Then you have to compensate by adding the difference between 120 and 136 divided by 4. Thus \(136 \div 4\) is \((120 \div 4) + (16 \div 4) = 30 + 4 = 34\).

---

4. \(1652 \div 28\) (compensation – thinking): To easily calculate, change 1652 to something that 28 divides easily into. Two options are 2800 (2800 ÷ 28 = 100) and 1400 (1400 ÷ 28 = 50). Let us choose 1400 because it is closer to the original number. Now 1652 is 1400 + 252, so to compensate for reducing 1652 to 1400, we have to add 252 ÷ 28. Since 252 is 28 \times 10 – 28, 252 ÷ 28 = 9, so the answer is 59.

---
Unit 4: Word Problems

One of the end points of multiplication and division is problem solving in difficult situations. When teaching this, students need to know how to interpret and construct word problems. To teach this, the following need to be incorporated:

- two new meanings for multiplication and division – inverse and combinations;
- determining whether a problem is multiplication/division or addition/subtraction;
- determining whether a problem is multiplication or division; and
- looking briefly at problem-solving strategies and plans of attack.

4.1 Multiplication and division as inverse and combinations

Use a variety of problems for students to learn problem solving. In Unit 1, we looked at the basic meanings of multiplication and division as:

<table>
<thead>
<tr>
<th>OPERATION</th>
<th>MEANING</th>
<th>REAL-WORLD PROBLEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiplication</td>
<td>Forward multiplication (combining equal groups)</td>
<td>4 people went out fishing. Each person caught 3 fish. How many fish were caught in total? The carpenter joined 4 pieces of wood. Each piece of wood was 3 m in length. How long was the joined wood?</td>
</tr>
<tr>
<td>Division</td>
<td>Backward division (inverse)</td>
<td>John caught some fish. He divided them equally between the 4 families. Each family got 3 fish. How many fish did John catch? The carpenter cut the wood into 4 equal length pieces. Each piece was 3 m. How long was the original piece of wood?</td>
</tr>
</tbody>
</table>

There are two additional meanings for multiplication and division which are: inverse and combinations.

**Inverse**

As an action, multiplication and division take place over time. Thus, we can know the start of the problem and want the end (forward problems), or we can know the end and want the start (backward problems).

Interestingly, if we run a combining (multiplication) backwards activity, we get a partitioning (division) and vice versa. This means that backward problems reverse or inverse the operation. In other words, **backward combining is division and backward partitioning is multiplication**.

These examples include set model and number-line problems.
Division

Forward division (partitioning into equal groups)
Joe caught 21 fish and shared them equally among his 7 friends. How many did each friend get?
The rope was 21 m long. Jill cut the rope into 7 equal pieces. How long was each piece?

Backward multiplication (inverse)
7 people went out fishing and all caught the same number of fish. In total, they caught 21 fish. How many fish did each person catch?
Fred joined 7 equal lengths of rope to make a 21 m length of rope. How long was each rope length?

The best way to see this is to think of multiplication and division as triads (the triad method). They have three components – so think of multiplication and division sums with all numbers present and then make three problems by making one of the numbers an unknown. This will mix forward and backward stories.

For example:

<table>
<thead>
<tr>
<th>PROBLEM (ALL NUMBERS)</th>
<th>THREE PROBLEMS (EACH WITH ONE UNKNOWN)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 women each ran 3 km of the relay race. The race was 12 km long. (a multiplication or combining equal groups problem)</td>
<td>Women ran a relay race where each ran 3 km. The length of the race was 12 km. How many women? (a division or inverse problem – backwards)</td>
</tr>
<tr>
<td></td>
<td>4 women ran a relay race. Each ran the same distance. The length of the race was 12 km. How far did each woman run? (a division or inverse problem – backwards)</td>
</tr>
<tr>
<td></td>
<td>4 women each ran 3 km of the relay race. How long was the race? (the multiplication problem – forwards)</td>
</tr>
</tbody>
</table>

Jess shared the $21 equally among 7 people. Each person got $3. (a division or partitioning problem)

Jess shared the money equally among 7 people. Each person got $3. How much was shared? (a multiplication or inverse problem – backwards)

Jess shared the $21 equally among a group of people. Each person got $3. How many people got a share? (a division problem but grouping – forwards)

Jess shared the $21 equally among 7 people. How much did each person get? (a division or sharing problem – forwards)

Mix up these problems and give them to students; this will really determine whether they understand multiplication and division.

Combinations

In this meaning, two things are put forward to make a composite. For example, 3 shirts can be mixed and matched with 5 pants to make 15 outfits; or a length of 3 m can be put with a width of 5 m to make a rectangle whose area is 15 m².

