Accelerated Inclusive Mathematics Project

YuMi Deadly Maths

AIM Module O1

Year A, Term 2

Operations:

Addition and Subtraction for Whole Numbers

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ACKNOWLEDGEMENT

The YuMi Deadly Centre acknowledges the traditional owners and custodians of the lands in which the mathematics ideas for this resource were developed, refined and presented in professional development sessions.

YUMI DEADLY CENTRE

The YuMi Deadly Centre is a research centre within the Faculty of Education at the Queensland University of Technology which is dedicated to enhancing the learning of Indigenous and non-Indigenous children, young people and adults to improve their opportunities for further education, training and employment, and to equip them for lifelong learning.

"YuMi" is a Torres Strait Islander Creole word meaning "you and me" but is used here with permission from the Torres Strait Islanders' Regional Education Council to mean working together as a community for the betterment of education for all. "Deadly" is an Aboriginal word used widely across Australia to mean smart in terms of being the best one can be in learning and life.

YuMi Deadly Centre's motif was developed by Blacklines to depict learning, empowerment, and growth within country/community. The three key elements are the individual (represented by the inner seed), the community (represented by the leaf), and the journey/pathway of learning (represented by the curved line which winds around and up through the leaf). As such, the motif illustrates the YuMi Deadly Centre's vision: *Growing community through education*.

The YuMi Deadly Centre (YDC) can be contacted at ydc@qut.edu.au. Its website is http://ydc.qut.edu.au.

DEVELOPMENT OF THE AIM MODULES

The Accelerated Inclusive Mathematics (AIM) modules were originally developed from 2010 to 2013 as part of the Accelerated Indigenous Mathematics project funded under the Commonwealth Government's *Closing the Gap: Expansion of Intensive Literacy and Numeracy* program for Indigenous students. The project aimed to assist secondary schools with beginning junior-secondary Indigenous students who were at Year 2/3 level in mathematics by developing a junior-secondary mathematics program that accelerates the students' learning to enable access to mathematics subjects in the senior-secondary years and therefore enhance employment and life chances. The project developed three years of modules (Years A to C) that are vertical sequences of learning to take students from their ability level to their age level in mathematics. The YuMi Deadly Centre acknowledges the role of the Department of Education, Employment and Workplace Relations in the development of the AIM project and these modules.

Contents

Module Overview

This is the first of the Operations modules, Module O1 *Addition and Subtraction for Whole Numbers*. It covers addition and subtraction from early childhood to early secondary years – moving from meanings and models for one-digit addition and subtraction through to algebraic representations. It will be followed by Module O2 *Multiplication and Division for Whole Numbers* focusing on multiplication and division and, in Years B and C, there are further Operations modules: Module O3 *Common and Decimal Fraction Operations* in Year B; and Module O4 *Arithmetic and Algebra Principles* and Module O5 *Financial Mathematics* in Year C.

Background information for teaching addition and subtraction operations

This section describes the two foci of operations (meaning and computation); outlines the major connections between addition and subtraction and other topics, and within addition and subtraction; and lists the big ideas for operations.

Two foci of operations

There are two components for operations. The first of these deals with **meaning or operating** – what do addition and subtraction mean (what are their concepts), where are they used in the world, and what are their properties. The second deals with **computation or calculating** – what is the answer to the operation, what ways can we work it out, and how accurate does this have to be.

Interestingly, problem solving is based on meaning, not computation, as is understanding algebraic sentences with addition and/or subtraction. Obviously algorithms are computation along with basic facts and estimation. As well, most of the operations work is based on strategies and properties of the operations (which this module will call principles). Addition and subtraction operate in the world in two ways – on individual objects, and on measures. The major components of operations are given below, followed by a description of how they interact.

Meaning

1. **Concepts** – these are meanings that define the operation in terms of the everyday world; they also cover the different models in which the meanings can be encapsulated (for addition and subtraction, there are two models – set, and number line or length); they relate story, action, drawing, language and symbols (these are called representations).

1. **Principles** – these are properties that hold no matter what the numbers are (i.e. they hold for whole numbers, decimals, fractions, and so on); for example, the commutative law, that first + second always equals second + first.

2. **Problem-solving strategies** – these are general rules of thumb that direct towards the answer; for example, the major ones in this module are “identify given and wanted”, “act it out”, “make a drawing”, “restate the problem”, “solve a simpler problem”, “check the answer”, and “learn from the solution”. The strategies will also include Polya’s plan of attack (See, Plan, Do, Check).

3. **Word problems** – being able to interpret problems given in words and determine the operation to use, and construct problems from equations.

4. **Extension to algebra** – being able to repeat concepts for when there are variables (i.e. relate algebra sentences to actions and stories).

Computation

1. **Basic fact strategies** – ways to find answers to addition and subtraction facts (albeit slowly); the major ones in this module are “count on”, “near doubles”, “near tens”, “turnarounds”, “families”, and “think addition”.

2. **Basic facts** – automated answers to addition and subtraction facts (one-digit numbers) from 0 + 0 to 9 + 9, from 18 − 9 to 0 − 0, and for multiples of 10 (e.g. 400 + 500, 13 000 − 8 000).

3. **Computation strategies** – ways of finding answers to addition and subtraction algorithms; the major ones in this module are “separation”, “sequencing”, “additive subtraction sequencing”, and “compensation”.

4. **Algorithms** – calculating answers to addition and subtraction algorithms (more than one-digit numbers – mental, written and calculator).

5. **Estimation strategies** – ways to find approximate answers to large-number computations; the major ones are “front end”, “rounding”, “straddling”, and “getting closer” (covered in Module O4).

6. **Estimation** – calculating approximate answers to large-number computations; uses principles to be accurate (covered in Module O4).

Connections

The major connections between operations and the other topics are to topics that use number and/or operations as the basis of their mathematics (e.g. algebra, measurement, statistics and probability). Major connections are as follows.

1. **Operations and number** – an obvious connection as operations need numbers to act on. In particular, the strategies for computation relate to the numeration concepts, that is: (a) separation strategy relies on a place-value understanding of 2- to 4-digit numeration; and (b) sequencing and compensation strategies rely on a rank understanding of numeration.

2. **Operations and algebra** – again an obvious relationship as algebra is generalisation of arithmetic activities. In particular, x + 3 relates to an example like 5 + 3. The difference is that 5 is an actual number while x is a variable.

3. **Operations and measurement** – measurement involves a lot of operations particularly with respect to formulae (e.g. perimeter, area).

4. **Operations and statistics and probability** – both of these involve operations (e.g. in calculating mean and chance).

As well as between operations and other topics, there are connections between topics within the two foci of addition and subtraction and multiplication and division; and between topics within operations. The major connections within operations are as follows.

1. **Addition with subtraction and multiplication** – subtraction is the inverse of addition and one meaning of multiplication is repeated addition.

2. **Subtraction with division** – one meaning of division is groups of or repeated subtraction.

3. **Concepts and problem solving** – the meanings of the operations are the basis of solving problems as they determine which operations relate to which situations.

4. **Calculation and estimation** – estimation requires calculation but they also have strategies in common (the calculation strategies help to estimate).

Big ideas

The big ideas for operations are global and come from the principles of a *field* and *equivalence class* (or extensions of these principles) – a field is a mathematical structure that is followed by operations on numbers, while an equivalence class is a mathematical structure that is followed by equals. The major big ideas are as follows.

Global principles

1. **Symbols tell stories**. The symbols of mathematics enable the world to be described succinctly and in a generalised way (e.g. 2 + 3 = 5 means caught 2 fish and then caught another 3 fish, or bought a $2 chocolate and a $3 drink, or joined a 2 m length of wood to a 3 m length, and so on).

2. **Relationship vs change**. Mathematics has three components – objects, relationships between objects, and changes/transformations between objects. All relationships can be perceived as changes and vice versa. This is particularly applicable to operations; 2 plus 3 can be perceived as relationship 2 + 3 = 5 or change $2 \xrightarrow{+3} 5$.

3. **Interpretation vs construction**. Things can either be interpreted (e.g. what operation for this problem, what properties for this shape) or constructed (write a problem for 2 + 3 = 5; construct a shape of four sides with two sides parallel).

4. **Accuracy vs exactness**. Problems can be solved accurately (e.g. find 5 275 + 3 873 to the nearest 100 – rounding and estimation) or exactly (e.g. 5 275 + 3 873 = 9 148 – basic facts and algorithms).

5. **Part-part-total/whole**. Two parts make a total or whole, and a total or whole can be separated to form two parts – this is the basis of numbers and operations (e.g. fraction is part-whole, ratio is part-to-part; addition is knowing parts, wanting total).

Field properties for operations

1. **Closure**. Numbers and an operation always give another number (e.g. 2.17 + 4.34 = 6.51 – for any numbers a and b, a + b = c which is another number; and 2.17 × 4.3 = 9.331 – for any numbers a and b, a × b = c, where c is another number).

2. **Identity**. 0 and 1 do not change things (+/− and ×/÷ respectively). Adding/subtracting zero leaves numbers unchanged (e.g. 9 +/− 0 = 9, where 0 can equal +1−1, +6−3−3, +11−14+3, and so on). Anything multiplied by 1 = itself (e.g. for any a, a × 1 = 1 × a = a). Anything multiplied by 0 = 0.

