Accelerated Inclusive Mathematics Early Understandings Project

YuMi Deadly Maths

AIM EU
Module O3

Operations: Calculating

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The YuMi Deadly Centre acknowledges the traditional owners and custodians of the lands in which the mathematics ideas for this resource were developed, written and refined and presented in professional development sessions.

YUMI DEADLY CENTRE

The YuMi Deadly Centre is a research centre within the Faculty of Education at the Queensland University of Technology which is dedicated to enhancing the learning of Indigenous and non-Indigenous children, young people and adults to improve their opportunities for further education, training and employment, and to equip them for lifelong learning. The YuMi Deadly Centre (YDC) can be contacted at ydc@qut.edu.au. Its website is http://ydc.qut.edu.au.

"YuMi" is a Torres Strait Islander Creole word meaning "you and me" but is used here with permission from the Torres Strait Islanders' Regional Education Council to mean working together as a community for the betterment of education for all. "Deadly" is an Aboriginal word used widely across Australia to mean smart in terms of being the best one can be in learning and life. YuMi Deadly Centre's motif was developed by Blacklines to depict learning, empowerment, and growth within country/community. The three key elements are the individual (represented by the inner seed), the community (represented by the leaf), and the journey/pathway of learning (represented by the curved line which winds around and up through the leaf). As such, the motif illustrates the YuMi Deadly Centre's vision: *Growing community through education*.

DEVELOPMENT OF THE AIM EARLY UNDERSTANDINGS MODULES

In 2009, the YuMi Deadly Centre (YDC) was funded by the Commonwealth Government's *Closing the Gap: Expansion of Intensive Literacy and Numeracy* program for Indigenous students. This resulted in a Year 7 to 9 program of 24 half-term mathematics modules designed to accelerate the learning of very underperforming Indigenous students to enable access to mathematics subjects in the senior secondary years and therefore enhance employment and life chances. This program was called Accelerated Indigenous Mathematics or AIM and was based on YDC's pedagogy for teaching mathematics titled YuMi Deadly Maths (YDM). As low-income schools became interested in using the program, it was modified to be suitable for all students and its title was changed to Accelerated Inclusive Mathematics (leaving the acronym unchanged as AIM).

In response to a request for AIM-type materials for early childhood years, YDC decide to develop an Early Understandings version of AIM for underperforming Years F to 2 students titled Accelerated Inclusive Mathematics Early Understandings or AIM EU. This module is part of this new program. It uses the original AIM acceleration pedagogy developed for Years 7 to 9 students and focuses on developing teaching and learning modules which show the vertical sequence for developing key Years F to 2 mathematics ideas in a manner that enables students to accelerate learning from their ability level to their age level if they fall behind in mathematics.

YDC acknowledges the role of the Federal Department of Education in the development of the original AIM modules and sees AIM EU as a continuation of, and a statement of respect for, the *Closing the Gap* funding.

Contents

List of Abbreviations

AIM EU	Accelerated Inclusive Mathematics Early Understandings
BAE	begin–act–end
RAMR	Reality–Abstraction–Mathematics–Reflection
SPDC	see–plan–do–check
YDC	YuMi Deadly Centre
YDM	YuMi Deadly Maths

Module Overview

This module is the last in a sequence of three, O1 *Thinking and Solving*, O2 *Meaning and Operating* and O3 *Calculating*, which cover Operations from before Year F to Year 2. This module is designed to ensure that teachers cover the before-Year-F work as well as the F-to-2 work. It follows on from the ideas in O2 and moves the focus from meaning (operating) to calculating, that is, from joining 4 cats and 5 cats meaning 4 + 5, to joining 4 cats and 5 cats meaning 9 cats.

The AIM EU modules are designed to provide support in Years F to 2 to improve Year 3 mathematics performance. They are based on the AIM Years 7 to 9 modules, which are designed to accelerate mathematics teaching and learning to where underperforming mathematics students (at around Year 3–4 level in Year 7) can learn six years of mathematics in three years and thus access Year 10 mathematics as mainstream students. The AIM EU modules are designed to accelerate learning in the early years so that students with little schooling cultural capital at the start of their Foundation year can learn the school mathematics understandings normally taught in home, plus those taught in Years F–2, in three years and reach the beginning of Year 3 with strong Year 2 mathematics knowledge. The nine AIM EU modules covering Number, Algebra and Operations Years F to 2, plus background on the modules, are described in **Appendix A**.

AIM EU uses the YuMi Deadly Mathematics (YDM) pedagogy, which is based around the structure of mathematics (sequencing, connections and big ideas) and a Reality–Abstraction–Mathematics–Reflection (RAMR) teaching cycle that is described in **Appendix B**. The YDM pedagogy endeavours to achieve three goals: (a) to reveal the structure of mathematics, (b) to show how the symbols of mathematics tell stories about our everyday world, and (c) to provide students with knowledge they can access in real-world situations to help solve problems. The YuMi Deadly Centre (YDC) argues that the power of mathematics is based on how the structure of connections, big ideas and sequences relates descriptively (with language) and logically (through problem-solving) to the world we live in.

This chapter introduces and overviews the module by: (a) looking at connections and big ideas with regard to calculating, (b) sequencing the teaching of calculation across basic facts to 3-digit numbers, (c) discussing teaching and cultural implications, and (d) summarising the structure of the module.

Connections and big ideas

The starting point for all YDC AIM modules is the connections between mathematics topics and using these connections to accelerate learning, in particular in the formation of big ideas whose learning will provide understanding across mathematics topics and across year levels. This section covers the importance of connections and big ideas in the early years, connections and big ideas underlying operations, and mathematical structure.

Importance of connections and big ideas in early years

YDC believes mathematics is best understood and applied in a schematic structured form which contains knowledge of when and why as well as how. Schema has knowledge as connected nodes, which facilitates recall and problem-solving. As O1 argues, understanding schematic structure enables teachers to:

(a) *determine what mathematics is important to teach* – mathematics with many connections or based on big ideas is more important than mathematics with few connections or little use beyond the present;

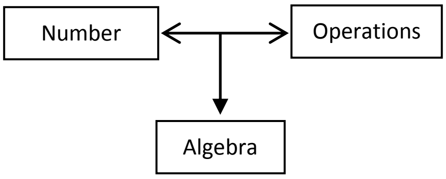
(b) *link new mathematics ideas to existing known mathematics* – mathematics that is connected to other mathematics or based on the one big idea is easier to recall and provides options in problem-solving;

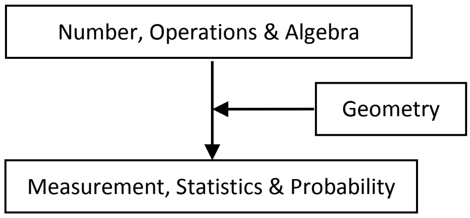
(c) *choose effective instructional materials, models and strategies* – mathematics that is connected to other ideas or based around a big idea can be taught with similar materials, models and strategies; and

(d) *teach mathematics in a manner that enables later teachers to more easily teach more advanced mathematics* – by pre-empting the knowledge that will be needed later, preparing linkages to other ideas, and building foundations for big ideas the later teachers will use.

Connections with respect to operations

Operations concepts are used with Number and lead into Algebra. Finally, through the inclusion of Geometry, Number, Operations and Algebra lead on to Measurement and, more directly, to Statistics and Probability. This can be diagrammatically represented as below:





The major connections between Operations and the other topics are to topics that use number and/or operations as the basis of their mathematics (e.g. Number, Algebra, Measurement, Statistics and Probability). Major connections are as follows.

- *Operations and Number* – an obvious connection as operations need numbers to act on. In particular, the strategies for computation relate to the numeration concepts, that is: (a) separation strategy relies on a place-value understanding of 2- to 4-digit numeration; and (b) sequencing and compensation strategies rely on a rank understanding of numeration.
- *Operations and Algebra* – again an obvious relationship as algebra is generalisation of arithmetic activities. In particular, $2x + 3$ relates to an example like $2 \times 5 + 3$. The difference is that 5 is an actual number while x is a variable.
- *Operations and Measurement* – measurement involves a lot of operations particularly with respect to formulae (e.g. perimeter, area).
- *Operations and Statistics and Probability* – both of these involve operations (e.g. in calculating mean and chance).

Big ideas underlying operations

Operations have many big ideas. They can be global or based on concepts, principles, and strategies. The following two lists cover **global** big ideas because these apply to all mathematics, and **operation** big ideas because these apply to the operation work in both O2 and O3. They lead to ideas that are the basis of calculating and strategies for calculation.

The ideas of schema are as crucial in the early years as later, maybe even more so, because the early years lay the foundations for later understandings. Strong foundations build acceleration in learning and powerful mathematics ideas. This is particularly so when big ideas are identified that cover a variety of topics and are useful across more than one year level, and also cover both pre-operational and operational thinking.

For YDM, **curricula should be taught so that big ideas and connections are emphasised**. Our aim is to construct teaching frameworks that specify and sequence topics and draw attention to connections, particularly those that result in big ideas.

Global big ideas

1. **Symbols tell stories**. The symbols of mathematics enable the world to be described succinctly and in a generalised way (e.g. 2 + 3 = 5 means caught 2 fish and then caught another 3 fish, or bought a $2 chocolate and $3 drink, or joined a 2 m length of wood to a 3 m length … and so on).
2. **Relationship vs change**. Mathematics has three components – objects, relationships between objects, and changes/transformations between objects. All relationships can be perceived as changes and vice versa. This is particularly applicable to operations; 2 plus 3 can be perceived as relationship 2 + 3 = 5 or change $2 \xrightarrow{+3} 5$.
3. **Interpretation vs construction**. Things can either be interpreted (e.g. what operation for this problem, what properties for this shape) or constructed (write a problem for 2 + 3 = 5; construct a shape of 4 sides with 2 sides parallel).
4. **Accuracy vs exactness**. Problems can be solved accurately (e.g. find 5 275 + 3 873 to the nearest 100 – rounding and estimation) or exactly (e.g. 5 275 + 3 873 = 9 148 – basic facts and algorithms).
5. **Part-part-total/whole**. Two parts make a total or whole, and a total or whole can be separated to form two parts – this is the basis of numbers and operations (e.g. fraction is part-whole, ratio is part to part; addition is knowing parts, wanting total).
6. **Triad**. Any relationship which as three components (like begin–act–end or BAE) has three types of problem. For example, 3 + 5 = 8 can be: (a) *There were some children, 5 joined them, this made 8, how many children to start with?* (3 unknown); (b) *There were 3 children, some joined them, this made 8, how many children joined?* (5 unknown); and (c) *There were 3 children, 5 joined them, how many children?* (8 unknown). Triads also apply to many other topics such as measures (object, unit, number) and percent (amount, percent, percentage).

Basic operation big ideas

1. **Concepts of the operations**. This covers the meanings of addition, subtraction, multiplication and division (the focus of Module O2). It also covers the knowledge that only addition and multiplication are true operations, with subtraction and division being their inverses.
2. **Concepts of equals and order**. This covers the meanings of equations and inequations (and equals and order). It covers the three principles of equals and order, that is: (a) *reflexivity/non-reflexivity*, which states that A = A but A is not > A; (b) *symmetry/antisymmetry*, which states that A = B → B = A while A > B → B < A and A < B → B > A; and (c) *transitivity*, which states that A = B and B = C → A = C, and A > B and B > C → A > C. It also covers that the true meaning of equals is equivalence (LHS is same value as RHS).
3. **The operation principles**. This covers the six properties of the operations: (a) *closure*, operations with numbers always gives a number; (b) *identity*, addition/subtraction of 0 and multiplication/division by 1 do not change things; (c) *inverse*, changes that undo other changes (such as +2 and −2, and ×3 and ÷3); (d) *commutativity*, order does not matter for + and × but does for and − and ÷ (e.g. 8 + 6 = 6 + 8; 4 × 3 = 3 × 4); (e) *associativity*, what is done first does not matter for + and × but does for and − and ÷ (e.g. (8 + 4) + 2 = 8 + (4 + 2), but (8 ÷ 4) ÷ 2 ≠ 8 ÷ (4 ÷ 2); and (f) *distributivity,* + and – act on like things while × and ÷ act on everything (e.g. 2 × (3 + 4) = 6 + 8; (6 + 8) ÷ 2 = 3 + 4). These principles are used in this module and lead on to strategies.
4. **Extensions of the equals and operation principles**. (a) *balance*, whatever is done to one side of the equation is done to the other for the equation to stay true; (b) *compensation*, ensuring that a change is compensated for so answer remains the same – related to inverse (e.g. 5 + 5 = 7 + 3; 48 + 25 = 50 + 23; 61 − 29 = 62 − 30); (c) *equivalence*, two expressions are equivalent if they relate by +0 and ×1 – also related to inverse, number, fractions, proportion and algebra (e.g. 48 + 25 = 48 +2 + 25 − 2 = 73; 50 + 23 = 73; $\frac{2}{3} = \frac{2}{3} \times \frac{2}{2} = \frac{4}{6}$); (d) *inverse relationship* for −, ÷ and *direct relationship* for +, ×, meaning a higher number can result in a lower answer for −, ÷ (e.g. 12 ÷ 2 = 6 > 12 ÷ 3 = 4; $\frac{1}{2} > \frac{1}{3}$), but a higher number always results

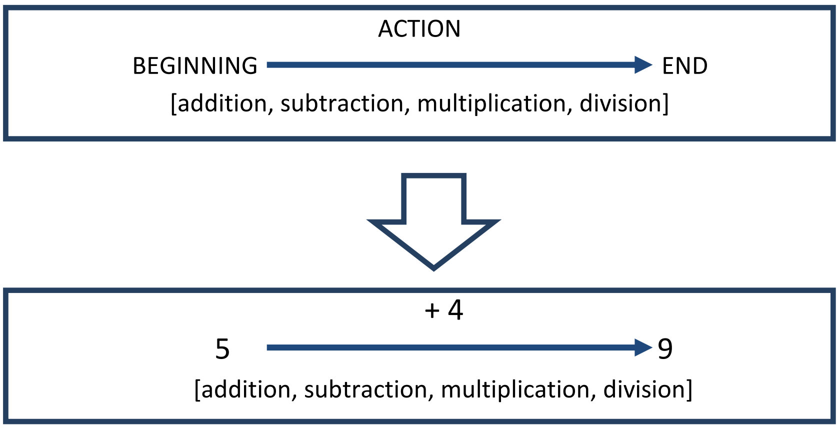
in a higher answer for +, × (e.g. 4 + 3 < 4 + 7), and vice versa for lower numbers; and (e) *backtracking*, using inverse to reverse and solve problems (e.g. $2y + 3 = 11$ means $y \times 2 + 3$, so answer is $11 - 3 \div 2 = 4$).