This type of multiplication is common in probability where tree diagrams are used.
Suppose that a die is thrown and then a coin is tossed. This gives the following diagram:

<table>
<thead>
<tr>
<th>DIE</th>
<th>COIN OUTCOMES</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>H 1H</td>
</tr>
<tr>
<td></td>
<td>T 1T</td>
</tr>
<tr>
<td>2</td>
<td>H 2H</td>
</tr>
<tr>
<td></td>
<td>T 2T</td>
</tr>
<tr>
<td>3</td>
<td>H 3H</td>
</tr>
<tr>
<td></td>
<td>T 3T</td>
</tr>
<tr>
<td>4</td>
<td>H 4H</td>
</tr>
<tr>
<td></td>
<td>T 4T</td>
</tr>
<tr>
<td>5</td>
<td>H 5H</td>
</tr>
<tr>
<td></td>
<td>T 5T</td>
</tr>
<tr>
<td>6</td>
<td>H 6H</td>
</tr>
<tr>
<td></td>
<td>T 6T</td>
</tr>
</tbody>
</table>

12 outcomes

It also means that the number of outcomes is the number of outcomes from a die (6) × the number of outcomes from the coin (2) and this is 12.

Examples of this meaning are:

**MULTIPLICATION**
5 men and 7 women meet. *How many possible couples?* (35)

**DIVISION**
A die is tossed and a spinner is spun. There are 18 outcomes. *How many outcomes from the spinner?* (3)

There can also be forward and backward combinations.

### 4.2 Determining whether multiplication-division or addition-subtraction

The first step in working out the operation for a word problem is to determine whether it is multiplication-division or addition-subtraction. *There is no need to give answers at this point.*

Look at various forms of addition, subtraction, multiplication and division stories. Particularly, compare addition and multiplication, and subtraction and division, using examples as follows.

**Addition**

\[ 3 + 4 = 7 \]

**Multiplication**

\[ 3 \times 4 = 12 \]

**Subtraction**

\[ 8 - 2 = 6 \]

**Division**

\[ 8 \div 2 = 4 \]

Students need to identify that:

(a) addition and multiplication are both joining/combining, while subtraction and division are both separating/partitioning;

(b) addition and subtraction can have different-sized groups but multiplication and division have same-sized groups; and

(c) in addition and subtraction, all numbers represent the same thing (here, black discs), while in multiplication and division, two numbers represent the same thing (black discs) but one number represents groups (not discs).
Thus, we can now differentiate between multiplication/division or addition/subtraction by looking for group size, and whether one number represents groups.

A worksheet (below) or oral questions will allow students to practise. Use the same numbers for addition and multiplication (e.g. $3 + 4$ and $3 \times 4$) and the same numbers for subtraction and division (e.g. $8 – 2$ and $8 \div 2$).

Addition and subtraction is when the answer is No for equal groups and No for one number referring to groups and multiplication and division is when the answer is Yes for equal groups and Yes for one number referring to groups.

<table>
<thead>
<tr>
<th>PROBLEM</th>
<th>Multiplication/division or Addition/subtraction?</th>
</tr>
</thead>
<tbody>
<tr>
<td>I went into a shop and bought a pie for $4 and a drink for $5. How much did I spend?</td>
<td>$+\ -\ \times\ \div$</td>
</tr>
<tr>
<td>Are groups the same size: Yes / No</td>
<td>Does one number represent number of groups: Yes / No</td>
</tr>
<tr>
<td>I went into a shop and bought 4 drinks for $5 each. How much did I spend?</td>
<td>$+\ -\ \times\ \div$</td>
</tr>
<tr>
<td>Are groups the same size: Yes / No</td>
<td>Does one number represent number of groups: Yes / No</td>
</tr>
</tbody>
</table>

### 4.3 Determining whether the problem is multiplication or division

After it is determined that the problem is multiplication/division, we now need to determine whether it is multiplication or division. This is done by interpreting problems in terms of factor-factor-product and it is assisted by constructing problems.

**Interpreting problems**

To interpret problems, look at all types of problems (including forward and backward) in terms of factor-factor-product. Students need to recognise what is a factor and what is a product, and that when the product is unknown it is multiplication and if a factor is unknown then it is division.

One way to do this is to set examples and ask students to put numbers and question mark (if unknown) beside the headings Factor and Product. Then use these numbers to circle which operation (multiplication or division) should be used (the answer is not required).