3. **Inverse**. A change that undoes another change. Addition is undone by subtraction and vice versa (e.g. + 5 − 5 = − 5 + 5 = 0, so 2 + 5 = 7 means 7 – 5 = 2). Multiplication's inverse is division and vice versa (e.g. × 5 ÷ 5 = ÷ 5 × 5 = 1, so 2 × 5 = 10 means 10 ÷ 5 = 2). This principle holds for fractions and indices (e.g. for fractions, the inverse of $\frac{2}{3}$ is the reciprocal $\frac{3}{2}$ (or 1 over $\frac{2}{3}$) because $\frac{2}{3} \times \frac{3}{2} = \frac{3}{2} \times \frac{2}{3} = 1$; for indices, the inverse of 6^3 is 6^{-3} and vice versa because 3 + −3 = 0 and $6^3 \times 6^{-3} = 6^{(3+-3)} = 6^0 = 1$).

4. **Commutativity**. Order does not matter for addition but does for subtraction (e.g. 3 + 4 = 4 + 3, but 7 – 5 ≠ 5 – 7). Order does not matter for multiplication but does for division (e.g. 12 × 4 = 4 × 12 but 12 ÷ 4 ≠ 4 ÷ 12; for any a, b and c, (a × b) × c = a × (b × c)). Also known as turnarounds.

5. **Associativity**. What is done first does not matter for addition and multiplication but does matter for subtraction and division (e.g. (8 + 4) + 2 = 8 + (4 + 2), and (8 × 4) × 2 = 32 × 2 = 64 and 8 × (4 × 2) = 8 × 8 = 64 but (8 ÷ 4) ÷ 2 ≠ 8 ÷ (4 ÷ 2)).

6. **Distributivity**. Multiplication and division are distributed across addition and subtraction and act on everything (e.g. 3 × (4 + 5) = (3 × 4) + (3 × 5); (21 – 12) ÷ 3 = (21 ÷ 3) – (12 ÷ 3)). Distributivity does hold for all operations (e.g. 7 × (8 – 3) = (7 × 8) – (7 × 3), (56 + 21) ÷ 7 = (56 ÷ 7) + (21 ÷ 7) and (56 – 21) ÷ 7 = (56 ÷ 7) – (21 ÷ 7)).

Extension of field properties

1. **Compensation**. Ensuring that a change is compensated for so the answer remains the same – related to inverse (e.g. 5 + 5 = 7 + 3; 48 + 25 = 50 + 23; 61 – 29 = 62 – 30).

2. **Equivalence.** Two expressions are equivalent if they relate by adding or subtracting 0 and multiplying or dividing by 1; also related to inverse (e.g. 48 + 25 = 48 + 2 + 25 – 2 = 73; 50 + 23 = 73; $\frac{2}{3} = \frac{2}{3} \times \frac{2}{2} = \frac{4}{6}$).

3. **Inverse relation**. The higher the second number in subtraction and division, the smaller the result (e.g. 12 ÷ 2 = 6 > 12 ÷ 3 = 4; $\frac{1}{2} > \frac{1}{3}$). For division, the more you divide by, the less you have (e.g. 24 ÷ 8 is less than 24 ÷ 6). This principle does not apply to addition or multiplication.

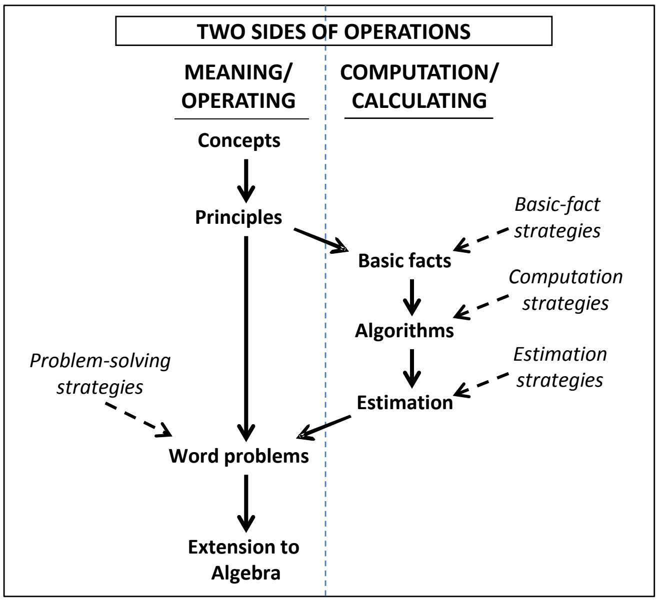
4. **Triadic relationships.** When three things are related there are three problem types where each of the parts are the unknowns. For example, 2 + 3 = 5 can have a problem for: ? + 3 = 5, 2 + ? = 5, 2 + 3 = ?. This principle holds for all four operations: a + b = c (? + b = c, a + ? = c, a + b = ?); a – b = c (? – b = c, a – ? = c, a – b = ?,); a × b = c (? × b = c, a × ? = c, a × b = ?); a ÷ b = c (? ÷ b = c, a ÷ ? = c, a ÷ b = ?).

Sequencing for operations

This section looks at sequencing in addition and subtraction and sequencing in this module.

Sequencing in addition and subtraction

The diagram on the right shows how the various components of meaning and computation interact. There are two columns. Concepts, principles, word problems, problem-solving strategies, and algebra are on the meaning side while basic facts, algorithms and estimation, along with their strategies, are on the computation side.



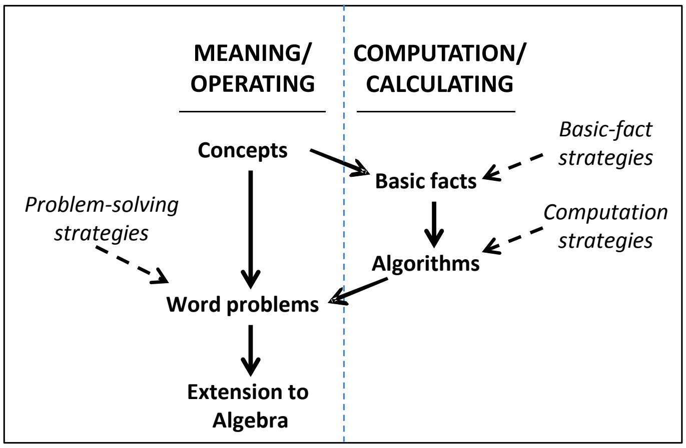
Previous versions of this module covered all the work in the diagram on right. However, this amount of material in one module caused problems. Covering all the ideas made the model too long for half a term. There were too many new ideas, particularly because the module also included a number of models associated with concepts, namely, number line, array, area, and tree diagrams.

The principles were the most difficult; these were more novel than concepts and strategies. This caused difficulty with estimation because it is based on knowing the principles, and because as a near-last idea, it tended to be rushed. Therefore, we restrict this version of the addition and subtraction module for whole numbers to concepts and strategies, giving the module a focus on getting answers as well as understanding operations as a language that describes the world. To allow this to happen, a special module, Module O4 *Arithmetic and Algebra Principles*, was developed for Level C. It was designed to cover the operation principles and includes a section on estimation before application to algebra.

Therefore, this module now focuses on concepts, word problems and algebra, and basic facts and algorithms, along with relevant strategies (see new diagram on next page).

Sequencing in this module

The sequence for the module is based on the figure on right. It starts with concepts and then moves to basic facts and algorithms, using a strategies approach. It then returns to word problem solving and concludes with extension to algebra. Concepts, word problems and algebra are part of meaning, while basic facts and algorithms are computation. Basic facts, algorithms and word problems use a strategy approach.



The figure on the right provides the basis of the vertical sequencing for the units in this module. The module has **seven sections** including five units as below:

Overview: Background information, sequencing and relation to Australian Curriculum

Unit 1: Basic concepts – initial meanings, models, and representations for addition and subtraction of whole numbers, and relationship to stories and activity

Unit 2: Basic facts – basic and multiple-of-ten facts, and strategies to learn them

Unit 3: Algorithms – mental, pen-paper and calculator form to add and subtract larger whole numbers, and strategies of separation, sequencing and compensation

Unit 4: Word problems – more detailed concepts for more difficult word problem types, and using these to interpret and construct word problems

Unit 5: Extension to variable – meanings, models, representations for addition and subtraction of variables and relationship to stories and activity when using variables

Appendices: RAMR cycle components and description; AIM scope and sequence showing all modules by year level and term.

The module covers:

(a) four meanings of addition/subtraction – (i) joining/taking away, (ii) comparison/difference, (iii) part-part-total, and (iv) inverse of joining/taking away (backward problem);

(b) two models – set and number line; and

(c) three major strategies – separation, sequencing and compensation.

The module has a very large amount of work to cover, so we have selected only the crucial aspects. It will be assumed that all activities will relate to the real world of the students as far as possible and that teaching will be active, involving the students acting out situations kinaesthetically.

The modules are designed to provide resources – ideas to teach the mathematics. Within each unit, there is a sequence for the teaching, and the stages are designed to be followed in sequence. Although it is expected that teaching this module will use the RAMR framework or cycle (see **Appendix A** and the *AIM Overview* booklet for more detail), many of the ideas in this module are not given in RAMR form.