5. **Strategies for calculation**. This covers (a) *basic fact* strategies – counting, doubles, near 10, patterns, connections, think addition, think multiplication; (b) *algorithm* strategies – separation, sequencing and compensation; and (c) *estimation* strategies – front end, rounding, straddling and getting closer.

The two important big ideas from O1

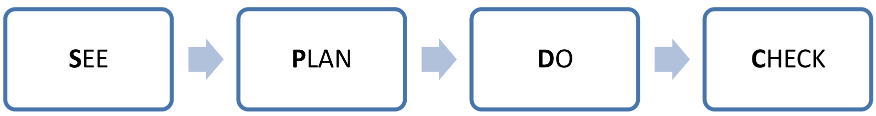
Module O3 builds on the knowledge from Modules O1 and O2 and requires little prior-to-school knowledge except for experiences students have with adults collecting objects, handling money and measuring. However, the two special big ideas that we met first in O1 are still relevant as they provide a framework from which to handle calculating.

1. **Begin–act–end (BAE)**. The basis to understanding operating and calculating is BAE because all operations have a beginning situation, an action that is the operation, and an end situation, as below. As well, all calculations have a beginning number acting on another number for an end result.



   We also looked at reversing the situations, that is, knowing the end and wanting the beginning for a given action. This is the same as reversing or inversing the operation.

2. **See–plan–do–check (SPDC)**. In Module O1, it was also argued that the basis of thinking about and solving problems is the Polya's four stages (SPDC) big idea, that powerful thinking when faced with a problem to solve is to work out what the problem is saying (**see**), make a plan to solve it (**plan**), apply that plan to get a solution (**do**) and then check the answer (**check**), as below:



   SPDC is an approach that can work for any problem that involves numbers and operating and therefore leads to calculating to get the answer. SPDC works with word problems to enable the answer to be a calculation.

Mathematical structure

As for operating, mathematical structure is the basis of understanding. This covers: **number-size principles** – how operations calculate (e.g. subtracting more gives less in the answer: 7 – 3 = 4 and 7 – 5 = 2); **operation principles** – the rules by which operations calculate; and **equivalence class structure** – rules about how equals acts (e.g. that 5 = 2 + 3 is as correct as 2 + 3 = 5 because the meaning of = is "same value as").

Interestingly, as stated before, these principles and structures mean that, **mathematically, only addition and multiplication are operations** – only these two operations obey all the operation principles in Big Ideas above. Subtraction and division are inverses of addition and multiplication respectively.

An important way to learn to understand mathematics structure is to use representations. In calculating, these representations can become predominantly symbols (e.g. 11 + 7 = 18). However, the relationship between stories, acting out with materials, drawings, language and symbols is still important as a halfway house between operating and calculating. As well, the use of models as representations of calculations is important.

The **Review** section of this module has a section on models and representations.

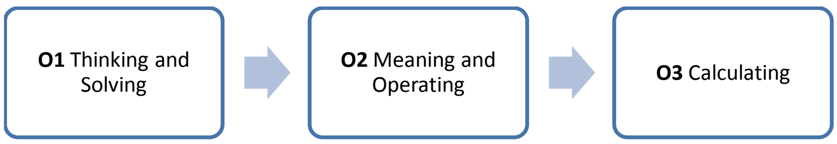
Sequencing

Teaching of calculation has in the past focused on rote memorisation of procedures and has been based on basic facts that have worked because of the nature of numbers, which is their positional place value. With the advent of calculators, there is less need for memorisation of algorithm procedures but more need for understanding and use of strategies that were often hidden behind the computational procedures.

This sequencing section discusses sequencing across modules, sequencing across units within Module O3, sequencing of principles and strategies in relation to calculation and, finally, sequencing within units.

Sequence across modules

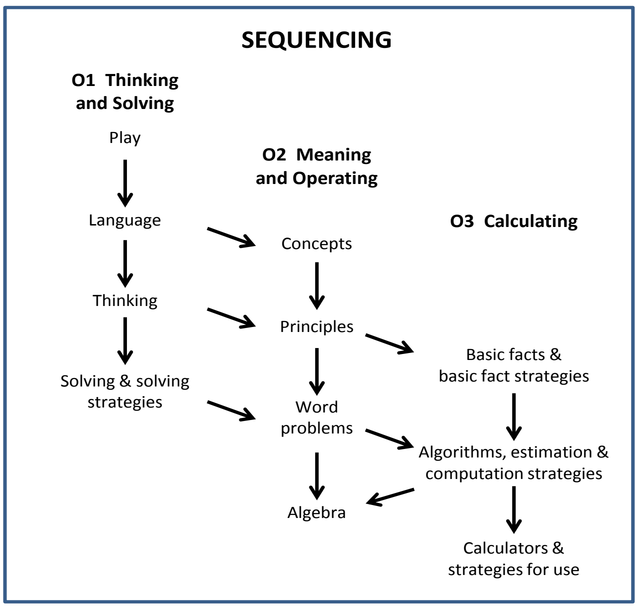
As shown in Module O1, pre-operations precedes operations, which precedes calculation, so that the three modules follow the sequence below:



To teach for rich schema, it is essential for teachers to know the mathematics that precedes, relates to and follows what they are teaching, because they are then able to build on the past, relate to the present, and prepare for the future. Thus, the information in this module is presented as sequences of ideas that relate to and connect with each other. This is particularly important for early understandings as this module is designed to cover some early work although its emphasis is on the Year 2 sequence.

In general, YDM advocates that operations be divided into **two columns or sides**: *meanings-operating* and *computation-calculating*. Meanings-operating covers concepts, principles, and word problems, while computation-calculating covers basic facts, computation, algorithms and estimation. This is done to highlight that problem-solving is based on the meanings-operating side (i.e. concepts and principles), not computation. Of course, expertise with respect to operations is a combination of both sides.

The three AIM EU modules on operations follow this separation in Modules O2 and O3 (see diagram on right). However, to cover pre-operational thinking and prior-to-school experiences, Module O1 (*Thinking and Solving*) is placed into the start of the sequence (the left of diagram on right); to cover the work on larger numbers in O3, algorithms, estimation and computation along with their strategies are placed on the right-hand side of the diagram, as well as calculators and calculator strategies. Algebra is also included to reinforce that it is the meaning and operating components of operations that provide the important understandings to algebraic manipulation.

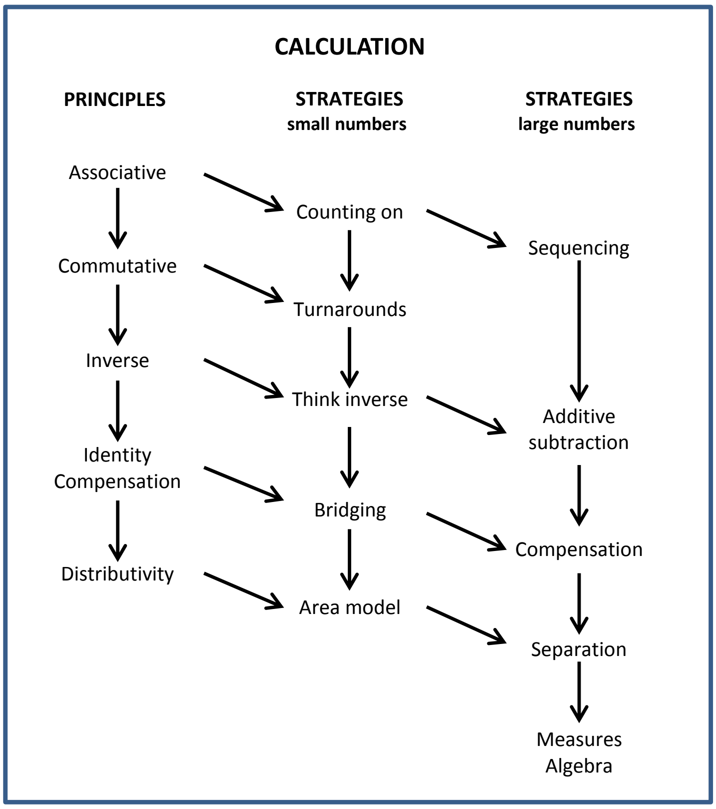


Finally, an important sequence – the move across O1, O2 and O3 was one from story to answer as is seen below and discussed in Unit 4 of O2:

	Operating		**Calculating**	
STORY		SYMBOLS		ANSWER
John had 2 apples, then he got 3 more	⟶	2 + 3	⟶	5

The problem of this sequence is if the move loses the story, we end up with operations and calculations where the symbols are decontextualised from other representations. This can lead to problems with understanding, because it can lead to number activity as 'modelling the world' changing to number activity as 'solving calculations through rote processes'.

Principles and strategies sequence



The strategies that are the basis of the work in this module are related to the principles discussed in the last unit of Module O2 (and in the big ideas section above). The diagram on the right lists the strategies used in this module (O3) – these strategies are discussed in Unit 1 (basic number facts) and unit 2 (calculation to 100) and they are separated into two groups (small numbers and larger numbers – middle and right of diagram) based on the units.

The diagram relates the strategies to the principles:

(a) *Associative* enables numbers to be broken up into parts and the parts added in any way – this is *sequencing*.

(b) *Commutative* are the turnarounds. *Inverse* leads to subtraction (e.g. 11 – 5) being considered as addition (e.g. *what + 5 gives 11?*).

(c) *Inverse* is undoing. It is how things can be reversed. It leads to the idea of doing subtraction by "think addition", that is changing 11 – 7 to "what added to 7 gives 11", and doing division by "think multiplication", that is changing 24 ÷ 8 to "what multiplied by 8 gives 24".

(d) *Identity* leads to bridging tens and the *compensation* strategies such as 9 + 6 is 10 + 5 with 1 added to the 9 and 1 subtracted from the 6 to compensate.

(e) *Distributivity* is the combined action of multiplication and addition. For example, 8 × 7 can be considered as 8 × (5 + 2). The distributive law says that 8 × (5 + 2) is 8 × 5 + 8 × 2. This is the basis of separation for large numbers where, for example, 32 × 6 is considered as (30 + 2) × 6= 30 × 6 + 2 × 6 = 180 + 12 = 192.

The diagram also sequences the strategies in the order that appears to be the most viable.

Sequences within units

The following summarises the activities within each of the five units in this Module O3 *Calculating*

Basic number facts

- Addition basic facts - strategies for learning addition facts (counting, doubling, near tens)
- Subtraction basic facts - strategies for subtraction facts (counting, think addition)
- Extended facts for addition and subtraction - strategies for extending facts to multiples of 10
- Multiplication basic facts - strategies for multiplication facts (patterns, connections
- Division basic facts - strategies for learning division facts (think multiplication)
- Automaticity - looking at ways can encourage students to practice facts to be automaticc

Calculation to 100

- Separation - separation strategy is the break both numbers up into place values and tackle each place value separately - this is the standard algorithms for operations to 100
- Sequencing strategy - this is the strategy where one numbers stays as is and parts of the other number are combined with it in sequence (builds around number line model)
- Compensation strategy - this is where something easy near the operation is used to get an answer and then this answer is compensated for the difference between original and easy calculation

Calculation written approaches

- Models, drawings and jottings/doodles- the ways students can use materials, drawings and doodlings to help in the calculation
- Student-created recording - discussion of student developed ways of using writing in calculation
- Formal recording - discussion of formal written methods and how they relate to the ways that place value materials assist with solving calculations
- Making sense of written calculations - students need to understand number sentences

Introducing and using calculators

- Exploring the calculator - looking at how calculators function and can be used in calculation
- Calculator language - discussing the languageof calculators
- Calculator as a recording device - showing how to use a calculator and record what is being done
- Explaining to others - exploring how to use the calculator to explain thinking to other students
- Computational and solving tool - using the calculator as a computational and problem solving tool

Teaching and culture

This section looks at teaching and cultural implications, including the Reality–Abstraction–Mathematics–Reflection (RAMR) framework and the impact of Western number teaching on Indigenous and low-SES students.

Teaching implications

Because the teaching in this module moves from before-school knowledge through to Year 2 knowledge, the teaching implications are as follows.

1. **Language teaching ideas**. The module will require you to do play and language activities that will build operations. Some ideas for this are: (a) use storytelling time to develop thinking and to model verbalised thinking; (b) create ideas for dialogue using thinking language, particularly opportunities for structured dialogue; (c) develop questioning, focusing on students' ability to ask questions; and (d) wonder together with students.
2. **Thinking–solving teaching ideas**. There will also be a large focus on starting students on thinking and solving. Some ideas for this are: (a) develop spatial, logical, creative and flexible thinking skills and skills in decision-making, plus metacognition; (b) provide opportunities to explore identifying and describing attributes, matching and sorting, and comparing and ordering; (c) create thinking times and develop the

ability to ask questions that stimulate children's thinking and encourage children to elaborate on their ideas; (d) recognise creativity in approach by students and need for solitary as well as social play; and (e) provide opportunities to plan and reflect/evaluate thinking with students and to solve problems.

3. **RAMR cycle**. The module will be based on teaching following the RAMR cycle. Each unit will have at least one RAMR exemplar lesson. The lessons will: (a) start with something students know and in which they are interested; (b) move on to creatively representing the new knowledge through the sequence body → hand → mind; (c) develop language and symbols, and practice and connect components; and (d) finally, reflect the new knowledge back into the lives of the children using problems and applications, and focus on ensuring flexibility, reversing and generalising (by changing parameters if needed). The RAMR cycle is in **Appendix B**.
4. **Models**. It is important that students connect symbols, language, real-world situations and models in many and varied contexts and forms. For models, it is important that there is a balance of set and number-line (or length) models. Set models are discrete items like money, fingers, counters and other objects, while the number-line model is a ruler or steps or jumps along a line (this means that 2 + 3 can be two books joining three books – set model; or two steps and then three more steps – number-line model). With new meanings, it is also important to begin to use array models and combinations models.

Cultural implications

In this section, we move on from just looking at teaching to the cultural implications in this teaching, because students who need AIM EU modules include Indigenous and low-SES students.