<table>
<thead>
<tr>
<th>PROBLEM</th>
<th>Multiplication or division?</th>
</tr>
</thead>
<tbody>
<tr>
<td>I took money out of bank, and shared it among my 9 nephews, each got $50, how much did I take out of the bank?</td>
<td>$\times\ \div$</td>
</tr>
<tr>
<td>Factor: _____________ Factor: ____________ Product: ____________</td>
<td></td>
</tr>
<tr>
<td>Joan went shopping, she bought 9 dresses, each cost the same, she spent $450, how much for each dress?</td>
<td>$\times\ \div$</td>
</tr>
<tr>
<td>Factor: _____________ Factor: ____________ Product: ____________</td>
<td></td>
</tr>
<tr>
<td>Sue and Jane went out together, Sue had 3 times the money Jane had, Sue had $42, how much did Jane have?</td>
<td>$\times\ \div$</td>
</tr>
<tr>
<td>Factor: _____________ Factor: ____________ Product: ____________</td>
<td></td>
</tr>
</tbody>
</table>
Constructing problems

The best way to become an expert at interpreting word problems is to learn how to construct them. To do this, give students an equation such as $5 \times 3 = 15$ and ask them to write forward combining, backward partitioning, separating, multiplicative change-comparison, and inaction/factor-factor-product problems for set, array and number-line models and for different day-to-day contexts (e.g. shopping, driving, walking, playing sport, and so on). The RAMR cycle can still be useful.

Some hints are as follows.

1. **Materials.** Give students materials to work with (e.g. the students in the class, toy models of people, cars, animals and so on, materials to set up a shop) and ask them to make up and act out a story for $5 \times 3 = 15$. *(Note: One of the best teaching methods seen for this approach was by a teacher who organised the students to do a slow motion animation of their story on computers.)*

2. **Social interaction roles.** Set up students in groups of three to make up and act out stories by assigning roles of director (leader – makes decisions when there is an impasse), continuity (continuously checks that the group is not making any errors), and script writer (records and reports on the story and how it will be acted).

3. **Triad approach.** If given $5 \times 3 = 15$, write a straightforward combining story with all numbers known, then rewrite with each of the numbers as the unknown – this will give three stories, one where the answer is found by multiplication and two where the answer is found by division. After this, write the associated division stories for $15 \div 3 = 5$ and $15 \div 5 = 3$, and then rewrite these with numbers unknown. Once again, there are three stories for each sum. And in the three cases, one of the stories is multiplication (the one with 15 unknown) and two are division. Students can be encouraged to see that 15 unknown for $5 \times 3 = 15$ is forward and 15 unknown for $15 \div 3 = 5$ and $15 \div 5 = 3$ is backward.

4. **Use factor-factor-product.** If given $5 \times 3 = 15$ and asked for a multiplicative comparison story, the 5 and 3 are factors so they have to be two factors in the story (e.g. the number of groups and the number in each group, or the start number and the change multiple, the things being combined, or the length and the width), while the 15 is the product (e.g. the total number, the end number, or the number of combinations or area) and is unknown. So have a start of 5, multiply by 3 and then find the end number. Then write this into the context as a story.

5. **Extend an existing problem.** Give students a problem, then ask the students to add further words to the problem and change the context of the problem to make it harder/easier.

Another interesting way to construct stories is to think of words that most people associate with an operation, then try to write problems using those words (and actions) that have the opposite operations. A possible way to do this with the RAMR cycle is described below.

**Reality**

Make a list of as many words the students can think of for multiplication and division that they use every day (e.g. “times”, “share”, “lots of”). Ensure that this list is flexible – sometimes, the same word may be used in different ways for multiplication and division.

**Abstraction**

Take each of these words and act out their normal meaning with two knowns and one unknown. Now give the unknown a number, and act out the problem with one of the other numbers unknown – does this change multiplication to division or vice versa? Draw diagrams of the two problems.

- *I was given 4 lots of books where each lot had 3 books, how many books do I have?* – first factor is 4, second factor is 3, product is unknown – so it is multiplication.
- *I was given 4 lots of books, each lot had the same number of books, I now had 12 books, how many books in each lot?* – first factor is 4, second factor is unknown, product is 12 – so it is division.
Mathematics

Write the two problems and determine the factors and the product and which is unknown. Relate the problems to the factor-factor-product approach, write the example down as a “sum”, and calculate the answers.

Reflection

Try to generalise the process. For example, if multiplication, how do we change that to division? Is there a pattern? Also this change from multiplication to division can be done on symbols – see on right.

4.4 Strategies and plan of attack

Solving word problems can be further improved with a few problem-solving strategies.

Some that appear useful are as follows.

1. **Act it out** and **Make a drawing** – helps visualise what is going on – but needs the right drawing.

2. **Given and wanted** – underline numbers that are given and the phrase that asks for the answer – the basis of factor-factor-product.

3. **Check answer** and **Learn from solution process** – helps to develop ideas for the next problem.