Relation to Australian Curriculum: Mathematics

AIM O1 meets the Australian Curriculum: Mathematics (Foundation to Year 10)						
Unit 1: Basic concepts Unit 2: Basic facts Unit 3: Algorithms	Unit 4: Word problems Unit 5: Extension to variable					
Content Description	**Year**	**O1 Unit**				
		1	**2**	**3**	**4**	**5**
Represent practical situations to model addition and sharing (ACMNA004)	**P to 2**	✓			✓	✓
Represent and solve simple addition and subtraction problems using a range of strategies including counting on, partitioning and rearranging parts (ACMNA015)		✓	✓		✓	✓
Explore the connection between addition and subtraction (ACMNA029)			✓		✓	
Solve simple addition and subtraction problems using a range of efficient mental and written strategies (ACMNA030)		✓	✓	✓	✓	✓
Solve problems by using number sentences for addition or subtraction (ACMNA036)					✓	✓
Recognise and explain the connection between addition and subtraction (ACMNA054)	**3**				✓	✓
Recall addition facts for single-digit numbers and related subtraction facts to develop increasingly efficient mental strategies for computation (ACMNA055)			✓			
Describe, continue, and create number patterns resulting from performing addition or subtraction (ACMNA060)					✓	✓
Investigate and use the properties of odd and even numbers (ACMNA071)			✓			
Use estimation and rounding to check the reasonableness of answers to calculations (ACMNA099)	**5**		✓		✓	✓
Use efficient mental and written strategies and apply appropriate digital technologies to solve problems (ACMNA291)				✓	✓	
Select and apply efficient mental and written strategies and appropriate digital technologies to solve problems involving all four operations with whole numbers (ACMNA123)	**6**			✓	✓	
Apply the associative, commutative and distributive laws to aid mental and written computation (ACMNA151)	**7**		✓		✓	
Compare, order, add and subtract integers (ACMNA280)						
Introduce the concept of variables as a way of representing numbers using letters (ACMNA175)						✓

Unit 1: Basic Concepts

This unit looks at the meanings of addition and subtraction – covering the different representations (stories, actions, drawings, language, and symbols), the initial real-world situations that are addition and subtraction (joining/separating, comparison and part-part-total), and the different models (set, number line). It begins by looking at the initial and most basic meanings (joining and separating) and using these to reinforce models and representations, then overviews the other basic meaning, comparison, and the part-part-total meaning. It concludes by looking in more detail at symbolic representations and the role of equals. More complex meanings (e.g. inverse) are left to Unit 4, Word Problems.

1.1 Addition and subtraction as joining and taking away

The initial and most basic meanings for addition and subtraction are actions, **joining** (e.g. 3 joins 4) and **separating** or taking away (e.g. 7 take away 3) – teaching these incorporates different representations from stories to symbols. Applying the RAMR cycle gives the following (using example 7 – 3 = 4).

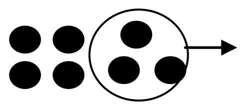
Reality

Act out problems using the students: addition – *There were 3 students sitting at a table and 2 more students joined them. How many students altogether?*; and subtraction – *7 students were eating their lunch together, 3 students had to go to a meeting and walked away. How many students were left?*

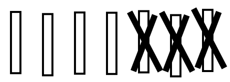
Abstraction

Set model

Start with acting out a problem (and finding the answer, if important) and getting students to model the acting out with counters: *John has 7 pencils, he gives 3 to Jane, he has 4 left*.

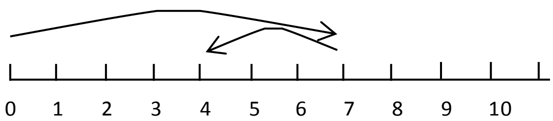


Then get students to draw a picture; *John has 7 pencils, he gives 3 to Jane, he has 4 left*.



Number-line model

Start with acting out a problem (and finding the answer if important) and getting students to model the acting out with a number line: *John walks 7 blocks and then walks 3 back, how much further to where he started?*

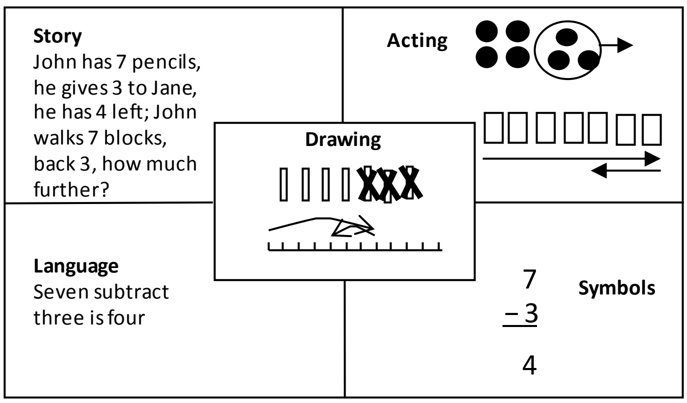


Mathematics

Introduce language: *seven subtract three is four*.

Finally introduce symbols and setting out – *How many pencils did John have?* [7] *How many did he give Jane?* [3] and *How many were left?* [4]

Then use thinkboard on right to practise the relationship between the five representations, namely, story, acting out with materials, drawing, language and symbols.



Begin with any of the representations and complete the others, that is, relate all five representations – stories ←→ act out ←→ pictures ←→ language ←→ symbols. Using the thinkboard, fill in one area (any of the areas) and ask for the other areas to be completed. It is important to ensure that:

(a) *all models are covered* – set and number line;

(b) *all connections are both ways* – students can write a story for language or symbols, and can interpret a drawing in a story or symbols; and

(c) stories are used for a *variety of situations* – shopping, sporting, fishing, driving, TV stories, and so on.

Reflection

In reflection, begin by getting students to apply their knowledge to their world – to find addition and subtraction in their world. A poster could be made of different examples.

Focus on seeing that *operations are generic* – that 7 – 3 = 4 means that 7 fish subtract 3 fish gives 4 fish, $7 subtract $3 gives $4, 7 metres subtract 4 metres gives 3 metres, and so on. Thus, 7 – 3 = 4 holds for every set of objects and every measure in the world.

Ensure that students understand all words associated with operations (e.g. sum, plus, minus) and explore a variety of ways of doing addition and subtraction. Make sure that students can reverse ideas – that is, can start with an equation and write stories, act out the equation and draw the equation. Give students materials and get them to make up a play with the materials and show it to the class.

1.2 Addition and subtraction as comparison and part-part-total

The other meanings that are introduced early are: comparison and part-part-total. These meanings should be taught similarly to the actions meanings above – using RAMR structure in the same manner, that is:

(a) covering all representations – telling stories ←→ acting out situations and modelling them with materials ←→ drawing the situations ←→ language ←→ symbols; and

(b) using both set and number-line models and ensuring stories are in a variety of situations for both these models.

Comparison

There are two ways of looking at operations – relationship and transformation. Relationship is the traditional way of looking at operations where three numbers are related, e.g. 7 – 3 = 4 or 5 + 4 = 9. However, transformation thinks of operations as movements, e.g. 7 goes to 4 by –3 and 5 goes to 9 by +4. They are best represented by arrowmath as on right. Thus it is important to think of operations as transformation as well as joining–separation and to use the arrowmath notation as an alternative to equations.

$$7 \xrightarrow{-3} 4$$

$$5 \xrightarrow{+4} 9$$

Operations as change will be part of the algebra A3 *Change and Function* module. For this module O1, we will focus on comparison. For addition and subtraction, comparison is where there is a difference between two numbers with the answer being one of the numbers. For example, addition comparison is a story like *John caught 6 fish, Jack caught 3 more fish than John, Jack caught 9 fish*; and subtraction comparison is difference: *Jack caught 9 fish, John caught 6 fish, the difference was 3 fish*.

Part-part-total

Part-part-total is one of the **most important meanings** as it can include all other meanings and prevent difficulties in interpreting word problems. To use it, you need to be able to analyse situations in terms of parts and totals and to be able to see what are the parts and what is the total in the situation and what is known and unknown. It can be used to determine the operation for more complex word problems. Then, the following holds:

- *addition* is knowing the parts (P, P) and wanting the total (T); and
- *subtraction* is knowing the total (T) and one part (P) and wanting the other part (P).

In particular, this meaning holds for situations where there is no action, for example: addition is where there are 3 Holdens and 5 Fords, and this makes 8 cars, while subtraction is where there are 8 cars and 3 were Holdens, and this leaves 5 Fords.

For both these meanings (comparison and part-part-total), the idea is to use a **thinkboard** to represent different problems that have an element missing.

Note: As will be shown in Unit 4, part-part-total can solve complex problems as the figure below shows.

OPERATION	**MEANING**	**PROBLEM**	**THINKING**
Addition	Know parts – want total	Straightforward problem: *I had $11 892 in my account, I put in $5 238, how much do I have now?*	"The $11 892 and $5 238 are parts. The wanted amount is the total. So, the operation is addition."
		Complex problem: *I took $5 238 from my account, this left $11 892, what was in the account to start with?*	"The $5 238 and $11 892 are parts. The wanted amount is the total. So, the operation is addition."
Subtraction	Know total – want a part	Straightforward problem: *I had a grant of $9 561, I spent $7 832, how much do I have left?*	"The $9 561 is the total. The $7 832 is a part. The wanted amount is a part. So, the operation is subtraction."
		Complex problem: *I added the grant to my $7 832 account, this gave me $9 561, how much was in the grant?*	"The $7 832 is a part. The $9 561 is the total. The wanted amount is a part. So, the operation is subtraction."

1.3 Role of equals

For many students the equals sign has become a symbol for "put the answer here" or "do something" when its real meaning is "same value as". Thus although 7 subtract 3 is 4 is represented with symbols as 7 – 3 = 4, it must be seen as **7 – 3 is the same value as 4**. This means that it is possible and equally correct to show 7 – 3 = 4 as 4 = 7 – 3. Thus many forms of equations are possible and all relate to stories, as the following shows.

STORY	**SYMBOLS**
There were 7 pencils, 3 were taken away.	7 – 3
There were 7 pencils, 3 were taken away. How many pencils are left?	7 – 3 =
There were 7 pencils, 3 were taken away, this left 4 behind.	7 – 3 = 4
There were 4 pencils. This was a result of 3 pencils being removed from 7 pencils.	4 = 7 – 3
Jack had 7 pencils. He gave 3 to Jane. This meant he had the same number of pencils as Bill who had 4.	7 – 3 = 4
Jack had 7 pencils. He gave 3 to Jane. Bill had 2 pencils. He got 2 more from his teacher. Therefore Jack and Bill had the same number of pencils.	7 – 3 = 2 + 2 or 2 + 2 = 7 – 3

The important point here is that the following teaching components must focus on the line in $\begin{array}{r} 7 \\ -3 \\ \hline 4 \end{array}$ and the equals sign as meaning “same as value as”:

(a) the sequence, stories → acting out/modelling → pictures → language → symbols;

(b) the reverse of this sequence; and

(c) the RAMR cycle, reality → abstraction → mathematics → reflection.