1. **Teaching Indigenous students**. Aboriginal and Torres Strait Islander students tend to be high context – their mathematics has always been built around pattern and relationships. Their learning style is best met by teaching patterning that presents mathematics structurally as relationships, without the trappings of Western culture. As Ezeife (2002) and Grant (1998) argued, Indigenous students should flourish in situations where teaching is holistic (from the whole to the parts). Thus, problems and investigations where there is opportunity for creativity and patterning should have positive outcomes for Indigenous students as long as the problems are realistic, make sense within the Indigenous students' context and matter to the students. In general, this means a lesser focus on algorithms and rules, and a greater focus on patterning, generalisations and applications to everyday life. It also means a strong language focus to translate the students' abilities to the world of standard English.
2. **Teaching low-SES students**. Interestingly, holistic teaching is also positive for low-SES students. Three reasons are worth noting. First, low-SES students tend to have strengths with intuitive–holistic and visual–spatial teaching approaches rather than verbal–logical approaches. Thus, a focus on solving problems that make contextual sense and for which the answers matter and with a strong language component should be positive for low-SES students. Second, many low-SES students in Australia are immigrants and refugees from cultures not dissimilar to Aboriginal or Torres Strait Islander cultures. They are also advantaged by holistic algebraic and patterning approaches to teaching mathematics. Third, many low-SES students and their families have long-term experience of failure in traditional mathematics teaching, resulting in learned helplessness. This can be overcome with a focus on investigations along with a strong language focus. Holistic-based problem-oriented teaching of mathematics through patterns is sufficiently different that students can escape their helplessness – particularly if taught actively and from reality as in the RAMR model.
3. **Prior-to-school knowledge**. Both Indigenous and low-SES students can come to school with lots of knowledge from their culture and background, but little knowledge that helps with school work. These AIM EU modules are designed to provide this prior-to-school school knowledge, but it is important that students' cultural and context knowledge is also equally appreciated and maintained. Pride in heritage and connection to heritage is important in learning 'school' knowledge.

Structure of module

Components

Based on the ideas above, this module is divided into this overview section, four units, a review section, test item types, and appendices, as follows.

Overview: This section covers a description the module's focus, connections and big ideas, sequencing, teaching and culture, and summary of the module structure.

Units: Each unit includes examples of teaching ideas that could be provided to the students, some in the form of RAMR lessons, and all as complete and well sequenced as is possible within this structure. This module differs from Modules O1 and O2 in that the ideas provided in the units are in a form that integrates background information and teaching ideas. Also, as this module is for Years 2 to 3, there is more emphasis on addition and subtraction calculation than on multiplication and division (especially with mental and written procedures for larger numbers).

Unit 1: Basic number facts. This unit covers basic fact strategies and automaticity for addition, subtraction, multiplication and division.

Unit 2: Calculation to 100. This unit looks at the separation, sequencing and compensation strategies for addition and subtraction calculations to 100.

Unit 3: Calculation written approaches. This unit explores methods of writing and recording calculations, including models, drawings, jottings, student-created recording and formal recording.

Unit 4: Introducing and using calculators. This explores the calculator, its language, and using it as a recording device as well as a computational and solving tool.

Module review: This section reviews the module, looking at important components across units. This includes the teaching approaches, models and representations, competencies and later activity (where the activity in this module leads to in Years 3 to 9).

Test item types: This section provides examples of items that could be used in pre- and post-tests for each unit.

Appendices: This comprises three appendices covering the AIM EU modules, the RAMR pedagogy, and proposed teaching frameworks for operations.

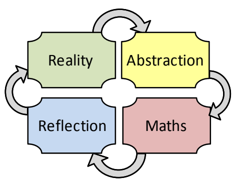
Further information

Sequencing the teaching of the units. The four units are in sequence and could be completed one at a time. However, each of the units is divided into sub-ideas (concepts and processes) that are also in sequence within the unit. Therefore, schools may find it advantageous to: (a) teach earlier sub-ideas in a later unit before completing all later sub-ideas in an earlier unit; (b) teach sub-ideas across units, teaching a sub-idea in a way that covers that sub-idea in all the units together; or (c) a combination of the above.

The AIM EU modules are designed to show sequences within and across units. However, it is always YDC's policy that schools should be free to adapt the modules to suit the needs of the school and the students. This should also be true of the materials for teaching provided in the units in the modules. These are exemplars of lessons and test items and schools should feel free to use them as they are or to modify them. The RAMR framework itself (see Appendix B) is also flexible and should be used that way.

Together, the units and the RAMR framework are designed to ensure that all important information is covered in teaching. Therefore, if modifying the order, try to ensure the modification does not miss something important (see Appendix C for detailed teaching frameworks).

RAMR lessons. We have included RAMR lessons or part lessons as exemplars wherever possible in the units of the module. Activities that are given in RAMR framework form are identified with the symbol on the right.



Suggestions for improvement. We are always open to suggestions for improvement and modification of our resources. If you have any suggestions for this module, please contact YDC.

Unit 1: Basic Number Facts

AIM EU modules are designed to accelerate the mathematics learning of students so that their understanding can support their aspirations. In modules O1 and O2, students have become familiar with operations in terms of concepts (meanings) and principles (properties). We now move into calculating and the first topic is single digit numbers and the basic facts.

Background information

Once the concepts and principles of the operations are introduced, it is time to teach ways to calculate the answers quicker than representing the operation with counters and counting to get the answer. This skill is called knowing basic facts and, for addition and subtraction, is being able to make unthinking, immediate calculations with numbers less than 10 for addition and the inverse for subtraction, that is. 0+0, 0+1, 0+2, .., 1+0, 1+1, 1+2, ..., 2+0, 2+1, 2+2 ..., 9+0, 9+1, 9+2, ..., 9+9, and 0−0, 1−0, 2−0, ..., 9−0, 1−1, 2−1, 3−1, ..., 10−1, 2−2, 3−2, 4−2, ..., 11−2, ..., 9−9, 10−9, 11−9, ..., 18−9, plus the multiples of ten or extended facts such 200+300, 6000+7000 and 1100−600.

Rationale for strategies

The aim for the teaching of basic facts is that they be "known off by heart" as automaticity with facts is essential to free up working memory space for more complex mathematical thinking and multi-step problem solving. However, it is not recommended to use rote learning and drill and practice to achieve understanding about the operation concepts and the calculation of basic facts. It is **recommended to build basic facts through the use of strategies** because:

(a) strategies can help develop students' understanding of the operation concepts as well as the basic facts themselves; and

(b) automaticity in basic facts is best achieved by speeding up strategy steps rather than making direct connections.

After introduction through strategies, repeated practice can lead to automaticity. Therefore, this subsection looks at basic fact strategies for addition, subtraction and multiples of ten, and briefly at diagnosing and practising these facts.

Strategies

The strategies on which to base automaticity are as follows:

(a) addition – counting on, doubles and tens;

(b) subtraction – counting back, think addition and rules;

(c) multiplication - connections, patterns;

(d) division – think multiplication.

The strategies are given below for each operation and followed by a sections on extended facts and automaticity.

1.1 Basic fact strategies for addition

If students do not know a lot of facts in a strategy, then the strategy should be the first step to help the student. If students are taught the strategy then they have a way of working out the facts (albeit slowly) that can be speeded up to automaticity by the practice activities. The major strategies and how to teach them are given below. At the end of each strategy, the principles that are needed and should precede these strategies are given.

The addition facts are provided first and are counting, turnarounds, use doubles, and use tens.

Counting

Counting on is a strategy used for addition facts where one number is 0, 1, 2 or 3, e.g. 6 + 2. The idea is to change counting from both numbers all together (called "SUM") to where only the 0, 1, 2 and 3 are counted and the other number is the start. Counting all is inefficient and students need to develop an ability to count on rather than count all. Counting on more than 3 is also inefficient and there will be other strategies which will work better for basic facts beyond +0, +1, +2 and +3. Students need to be able to subitise (recognise a quantity by sight without counting) and trust that this number does not change to be able to count on. A child that cannot recognise 5 or a collection of 5 objects as "five" or do not trust that this is "five" will feel the need to count to check. This is inefficient. A child who trusts 5 as "five" will be able to count on to work out 5 + 2 by using the counting on strategy from 5, for example "five; six, seven".

To develop this strategy, use a collection of objects. Make the larger number with materials (e.g. for 6 + 2 make a collection of 6). Ensure the student trusts that this is six. Cover the six objects with a hand or a container, recall its number and then count on the 0, 1, 2 or 3. For example: *Put 6 counters into your left hand. Put 2 counters in your right hand. Say "six" showing the left hand and then drop in the counters one at a time from the right to the left hand, saying "seven, eight".* A tin labelled with a symbol into which marbles can be dropped for counting on is also good for this.

This strategy is also used when subtracting 0, 1, 2 and 3 (counting back). For example, 7 − 2 is "seven, six, five". This can be taught by dropping counters out of a hand or a container: *Put 5 counters in the left hand, show hand and say "five", drop out 3 counters one at a time into the right hand saying "four, three, two".*

This strategy is based on the *Associative principle.*

Turnarounds

This strategy is for all facts. It nearly halves the number of facts to be learnt by showing that "bigger + smaller" (e.g. 5 + 2) is the same as "smaller + bigger" (e.g. 2 + 5). For example, 4 + 7 is 7 + 4 equals 11. This is taught by showing that the counters can be joined either way. For example: *Put out 6 counters, add 3 counters to it. Put out 3 counters, add 6 counters to it. Are the final amounts the same? Put out 9 counters. Separate into 6 and 3. Remove and add the 6. Repeat for the 3. Say "3 + 6 is the same as 6 + 3".* (This is based on the **commutative principle**.)

Doubles and near doubles

This strategy is for doubles and for facts that are 1 or 2 from doubles (e.g. 4 + 5 is "double four, eight, plus one, eight, nine", and 6 + 8 is "double six, twelve, plus two, twelve, thirteen, fourteen"). The first teaching step is to learn the doubles. This can be taught as follows.

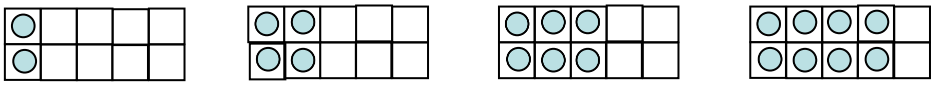
Doubles are addition facts where two of the same number are added, e.g. 4 + 4. Using mental images of real situations that involve doubles can be helpful. For example:

NUMBER TO DOUBLE	MENTAL IMAGE
1	2 feet or 2 hands on a person
2	4 tyres on a car
3	6 wickets in cricket
4	8 legs of a spider
5	10 fingers on our hands

6	12 eggs in an egg carton
7	14 days in a fortnight (use a calendar to reinforce)
8	16 legs in two octopi (or spiders)
9	the 18 dots in two Channel 9 symbols

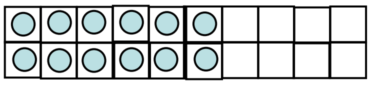
If students have their own images or ideas of mental images for doubles these may also be useful. Allow for creativity and flexibility through discussion and drawings.

Doubles with totals to ten can be modelled using a ten frame where counters are added in 1:1 correspondence.

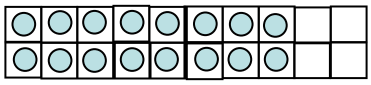


etc.

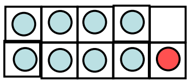
Doubles with totals to twenty can be modelled using two ten frames end to end.



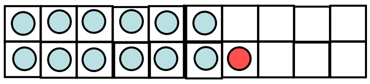
or



The Use Doubles strategy also applies to facts that are 1 or 2 from doubles (e.g. 4 + 5 is "double four, eight, plus one, eight, nine", and 6 + 8 is "double 6, twelve, plus two, twelve, thirteen, fourteen").



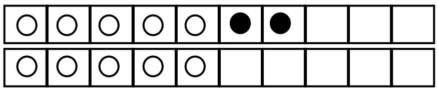
4 + 5 as double 4 + 1



6 + 7 as double 6 + 1

The students can see a double as the two rows of counters and the extra (shown above in red).

Doubles +2 can be modelled similarly. The example below shows 7 + 5 as double 5 + 2. The students can see the double five and the two extras and count or visualise that this is 12, e.g. "ten; eleven, twelve".



so 7 + 5 = (5 + 5) + 2 i.e. 7 + 5 is double 5 = 10, 11, 12.

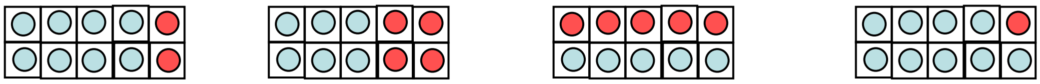
It should be noted that counting on is not the only way to use this strategy. Some children *count back* (e.g. 6 + 8 is "double eight, sixteen, back two, sixteen, fifteen, fourteen"), while some children *level pairs*, for the "two" case (e.g. 6 + 8 is "7 + 7 by adding 1 to 6 and taking 1 from 8, is double 7, 14"). This strategy is based on the *associative principle*.

The related subtraction doubles facts relate to halving. Once students are familiar with the addition doubles they will be able to relate to the halves of known doubles. For example, once a student knows that double 7 is 14 they will also be able to work out that half of 14 is 7 or that 14 – 7 = 7. This strategy is also described in subsection 1.2 (basic fact strategies for subtraction – "think addition") where students work out subtraction basic facts by using the related addition facts.

Ten and near ten

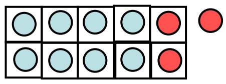
The "use ten" strategy focuses on all the facts that total ten, e.g. 5 + 5 and 7 + 3 as well as the ones that are close to these, e.g. 5 + 6 and 7 + 4. As our number system is based on ten this set of facts is very important and will become valuable when calculating with larger numbers using computation strategies involving separation, sequencing or compensation (see subsections 2.1, 2.2 and 2.5).

The first thing to be taught is the difference between each number 1 to 9 and the number 10. This can be done on the fingers: *Show 10 fingers on your 2 hands. Drop your first 7 fingers. How many left? How many to the ten? Repeat for 4, 6 and 8 fingers.* It can also be done using ten frames. Students place a number of counters in the ten frame and then fill the frame with another colour. The students know the frame holds ten so they see the basic facts that make ten clearly.



8 + 2 = 10 6 + 4 = 10 5 + 5 = 10 9 + 1 = 10

Students can also use ten to work out facts that are 1 or 2 from the ten facts, e.g. 8 + 3 is 8 + 2 and 1 more:



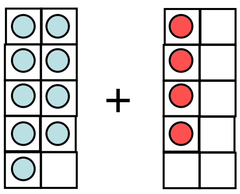
8 + 3 = 11

This understanding can be used in "build to 10" mode for basic facts involving numbers close to ten, e.g. 8 or 9. For example, the basic fact 9 + 5 can be thought of as "9 + 1 to make 10 plus another 4, 14". This can be modelled using ten frames as well. In this thinking, the 5 has been partitioned into 1 and 4 for convenience as it makes a ten.

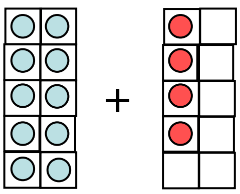
9 + 5 9 + 1 + 4 = 10 + 4 = 14

This understanding can also be used to model the compensation situation where students can understand that 9 + 5 = 10 + 4. The partition strategy and the compensation strategy are valuable for computation beyond basic facts. Understanding of these strategies can begin with basic facts if these are taught using a strategy approach.