4. **Solve a simpler problem** – make numbers smaller in the problem, work out what to do with these smaller numbers, then replace them with the large numbers and calculate (called “calculator codes”).

It is also useful to have a **plan of attack** – a metacognitive process for attacking the problem. Polya found that good problem solvers nearly always used the following plan (called Polya’s four stages):

1. **See** – spend time simply working out what the problem means.

2. **Plan** – make up a plan based on strategies and knowledge to tackle the problem.

3. **Do** – implement the plan, see if it works and calculate the answer.

4. **Check** – check the answer for sensibleness and see what you can learn from what you have done.

**Allow students to solve problems, but you will need to direct them to follow these steps.**
Unit 5: Extension to Variable

Extending the meanings of multiplication and division to variables, that is, algebra, is done for one multiplication or division problem first and then for problems with two operations from multiplication, division, addition and subtraction. Note: Two-operation problems often involve more than one step. Make these problems straightforward – two or more step problems will be covered in a later module.

5.1 Introducing variable for one operation

An effective method for introducing understanding of multiplication and division to variables is to give problems with more than one unknown. This can be reinforced with physical materials such as cups and counters (and other materials which have ones and something which can be said to hold or represent any number). See RAMR lesson below.

Reality

Revise turning symbols into stories, e.g. 4 × 7 could be: I have 4 bags each with 7 chocolates.

Abstraction

Discuss with students scenarios that don’t have all the information, e.g. I bought 4 boxes of chocolates. How much did I spend?

Discuss the following: Why can’t this problem be answered?
<Don’t know the cost of chocolates>

What information would I need to be able to calculate something?
<Calculate spending if given the cost of chocolates>

What else is possible?
<Calculate cost of chocolates if given the total amount of spending>

Mathematics

Let students give amounts and calculate some answers (e.g. “what if ...“) and write the equations.

What if the chocolate cost $3? – the equation is 4 × 3 = ? (answer is $12)
What if the chocolate cost $1.50? – the equation is 4 × 1.50 = ? (answer is $6)

Discuss if it is possible to write the equation if the spending is known and the cost of the chocolate isn’t. Allow students to devise their own ways of representing this before introducing letters as variables.

What if the total spending is $24 – the equation could be:

4 × chocolate cost = 24
4 × ≡ 24
4 × C = 24 or 4C = 24 (introduce the notation where the sign is not used)

If students are struggling to write an equation, have the students write the problem in words on the thinkboard.

Allow student to construct problems using their own method before introducing letters as the Western mathematics symbol for unknown and variable.

Ensure students experience simple expressions such as 3 × a (3 bags with the same number of apples in each bag) and equations 3a = 15.
Reflection

Use these symbols to do two types of activities:

- from symbols with unknowns/variable, write the story without all the numbers; and
- from the story without all the numbers, write the symbols with unknowns/variable.

Have students give the three stories for \(5 \times 9 = 45\) where each of the stories has a different part of the equation “unknown”:

1. There were 5 teams of 9 players, how many players?
2. There were 5 teams all with the same number of players, altogether there were 45 players, how many players on each team?
3. There were teams of 9 players, there were 45 players altogether, how many teams?

Do this for more than two unknowns – I bought a number of pies, how much did I spend?

\[
c = \text{cost of one pie} \\
n = \text{number of pies} \\
T = \text{total cost}
\]

Equation \(n \times c = T\)

The above can also be done for division. For instance, I share $24 equally among 6 friends, is \(24 \div 6\) (which has an answer of 4). Thus, I share $24 equally among my friends, where I do not give the number of friends, is \(24 \div n\) or \(24 \div x\) (depending what I choose for the letter). Likewise, I share the money equally among 6 friends, is \(m \div 6\).

5.2 Cups and counters meaning of variable – one and two operations

It is possible to support students’ understanding of algebraic terms like \(4x\) by using cups and counters. In this, the counters represent numbers and the cups represent an unknown or variable amount of counters depicted by letters. As a letter represents any number, it can be considered as a cup of counters (and number). So using cups and counters can represent variable. The following activity uses 10 cups and 10 counters per student and can be used for problems with one or two operations.

1. One operation. The counters represent numbers and the cups represent an unknown or variable amount of counters depicted by letters. Thus, \(4x\) is 4 unknowns or 4 cups, as on right.

Further questions can be asked such as: show \(7x\), or \(6d\).