The wide variety of problems given to students should give precedence to equals as “the same value as” because it is the long-term meaning used in algebra.

Unit 2: Basic Facts

Once the concepts of the operations are introduced, it is time to teach ways to calculate the answers more quickly than representing the operation with counters and counting to get the answer. The first of the calculations to teach are those that form the basis of the later algorithms and estimation – the basic facts. While it is widely accepted that these facts have to be learnt off by heart, that is, automated by practice (drill), it is NOT something that, at this stage of these students' schooling, should have an inordinate amount of time spent on memorising to the detriment of time for other concepts that must be accelerated. The reason for still automating facts is that automated facts are available in task situations without taking any thinking away from the task – automated facts have no cognitive load.

The basic facts are all the calculations with numbers less than 10 for addition and the inverse operations for subtraction, that is:

0+0, 0+1, 0+2, ..., 0+9; 1+0, 1+1, 1+2, ..., 1+9; 2+0, 2+1, ..., 2+9; … ; 9+0, 9+1, 9+2, ..., 9+9

0−0, 1−0, 2−0, ..., 9−0; 1−1, 2−1, ..., 10−1; 2−2, ..., 11−2; ... ; 9−9, 10−9, ..., 18−9.

2.1 Diagnosing and practising facts

As we are coming from Year 8 back to the facts, the first step is to diagnose what facts are not known and to set up regular speed practice for the unknown facts.

Diagnosis

Give students a list of random basic facts to complete, keep all students together on the facts by reading each fact with a short time to write the answer. Mark the results and place on an addition and subtraction table as below.

+	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										

−	18	17	16	15	14	13	12	11	10	9	8	etc
0												
1												
2												
3												
4												
5												
6												
7												
8												
9												

	Counting on
	Turnarounds
	Doubles
	Near doubles
	Build to ten/Near ten
	Think addition

These grids can be used to determine **both the facts with which students make errors and the strategies needed to help students with their errors**. For instance, the count on, near doubles and near ten facts can be placed on an addition grid using different colours. Then, if a student's errors are placed on the grid, the position of the errors will determine which strategy or strategies are needed.

Practice

Determine the facts that students do not know from the above and set up practice. Use the student tracking worksheet to aid students with the process. Set up a regular daily practice program – 10 minutes per day with speed practice that uses different practice for each student depending on what students do not know (e.g. 4 minute mile, flash cards, bingo). Where possible, for each student, mark facts and graph the correct number of answers each day to compare with previous days. As well, record errors for special practice.

2.2 Teaching basic facts using strategies

If students do not know a lot of facts in a strategy, then the strategy should be the first step to help the student. If students are taught the strategy then they have a way of working out the facts (albeit slowly) that can be speeded up to automaticity by the practice activities. The major strategies and how to teach them are given below. At the end of each strategy, the principles that are needed and should precede these strategies are given.

Counting on

Counting on is used for addition facts where one number is 0, 1, 2 and 3. The idea is to change counting from both numbers all together (called "sum") to where only the 0, 1, 2 and 3 are counted and the other number is the start. For example, 6 + 2 is "six, seven, eight". One way to teach this is to cover the larger number, recall its number and then count on the 0, 1, 2 or 3. This can be done with a hand, or a container. For example: *Put 4 counters into your left hand. Put 2 counters in your right hand. Say "four" showing the left hand and then drop in the counters one at a time from the right to the left hand, saying "five, six".* The strategy is also used when subtracting 0, 1, 2 and 3 (counting back). For example, 7 − 2 is "seven, six, five". This can be taught by dropping counters out of a hand or a container: *Put 5 counters in the left hand, show hand and say "five", drop out 3 counters one at a time into the right hand saying "four, three, two".* (This is based on the **associative principle**.)

Turnarounds

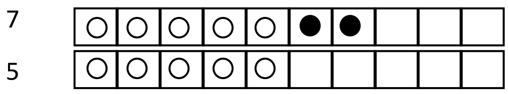
This strategy is for all facts. It nearly halves the number of facts to be learnt by showing that "bigger + smaller" (e.g. 5 + 2) is the same as "smaller + bigger" (e.g. 2 + 5). For example, 4 + 7 is 7 + 4 equals 11. This is taught by showing that the counters can be joined either way. For example: *Put out 6 counters, add 3 counters to it. Put out 3 counters, add 6 counters to it. Are the final amounts the same? Put out 9 counters. Separate into 6 and 3. Remove and add the 6. Repeat for the 3. Say "3 + 6 is the same as 6 + 3".* (This is based on the **commutative principle**.)

Near doubles

This strategy is for doubles and for facts that are 1 or 2 from doubles (e.g. 4 + 5 is “double four, eight, plus one, eight, nine”, and 6 + 8 is “double six, twelve, plus two, twelve, thirteen, fourteen”). The first teaching step is to learn the doubles. This can be taught as follows. (This is based on the **associative principle**.)

NUMBER TO DOUBLE	MENTAL IMAGE	NUMBER TO DOUBLE	MENTAL IMAGE
1	The 2 feet or 2 hands on a person	6	The 12 eggs in an egg carton
2	The 4 tyres on a car	7	The 14 days in a fortnight
3	The 6 wickets in cricket	8	The 16 legs in 2 octopi (or spiders)
4	The 8 legs of a spider	9	The 18 dots in 2 Channel 9 symbols
5	The 10 fingers on our hands		

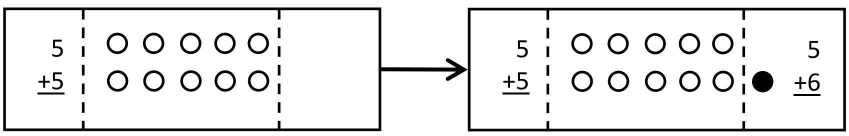
Then “doubles + 1” and “doubles + 2” are introduced. This can be taught by putting counters for the two numbers in rows in 1:1 correspondence and covering the extras, e.g. for 6 + 7 putting out a row of 6 and a row of 7 under it in line, the hand covers the extra object in the 7. The students can see a double. Lifting the hand will enable the extras to be counted. For example (double + 1): *Use sheet with two rows of ten 2 cm squares. Place 5 Unifix on the top row. Place 6 on the bottom row. Cover the extra Unifix so double 5 is showing, say “double five is 10”, reveal extra cube, and say “count on, ten, eleven”*. Similarly, another example (double + 2): *Use sheet with two rows of ten 2 cm squares. Place 6 Unifix on the top row. Place 8 on the bottom row. Cover the extra Unifix so double 6 is showing, say “double six is 12”, reveal extra 2 cubes, and say “count on, twelve, thirteen, fourteen”.* A special diagram can be used as below (using example, 7 + 5):



so 7 + 5 = (5 + 5) + 2

i.e. 7 + 5 is double 5 = 10; 11, 12

There is also a special sheet for reinforcing this strategy: Take an A4 sheet lengthways, fold both ends back 5 cm. On the folded left end, write 5 + 5 vertically. On the folded right end write 5+6 vertically. Put two rows of 5 circles on the unfolded section and one extra circle under right fold on bottom row. Fold back the right end, show left end and main section and say “5 + 5, double 5, ten”, fold back the left end, unfold the right end and say “5 + 6, double 5 and 1, ten, eleven”. The diagram below shows this aid for 5 + 6.



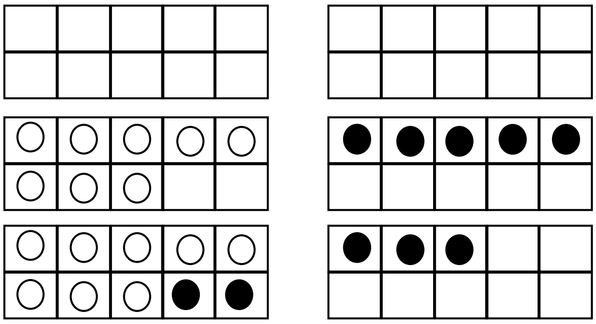
It should be noted that counting on is not the only way to do this strategy. Some students *count back* (e.g. 6 + 8 is “double eight, sixteen, back two, sixteen, fifteen, fourteen”), while some students *level pairs*, for the “two” case (e.g. 6 + 8 is “7 + 7 by adding 1 to 6 and taking 1 from 8, is double 7, 14”).

Near ten

This strategy is for all the remaining addition facts – where one number is not 0, 1,2 or 3 and not a near double (e.g. 8 + 5 and 9 + 4). Usually, one of the numbers is 7, 8 or 9. The first thing to be taught is the difference between each number 1 to 9 and the number 10. This can be done on the fingers: *Show 10 fingers on your two hands. Drop your first 7 fingers. How many left? How many to the 10? Repeat for 4, 6 and 8 fingers.* Once this is known, the strategy can be used in “build to 10” mode. For example, 9 + 5 is “9 + 1 to make 10 plus another 4, 14”. This can be taught on a sheet with two 2 × 5 arrays of 2 cm squares: *Place 8 Unifix on the first array and 5 Unifix on the second array. Say “5 + 8”. Move counters from second to first array until all 10 squares are* *covered. Say "5 + 8 is 10 + 3 is 13". Repeat this for 7 + 4 and 9 + 6.* This can be shown diagrammatically as below (for example 8 + 5):

10 grids

8 + 5 is the same as
10 + 3 if 2 is moved
from the 5 to the 8



There is another "near ten" strategy called "add ten" in which 9 + 4 is considered as 4 + 9 and thought of as 4 + 10 = 14 − 1. This could be introduced with MAB by putting out 4, adding 9, and then trading to 10 and 3. This could be followed by discussion of how this could be short circuited by directly taking the 10 and handing in a 1 (instead of adding the 9 first). A 99 board can also help, adding 9 can be seen as adding 10 and going back 1, while adding 8 can be seen as adding 10 and going back 2. (This is based on the **identity** and **inverse principles** – really adding and subtracting 2, so adding 0.)