Another "near ten" strategy utilises student understandings of ten. This strategy is called "add ten". Students identify basic facts that involve a number near ten and they choose to add ten instead and then compensate for the extra by subtracting. If students have been introduced to teen numbers using two ten frames, e.g. 14 is a full ten frame and 4 more (1 ten and 4 ones), they can use this number sense understanding to solve basic facts involving 9 or 8. So 9 + 4 can be thought of as 10 + 4 and then subtracting 1.



is thought of as



10 + 4 is 14 then subtract 1 = 13

This strategy can be modelled on a number board (100 or 99 board) where +9 is seen as adding ten (one row down) and then going back 1. This strategy is based on the *identity* and *inverse* principles.

The "use ten" strategy also relates to the subtraction basic facts where a number is subtracted from ten. Students can use knowledge of the tens addition basic facts to work out the related subtraction facts. For example, if students know that 6 + 4 = 10 they will also be able to work out that 10 – 6 = 4. This can be modelled using ten frames by taking a full ten frame and removing or covering a quantity of counters showing what remains. Using the "use ten" strategy for subtraction also relates to the "think addition" strategy described in subsection 1.2.

Adding zero rule

The *identity* principle which is about leaving things unchanged applies to addition in that the addition of zero will leave any number unchanged. This is easy to conceptualise. When zero is added there are no more and therefore the number stays the same. The "use counting" basic fact strategy covered the addition of zero as count on zero. It is worth considering this special basic fact as using a rule as well.

1.2 Basic fact strategies for subtraction

The "think addition" strategy is used for subtraction facts. The idea is not to do subtraction but to think of the facts in addition terms. When students want to work out a subtraction basic fact they can think of the subtraction as difference and relate the subtraction fact to an addition fact and strategy. For example 8 – 3 can be thought of as "what is added to 3 to make 8". To use this strategy, students need to understand that subtraction and addition are inverses of each other. This is another situation where ten frames can assist student understanding. By using the frame the empty cells left after counters are removed are still visible. For example: using a ten frame to show that 6 + 4 = 10 (and that 4 + 6 = 10) can also show that 10 – 6 = 4 and that 10 – 4 = 6. Students can see that when they have 4 counters they will need 6 more to make ten.

This situation can also be modelled using numbers other than ten. For example take 7 counters and 4 counters. Combine them to make 11. Separate them back to 7 and 4. Repeat this for 3 and 6 counters and 5 and 8 counters. The notion of adding on to get a subtraction can also be directly modelled: Put out 11 counters. Below them put 7 counters. Add counters to the bottom group until both groups are the same. Repeat for 8 and 13.

This strategy is to reinforce "think addition" and to relate + and –. It is based on the *inverse* principle. For each addition/subtraction fact, there are four members of the fact family, e.g. 3 + 5 = 8, 5 + 3 = 8, 8 – 5 = 3, and 8 – 3 = 5. Families for 4 + 7 and 15 − 9 are:

4 + 7 = 11, 7 + 4 = 11, 11 – 4 = 7, 11 – 7 = 4

9 + 6 = 15, 6 + 9 = 15, 15 – 9 = 6, 15 – 6 = 9

1.3 Addition and subtraction extended facts

Extended facts are where the strategies used to develop and complete basic facts are used for other computations beyond the basic facts. Generally, extended facts relate to *multiples of ten* so the basic fact strategy is used for the tens or hundreds (and so on) as if they were ones. For example, the basic fact of 6 + 4 = 10 can be used to work out 60 + 40 = 100. The basic fact strategies can also be used for *extensions* which are other numbers where the number of tens/hundreds can alter and the basic fact strategy can be applied. For example, the "use doubles" strategy used for 4 + 4 = 8 can also be used for 54 + 4 or 540 + 40.

Multiple-of-ten facts for addition and subtraction

Students can be shown how strategies for basic facts (single digit numbers) also work for the related multiples of ten. Students can think tens for computations that use multiples of ten. For example the count on basic fact of 5 + 2 also works for 50 + 20. Students can think of this as 5 tens + 2 tens and the basic fact strategy will work. This also works for "use doubles" facts, e.g. 6 + 6 = 12 so 600 + 600 = 1200 (double 6 hundred).

Complete activity sheets with related computations like below and discuss the strategies used, highlighting the transferability of known strategies to the larger numbers:

4 + 2 = ______	40 + 20 = ______	400 + 200 = ______
7 + 6 = ______	70 + 60 = ______	7000 + 6000 = ______
11 − 8= ______	110 − 80 = ______	1100 − 800 = ______

Discuss patterns that would enable multiples of ten facts to be determined from basic facts. (*Note*: this can also be completed using the "Do these with a calculator – Do these without!" approach.)

Extensions

Extensions are computations beyond basic facts, other than straight multiples of ten, where basic fact strategies can be applied. The basic fact will be evident in the digits of the computation and the other digits can be considered as multiples of ten. For example the basic fact 5 + 3 = 8 which could be solved using a "count on" or a "near double" strategy can be used to assist in completing computations like 55 + 3 or 550 + 30. Students can practise related computations and discuss the patterns they find in the answers and in the strategies that they use.

5 + 6 = _______	45 + 6 = _______	185 + 6 = _______
9 + 9 = _______	89 + 9 = _______	490 + 90 = _______
15 – 9 = _______	175 – 9 = _______	150 – 90 = _______

Discuss patterns that would enable extension facts to be determined from basic facts. (*Note*: this can also be completed using the "Do these with a calculator – Do these without!" approach.)

1.4 Basic fact strategies for multiplication

This section looks at how to teach ways to calculate the answers more quickly than representing the operation with counters and counting to get the answer. The first of the calculations to teach are those that form the basis of the later algorithms and estimation – the basic facts. While it is widely accepted that these facts have to be learnt off by heart, that is, automated by practice (drill), it is NOT something that, at this stage of these students' schooling, should have an inordinate amount of time spent on memorising to the detriment of time for other concepts that must be accelerated. The reason for still automating facts is that automated facts are available in task situations without taking any thinking away from the task – automated facts have no cognitive load.

The basic facts are all the calculations with numbers less than 10 for multiplication and the inverse operations for division:

0×0, 0×1, 0×2, ..., 0×9; 1×0, 1×1, 1×2, ..., 1×9; 2×0, 2×1, ..., 2×9; … ; 9×0, 9×1, 9×2, ..., 9×9

1÷1, 2÷1, ..., 10÷1; 2÷2, 4÷2, ..., 18÷2; 3÷3, ..., 27÷3; ... ; 9÷9, 18÷9, ..., 81÷9.

For Years F–2 do multiplication facts for 1, 2, 3, 5, 10.

There are five types of strategies for multiplication and division basic facts:

- turnarounds
- patterns
- connections
- think multiplication
- families

Strategies are used differently for multiplication and division than they are for addition and subtraction. In addition and subtraction, strategies covered a variety of facts and there was no need to focus on tables. In multiplication and division, there is more focus on tables (e.g. 4× and 7× tables).

The big ideas relevant for the facts are *identity* and *inverse* and the *commutative, associative* and *distributive* principles. These big ideas were covered in the Module Overview section.

Turnarounds (commutative principle)

This strategy is applied to all facts. It means that "larger × smaller" is the same as "smaller × larger", for example, $7 \times 3 = 3 \times 7$.

Body. Use the Maths Mat to make rectangles, for example, 3 squares by 4 squares, and discuss the rectangles as being representations of 3×4 AND 4×3. Include activities where the answer is 12 squares and ask: *What does the array look like? Is there only one answer?*

Hand. Compare groups of counters by rotating the array model of multiplication: Put out 3 groups of 5 counters, and put out 5 groups of 3 counters. *Which is larger? Are they the same?* Construct an array for 4×7. Turn this array 90 degrees. *Has the amount changed? What does this mean?*

Mind. Using two dice, coloured pencils and some squared paper (an A4 sheet of 1 cm grid paper is ideal), this game reinforces the multiplication facts and the concept of area. Students can work in pairs or threes, each taking a turn.

A student rolls the two dice, and with coloured pencils, shades the rectangle formed by that number of rows and columns. For example, a dice roll of 3 and 6 means they shade a 3×6 rectangle. Write the numbers for the lengths in, and the total number of squares (3×6 is 18 squares).

The next student/s repeats the process and shades their rectangle a different colour. Continue until a student cannot fit their rectangle onto the grid. The winner is the student who last fitted their rectangle on the sheet.

For older students, vary the numbers on the dice (to develop multiplication of numbers greater than 6).

Patterns

This is one of the two major strategies for multiplication. This strategy applies to any of the tables for which there is a pattern that could help students remember the facts. The following tables have patterns:

TABLE	PATTERN
0×	Gives zero for all multiplication, e.g. $3 \times 0 = 0 \times 3 = 0$; $8 \times 0 = 0 \times 8 = 0$
1×	Gives the number (identity), e.g. $3 \times 1 = 1 \times 3 = 3$; $8 \times 1 = 1 \times 8 = 8$
2×	Doubles: 2, 4, 6, 8, 0, ... and so on
5×	Fives: 5, 0, 5, 0 , 5, ... and so on; 5, 10, 15, ...; half the 10× tables; hands; clockface (minutes in one hour)
9×	Nines: tens are one less than number to be multiplied by 9, ones are such that tens and ones digits add to 9; 9, 18, 27, ... and so on

Teaching

These patterns can most easily be seen with a calculator, Unifix, and large and small 99 boards.

The table for the pattern is chosen (e.g. 4×). The number of the table is entered on the calculator and [+] [=] pressed (e.g. [4] [+] [=]). The result (4) is covered on the 99 board with a Unifix.

From there, [=] is continually pressed (adding 4) and the number shown is covered. Once sufficient numbers are covered to see the visual pattern on the 99 board, this pattern is transferred to the small 99 board by colouring squares.

The numbers coloured are discussed to arrive at the pattern.

If more reinforcement is needed, [number] [+] [=] [=] [=] [=] [=] ... is pressed on the calculator and the ones or tens called out at each [=] press (see below). This enables students to verbally hear patterns. The numbers could also be written down for inspection for pattern.

Press [5] [+] [=] [=] [=] ... stating the ones position

Press [9] [+] [=] [=] [=] ... stating the ones position, then repeat, stating the tens position

Press [4] [+] [=] [=] [=] ... stating the ones position

Some teachers also do 4× and 3× with patterns on a 99 board as below:

4× Fours: This is a pattern in the ones position, 0, 4, 8, 2, 6, 0, 4, ... and so on; odd tens is 2 and 6 for ones and even tens is 0, 4 and 8 for ones, that is,

0 4 8
12 16
20 24 28
32 36

3× Threes: This is a "one back" pattern, that is,

0 3 6 9
12 15 18
21 24 27

Connections

The other major strategy is for all the tables not covered by patterns. Here, the unknown table is connected to a known table using the distributive principle.

We can use counters, Unifix, dot paper and graph paper for the models. The idea is that the answers to the unknown table are found from the known tables.

UNKNOWN TABLE	KNOWN TABLE(S)	CONNECTION	EXAMPLE	DIAGRAM
3×	2×	3× is 2× + 1×	3×7 is the same as 2×7 + 1×7 = 14+7 = 21	o o o o o o o o o o o o o o o o o o o o o
4×	2×	4× is double 2× or 4× is double doubles	4×7 is the same as 2×7 + 2×7, i.e. double 2×7 is double 14 = 28	o o o o o o o o o o o o o o o o o o o o o o o o o o o o
6×	2×, 3×	6× is double 3× or 6× is 3× + 3×	6×7 is the same as 3×7 + 3×7, i.e. double 3×7 is double 21 = 42	o o o o o o o o o o o o o o o o o o o o o o o o o o o o o o o o o o o o o o o o o o
6×	5×	6× is 5× + 1×	6×7 is the same as 5×7 + 1×7 = 35+7 = 42	o o o o o o o o o o o o o o o o o o o o o o o o o o o o o o o o o o o o o o o o o o

7×	2×, 5×	7× is 5× + 2×	7×7 is the same as 5×7 + 2×7 = 35+14 = 49	o o o o o o o o o o o o o o o o o o o o o o o o o o o o <u>o o o o o o o</u> o o o o o o o o o o o o o o
8×	2×, 4×	8× is double 4× or 8× is double double doubles	8×7 is the same as 4×7 + 4×7 = 28+28 = 56	o o o o o o o o o o o o o o o o o o o o o <u>o o o o o o o</u> o o o o o o o o o o o o o o o o o o o o o o o o o o o o

An excellent teaching sequence, taking into account patterns and connections, is: 2×, 5×, 9×, 4×, 8×, 3×, 6×, and 7×. This, of course, is not the only correct or appropriate sequence. For example, 2×, 4×, 8×, 5×, 3×, 6×, 9×, 7× is also efficient.

1.5 Basic fact strategies for division

Think addition

This strategy is for all division facts. The division facts are reversed in thinking to multiplication form; for example, 36 ÷ 9 is rethought as "what times 9 equals 36". It can be taught by looking at combining and partitioning: *Take 3 groups of 5 counters and combine. Partition 15 into groups of 5. Repeat. State "15 divided by 5 is the same as 5 multiplied by what is 15". Do the same for 4 × 7 = 28.*

Families

This strategy reinforces "think multiplication" and relates multiplication to division. For each multiplication/division fact, there are four members of the fact family, for example:

3 × 5 = 15, 5 × 3 = 15, 15 ÷ 5 = 3, and 15 ÷ 3 = 5.

Families for 4 × 7 and 36 ÷ 9 are:

4 × 7 = 28, 7 × 4 = 28, 28 ÷ 7 = 4, 28 ÷ 4 = 7
4 × 9 = 36, 9 × 4 = 36, 36 ÷ 9 = 4, 36 ÷ 4 = 9.

1.6 Automaticity

Addition and subtractions facts

The goal of learning basic facts is for students to develop automaticity. Students who have automated instant recall of basic facts will have these facts available in task and problem situations without needing to waste thinking on working them out. This way automaticity does not take away any thinking from the task – automated facts have no cognitive load. YDM believes that developing basic fact automaticity should be the focus after learning of the strategies and investigation of the patterns and connections within the basic facts has occurred. A student who has learnt the various multiplication and division basic fact strategies will have a fall-back method to work out facts they can't remember where a student who has only had learning focussing on rote memorisation will not have the same support. The added benefit to learning the strategies is their applicability to computations beyond basic facts

Diagnosis

If teaching upper primary students, the first step with basic facts is to diagnose what facts are not known and to set up regular speed practice for the unknown facts. To do this, give students a list of random basic facts to complete, keep all students together on the facts by reading each fact with a short time to write the answer. Mark the results and place on an addition and subtraction table as below.

+	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										

–	18	17	16	15	14	13	12	11	10	9	8	etc
0												
1												
2												
3												
4												
5												
6												
7												
8												
9												

	Counting on		Use ten
	Doubles		Think addition
	Near doubles		Use a rule

These grids can be used to determine both the facts with which students make errors and the strategies needed to help students with their errors. For instance, the counting on, near doubles and near ten facts can be placed on an addition grid using different colours. Then, if a student's errors are placed on the grid, the position of the errors will determine which strategy or strategies are needed. However, many addition and subtraction basic facts can be completed using more than one of the strategies. For example 2 + 2 is a double but it could be answered using a count-on 2 strategy. Students can choose which strategy they prefer for these. (*Note*: Generally counting strategies have less transferability to other computations than other strategies.)