2. Two operations. This is an extension of problem solving with two operations in Unit 4 and is introduced the same way, by leaving one number out. Thus, we have the following:

\[
\begin{align*}
\text{I bought a chocolate for$6 and 3 pies for$5 each} & \quad 6 + 3 \times 5 \\
\text{I bought a chocolate for$6 and 3 pies} & \quad 6 + 3p \\
\text{I received a third of the$18 and gained an extra$5} & \quad 18 \div 3 + 5 \\
\text{I received a third of a sum of money and gained an extra$5} & \quad m \div 3 + 5
\end{align*}
\]

Again this can be reinforced for multiplication with cups and counters. This allows different meanings to be explored as follows:

\[
\begin{align*}
4y + 1 \text{ is} & \quad \includegraphics[width=5cm]{cups-counter-1} \\
4(y+1) \text{ is} & \quad \includegraphics[width=5cm]{cups-counter-2}
\end{align*}
\]
5.3 Interpreting and constructing

This can proceed in the same way word problems were interpreted in Unit 4 but now unknowns are given letters. For example, *Each horse was given 3 bales of hay, how many bales were needed to feed all the horses?* This problem can be stated now as $H \times 3 = B$ or as $B ÷ 3 = H$, depending on whether we interpret it as product unknown or factor unknown.

We can even have all numbers unknown. For example, *Each horse was given some bales of hay, how many bales of hay?* This problem can be stated now as $H \times B = T$, or as $T ÷ H = B$, or $T ÷ B = H$, depending on whether we interpret it as product unknown or factor unknown.

In fact, algebra allows much more flexibility on whether it is multiplication or division.

Finally, constructing problems is very flexible. *Write a story for $x \times y = z$, can be anything: I bought a pie for each of my friends and paid money for them.*
Test Item Types

This section presents instructions and the test item types for the subtests associated with the units. These will form the bases of the pre-test and post-test for this module.

Instructions

Selecting the items and administering the pre-post tests

This section provides an item bank of test item types, constructed around the units in the module. From this bank, items should be selected for the pre-test and post-test; these selected items need to suit the students and may need to be modified, particularly to make post-test items different to pre-test items. The purpose of the tests is to measure students’ performance before and after the module is taught. The questions should be selected so that the level of difficulty progresses from easier items to more difficult items. In some modules this will follow the order of the units and subtests, and in other modules it will not, depending on the sequencing across the module and within units. The pre-test items need to allow for students’ existing knowledge to be shown but without continual failure, and the post-test items need to cover all the sections in a manner that maximises students’ effort to show what they can do.

In administering the pre-test, the students should be told that the test is not related to grades, but is to find out what they know before the topic is taught. They should be told that they are not expected to know the work as they have not been taught it. They should show what they know and, if they cannot do a question, they should skip it, or put “not known” beside questions. They will be taught the work in the next few weeks and will then be able to show what they know. Stress to students that any pre-test is a series of questions to find out what they know before the knowledge is taught. They should do their best but the important questions come at the end of the module. For the post-test, the students should be told that this is their opportunity to show how they have improved.

For all tests, teachers should continually check to see how the students are going. Items in later subtests, or more difficult items within a particular subtest, should not be attempted if previous similar items in earlier subtests show strong weaknesses. Students should be allowed to skip that part of the test, or the test should be finished. Students can be marked zero for these parts.

Information on the multiplication and division item types

For multiplication and division of whole numbers, the items are divided into five subtests that align with the five units in the module. The pre-test should focus on the first three units – basic concepts, basic and extended facts, and algorithms. However, the expectation at the post-test will be that understandings would have improved to cover the more advanced concepts in the word problem unit and the extension to algebra.

The item types are also organised so that priority is given to understanding concepts and strategies and not to getting answers for numerical examples (something that calculators would be able to do). The item types also cover all the concepts and strategies – this is based on the belief that it is important for students to know all meanings and all methods as it is the methods that help in later years and which are the big ideas that recur across years and topics.
### Subtest item types

#### Subtest 1 items (Unit 1: Basic concepts)

1. Draw a circle around the equations that are true

   **If the animals were numbers, they would keep the same value**

   
   ![Equations](image)

   (a) $\times \times = \times \times \\
   (b) \times = \times \\
   (c) \times 1 = \\
   (d) \times + \times = \times \times + \times \times$
2. Using lines, join the stories to the pictures to the symbols that tell the same story.