Think addition

This strategy is used for subtraction facts. The idea is not to do subtraction but to think of the facts in addition terms. For example 8 − 3 is thought of as "what is added to 3 to make 8". To teach this, we need to show that subtraction and addition are inverses of each other: *Take 7 counters and 4 counters. Combine them to make 11. Separate them back to 7 and 4. Repeat this for 3 and 6 counters and 5 and 8 counters.* The notion of adding on to get a subtraction can also be directly modelled: *Put out 11 counters. Below them put 7 counters. Add counters to the bottom group until both groups are the same. Repeat for 8 and 13.* This strategy is to reinforce "think addition" and to relate + and −. For each addition/subtraction fact, there are four members of the family: 3 + 5 = 8, 5 + 3 = 8, 8 – 5 = 3, and 8 – 3 = 5. Families for 4 + 7 and 15 − 9 are: 4 + 7 = 11, 7 + 4 = 11,
11 – 4 = 7, 11 – 7 = 4; and 9 + 6 = 15, 6 + 9 = 15, 15 – 9 = 6, 15 – 6 = 9. (This is based on the **inverse principle**.)

2.3 Teaching multiple-of-ten facts

Multiple-of-ten facts are based on the basic strategies that are outlined above, combined with place-value concepts. This approach allows students to recognise the pattern by repeating the process. Calculators may be used to begin with to find the pattern but students should be continually looking for a pattern and then applying it when they can and moving to not requiring the calculator as soon as they are able.

Introducing the place-value language will help students generalise the process. This method can be done using a worksheet that students can work on individually or in groups.

Use a calculator to complete activity sheets with questions like below:

4 + 2 = ______	40 + 20 = ______	400 + 200 = ______
7 + 6 = ______	70 + 60 = ______	7000 + 6000 = ______
11 − 8= ______	110 − 80 = ______	1100 − 800 = ______

The first one should be relatively easy: 4 + 2 = 6. The second one, using place-value renaming, is "4 tens + 2 tens is 6 tens". The third one is "4 hundreds + 2 hundreds is 6 hundreds". Discuss patterns that would enable multiple-of-ten facts to be determined from basic facts. Using the patterns identified, move on to activities without a calculator.

Unit 3: Algorithms

Addition and subtraction algorithms for whole numbers are accurate (not approximate) addition and subtraction for numbers of two or more digits. Today, they can be completed mentally, with pen and paper, and with calculators. This module looks at them through the three strategies that can be used, which we call: **separation**, **sequencing** and **compensation**. We recommend that these strategies be reserved for computations of up to three digits – over this size we recommend estimation and calculators.

3.1 Separation

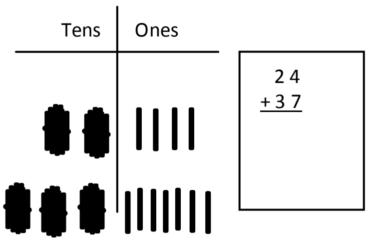
The separation strategy is separating the problem into parts, completing each part separately and then combining – that is, it is the strategy behind the traditional algorithm, although it does not need the lowest place value to be completed first in new approaches. As a strategy, separation (separate, calculate separately, and combine) is used in a variety of contexts – it is useful for whole numbers (e.g. 346 + 238), decimal numbers (e.g. 4.65 + 23.8), measures (e.g. 3 m 342 mm + 2 m 302 mm), mixed numbers (e.g. $3\,{}^{1}/_{6} + 2\,{}^{4}/_{6}$), and algebra (e.g. 3a + 2b + 5a + 3b).

The separation strategy for addition and subtraction is best taught for whole numbers with place value charts (PVCs) and size materials such as bundling sticks, MAB and money placed on top of these PVCs. Below are plans of activities on how to use bundling sticks to teach this separation method for addition. MAB and money can be used similarly to the bundling sticks. It is crucial that students experience addition with real bundling sticks, MAB and money before moving to do the virtual bundling stick, MAB and money activities.

Addition

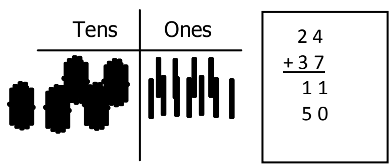
Reality

Suppose it costs $24 for food and $37 for clothes, how much do we pay? Act this out with money or other materials.



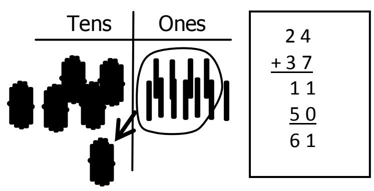
Abstraction

Put out 24 and 37 in tens and ones. Combine the tens and ones separately. Trade 10 ones for 1 ten and move to tens PV position. Record as you go and write the answer at the end (see diagrams/setting out on right). Record as students do the materials work.



Mathematics

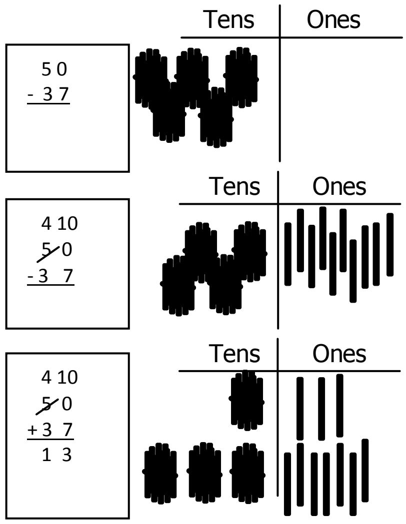
Get students to imagine activity with sticks, MAB and money in mind and to just record with numbers on pen-paper or in the mind.



Subtraction

Reality

You pay a $37 bill with $50; how much change? Act this out in a shop situation.



Abstraction

Put out 50 (**not** 37 as well) – 5 tens and 0 ones. Students need to realise they have to remove 3 tens and 7 ones. Check if they have sufficient tens and ones. Exchange or trade to have sufficient ones to subtract 7 ones. Remove the 3 tens and 7 ones – slide them down the chart. What remains at the top is the answer (here 13) – or 1 ten and 3 ones. Check by joining – should get back to 50 (see diagrams/setting out on right). Record as students do the materials work.

Mathematics

Students imagine materials in their mind and then complete algorithm without material – either pen-paper or in the mind.

3.2 Sequencing and compensation

The sequencing and compensation strategies are often referred to as the mental-computation strategies but they can be completed with recordings. Again, they are useful strategies with wide application. Further, they are best taught from the **rank** understanding of number, that is, by 99 boards and number lines. For the 99 board, operations are based on adding being across and down and subtracting being back and up. For the number line, operations are based on adding being to the right and subtracting being to the left (towards the zero). The method will be used with the number line. Note that sequencing can also be used for subtraction **additively** – where, for example, 76 − 28 is thought of as how far from 28 to 76.

Sequencing for addition (23 + 45)

Sequencing involves one number being left as is and the other number being separated, so that parts of it are added in sequence. For example 27 + 48, the 48 is separated into 40, 3 and 5 and these numbers are added in sequence 27 + 40 = 67, 67 + 3 = 70, and 70 + 5 = 75.

Reality

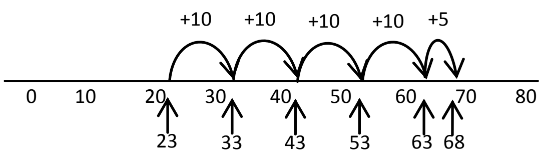
Look for real-world instances of addition and subtraction that have a length orientation so that the number line is appropriate as a model. Act out the number line by walking the students – start at the first number and then move forward the second number – large steps for tens and small steps for ones – so 23 + 45 is 23, 33, 43, 53, 63 (large steps), 64, 65, 66, 67, 68 (small steps).

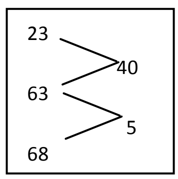
Abstraction

For example, 23 + 45 = 68, use a number line from 1 to 100 with 10s marked in. Start on 23 and move 45 to the right, 4 tens forward then 5 ones forward to get to 68.

Mathematics

A recording procedure as on right can be used to imitate what happens on the line.





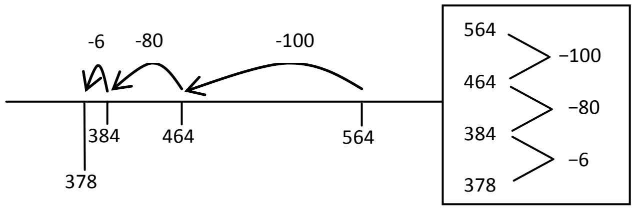
Sequencing for subtraction (564 − 186)

Reality

Again look for real-world instances and walk a subtraction.

Abstraction

Use the number line on right as shown. Because numbers are large, there are no markings on the line.



Mathematics

Use the recording procedure on right.