Practice

Once students have been taught the various basic fact strategies, practising facts that use these strategies will assist in strengthening their understanding and application of the strategies as well as boosting their automaticity and recall of the basic facts. Activities that focus on speed can be effective for students with knowledge of the strategies. The following appear to work:

- To practise a particular strategy, provide **multiple examples** of facts **that use this strategy**.
- To practise speed and overall automaticity provide a **mix of basic facts** so different strategies are used.
- Use student **tracking worksheets** to aid students with the process and enable them to gauge their own progress – and to make students complicit in the process. Get each student to **mark facts and graph the correct number of answers** each day to compare with previous days. Get students to also **record any errors** they make.
- Set up a regular daily practice program – 10 minutes per day (e.g. 4 minute mile, flash cards, bingo) for **random speed practice** and a later time where students can **practise their errors** recorded at the earlier practice.

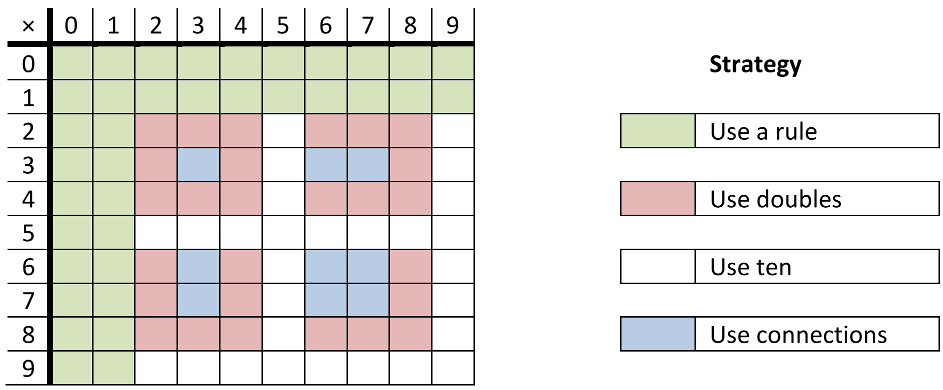
Multiplication and division facts

As for addition and subtraction, the goal of learning basic facts is for students to develop automaticity to reduce cognitive load. We believe that developing basic fact automaticity should be the focus after learning of the strategies and investigation of the patterns and connections within the basic facts has occurred. The added benefit to learning the strategies is their applicability to computations beyond basic facts.

All the multiplication and division basic facts can be completed using one or more of the strategies described in this section. Some basic facts can be completed using more than one basic fact strategy. As multiplication is **commutative**, the **turnarounds** can lead to the use of different strategies. For example 5 × 6 can be worked out with a Use Ten strategy (5 × 6: think 10 × 6 then halve it; half of 60 is 30) but it can also be thought of as 6 × 5 which enables a connection strategy to be used: 6 × 5 is 5 × 5 + 1 × 5. Depending on the student's number sense they may prefer one of these strategies over the other. Once students are confident with the commutativity principle this can be an advantage for working out basic facts but also for multiplication computations with numbers beyond basic facts.

Give students a list of random multiplication basic facts to complete. Keep all students together on the facts by reading each fact with a short time to write the answer. Mark and record the results on a multiplication grid.

The multiplication grid can be used to determine both the facts with which students make errors and the strategies needed to help students with their errors. If a student's errors are placed on the grid, the position of the errors will determine which strategy or strategies are needed. For example, if students know one fact but not its turnaround, they need to be taught "turnarounds".



Practising basic facts

Once students have been taught the various basic fact strategies, practising facts that use these strategies will assist in strengthening their understanding and application of the strategies as well as boosting their automaticity and recall of the basic facts. Activities that focus on speed can be effective for students with knowledge of the strategies. Ideas for practice mirror those listed in Addition and Subtraction including speed and overall automaticity with a mix of basic facts, student tracking worksheets, daily practice program and recording of successes and errors to guide future practice.

1.7 Activities for basic number facts

1. The four operations involve situations with parts that are known and parts that need to be found. Addition and subtraction situations involve two parts and a total. (Multiplication and division situations involve two parts (factors) and a product.)
   (a) Use imagery and mental counting strategies to add and subtract small numbers where the numbers are generated by a story e.g. Edward the Emu
   (b) Use gumnuts or bottle tops to model an addition story involving change and then compare their answers. E.g. there are five emus in a paddock. If three more come back to the paddock how many will there be? Ask: what if we counted them another way? Suppose the emus move around? Ask students to count the total in different ways.

(c) Have children mentally add or take away from a small collection of objects e.g. 5 plastic bears, or 5 cars, or 5 farm animals. Ask children to imagine two animals have been taken away. Ask: how many animals will be left? Try with adding animals as well, first two, then three etc.

2. Addition is a situation when parts are known and a total is wanted.

   (a) Encourage students to build their own table of addition facts first to 5+5 then over time build further.

   (b) model and record patterns for individual numbers up to and including 9 by making all possible whole-number combinations, e.g.

   5+0=5

   4+1=5
   3+2=5
   2+3=5
   1+4=5
   0+5=5

   (c) Recognise, recall and record combinations of two numbers that add to 10

   (d) Play games with dice with students adding numbers together and keeping a running total. Ask how did you add your numbers together?

   (e) Play board games to describe combinations for numbers using words such as 'more', 'less' and 'double', e.g. describe 5 as 'one more than four', 'three combined with two', 'double two and one more' and 'one less than six'

   (f) Create, record and recognise combinations of two numbers that add to numbers from 11 up to and including 20. Use 2 x ten frames to find ways of breaking up numbers to calculate

   (g) Use combinations for numbers up to 10 to assist with combinations for numbers beyond 10. Use beads and bead strings to make numbers and number stories to assist children to 'see' in their mind's eye and visualise parts of collections.

   (h) Use dominos to play count on one, two, three games and build associated number facts.

   (i) Toss a hand full of two colour sided counters. How many of each colour? Have students toss and decide. Make a record of the combinations that come up. Ask: What stays the same? have we got all the combinations? How do you know?

3. Subtraction is a situation when total and one part is known and the other is wanted

   (a) Use story books , poetry, counting rhymes, sports equipment, class collections to relate addition and subtraction facts for numbers to 10 then to at least 20

   (b) Use think addition

4. Use a range of mental strategies regularly:

   (a) Counting *counting on from the larger number to find the total of two numbers;

   (b) *counting back from a number to find the number remaining;

   (c) *counting on or back to find the difference between two numbers;

   (d) Doubles *using doubles and near doubles;

   (e) Use 10 *combining numbers that add to 10- Use the 10 frames to play compensate to ten.

   (f) *bridging to 10, e.g. 17 + 5: 17 and 3 is 20, then add 2 more; *using place value to partition numbers, e.g. 25 + 8: 25 is 20 + 5, so 25 + 8 is 20 + 5 + 8, which is 20 + 13

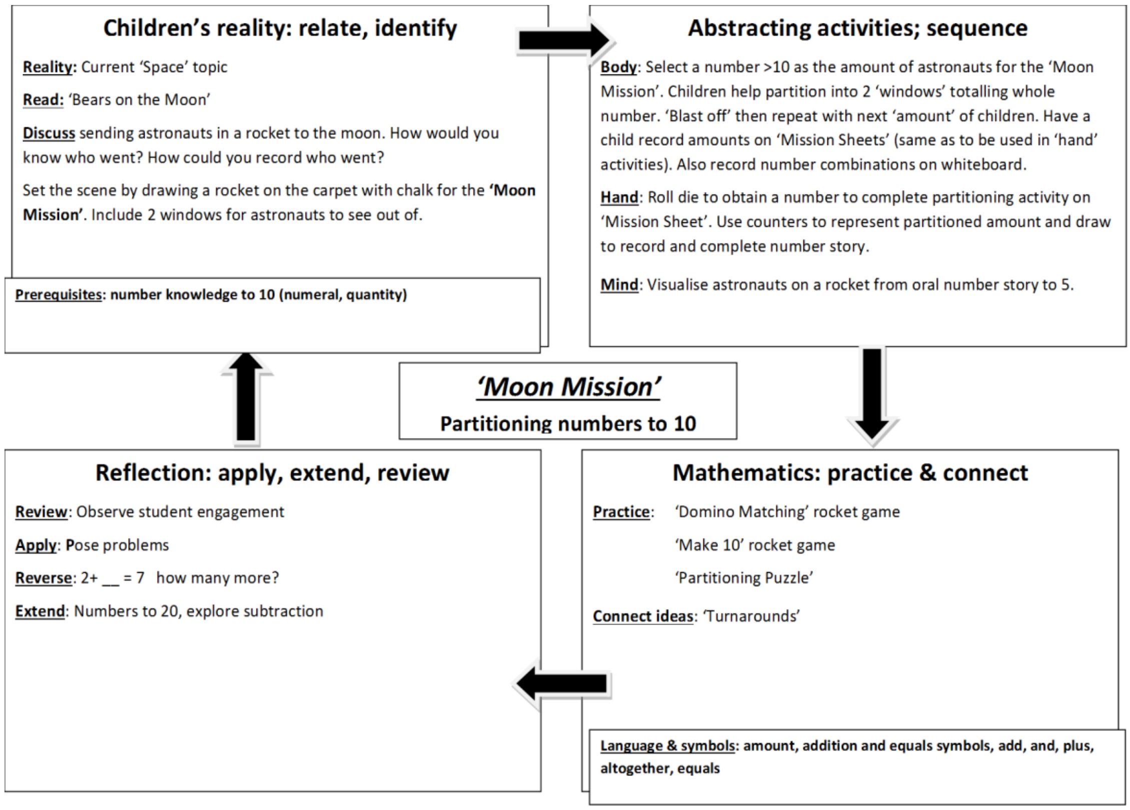
   (g) Use a rule *Investigate and generalise about the effect of adding zero to any number

(h) *Turnarounds- explore these with dominos

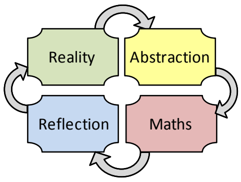
5. Multiplication is a situation when parts (factors) are known and product is wanted
   (a) Use of patterns- auditory, visual, counting, with 0, 1 2, 5 10, first; then others as capable
   (b) Play animal patterns where students create an animal using pattern blocks and then say how many blocks were used. Student might say ‘My dog was made with 5 triangles. Ask: How many triangles would you need for 3 dogs? Have students solve and share how they worked it out.
   (c) Play doubles and halves with of collections

6. Division is a situation when product and one part (factor) is known and the other part (factor) is wanted. For division the unknown may be a number of groups (called grouping or repeated subtraction) or the number in each group (called sharing).
   (a) Use “think multiplication”.
   (b) Have children work with materials that are a fixed representations of number (e.g. Unifix cubes), using different colours to show equal groups. Ask the students to investigate and record the different equal groups that can be made for each number.
   (c) Read *The Doorbell Rang* (Hutchins, 1987) and ask students to draw and explain groupings in the story.

The diagram below shows a teacher-constructed RAMR for partitioning numbers to 10, which is suitable for Prep students. We have expanded this to create the RAMR lesson in section 1.8.



1.8 RAMR lesson for basic number facts



Learning goal: Add and subtract collections.

Big idea: Symbols tell stories; interpretation vs construction; part-part-total; triad; accuracy vs exactness.

Resources: Collections of items; dice; blocks; bears and so on; ten frames; laminated 0–99 boards; mat; whiteboards; calculators.

Reality

Local knowledge: When do you need to know/work out numbers of people/things? e.g. my weekend – how many people seated on a plane/bus?

Prior experience: Awareness of numbers in the environment; counting language; quantifying personally significant collections.

Kinaesthetic: Play modelling on and off the plane/bus with students on chairs.

Abstraction

Body: Students act out own modelled on-and-off stories (bus, plane, train, 4WD car, ferry, etc.)

Hand: Ten frames and counters; game boards, dice and blocks to mirror transport activity.

Mind: Ten on a plane, four get off. Students take turns to tell action for others to see in mind's eye.

Creativity: Create own basic fact scenario to share with class using props/whiteboards/models.

Mathematics

Language/symbols: add, subtract, on, off, tens, ones, counting on, counting back, turnarounds, doubles, near ten.

Practice: Read Dr Seuss book *Ten on Top*. Play ten on, three off and as many combinations necessary for children to be comfortable and confident to add/subtract using mat/counters/models/ten frames. Remember to put several planes/trains/ten frames together to make 10, 20, 30, 40 and so on. If your students are not confident to write their facts yet, choose a willing recorder first, then encourage others to take the role every day you repeat the activity. Have them record on a small whiteboard/s to share with each other.

Connections: Bead strings and Unifix partitioning.

Reflection

Validation: Have pairs of students ask their partner: *What did we do? Who needs to know this? When do you see it?*

Applications/problems: Show two tens and three ones with any materials and three tens and two ones with materials. Ask: *How are they the same? How are they different?* Encourage students to draw or make it themselves if they do not want to/cannot explain/discuss.

Extension: Beyond 10 beads and make numbers into the 100s counting on and counting back.

Generalising: Ask: *What does 10 look like? What does 100 look like?* And so on.

Unit 2: Calculation to 100

This unit follows on from Unit 1 basic facts. Basic facts calculations were 0 to 100. This unit's calculations are 0-100. Thus they have to take into account place value as well as basic facts to make accurate calculations.

Because this is a Foundation to Year 2 program, this unit only focuses on addition and subtraction.

Background information

Addition and subtraction algorithms for whole numbers are accurate (not approximate) addition and subtraction for numbers of two or more digits. Today, they can be completed mentally, with pen and paper, and with calculators. This module looks at them through the three strategies that can be used, which we call: separation, sequencing and compensation. We recommend that these strategies be reserved for computations of up to three digits – over this size we recommend estimation and calculators.

Computation is the calculation of answers to the application of operations and the strategies used to complete these calculations. Calculation has always been a significant component of school mathematics programs. The National Research Council back in 1989 noted that "the teaching of mathematics is shifting from a preoccupation with inculcating routine skills to developing broad-based mathematical power" (p. 82). A key element of this mathematical power is the ability to calculate exact answers efficiently and with understanding. Australian curriculum documents in recent years have shown a shift in emphasis from written algorithms for computation to mental computation, the use of computation strategies and electronic tools like calculators and computers (Australian Curriculum, Assessment and Reporting Authority, 2011; Australian Education Council, 1991; Queensland Studies Authority, 2004).