<table>
<thead>
<tr>
<th>STORY</th>
<th>PICTURE</th>
<th>SYMBOL</th>
</tr>
</thead>
<tbody>
<tr>
<td>There were three cars each with 5 students, this made 15 students</td>
<td><img src="image1" alt="Cars" /></td>
<td>6 ÷ 2 = 3</td>
</tr>
<tr>
<td>6 students were put into 2 rows. There were 3 students in each row</td>
<td><img src="image2" alt="Students" /></td>
<td>3 × 7 = 21</td>
</tr>
<tr>
<td>Each of the 4 team members ran 3 blocks, this was 12 blocks in all.</td>
<td><img src="image3" alt="Blocks" /></td>
<td>4 × 3 = 12</td>
</tr>
<tr>
<td>10 metres of wood was cut into 2 m sections. This made 5 sections.</td>
<td><img src="image4" alt="Wood" /></td>
<td>20 ÷ 4 = 5</td>
</tr>
<tr>
<td>20 cakes were put on 4 plates, there were 5 cakes on each plate.</td>
<td><img src="image5" alt="Cakes" /></td>
<td>3 × 5 = 15</td>
</tr>
<tr>
<td>There were 3 rows of chairs. Each row had 7 chairs. Overall there were 21 chairs.</td>
<td><img src="image6" alt="Chairs" /></td>
<td>10 ÷ 5 = 2</td>
</tr>
</tbody>
</table>
Subtest 2 items (Unit 2: Basic facts)

Use the example to complete the following problems.

Example: \(7 \times 2 = 14\)
If the 7 gets bigger and the 2 stays the same, what will happen to the 14?
Answer: The 14 will get bigger

1. \(4 \times 7 = 28\)
   (a) If the 4 gets bigger and the 7 stays the same, what will happen to the 28?
       ______________________________
   (b) If the 4 gets smaller and the 28 stays the same, what will happen to the 7?
       ______________________________

2. \(8 \div 2 = 4\)
   (a) If the 8 gets smaller and the 2 stays the same, what will happen to the 4?
       ______________________________
   (b) If the 8 gets bigger and the 2 gets smaller, what will happen to the 4?
       ______________________________

3. \(6 \times 4 = 24\)
   (a) If the 4 gets bigger and the 6 stays the same, what will happen to the 24?
       ______________________________
   (b) If the 4 gets smaller and the 24 stays the same, what will happen to the 6?
       ______________________________

4. \(12 \div 3 = 4\)
   (a) If the 12 gets smaller and the 4 stays the same, what will happen to the 3?
       ______________________________
   (b) If the 12 gets bigger and the 3 gets smaller, what will happen to the 4?
       ______________________________
Subtest 3 items (Unit 3: Algorithms)

1. Answer each problem in the space provided – show all working

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>36 × 5</td>
<td>(b)</td>
<td>91 ÷ 7</td>
<td></td>
</tr>
<tr>
<td>(c)</td>
<td>12 × 23</td>
<td>(d)</td>
<td>812 ÷ 4</td>
<td></td>
</tr>
<tr>
<td>(e)</td>
<td>34 × 5</td>
<td>(f)</td>
<td>105 ÷ 5</td>
<td></td>
</tr>
<tr>
<td>(g)</td>
<td>21 × 25</td>
<td>(h)</td>
<td>612 ÷ 3</td>
<td></td>
</tr>
</tbody>
</table>
Subtest 4 items (Unit 4: Word problems)

1. If you were using a calculator, which key (\(\times\) or \(\div\)) would you press to solve each of these problems? Tick the column. Do not find the answer.

<table>
<thead>
<tr>
<th>Tick the box for the operation for each story:</th>
<th>(\times)</th>
<th>(\div)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Mark has 3 times as many pencils as Kate. Mark has 6 pencils. How many does Kate have?</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(b) 5 students had shared a packet of lollies. There were 85 lollies in the packet. How many lollies did each student get?</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(c) 8 pieces of ribbon that were each 58 cm long were cut from a roll of ribbon. How much ribbon was cut from the roll altogether?</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. If you were using a calculator, which key (+, −, \(\times\) or \(\div\)) would you press to solve each of these problems? Tick the column. Do not find the answer.

<table>
<thead>
<tr>
<th>Tick the box for the operation for each story:</th>
<th>+</th>
<th>−</th>
<th>(\times)</th>
<th>(\div)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) I went to the shop and bought 2 drinks. One cost $2 and the other one cost $3. How much did I spend?</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(b) Peter bought 4 sausage rolls at the shop, each one cost $2.50. How much did he spend?</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(c) Sue cut a rope into five even pieces. The rope was 20 m long to begin with. How long was each piece?</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
3. In each story, draw a line under the **factors**; draw a circle around the **product**.

**Do not solve the problems.**

**Examples:**

(a) Mother bakes 3 **cakes** for each of her 2 **sisters**. How many cakes altogether?

(b) There were 82 **fish** in the net to be shared among 4 **people**. How many fish did they get each?

(a) In Year 8 there are 3 classes with 18 students in each class. How many students in year 8 altogether?

(b) I bought 5 drinks and spent $9.60 in total. How much did each drink cost?