Sequencing for additive subtraction (620 − 332)

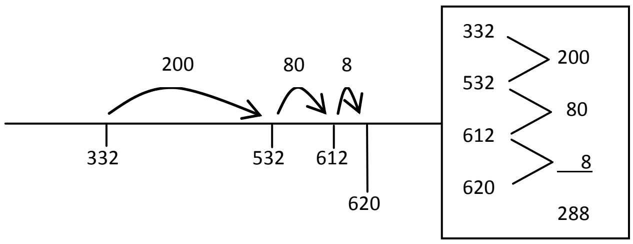
This is where you think of subtraction in terms of addition and move from the second number to the first number.

Reality

Again look for real-world instances and walk an additive subtraction from small to large number (have to pick a path).

Abstraction

Use the number line on right as shown – note the idea here is to follow a path from small to large number – note also that you can underestimate how much to increment in each leap as you can just simply do another leap. The answer is found by adding up the leaps.



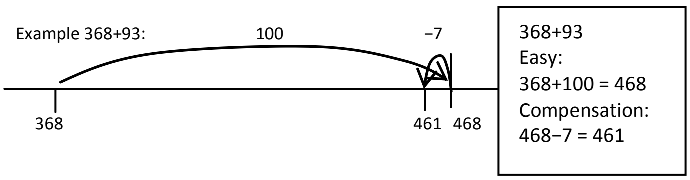
Mathematics

Use the recording procedure on right.

Compensation

The compensation method leaves both numbers unseparated, but changes the example to an easy one and then compensates for the change. For the example 27 + 48, it is seen that adding 50 is easier so this is done and then 2 is subtracted to compensate for 50 being 2 too large, that is, 27 + 50 = 77 and 77 – 2 = 75. 99 boards and number lines are again the best materials.

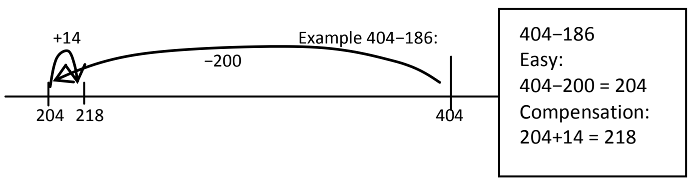
Reality



Similar to sequencing, try to find real-world situations that involve length so that the number line is appropriate. Again walk number lines with bodies before move to hand-drawn lines.

Abstraction

Use the lines as on right for addition and subtraction – do the easy sum and then compensate.



Mathematics

Use the recording procedure as on the right.

3.3 Recording

Schools need to plan what they want from algorithms. The position of YDM is that there should be opportunities for students to create their own ways of recording, and that they should be able to show their working in some way. This requires students to examine their own thinking and, thus, develop metacognition. These creations need not be the same for each learner.

In general, the position of YDM is that all methods should be taught to all students as the methods are more important than getting answers which can be done with a calculator or, close enough, by estimation. However, in a remedial situation, as in AIM, only one method is needed. YDM's recommendation would be to ask the students how they would do a sum – determine which method it is and, if they are happy with it, support the student to be accurate with that method. If students have no method, choose one which is common across your school and teach that.

With regard to recording, there are three ways it could be done: (a) answer only – when full mental methods used or when calculators are used; (b) informal writing or doodling – numbers and drawings that assist the mental processes, mostly idiosyncratic to the learner; and (c) pen-paper recordings that imitate the material manipulation – ways of recording that lead on from materials and can replace the material thinking.

Unit 4: Word Problems

This unit looks at one of the end points of addition and subtraction and that is problem solving in difficult situations. It discusses how to teach students how to interpret and construct word problems. It also looks at problem-solving strategies and plans of attack.

4.1 Addition and subtraction as inverse actions

In Unit 1, we looked at the basic meanings of addition and subtraction as follows:

ADDITION	SUBTRACTION
Joining	Separation or taking away
Comparison (the larger number)	Comparison (the smaller number or difference)
Part-part-total (know parts and not total)	Part-part-total (know total and one part and not other part)

We now look at a further meaning and how these meanings can be applied to problem solving.

Forward and backward

Addition and subtraction are inverses of each other, for example, 4 add 3 is 7 and 7 subtract 3 is 4. Thus, addition is joining and subtraction is the inverse of joining (and subtraction is take-away and addition is the inverse of take-away).

The question is how do we think of inverses? The first point is that as an action, addition and subtraction take place over time. So we can know the start of the problem and want the end, or we can know the end and want the start – we shall call the former "forward problems" and the latter "backward problems". The backward problems are the inverses.

Thus, if we run a joining backwards we get a separation and vice versa. This means that backward problems reverse the operation. In other words, backward joining is subtraction and backward separating is addition. These examples may help (they include set model and number-line problems).

OPERATION	MEANING	REAL-WORLD PROBLEM
Addition	Forward addition	Joe caught 3 fish in the morning and 5 fish in the afternoon. How many fish did he catch?
		The crane lifted 3 extra storeys onto the top of the 2 storey building. How high is the building now?
	Backward subtraction	Joe caught some fish. He gave away 5 fish and this left 3 fish. How many fish did he catch in total?
		The crane knocked off 3 storeys, 2 storeys are left. How high was the building to start with?
Subtraction	Forward Subtraction	Joe caught 7 fish and gave away 4. How many fish did Joe keep?
		The building had 5 storeys, the crane knocked off the 2 top storeys, how many storeys are left?
	Backward addition	Joe caught 4 fish in the morning and some more in the afternoon. He caught 7 fish overall. How many were caught in the afternoon?
		The crane lifted some extra storeys onto the top of the 2 storey building, the building is now 5 storeys high, how many extra storeys were lifted by the crane?

Using triads

The best way to see this is to think of addition and subtraction as **triads**. Addition and subtraction have three components (part-part-total) – so think of addition and subtraction sums with all numbers present and then make three problems by making, in turn, each of the three numbers an unknown. For example:

PROBLEM (ALL NUMBERS)	**THREE PROBLEMS (EACH WITH ONE UNKNOWN)**
Sue ran 3 km, then she ran another 4 km, altogether she ran 7 km.	Sue ran some km, then she ran another 4 km, altogether she ran 7 km, how many km in the first run? [backward joining – inverse – subtraction] Sue ran 3 km, then she ran some more km, altogether she ran 7 km, how many extra km did she run? [backward joining – inverse – subtraction] Sue ran 3 km, then she ran another 4 km, altogether how many km did she run? [forward joining – addition]
There were 8 boys playing, 3 ran away, there are now 5 boys playing.	There were some boys playing, 3 ran away, there are now 5 boys playing, how many at the start? [backward separating– inverse – addition] There were 8 boys playing, some ran away, there are now 5 boys playing, how many ran away? [forward separating – subtraction] There were 8 boys playing, 3 ran away, how many are left playing? [forward separating – subtraction]

Mix up these problems and give them to students; this will really determine whether they understand addition and subtraction.

Interpreting problems

To interpret problems, look at all types of problems (including forward and backward) in terms of part-part-total. Get students to recognise what is a part and what is total and to use the rule that total unknown means addition and part unknown means subtraction.

One way to do this is to set examples like below, then ask students to put numbers and question mark (if unknown) in Parts and Total, then use this to circle which operation should be used (answers not required).

PROBLEM	**Addition or subtraction?**
I took $458 out of bank, this left $1 045 in the bank. How much money did I have in the bank to start? Part: ______________ Part: ______________ Total: ______________	+ –
I had a lot of books, my friend gave me 27 more, this made 82 books, how many did I have at the start? Part: ______________ Part: ______________ Total: ______________	+ –
And so on	

4.2 Constructions

The best way to become an expert at interpreting word problems is to learn how to construct them. To do this, give students an equation such as 5 + 8 = 13 and ask them to write forward joining, backward separating, change-comparison, and inaction problems for set and number-line models and for different day-to-day contexts (e.g. shopping, driving, walking, playing sport, and so on). The RAMR cycle can still be useful. Some hints are as follows.

1. **Materials**. Give students materials to work with (e.g. toy models of people, each other, set up a shop) and ask them to make up and act out their story. One of the best teaching methods for this approach was by a teacher who organised the students to do a claymation animation of their story on computers.

2. **Social interaction roles**. Set up students in groups of three to make up and act out stories by assigning roles of director (leader – makes decisions when there is an impasse), continuity (continuously checks not making any errors), and script writer (records, and reports on the story and how it will be acted).

3. **Triad approach**. Use the triad method from 4.1. If given 5 + 8 = 13, ask students to write a straightforward joining story with all numbers known, then rewrite with one unknown. This will give three stories, one for 5 + 8 = ?, one for ? + 8 = 13, and one for 5 + ? = 13; the first will be forward and the last two will be backward. After this, write the associated subtraction stories for 13 – 5 = 8 and 13 – 8 = 5 and then rewrite each of these with one unknown. The students can be encouraged to see that (a) each equation will give three problems, one for each unknown; (b) 13 unknown means addition is the operation to get the answer; and (c) right-hand side unknown means problems forward, otherwise problems backward.

4. **Use part-part-total**. If given 5 + 8 = 13 and asked for a comparison, the 5 and 8 are parts so they have to be the initial number and the increase while the 13 is the large number and is unknown. So have a start of 5 and have an increase of 8 and then find the end. Then write this into the context as a story.

5. **Extend an existing problem**. Give students a problem, then ask the students to add further words to the problem and change the context of the problem to make it harder/easier.

Another interesting way to construct stories is to think of words that most people associate with an operation, then try to write problems using those words (and actions) that have the opposite operations. A possible way to do this with the RAMR cycle is described below.

Reality

Make a list of as many words as the students can think of for addition and subtraction that they use every day. Ensure that this list is flexible – sometimes, the same word may be used in a different ways for addition and subtraction.