Threlfall (2000) described mental computation strategies as requiring students to "construct a sequence of transformations of a number problem to arrive at a solution as opposed to just knowing, simply counting or making a mental representation of a paper and pencil method" (p. 30). Three strategies are focussed on in this booklet. Each of these strategies applies to the four operations. YDM recommends that students learn and be exposed to a range of computation strategies and methods (mental, written and calculator). YDM advocates that mental computation strategies be used for numbers up to 3-digits and operations on larger numbers be carried out using calculators.

Each of the three computation strategies described in this unit provides a means for students to transform calculations to make them more manageable. Each of the strategies applies to each of the operations. The three strategies are: **separation**, **sequencing** and **compensation**. Each subsection below describes these strategies and provides sample activities for using the strategy for addition and subtraction.

2.1 Separation

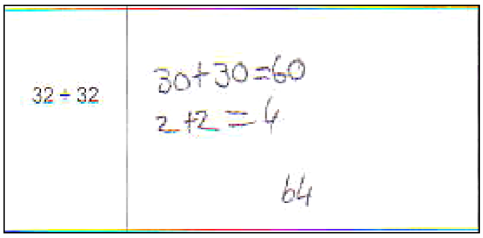
The separation strategy works on transforming the numbers in a calculation by partitioning them or breaking them into parts. The parts are then worked on and the result is found by recombining the parts. There are three ways that numbers in a calculation can be separated: (a) into place-value parts, (b) into compatible parts, or (c) into a combination of place-value and compatible parts.

In the separation strategy both (or all) of the numbers in the calculation are separated. Students need to understand the structure of our number system and how place values are multiplicatively related so that when numbers are separated they are considered in terms of their value, not as digits. The separation strategy is the basis of the traditional written algorithm. The traditional algorithm has a tendency for students to work with digits in columns rather than numbers that have value. Because of this it is valuable to teach and practise separation strategies using Place Value Charts (PVCs) and size materials such as bundling sticks, MAB and money placed on top of these PVCs. MAB and money can be used similarly to the bundling sticks. It is crucial that students experience addition with real bundling sticks, MAB and money before moving to do the virtual bundling stick, MAB and money activities. The separation strategy is used in a variety of contexts – it is useful for whole numbers (e.g. 346 × 8), decimal numbers (e.g. 4.65 + 0.8), measures (e.g. 3 m 342 mm × 5), mixed numbers (e.g. $3\,^{1}/_{6} \times 4$), and algebra (e.g. 2a × (3a + 2b)).

Separation strategy for addition

Without materials

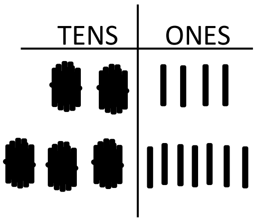
The separation strategy sees both (or all) numbers in a computation separated into their place-value parts. An example of this strategy for addition is 32 + 32 as 30 + 30 + 2 + 2 (see figure on right). The values of the tens in each number are added followed by the values of the ones. This strategy can be completed where the ones are combined first, as is the focus in the traditional written algorithm, but there is a risk that students will consider the tens as 3 + 3 rather than 30 + 30.



Using set materials

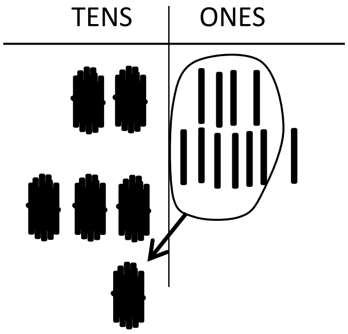
An example of the separation strategy for addition using bundling sticks and a PVC shows how the concept of regrouping to make a ten is needed in this version of this strategy. By using the bundling sticks the tens are clearly visible and not likely to be treated as ones. The number of tens (5) is recorded as 50 in the first step. This could be done horizontally just as easily as vertically.

Put out 24 and 37 in tens and ones.

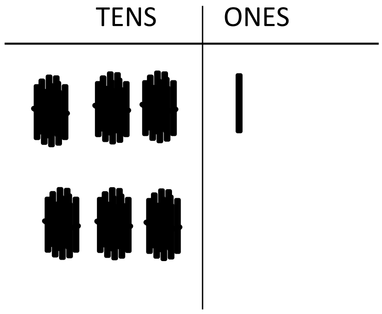


$$\begin{array}{r} 24 \\ +\,37 \\ \hline 50 \end{array}$$

Combine the tens and ones separately. This example combines the tens first, then combines the ones.



$$\begin{array}{r} 24 \\ +\,37 \\ \hline 50 \\ 11 \end{array}$$



$$\begin{array}{r} 24 \\ +\,37 \\ \hline 50 \\ 11 \\ \hline 61 \end{array}$$

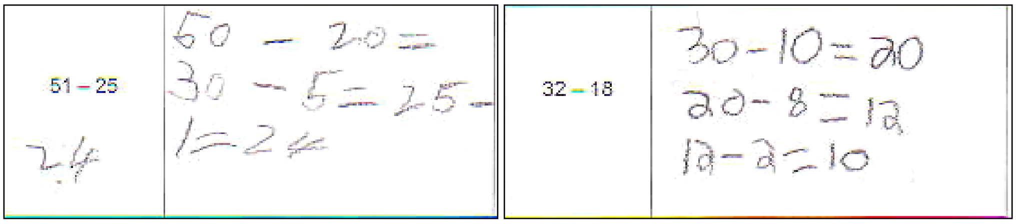
Trade 10 ones for 1 ten and move to tens place-value position. Record while manipulating the materials and write the answer at the end (see symbolic representations to the right of each diagram).

This example could also be recorded as 24 + 37 = 50 + 11 = 61. Students who understand place value and the concept of addition can manage the need to regroup as an addition.

Separation strategy for subtraction

The separation strategy for subtraction is more complex and research has shown that many students do not manage the separation of both numbers in a subtraction as well as they do for addition (Klein, Beishuizen, &

Treffers, 1998). This is the case particularly when there needs to be regrouping. The students separate both numbers into their place-value parts and while the subtraction of the tens is straightforward, subtraction of the ones causes confusion. The figure below shows student work samples where this error is evident.



An alternative way of completing subtraction using a separation strategy is to partition the numbers in a different way that allows the subtraction of the ones to be completed. This is particularly useful for computations that require bridging (regrouping). For example, for the 51 − 25 example on right, separating the 51 into non-place-value parts can make the computation more manageable, i.e. 51 as 40 and 11 allows the 20 and 5 to be subtracted more easily. This separating requires knowledge of place value and the relationship between place values. Completing computations using this strategy could enhance student understanding of place value.

<u>51 – 25</u>
Separate 51 into 40 + 11
Separate 25 into 20 + 5

40 – 20 = 20
11 – 5 = 6
20 + 6 = 26 (sum of parts)

Effectively this replicates the traditional algorithm but this method requires the student to use number-sense understandings about 51 to choose how to separate it rather than a procedural approach. There are also many other ways the 51 could be separated; for example, 26 + 25 (using knowledge that double 25 is 50) makes it very easy to subtract the 25 leaving the 26. Students with well-developed number-sense understandings can separate numbers in a wide variety of ways depending on the numbers in the computation.

The traditional written algorithm for subtraction uses regrouping of tens to manage the subtraction of the ones. The RAMR activity (2.5) at the end of this unit shows this strategy using a set material – bundling sticks.

2.2 Sequencing

The sequencing strategy is different from the separation strategy as follows: separation breaks both components into parts (usually on place value), while the sequencing strategy breaks one number up (can be place value but need not be) and keeps the other number whole. The parts are done with the whole number in sequence. In these examples, we will just show the steps through abstraction to mathematics. So the sequencing strategy works on the partitioning of numbers like the separation strategy. The operation being applied to the parts will vary depending on the operation of the overall computation. The sequencing strategy is often more efficient than the separation strategy as there are fewer parts when only one number is partitioned and so the entire computation will involve fewer steps.

As with separation strategies, there are three ways that the number in a calculation being partitioned can be separated:

(a) into place-value parts;

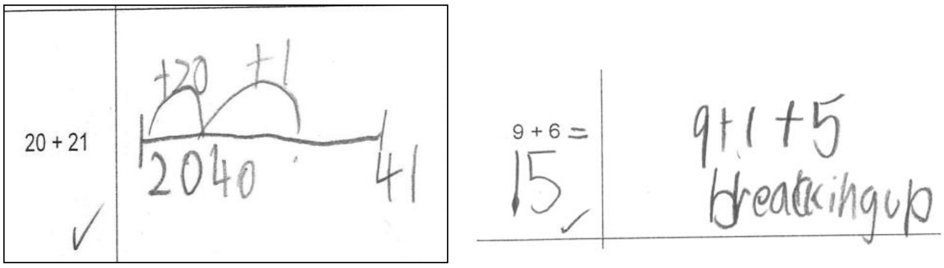
(b) into compatible parts; or

(c) into a combination of place-value and compatible parts.

The sequencing strategy works for each operation. This strategy is best taught from the rank understanding of number, that is, by using number boards and number lines. When the number line is used the operations are based on adding being to the right and subtracting being to the left (towards the zero).

Sequencing strategy for addition

Addition is commutative and therefore it makes little difference which number in an addition computation is partitioned and which number is left. The student work samples in the figure below show different ways students have partitioned one number to make an addition computation more manageable. In the first example the student has partitioned one number into place-value parts, in the second the student has partitioned into compatible parts so as to make a ten. This second example also shows the student's understanding of the number-sense concept of making ten and the Use Ten basic addition fact strategy.



Using a number board

Partitioning one number into place-value parts and progressively adding the parts (sequencing) can be well represented on a number board (99 or 100 or other board that is 10 columns wide). The following example shows how this can be used to add 33 + 22: 33 is the start number and the 22 is partitioned into place-value parts – 20 (2 tens) and 2 (2 ones). The tens jumps are completed first in this example. It is also possible to complete the ones jumps first.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

The jumps down the number board are in multiples of ten so in this example there are 2 tens hence the two jumps down (+20). The jumps to the right add 1 one per jump. There are two jumps to the right (+2).

Computations involving bridging (regrouping) on a number board require jumps off the right-hand end of the board which can prove difficult to manage. Using a different number board that still has ten columns but starts on a different number can alleviate these problems. The following example shows how this is done for 89 + 22.

34	35	36	37	38	39	40	41	42	43
44	45	46	47	48	49	50	51	52	53
54	55	56	57	58	59	60	61	62	63
64	65	66	67	68	69	70	71	72	73
74	75	76	77	78	79	80	81	82	83
84	85	86	87	88	89	90	91	92	93
94	95	96	97	98	99	100	101	102	103
104	105	106	107	108	109	110	111	112	113
114	115	116	117	118	119	120	121	122	123
124	125	126	127	128	129	130	131	132	133

Using a number line

Another representation of this strategy is the number line. The initial number which is being left untouched is the start number and the other number is partitioned into parts – either place-value parts or compatible parts and these parts are added sequentially as jumps along the number line. The number at the end is the answer. The following RAMR activity shows the sequencing strategy used with a number line.

Sequencing strategy for subtraction

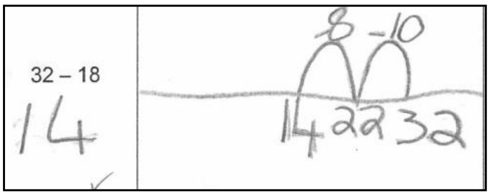
Subtraction is not commutative so the order of the numbers in a subtraction operation cannot be reversed. With the sequencing strategy for subtraction it is the subtrahend (the number being subtracted) which should be partitioned. This builds on the operation concept of subtraction as taking away and that there is a total and a part being subtracted to work out the other part.

Using a number board

The use of number boards can facilitate representation of the sequencing strategy for subtraction as it does for addition. The partitions will be place-value based and the jumps around the board are reversed (a jump up subtracts 10, a jump to the left subtracts 1). The example below shows 44 – 21 which is modelled using two jumps up (2 × 10 or 20) and one jump left (–1)

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Using a number line

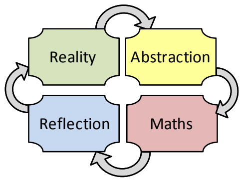


The number line also works for subtraction with the jumps represented from right to left rather than left to right. The figure on right shows a student work sample using a number line for subtraction using a sequencing strategy.

The sequencing strategy can be used to complete subtraction using an additive process. This requires students to understand subtraction as difference and that a move from the second number to the first number will enable us to find the difference between the two numbers. This makes contextual sense in relation to money to work out how much more is needed for a purchase or for counting back change. For example, 76 – 28 can be thought of as how far is it from 28 to 76 or what do I need to add to 28 to get 76?

The following RAMR activity shows how a number line can be used to keep track of the jumps and the progressive total. This example is working out 620 – 332. This strategy is particularly effective as it eliminates the need to deal with regrouping in computations that require bridging. The answer is found by adding the jumps. This represents the difference between the two numbers.

2.3 RAMR lesson outline for sequencing strategy for addition on number line



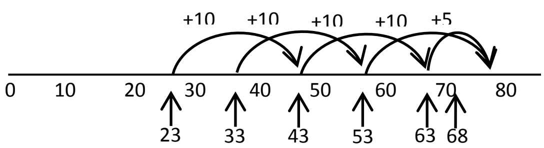
Sequencing involves one number being left as is and the other number being separated, so that parts of it are added in sequence. For example 27 + 48, the 48 is separated into 40, 3 and 5 and these numbers are added in sequence 27 + 40 = 67, 67 + 3 = 70, and 70 + 5 = 75.

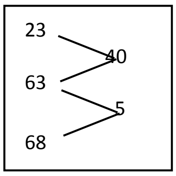
Reality

Look for real-world instances of addition and subtraction that have a length orientation so that the number line is appropriate as a model.

Abstraction

Act out the number line by walking the students – start at the first number and then move forward the second number – large steps for tens and small steps for ones. For the example 23 + 45 = 68, use a number line from 1 to 100 with 10s marked in. Start on 23 and move 45 to the right, 4 tens forward then 5 ones forward to get to 68 – so 23 + 45 is 23, 33, 43, 53, 63 (large steps), 64, 65, 66, 67, 68 (small steps).





Mathematics

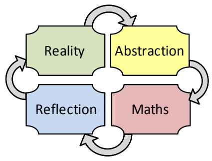
A recording procedure as on the right in the above diagram can be used to imitate what happens on the line. Students should be working towards not needing to rely on a physical number line.

The sequencing strategy can be recorded as a set of steps. The following example (on right) shows the addition of two three-digit numbers using the sequencing strategy. The 455 has been left alone and the 278 has been partitioned into a combination of place-value and compatible parts.

455 + 278
Start with 455. Separate 278 and add progressively.
+ 200 = 655
+ 50 = 705
+ 20 = 725
+ 5 = 730
+ 3 = 733
+ 278

2.4 RAMR lesson outline for sequencing strategy for additive subtraction

This is where we think of subtraction in terms of addition and move from the second number to the first number. For example, 76 − 28 is thought of as how far from 28 to 76.