(c) Each student has 5 marbles which they put in a bag. There were 125 marbles in the bag. How many students put their marbles in the bag?

(d) There were 5 children looking after 4 birds each. How many birds were there altogether?

(e) I bought 3 phones for $120 in total. How much did each phone cost?

(f) Each student paid $6 for the class excursion. The class collected $126 altogether. How many students put in their money?
### Subtest 5 items (Unit 5: Extension to algebra)

1. Fill in the empty boxes in the table. Write the story in symbols or write a story that fits the symbols – **the first one is done for you.**

<table>
<thead>
<tr>
<th>STORY</th>
<th>SYMBOLS</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Example:</strong> I bought a pie for $4, and I also bought a drink. How much did I spend? <em>(Cost of drink is d)</em></td>
<td>d + 4</td>
</tr>
<tr>
<td>(a) I received my pay and then spent $10. How much did I have left? <em>(Amount of pay is p)</em></td>
<td></td>
</tr>
<tr>
<td>(b) Write a story to show 5 × c</td>
<td>5 × c</td>
</tr>
<tr>
<td>(c) Write a story to show b ÷ 7</td>
<td>b ÷ 7</td>
</tr>
<tr>
<td>(d) I bought 3 pairs of shorts. I also bought a coat that cost $50. How much money did I spend? <em>(Cost of shorts is S)</em></td>
<td></td>
</tr>
</tbody>
</table>
Challenge question

2. Follow a path, calculating along the way until you have finished your turn and come out of the grid.

Example: \[ 16 \div 2 \times 4 - 10 \times 5 = 110 \]

or \[ 16 \div 4 - 3 + 6 = 7 \]

Rules:
- Start at 16
- You may move down, to the right or diagonally but NOT backwards
- You may move 3, 4, 5, or 6 places each turn but do not go back once you have moved
- You may use a calculator

\[ \begin{array}{cccc}
16 & \div 4 & -3 & +6 \\
\div 2 & \times 4 & -10 & \times 5 \\
+10 & \div 10 & +4 & -5 \\
\times 6 & -4 & \div 5 & \times 10 \\
\end{array} \]

Challenges:

(a) Find the highest possible solution.

(b) Find the lowest possible solution.

(c) Find the solution nearest to 49.
Appendix A: RAMR Cycle

AIM advocates using the four components in the figure on right, reality–abstraction–mathematics–reflection (RAMR), as a cycle for planning and teaching mathematics. RAMR proposes: (a) working from reality and local culture (prior experience and everyday kinaesthetic activities); (b) abstracting mathematics ideas from everyday instances to mathematical forms through an active pedagogy (kinaesthetic, physical, virtual, pictorial, language and symbolic representations, i.e. body → hand → mind); (c) consolidating the new ideas as mathematics through practice and connections; and (d) reflecting these ideas back to reality through a focus on applications, problem solving, flexibility, reversing and generalising (see figure on right).

The innovative aspect of RAMR is that the right half develops the mathematics idea while the left half reconnects it to the world and extends it. For example, whole-number place value built around the pattern of threes where hundreds-tens-ones applies to ones, thousands, millions, and so on, can be easily extended to metrics by considering the ones to be millimetres, the thousands to be metres and the millions to be kilometres.

Planning the teaching of mathematics is based around the RAMR cycle if it is deconstructed into components that are applied to a mathematical idea. By breaking instruction down into the four parts and taking account of the pedagogical approaches described above, the cycle can lead to a structured instructional sequence for teaching the idea. The table below briefly outlines how this can be done. Prerequisite mathematical ideas are considered in the Reality and Mathematics components of the cycle, while extensions and follow-up ideas are considered in the Reflection component.

<table>
<thead>
<tr>
<th>REALITY</th>
<th>ABSTRACTION</th>
<th>MATHEMATICS</th>
<th>REFLECTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>Local knowledge: Identify local student cultural-environmental knowledge and interests that can be used to introduce the idea.</td>
<td>Representation: Develop a sequence of representational activities (physical to virtual to pictorial materials to language to symbols) that develop meaning for the mathematical idea.</td>
<td>Language/symbols: Enable students to appropriate and understand the formal language and symbols for the mathematical idea.</td>
<td>Validation: Facilitate reflection of the new idea in terms of reality to enable students to validate and justify their new knowledge.</td>
</tr>
<tr>
<td>Prior experience: Ensure existing knowledge and experience prerequisite to the idea is known.</td>
<td>Body-hand-mind: Develop two-way connections between reality, representational activities, and mental models through body → hand → mind activities.</td>
<td>Practice: Facilitate students’ practice to become familiar with all aspects of the idea.</td>
<td>Applications/problems: Set problems that apply the idea back to reality.</td>
</tr>
<tr>
<td>Kinaesthetic: Construct kinaesthetic activities, based on local context, that introduce the idea.</td>
<td>Creativity: Allow opportunities to create own representations, including language and symbols.</td>
<td>Connections: Construct activities to connect the idea to other mathematical ideas.</td>
<td>Extension: Organise activities so that students can extend the idea (use reflective strategies – flexibility, reversing, generalising, and changing parameters).</td>
</tr>
</tbody>
</table>