Abstraction

Take each of these words and act out their normal meaning with two knowns and one unknown. Now give the unknown a number, and act out the problem with one of the other numbers unknown – does this change addition to subtraction or vice versa? Draw diagrams of the two problems. For example,

I had 4 books, I got 3 more, how many books do I have?	*Addition, part = 4, part = 3, total = unknown*
I had 4 books, I now have 7, how many more did I get?	*Subtraction, part = 4, total = 7, part = unknown*

Mathematics

Write the two problems and determine the parts and the total and which is unknown. Relate the problems to the part-part-total approach, write the sums vertically and calculate the answers.

Reflection

Try to generalise the process. For example, if addition, how do we change that to subtraction? Is there a pattern? Also this change from addition to subtraction can be done on symbols – see on right.

$$\begin{array}{r} 47 \\ \underline{+34} \\ \underline{\ ??} \end{array} \longrightarrow \begin{array}{r} ?7 \\ \underline{+3?} \\ \underline{\ 81} \end{array}$$

4.3 Strategies and plan of attack

Solving word problems can be improved with better understanding of the meanings of operations. However, it can be also improved with a few **problem-solving strategies**. The ones that appear useful are as follows.

1. **Act it out** and **Make a drawing** – helps visualise what is going on – but needs the right drawing.
2. **Given and wanted** – underline numbers that are given and the phrase that asks for the answer – the basis of part-part-total.
3. **Check answer** and **Learn from solution process** – helps to develop ideas for the next problem.
4. **Solve a simpler problem** – make numbers smaller in problem, work out what to do with these smaller numbers, then replace them with the large numbers and calculate (called “calculator codes”).

It is also useful to have a **plan of attack** – a metacognitive process for attacking the problem. Polya found that good problem solvers nearly always used the following plan (called Polya’s four stages):

1. **See** – spend time simply working out what the problem means.
2. **Plan** – make up a plan based on strategies and knowledge to tackle the problem.
3. **Do** – implement the plan, see if it works and calculate the answer.
4. **Check** – check the answer for sensibleness and see what you can learn from what you have done.

Unit 5: Extension to Variable

The final step of operations in this module is extending the meanings of addition and subtraction to variables, that is, algebra. (*Note*: Two or more step problems will be covered in a later module.)

5.1 Introducing variable

An effective method for introducing understanding of addition and subtraction to variables is to give problems with more than one unknown as the following RAMR model shows.

Reality

Revise turning symbols into stories: 4 + 7 = ? could be: *I caught 4 fish and my friend caught 7 fish. How many fish did we catch altogether?*

Discuss with students scenarios that don't have all the information – *I bought a box of chocolates and then a meal that cost $9. How much did I spend?*

Discuss the following: Why can't this problem be answered?
<Don't know the cost of chocolates>

What information would I need to be able to calculate something?
<Could calculate spending if given the cost of chocolates>

What else is possible?
<Calculate cost of chocolates if given the total amount of spending>

Mathematics

Let students give amounts and calculate some answers (e.g. "what if ....") and write the equations.

What if the chocolate cost $3 – the equation is 3 + 9 = ?
What if the chocolate cost $1.50 – the equation is 1.50 + 9 = ?

Discuss if it is possible to write the equation if the spending is known and the cost of the chocolates is not known. Allow students to devise their own ways of representing this.

What if the total spending is $11 – the equation could be:

<u>chocolate cost</u> + 9 = 11 *or*

C + 9 = 11 *or*

 + 9 = 11

If students are struggling to write an equation, have the students write the problem in words on the thinkboard.

Allow students to construct problems using their own method before introducing letters as the Western mathematics symbol for unknown and variable. Connect to cups and counters meaning of variable as per the activity 5.2.

Reflection

Use these symbols to do two types of activities:

- from symbols with unknowns/variable, write the story without all the numbers; and
- from the story without all the numbers, write the symbols with unknowns/variable.

Have students give the three stories for 5 + 9 = 14; each of the stories has a different part of the equation "unknown":

I scored 5 goals in the first half of the match and then 9 goals in the second. How many goals did I score altogether?

I scored 5 goals in the first half and I scored 14 goals in total. How many goals in the second half?

I scored some goals in the first half and 9 goals in the second half. I scored 14 goals altogether. How many did I score in the first half?

Do this for more than one unknown – I bought a pie and a can of coke. How much did I spend?

p = cost of pie
c = cost of coke
T = total cost Equation p + c = T

5.2 Interpreting and constructing

This can proceed in the same way word problems were interpreted in Unit 4 but now unknowns are given letters. For example, *There were cows, 5 sheep were put in the same pen. How many animals?* This problem can be stated now as C + 5 = A or as A – 5 = C, depending on whether we interpret it as total unknown (number of animals) or part unknown (number of cows).

We can even have all numbers unknown. For example, *There were cows, sheep were put in the same pen. How many animals?* This problem can be stated now as C + S = A or as A – S = C or A – C = S, depending on whether we interpret it as total unknown or part unknown.

In fact, algebra allows much more flexibility on whether it is addition or subtraction.

Finally, constructing problems is very flexible. *Write a story for* $x + y = z$, can be anything – *I bought a pie and a cake and paid money for them.*

Test Item Types

This section presents instructions and the test item types for the subtests associated with the units. These will form the bases of the pre-test and post-test for this module.

Instructions

Selecting the items and administering the pre-post tests

This section provides an item bank of test item types, constructed around the units in the module. From this bank, items should be selected for the pre-test and post-test; these selected items need to suit the students and may need to be modified, particularly to make post-test items different to pre-test items. The purpose of the tests is to measure students' performance before and after the module is taught. The questions should be selected so that the level of difficulty progresses from easier items to more difficult items. In some modules this will follow the order of the units and subtests, and in other modules it will not, depending on the sequencing across the module and within units. The pre-test items need to allow for students' existing knowledge to be shown but without continual failure, and the post-test items need to cover all the sections in a manner that maximises students' effort to show what they can do.

In administering the pre-test, the students should be told that the test is not related to grades, but is to find out what they know before the topic is taught. They should be told that they are not expected to know the work as they have not been taught it. They should show what they know and, if they cannot do a question, they should skip it, or put "not known" beside questions. They will be taught the work in the next few weeks and will then be able to show what they know. Stress to students that **any pre-test is a series of questions to find out what they know** before the knowledge is taught. They should do their best but the important questions come at the end of the module. For the **post-test**, the students should be told that **this is their opportunity to show how they have improved**.

For all tests, **teachers should continually check to see how the students are going.** Items in later subtests, or more difficult items within a particular subtest, should not be attempted if previous similar items in earlier subtests show strong weaknesses. Students should be allowed to skip that part of the test, or the test should be finished. Students can be marked zero for these parts.

Information on the addition and subtraction item types

For addition and subtraction of whole numbers, the items are divided into five subtests that align with the five units in the module. The pre-test should focus on the first three units – basic concepts, basic and extended facts, and algorithms. However, the expectation at the post-test will be that understandings would have improved to cover the more advanced concepts in the word problem unit and the extension to algebra.

The item types are also organised so that priority is given to understanding concepts and strategies and not to getting answers for numerical examples (something that calculators would be able to do). The item types also cover all the concepts and strategies – this is based on the belief that it is important for students to know all meanings and all methods as it is the methods that help in later years and which are the big ideas that recur across years and topics.

Subtest item types

Subtest 1 items (Unit 1: Basic concepts)

1. Put a circle around the operations that are true:

   (a) 1001 − 1205 = 1205 − 1001

   (b) 3691 + 678 − 3691 = 678

   (c) 0 + 120 = 0

   (d) 117 + 38 = 38 + 117

2. Put a circle around the operations that are true:

   (a) 78 + 96 = 96 + 78

   (b) 678 – 678 + 567 = 567

   (c) 5 + 0 = 0

   (d) 11 − 3 = 4 + 4

Subtest 2 items (Unit 2: Basic facts)

1. Solve the following:

   (a) 8 + 7 = ___________

   (b) 80 + 70 = ___________

   (c) 800 + 700 = ___________

   (d) 9 – 5 = ___________

   (e) 90 – 50 = ___________

   (f) 900 – 500 = ___________

2. Solve the following:

   (a) 4 + 2 = ___________

   (b) 40 + 20 = ___________

   (c) 400 + 200 = ___________

   (d) 9 – 2 = ___________

   (e) 90 – 20 = ___________

   (f) 900 – 200 = ___________

Subtest 3 items (Unit 3: Algorithms)

1. Calculate answers to the following questions. Show as much working as you can.

   (a) 64 + 15 =

   (b) 85 + 308 =

   (c) 253 − 74 =

2. Calculate answers to the following questions. Show as much working as you can.

   (a) 36 + 12 =

   (b) 454 − 37 =

   (c) 67 + 48 =

Subtest 4 items (Unit 4: Word problems)

1. Write a story for this sum: 18 + ? = 41

____

____

____

2. Write an equation for this situation:

The building had 5 storeys, the crane knocked off the 2 top storeys, how many storeys are left?

____

3. If you were using a calculator, which operation key (+, –) would you press to solve each of these problems? (Tick the column.) **Do not solve the problems**.

Tick the box for the correct operation for each story:	+	–
(a) A 1.6 m length was cut from a plank of wood. There was 2.8 m left. How long was the plank of wood at the start?		
(b) Jack had $13.45 and his mum gave him $6.30. How much money does he have altogether?		
(c) 18 horses were left in a yard after 3 were taken out. How many horses were there to begin with?		
(d) An elephant weighed 1245 kg and a kangaroo weighed 675 kg. How much heavier was the elephant?		
(e) 18 people walked out of a meeting. There were 27 people left in the meeting. How many people were at the meeting to begin with?		
(f) George is a plumber. He is to make a drain that is 5 m long. He has a piece of pipe that is 3.4 m long. How much more pipe does he need to join on to this to make the whole drain?		
(g) 8 dogs were left in a yard after 3 were taken out. How many dogs were there to begin with?		
(h) A box with books weighs 10.5 kilograms and a carton full of packaged drinks weighs 8.9 kilograms. How much more do the books weigh than the drinks?		