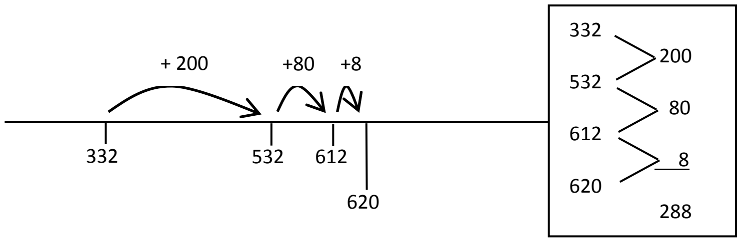


Reality

Again look for real-world instances, e.g. giving back change, and walk an additive subtraction from small to large number along a number line.

Abstraction

Use the number line as shown below. The idea here is to follow a path from small to large number and also to underestimate how much to increment in each leap (as you can simply do another leap). The answer is found by adding up the leaps. For example, 620 – 332 is found by starting at the 332 and working towards the 600.



Mathematics

Use the recording procedure on the right in the diagram above.

2.5 Compensation strategy

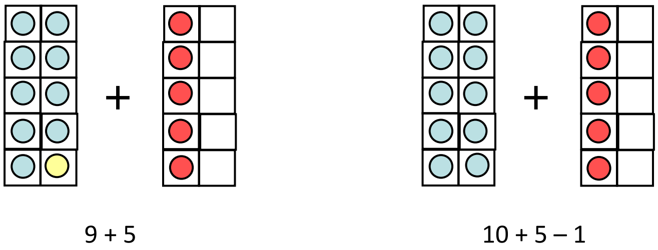
The compensation strategy does not partition the numbers involved in a computation but adjusts or changes the computation to use numbers that are more manageable. Sometimes there is a compensation for the change as an extra step after the change and sometimes the compensation is managed as part of the change. The focus is on making the computation more manageable. Often the changes that are made involve multiples of ten as it is easier to calculate with these numbers than others. The compensation strategy works for all four operations. Many of the compensation strategies use number-sense and basic fact understandings that students develop in early years of schooling. For example, knowing the numbers that add to make ten and the extensions of this Use Ten addition basic fact strategy are very useful with this strategy for larger numbers.

Compensation strategy for addition

It is easier to add multiples of ten than other numbers and numbers that add to form ten or multiples of ten can also be helpful when simplifying a computation. There are many different materials as well as good number sense that can assist to model this strategy.

Using ten frames

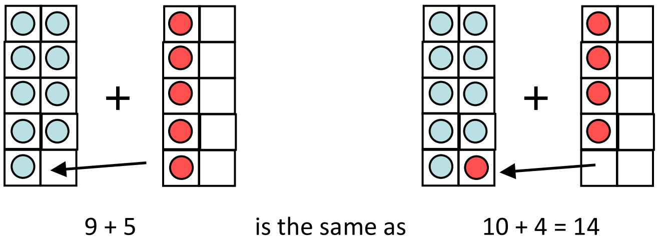
In the early years the compensation concept can be modelled using ten frames. With experience students soon learn that the frame has ten spaces and when full represents ten. By using these to model teen numbers (as was described in section 2.3, Basic fact strategies) students see that a full ten frame and a part full one is both a teen number and an addition. For example, the basic fact 9 + 5 can be thought of as 10 + 5 which is easier to work out (=15) but this is one too many. The answer to 9 + 5 must be one less (15 – 1 = 14). The students can identify the counter that needs to be removed when it is modelled with this material. This shows an adjustment followed by another action for the compensation.



A further example of this compensation strategy using decimal numbers is provided on the right.

35.8 + 34.3
Change 35.8 to 36 (by adding 0.2)
36 + 34.3 = 70.3
Compensate by –0.2
70.3 – 0.2 = **70.1**

Students can also be introduced to the idea of a double adjustment which changes the computation in such a way that there is no need for compensation. This works on the balance concept. The computation is modelled in the ten frames using the numbers given (e.g. 9 + 5). One of the counters making the 5 is moved to the other ten frame to complete the ten. The total has not changed but the computation now is easier to work out because one of the numbers is now 10.



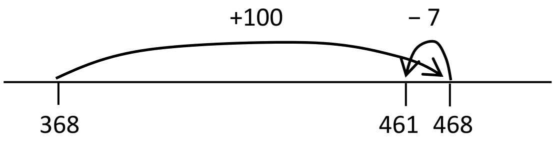
This strategy can be extended to larger numbers and decimals and can be generalised to state that you can change any addition computation by doing the opposite to each number. In the example above one number was changed by +1 and the other was changed by –1. The example on right shows this strategy being used with larger numbers. This example also shows how there can be more than one change made. It will only take a few changes to make the computation very easy.

1878 + 674
Change to 1880 + 672 (+2 and –2)
Change to 1900 + 652 (+20 and –20)
Change to 2000 + 552 (+100 and –100)
= **2552**

Using the number line

Addition can be modelled as jumps along a number line. One of the numbers in the computation is the start number and the other is the amount to be added which can be done in jumps of any size. In the sequencing strategy one of the numbers was partitioned and added in parts to the first number. The number line was used to record the jumps. With the compensation strategy the number line is used the same way but the number being added is changed to a nearby multiple of ten and an over jump is completed, because it is easier, and then the compensation is the jump back so the answer is accurate. It should be noted that if an approximate answer was all that was required then the compensation step can be left out.

Example 368 + 93



368 + 93
Easier:
368 + 100 = 468
Compensation:
468 – 7 = **461**

Using a number board

A similar use can be made of the number board. A multiple of ten can be added easily and the structure of the number board assists with this and then another jump can be made to compensate.

Example: 56 + 19:

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

<u>**56 + 19**</u>

Instead of +19
Easier to +20

Compensate by –1

56 + 20 = 76

76 – 1 = **75**

Compensation strategy for subtraction

The compensation strategy works for subtraction for similar reasons as it does for addition. It is easier to subtract multiples of ten than to subtract other numbers. Students need to realise that it is easier to subtract multiples of ten from another number than it is to subtract from a multiple of ten. So when looking to use the compensation strategy for subtraction, it is the subtrahend (the number being subtracted) that should be adjusted. Then, as with addition, a compensation action can be done to ensure the answer to the computation is exact.

Using a number board

In the example below (65 – 19) two jumps back of ten are made (–20) and the compensation this time is to +1. With the number board the students can do a comparison by jumping back 1 ten and then 9 ones and they will end up on the same square. They can also conceptualise the compensation needed in terms of "by taking off 20 instead of 19 I took away one too many, so I need to put that one back on".

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

<u>**65 – 19**</u>

Instead of –19
Easier to –20

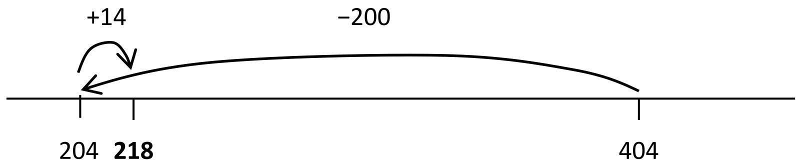
Compensate by +1

65 – 20 = 45

45 + 1 = **46**

Using a number line

The use of over jumps as an adjustment to a subtraction computation followed by a compensation action is another way to model the compensation strategy on a number line. The example below shows 404 – 186:

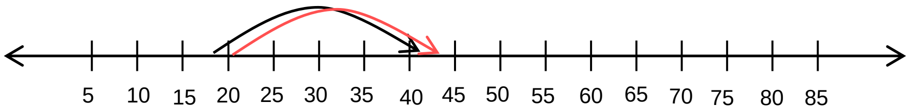


404 – 186
Easy:
404 – 200 = 204
Compensation:
204 + 14 = **218**

Using knowledge of adjusting for addition

The compensation strategy for addition was shown to enable a change to be made to both numbers and as long as the total remained the same the easier computation could be completed instead of the original. The same concept can be applied to subtraction computations but instead of the total needing to remain the same it is the difference that needs to remain the same. This can be shown using a number line.

The number line below shows the difference between 18 and 41 using an arrow (black). As a subtraction this would be represented as 41 – 18. To make this computation easier it is best to adjust the subtrahend (the number being subtracted) to be a multiple of ten. If this was changed to 20 (by +2), the difference needs to be kept the same which means that the first number also needs to be changed by +2. The second (red) arrow on this number line shows the new computation and that the difference is still the same. The new computation is 43 – 20. This compensation to maintain equivalence can be generalised to state that to change a subtraction computation you need to do the same to each number.



This strategy can then be applied to larger numbers. As with addition, the strategy can be applied several times until the computation is quite easy. Students need to remember to focus on changing the subtrahend to make the computation easier.

1878 – 674
Change to 1874 – 670 (–4 and –4)
Change to 1904 – 700 (+30 and +30)
= **1204**

2.6 Activities for using facts to calculate to 100

1. Split strategy, standard and non- standard:
   (a) Encourage children to talk about how they break numbers up to work with them. Look for responses that show use of basic facts.
   (b) Use groups of ten to count large collections. Use counting boxes regularly for variety and practice.
2. Jump strategy on the number line:
   (a) Have students skip count to a given number along a number line using groupings to see the count always reaches that number. E.g. 12 has equal groups of one, two, three, four and six. Have students draw this on the cement with chalk.
3. Use a range of mental strategies regularly:
   (a) Counting *counting on from the larger number to find the total of two numbers;
   (b) *counting back from a number to find the number remaining;
   (c) *counting on or back to find the difference between two numbers;
   (d) Doubles *using doubles and near doubles;
   (e) Use 10 *combining numbers that add to 10- Use the 10 frames to play compensate to ten.
   (f) *bridging to 10, e.g. 17 + 5: 17 and 3 is 20, then add 2 more; *using place value to partition numbers, e.g. 25 + 8: 25 is 20 + 5, so 25 + 8 is 20 + 5 + 8, which is 20 + 13

(g) Use a rule *Investigate and generalise about the effect of adding zero to any number

(h) Extended facts *Multiple-of-ten facts

(i) extensions using patterns to larger numbers

(j) Have students investigate arrays from different positions and describe the groupings they see; e.g. muffin tins, egg trays etc.

Unit 3: Calculation Written Approaches

Modules O2 and O3 complement each other. Module O2 develops students' ability to translate stories/problems into operations orally and in symbol form, as follows:

O2: Operating story → operation

Module O3 develops students' ability to calculate the answer once the operation is known, as follows:

O3: Calculating operation → calculation/answer

Putting them together, we have the following:

O2 → O3: story to operation → operation to calculation

The question arises as to how the calculation should be recorded. In the past, calculation was recorded by formal algorithms which had a common structure. This was essential because the algorithm was the only form of calculation and the process had to be checked, requiring a common process.

With the advent of calculators/computers, the need for a common written algorithm has diminished. There has remained a need for some recording to assist accuracy in calculation, but is this to be informal, child chosen, common to the school, or a formal algorithm?

Thus, schools need to plan what they want from calculation. The position of YDM is that there should be opportunities for students to create their own ways of recording and that they should be able to show their working in some way. This requires students to examine their own thinking and thereby develop metacognition. These creations need not be the same for each learner.

We also advocate that the crucial aspect of calculation should be understanding of the process. This means looking at the process as a strategy rather than an algorithm. Strategies are "rules of thumb" that point towards an answer, while algorithms are sequences of actions that will give an answer. We also believe that the calculation process should integrate with other topics, e.g. place value, concept of operation and principles for operations, and that this integration should be understood as well. For example:

$$\begin{array}{r} 32 \\ +\,3 \\ \hline 35 \end{array} \text{ because joining like things} \qquad \begin{array}{r} 32 \\ \times\,3 \\ \hline 96 \end{array} \text{ because 3 groups of 32}$$

This unit explores written calculation, looking at options and discussing approaches. It also looks at calculation algorithms or processes in terms of strategies and points out how algorithms as strategies can provide a foundation for understanding higher level mathematics.

Background information

In general, the position of YDM is that all methods should be taught to all students as the methods and strategies are more important than getting answers, which can be done with a calculator or, close enough, by estimation. However, in a remedial situation, as in AIM, only one method is needed. YDM's recommendation would be to ask the students to be accurate with that method. If students have no method, choose one which is common across your school and teach that.

With regard to recording, there are three ways it could be done: (a) *answers only* – when full mental methods or calculators are used; (b) *informal writing or doodling* – numbers and drawings that assist the mental processes, most idiosyncratic to the learner; and (c) *pen-paper recordings* that imitate the material manipulation – ways of recording that lead on from materials and can replace the material thinking.

YDC believes that the mental ↔ writing dichotomy is not helpful and would advocate a variety of possibilities.

First, it is useful for students to develop their own thinking/recording processes for computation as this fits in with their individual understanding. We think that such recordings are useful because they:

(a) assist with external memory (e.g. when adding 365 +4 27 by sequencing it is useful to write down the 427 so it is not forgotten in the sequencing process which starts with 365, adds hundreds = 765, adds tens = 785, and finally adds ones = 792;

(b) assist to extend the mental processes to other topics, for example, extending algorithm processes to algebra as follows:

24 ×2 is 24		and so $2(3x+2)$ is $3x + 2$	
$\underline{\times 2}$		$\underline{\times\ \ 2}$	
8	(4 × 2)	4	(2 × 2)
$\underline{40}$	(20 × 2)	$\underline{6x}$	($3x$ × 2)
48		$6x + 4$	

(c) enable some calculation to be done using recording if students forget to bring their calculator or the batteries fail;

(d) assist remembering because we have mental and writing memory; and

(e) assist students to communicate and explain their thinking processes.

Overall YDC's position is to have some other way than with a machine to calculate to 1000 and to use calculators and estimations for larger numbers.

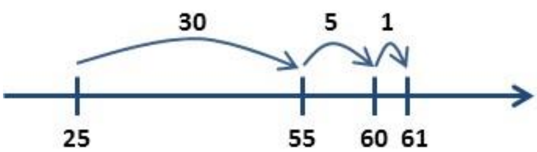
3.1 Models, drawings and jottings/doodles

1. The first step on the road to some form of writing with calculation is to allow students to use materials and drawings to help with calculation. The secret is to let the students play with a variety of things to assist with operations and then encourage them towards being more efficient. For example, in adding 17 and 14, a student may use a ruler to find 17 and then count on 14 (if the ruler is large enough). The idea here is to direct the student to, for example, the sequencing strategy – add on the 10, then count on 4. As another example, students often show 23 × 4 as 4 lots of 23 matches. This is okay for a start but if they could see it as 4 lots of 20 and 4 lots of 3, it would be moving towards a more efficient thinking.

2. Experience problems and calculating and try to replace materials with drawings and then numbers. For example: 23 × 4 could become a combination of drawings and numbers as on right:

20	III
20	III
20	III
20	III
80	12

3. The strategies in Unit 2 have good opportunities in drawings, for example, sequencing 25 + 36 is as below:



This is a quick guide drawing that helps the calculation. Try to encourage less drawing. For example, 25 → 55 → 60 → 61 could be sufficient as a jotting to replace the line.