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## Appendix B: AIM Scope and Sequence

<table>
<thead>
<tr>
<th>Yr</th>
<th>Term 1</th>
<th>Term 2</th>
<th>Term 3</th>
<th>Term 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>N1: Whole Number Numeration Early grouping, big ideas for H-T-O; pattern of threes; extension to large numbers and number system</td>
<td>O1: Addition and Subtraction for Whole Numbers Concepts; strategies; basic facts; computation; problem solving; extension to algebra</td>
<td>O2: Multiplication and Division for Whole Numbers Concepts; strategies; basic facts; computation; problem solving; extension to algebra</td>
<td>G1: Shape (3D, 2D, Line and Angle) 3D and 2D shapes; lines, angles, diagonals, rigidity and properties; Pythagoras; teaching approaches</td>
</tr>
<tr>
<td></td>
<td>N2: Decimal Number Numeration Fraction to decimal; whole number to decimal; big ideas for decimals; tenths, hundredths and thousandths; extension to decimal number system</td>
<td>M1: Basic Measurement (Length, Mass and Capacity) Attribute; direct and indirect comparison; non-standard units; standard units; applications</td>
<td>M2: Relationship Measurement (Perimeter, Area and Volume) Attribute; direct and indirect comparison; non-standard units; standard units; applications and formulae</td>
<td>SP1: Tables and Graphs Different tables and charts; bar, line, circle, stem and leaf, and scatter graphs; use and construction</td>
</tr>
<tr>
<td>B</td>
<td>M3: Extension Measurement (Time, Money, Angle and Temperature) Attribute; direct and indirect comparison; non-standard units; standard units; applications and formulae</td>
<td>G2: Euclidean Transformations (Flips, Slides and Turns) Line-rotation symmetry; flip-slides-turns; tessellations; dissections; congruence; properties and relationships</td>
<td>A1: Equivalence and Equations Definition of equals; equivalence principles; equations; balance rule; solutions for unknowns; changing subject</td>
<td>SP2: Probability Definition and language; listing outcomes; likely outcomes; desired outcomes; calculating (fractions); experiments; relation to inference</td>
</tr>
<tr>
<td></td>
<td>N3: Common Fractions Concepts and models of common fractions; mixed numbers; equivalent fractions; relationship to percent, ratio and probability</td>
<td>O3: Common and Decimal Fraction Operations Addition, subtraction, multiplication and division of common and decimal fractions; models, concepts and computation</td>
<td>N4: Percent, Rate and Ratio Concepts and models for percent, rate and ratio; proportion; applications, models and problems</td>
<td>G3: Coordinates and Graphing Polar and Cartesian coordinates; line graphs; slope and y-intercept; distance and midpoints; graphical solutions; nonlinear graphs</td>
</tr>
<tr>
<td>C</td>
<td>A2: Patterns and Linear Relationships Repeating and growing patterns; position rules, visual and table methods; application to linear and nonlinear relations and graphs</td>
<td>A3: Change and Functions Function machine; input-output tables; arrowmath notation, inverse and backtracking; solutions for unknowns; model for applications to percent, rate and ratio</td>
<td>O4: Arithmetic and Algebra Principles Number-size, field and equivalence principles for arithmetic; application to estimation; extension to algebra; simplification, expansion and factorisation</td>
<td>A4: Algebraic Computation Arithmetic to algebra computation; modelling-solving for unknowns; simultaneous equations, quadratics</td>
</tr>
<tr>
<td></td>
<td>N5: Directed Number, Indices and Systems Concept and operations for negative numbers; concept, patterns and operations for indices; scientific notation and number systems</td>
<td>G4: Projective and Topology Visualisation; divergent and affine projections; perspective; similarity and trigonometry; topology and networks</td>
<td>SP3: Statistical Inference Gathering and analysing data; mean, mode, median, range and deviation; box and whisker graphs; large data sets, investigations and inferences</td>
<td>O5: Financial Mathematics Applications of percent, rate and ratio to money; simple and compound interest; best buys; budgeting and planning activities</td>
</tr>
</tbody>
</table>

Key: N = Number; O = Operations; M = Measurement; G = Geometry; SP = Statistics and Probability; A = Algebra.