4. Identify the **part/parts** and the **total** in each story. Draw a line under the part/parts; Draw a **circle** around the total part. **Do not solve the problems**.

Example:

> (a) John bakes 8 cakes. Then he bakes another 3 cakes. How many cakes altogether?
>
> (b) There were 10 ducks in the backyard. A carpet snake ate 4 ducks. How many ducks are left?

(a) The football and netball teams travelled together on one bus. There were 16 players on one team and 14 players on the other team. How many players were there altogether on the bus?

(b) Lisa saved $145, and her Grandma gave her $35. How much does she have altogether?

(c) A 3.4 m piece of wood is cut from a 4.6 m plank of wood. How long was the remaining piece of wood?

(d) Pedro has $61.25 and wants to buy a stereo that costs $129.95. How much more money does he need?

(e) On one team there were 16 players and on the other team there were 11 players. How many players altogether?

(f) 14 balls were left in the sports room after some were taken out. There were 22 balls to begin with. How many were taken out?

(g) On Monday during the sport lesson, 17 students played football and 13 students went and played handball. How many students were playing sport that lesson?

(h) Josie has $34.25 and wants to buy a shirt that costs $49.55. How much more money does she need?

5. Solve the following problems:

(a) The Year 8 and Year 9 students were together to go on an excursion. There were 18 Year 8 girls and 17 Year 9 girls. There were 24 Year 8 boys and 18 Year 9 boys.

How many Year 9 students were there altogether?

(b) George held a fundraising stall where he sold cans of drink. He had started the day with 96 and had 17 left at the end of the day. How many cans of drink did he sell?

(c) Jenny had saved $32 and her aunt gave her $15. She wanted to buy an iPod shuffle that cost $59. How much more does Jenny need to save?

(d) Amy held a fundraising stall where she sold iceblocks. She had started the day with 80 iceblocks and had 13 left at the end of the day. How many iceblocks did she sell?

(e) Frank had saved $57 and his uncle gave him $15. He wanted to buy a phone that cost $98. How much more does Frank need to save?

Subtest 5 items (Unit 5: Extension to variable)

1. (a) Write a story for this equation: $R - 17 = 32$

____

____

____

(b) What does the variable 'R' represent? ____

2. (a) Write a story for this equation : $12 + P = 54$

____

____

____

(b) What does the variable 'P' represent? ____

3. Write an equation for the following story:

At the shop I bought a shirt which cost $18, and a pair of runners. The total cost was $53.

4. Write the number sentence for the following story:

At the shop I bought a hat which cost $18, and a shirt. The total amount was $65.

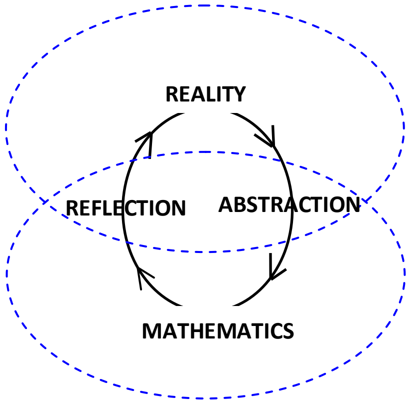
Challenge questions

5. (a) The answer to an operations question is 265. What might the question be?

(b) What other questions could have the same answer?

Appendix A: RAMR Cycle

AIM advocates using the four components in the figure on right, reality–abstraction–mathematics–reflection (RAMR), as a cycle for planning and teaching mathematics. RAMR proposes: (a) working from reality and local culture (prior experience and everyday kinaesthetic activities); (b) abstracting mathematics ideas from everyday instances to mathematical forms through an active pedagogy (kinaesthetic, physical, virtual, pictorial, language and symbolic representations, i.e. body → hand → mind); (c) consolidating the new ideas as mathematics through practice and connections; and (d) reflecting these ideas back to reality through a focus on applications, problem solving, flexibility, reversing and generalising (see figure on right).



The innovative aspect of RAMR is that the right half develops the mathematics idea while the left half reconnects it to the world and extends it. For example, whole-number place value built around the **pattern of threes** where hundreds-tens-ones applies to ones, thousands, millions, and so on, can be easily extended to metrics by considering the ones to be millimetres, the thousands to be metres and the millions to be kilometres.

Planning the teaching of mathematics is based around the RAMR cycle if it is deconstructed into components that are applied to a mathematical idea. By breaking instruction down into the four parts and taking account of the pedagogical approaches described above, the cycle can lead to a structured instructional sequence for teaching the idea. The table below briefly outlines how this can be done. Prerequisite mathematical ideas are considered in the Reality and Mathematics components of the cycle, while extensions and follow-up ideas are considered in the Reflection component.

REALITY • **Local knowledge**: Identify local student cultural-environmental knowledge and interests that can be used to introduce the idea. • **Prior experience**: Ensure existing knowledge and experience prerequisite to the idea is known. • **Kinaesthetic**: Construct kinaesthetic activities, based on local context, that introduce the idea.
ABSTRACTION • **Representation**: Develop a sequence of representational activities (physical to virtual to pictorial materials to language to symbols) that develop meaning for the mathematical idea. • **Body-hand-mind**: Develop two-way connections between reality, representational activities, and mental models through body → hand → mind activities. • **Creativity**: Allow opportunities to create own representations, including language and symbols.
MATHEMATICS • **Language/symbols**: Enable students to appropriate and understand the formal language and symbols for the mathematical idea. • **Practice**: Facilitate students' practice to become familiar with all aspects of the idea. • **Connections**: Construct activities to connect the idea to other mathematical ideas.
REFLECTION • **Validation**: Facilitate reflection of the new idea in terms of reality to enable students to validate and justify their new knowledge. • **Applications/problems**: Set problems that apply the idea back to reality. • **Extension**: Organise activities so that students can extend the idea (use reflective strategies – *flexibility*, *reversing*, *generalising*, and *changing parameters*).

Appendix B: AIM Scope and Sequence

Yr	Term 1	Term 2	Term 3	Term 4
A	**N1: Whole Number Numeration** Early grouping, big ideas for H-T-O; pattern of threes; extension to large numbers and number system **N2: Decimal Number Numeration** Fraction to decimal; whole number to decimal; big ideas for decimals; tenths, hundredths and thousandths; extension to decimal number system	**O1: Addition and Subtraction for Whole Numbers** Concepts; strategies; basic facts; computation; problem solving; extension to algebra **M1: Basic Measurement (Length, Mass and Capacity)** Attribute; direct and indirect comparison; non-standard units; standard units; applications	**O2: Multiplication and Division for Whole Numbers** Concepts; strategies; basic facts; computation; problem solving; extension to algebra **M2: Relationship Measurement (Perimeter, Area and Volume)** Attribute; direct and indirect comparison; non-standard units; standard units; applications and formulae	**G1: Shape (3D, 2D, Line and Angle)** 3D and 2D shapes; lines, angles, diagonals, rigidity and properties; Pythagoras; teaching approaches **SP1: Tables and Graphs** Different tables and charts; bar, line, circle, stem and leaf, and scatter graphs; use and construction
B	**M3: Extension Measurement (Time, Money, Angle and Temperature)** Attribute; direct and indirect comparison; non-standard units; standard units; applications and formulae **N3: Common Fractions** Concepts and models of common fractions; mixed numbers; equivalent fractions; relationship to percent, ratio and probability	**G2: Euclidean Transformations (Flips, Slides and Turns)** Line-rotation symmetry; flip-slides-turns; tessellations; dissections; congruence; properties and relationships **O3: Common and Decimal Fraction Operations** Addition, subtraction, multiplication and division of common and decimal fractions; models, concepts and computation	**A1: Equivalence and Equations** Definition of equals; equivalence principles; equations; balance rule; solutions for unknowns; changing subject **N4: Percent, Rate and Ratio** Concepts and models for percent, rate and ratio; proportion; applications, models and problems	**SP2: Probability** Definition and language; listing outcomes; likely outcomes; desired outcomes; calculating (fractions); experiments; relation to inference **G3: Coordinates and Graphing** Polar and Cartesian coordinates; line graphs; slope and y-intercept; distance and midpoints; graphical solutions; nonlinear graphs
C	**A2: Patterns and Linear Relationships** Repeating and growing patterns; position rules, visual and table methods; application to linear and nonlinear relations and graphs **N5: Directed Number, Indices and Systems** Concept and operations for negative numbers; concept, patterns and operations for indices; scientific notation and number systems	**A3: Change and Functions** Function machine; input-output tables; arrowmath notation, inverse and backtracking; solutions for unknowns; model for applications to percent, rate and ratio **G4: Projective and Topology** Visualisation; divergent and affine projections; perspective; similarity and trigonometry; topology and networks	**O4: Arithmetic and Algebra Principles** Number-size, field and equivalence principles for arithmetic; application to estimation; extension to algebra; simplification, expansion and factorisation **SP3: Statistical Inference** Gathering and analysing data; mean, mode, median, range and deviation; box and whisker graphs; large data sets, investigations and inferences	**A4: Algebraic Computation** Arithmetic to algebra computation; modelling-solving for unknowns; simultaneous equations, quadratics **O5: Financial Mathematics** Applications of percent, rate and ratio to money; simple and compound interest; best buys; budgeting and planning activities

Key: N = Number; O = Operations; M = Measurement; G = Geometry; SP = Statistics and Probability; A = Algebra.

Accelerated Inclusive Mathematics Project

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