4. Ensure drawings and doodlings lead to symbols that reflect the manipulation of ideas in the strategy. For example:

Separation (24 + 48)		Additive subtraction sequencing (82 − 37)	
2 4		37	
+4 8		40	3
1 2	1s	80	40
6 0	10s	82	2
7 2			45

Guide the students progressively to more abstract ways – utilising operation signs, adding in zeros, and drawings of MAB or bundling sticks and so on.

5. Enable students to experience the different ways calculations can be written. Get students to share their doodlings, drawings or writings. Do number studies, ask a problem and get children to come out who have different thinking and doodlings to get answers.

3.2 Student-created recording

1. Undertake a variety of activities such as:
   (a) playing with numbers and expressions for operations;
   (b) using thinkboards where the focus is on drawings and ways to calculate;
   (c) journals and jottings and graffiti pages to encourage creativity (one teacher had a good book and a messy book for maths and she only marked what was in the good book).
   (d) group work where students argue over the best way to do a calculation and record it.
2. Get students to justify why their method works. Encourage students to change method if inefficient.
3. Get students to work with other students' methods and recording.
4. Get students to work out how another student calculated from that student's written record. Discuss how to make this recording more easily understood by others.

3.3 Formal recording

1. Sometimes there has to be formal recording of the operation and calculation process. Introduce notion of number sentence and equation for calculations.
2. Get students to use their own calculation process and doodling but ask for a correct equation at the end. For example, a student given: *You buy pants for $42 and a shirt for $29, how much change do you get from $100?* doodled the answer as on right. Although formally wrong in setting out, this would be allowed but would require a final sentences such as:

42
+29
11
60
71
−100
29

   Answer = 42 + 29 = 71, 100 – 71 = 29.

   Explore different ways a number sentence could be written.
3. Ensure students understand equations, that = means "same value as". Check students understand that 71 = 42 + 29 is just as good as 42 + 29 = 71. Go back to Module A2 and look at work on "equivalence and equations".
4. Look at the number sentence and equations in SEE–PLAN–DO–CHECK. Sometimes thinking of a problem as a number sentence can help solve it. Remember, there may be equivalent number sentences which mean the same thing, e.g. 42 + 29 = 71 followed by 71 + 29 = 100 means the same as 42 + 29 = 71 followed by 100 – 71 = 29.

5. Look at properties of operations and relationships between them (and common sense) to help to decide whether number sentences are true. For example, if a problem asks for change from $50 and the student has calculated the sum as 23 + 17 = 58 (adding digits of numbers), it is likely their calculation is incorrect.

3.4 Making sense of written calculations

1. As stated at the start of this unit, Modules O2 and O3 represent two steps:

Story problem	→	Operation Number sentence	→	Calculation Answer

   Thus, a number sentence like 3+4 is the middle of this 2 step process while the calculation 7 is the end. An example is as follows

Problem	**Operation (Number sentence)**	**Calculation**
"John received $26 from Auntie and $32 from Uncle. How much money did he get?	$26 + $32	$58

   Experience this 2-step process for stories made up by students.

2. Reverse the process

Problem	**Number Sentence**	**Calculation**
?	?	68 fish

   (a) Experience this reverse with a variety of answers. What could the number sentence and problem be for the above example? Check that the results are reasonable by considering context, mathematics strategies and common sense.

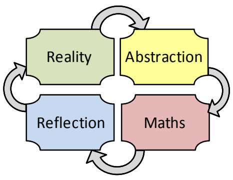
   (b) Experience situations where calculators are central – board games and card games.

   (c) Reverse calculation as on right. 4 $\boxed{}$ 4 $\boxed{}$

   Find the box numbers? 2 7 or + $\boxed{}$ 7

   7 1 7 1

3.5 RAMR lesson for using written methods to calculate



Unit 4: Introducing and Using Calculators

Technology devices are a part of everyday life and the classroom- iPad; phones; computers; and the stand alone calculator. These tools can be helpful when working mathematically.

This unit explores how to introduce calculators for best results.

Background information

Calculators fulfil two roles in the classroom – as an instructional aid and a computational aid. They provide: technology knowledge; enjoyment; accuracy, comparison and validation. It is important that students are taught how to use a calculator through considering keypad layouts; how to operate various keys; their purpose and function.

For young children, this first learning is:

- identifying where each number key is i.e. all calculator keypads show the digits 0 and 1 on the bottom left (most phones show 1 on top left of key pad);
- entering numbers and keying accurately;
- function keys Clear Enter = on/off;
- using operations keys + − × ÷ ;
- and reading and interpreting numbers from the display.

Following this students need to understand the limitations of calculators. Most basic calculators have an eight-digit display. The largest number that may be entered into an eight-digit calculator is 99 999 999. As you enter digits they start on the right and move to the left. Each time you press nine and the digit moves one place to the left it increases by a power of ten.

Later, in order to make use of a calculator properly students need to have a good understanding of place value. They eventually gain an understanding of when it is sensible to use a calculator, for example, to monitor calculations. This would include a variety of checking techniques one of which is estimation.

4.1 Exploring the calculator

The first skills students need are the ability to recognise and use the following keys: digits 0–9; operations + − × ÷, clearing the display, and using and understanding calculator language.

1. Talking about calculators

What are calculators used for? Who uses them? What have you learned to do using a calculator? These questions will promote discussion about the uses of a calculator. It will also provide opportunity for the children to demonstrate their current level of understanding of its functions. In this and subsequent lessons they will also use the constant addition function for counting and for counting on from a given number, forwards and backwards in ones, twos, threes, fives, tens, etc.

2. Playing with calculators

Calculator digits

Look carefully at the numbers on the display of any calculator and you will note that they are made up of little bars.

Enter each of the digits from 0 to 9 into your calculator and, using these as a guide make a set of calculator digits using match sticks.

- Which digit took the least number of matchsticks to build?
- Which digit took the most matchsticks?
- Which digit needed the same number of matchsticks to build?
- Find the largest number you can make with 8 match sticks.
- What numbers can be shown using exactly 7 matchsticks, 9 matchsticks?

Calculator queue

This activity can be adapted to many age levels simply by altering the size or range of numbers used, or by including decimals. It is ideal for testing children's understanding of place value and how well they read numbers. For young children begin with small whole numbers.

Ask children to enter a number into their calculator and then line up from smallest to largest.

Young children could be asked to enter a number between 1 and 10. 1 and 100, their house number etc.

3. How do calculators work?

Have students use calculators to explore, press keys and discover what happens. Focus students on watching the display to notice what happens each time they press a number key. Have them take turns to read out the digits on the display for others to make the same numbers.

Clearing

Encourage students to become familiar with the function of particular keys, e.g. C/CE. After a student discovers how to clear numbers, ask them to show all students the steps. To practise, they key in 1, press the clear key to clear it back to zero. Continue in order through the numbers to 12. Later as they enter calculations, e.g. 24 + 36, show them how pressing this key once clears the 36 and pressing it twice clears all back to zero. Practise this through games such as "Simon Says".

4.2 Calculator language

Develop young children's language in the following sequence:

Clear keys- AC (all clear); C (Clear); On/off

Digit keys- Ten (0-9). The digit keys are set out on the keyboard with the greatest numbers at the top.

Display screen – space for eight digits depending on calculator used. The digits appear from the right hand side of the screen and they move across the screen towards the left hand side

Operation keys- the four operation keys often appear in vertical order: division at the top through to addition at the bottom

Decimal point- this usually appears next to the zero key

Special purpose keys

Memory

- Correct terms

Ask students to use common correct terms to describe what they do on the calculator, and see what happens, e.g. press, key in, display, clear. At first students will use their natural language to describe the function keys that are beyond everyday use, e.g. for the square root sign they might say' the squiggly thing'. Over time, support them to rename terms.

4.3 Calculator as a recording device

Students need to be able to:

- -enter whole numbers correctly into the calculator
- -read the display for whole numbers
- -use the constant function to generate + and - sequences

- Recording device

Have students use the calculator as an informal recording device to enter multi-digit numbers such as telephone numbers, dates, and game scores, to read and for others to copy. Ask: Which number (digit) did you put in first? Why?

- Counting device

Ask the students to use the calculator as a counting device. For example, use the constant function , counting by twos : 2 + = = = = Ask: Will it work for other numbers? Or as one student said after entering 1 + = = = It works for 'real' counting too! For some students counting will mean making sequences of increasingly larger numbers; others will recognise it as 'counting by' particular numbers, or skip counting. Only a few in the beginning see any connection between the counting sequence they see on the display and the process of counting how many.

- Constant function

Have students watch numbers increasing into the hundreds using the constant function to skip count.

-Number scrolls: students record the numbers vertically on long strips of paper. Interrupt the count at times for students to predict the next number.

-Number Scrolls: When skip counting with a calculator, ask students to record the numbers on paper strips and say what number will come next and what number they might reach at the end of the strip. Ask: How did you decide? Also can relate to developing a number line.

-Develop' number lines', e.g. lily pads for a frog to jump a long

Have students work in pairs and practice by counting items such as chair legs (4's); bicycle wheels (2's); triangle corners in a set of pattern blocks (3's); and other groups of items they can find. Make sure they compare each constant function count.

4.4 Explaining to others

Students need:

-the skill to read and interpret the numbers in the display

-make use of and explain the effect of the operation keys

- Explaining to others

When students find something that always happens on the calculator, have them explain it step-by-step to others. For example: When you have a sequence of numbers and press C/CE, it changes the numbers to zero.

- Next Number

Ask students to read out loud the numbers on their calculators as they use the constant function to count. Stop students at 9, and then ask: What number will be next? Check to see if you are correct. What is different about 9 and 10? Has the calculator used these single numbers before? Use the students' responses to discuss the number of digits and the difference the place makes.

- Number Rolls

Ask students to generate number sequences using the constant function on their calculator over the decades and the hundreds. Then have students read say and verify the numbers from the calculator display. For older children have them predict the next number when bridging hundreds and thousands and larger numbers then check using the constant function on the calculator.

- Counting in hundreds (this is really a middle year's activity but little kids love playing with '100s')

Have students use constant addition on their calculators to count in 100s. Have them predict which number will come next, then press = to verify. How many hundreds did you put in to make 900? How many hundreds are in 1000(2000)?

- Guess my number

In pairs one student puts in a 2 (or 3) digit number. The other student has to guess it (work it out). Students record the number of times/steps it takes. Use simpler or more complex numbers depending on the understanding level you are working with.

4.5 Computational and solving tool

Students need to understand when it is sensible to use a calculator amongst other computational choices.

- Computational tool

Use the calculator as a computational tool. Have students choose which operation key to use when numbers are too large to solve mentally. Students may begin to use the ÷ key for sharing, e.g. eight bears need to share 74 cakes. How many cakes will each bear get? Share 74 cakes among 8 bears.

- Secret Number

How to play this calculator game. Organise yourself into pairs. One student secretly enters a number into the calculator and then adds a number they both agree on, e.g. five. The student with the calculator shows their partner what the new number is. The partner says what the original number was and checks it on the calculator by entering a subtraction. For example: the first student secretly enters seven, adds five and gets 12. The partner says 'the number was seven', but must check the calculator by entering a subtraction (12-5) and not by entering 7+5.

- Dicey Games

Play games such as three- dice game. Organise students into pairs. Give each pair three dice. Have students take turns to throw three dice, add the numbers together and keep a running total on a calculator. During the game ask: Which two numbers did you add together first? Why? At the end of the game (when one reaches a total of, say 50 or more) ask students to use number sentences to show their working out for at least one turn.

- Thinkboard

Use a thinkboard to create three different representations of one operation starting from a story (e.g. model with materials, draw the situation in a picture, use numbers and symbols). For example: there were 15 bears in the zoo enclosure, but I could only count nine. How many were hiding in their cave? Compare your representations with the others in your group.

What is the same/different about the things/pictures? What is the same different about the number sentences? Can you use both addition and subtraction to show what you did?

- Calculator Nim

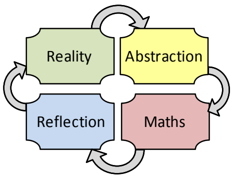
Nim is a strategy game. The main reason behind the game is for the children to discover a winning strategy. In particular Nim may be used to encourage children to use the 'working backwards' problem solving strategy. Once a strategy is found, there is little point in continuing with the game.

Players may use only the following keys 1 2 3 +. Players take turns entering either 1 2 or 3 and then the + key. The player who reaches 21 exactly wins the game.

- Beat the calculator

Ask your students to write down some addition and subtraction examples that they can do faster in their head than on a calculator and vice-versa. Have them test their suggestions. Discuss the outcomes. To further emphasise sensible computation choice pair students and give them some calculations to complete. One student must use the calculator and the other mental strategies to solve the problems. List the calculations that are faster in the head and those faster on a calculator. Ask what do you notice?

4.6 RAMR lesson for using calculators



Learning goal: To use a calculator and to be able to enter and interpret the information correctly

Big ideas: Technology and tools can help us think

Resources: counting resources, calculators, pens, paper,

Reality

Local knowledge: Where possible, find real-life contexts to embed the activities in; for example, using relevant objects or situations.

Prior experience: 1:1, counting, sorting

Kinaesthetic: Play Exploring Calculators

Have students use calculators to explore, press keys and discover what happens. Focus students on watching the display to notice what happens each time they press a number key. Have them take turns to read out the digits on the display for others to make the same numbers.

Use a student example to draw the calculator key sequence on the board e.g.

$\boxed{3}$ $\boxed{+}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$

Ask: what is happening here? Can each of you make your calculator 'count' like student 'J'?

Ask: will it work for 2 as well? How far can you count? What other numbers can you try?

What happens when we use the number 1? Observe how different students use the word 'count'. Do students see a connection between the counting sequence they see on the display and the process of counting to find out how many?

Abstraction

Body: Have students count each other using the calculator and making connection between the counting sequence and how many. Have them do this in groups of 2 and 3 and 4.

Hand: Have students make calculator key sequences on white boards or with number and symbol cards for others to interpret and follow.

Mind: Have students tell each other sequences to follow on their calculators e.g. have materials nearby, 'I want you to count by two's. Key in two plus. Fingers off! What do we have on the display? So the calculator is telling you it is ready to count by two's. Let's count the first lot of two…press equals. What do you have on the display? Now count another lot of 2.. Press equals. What do you have on the screen now? When students are familiar with this activity have them use pretend calculators and see in their mind's eye.

Creativity: Have students create their own sequences using materials and mediums of their choice and share them with each other. Ask them what strategies they devised to make sure they counted each object or groups of objects once only.

Mathematics

Language/symbols: digits, key, symbols, =, clear, display, key sequence, key in.

Practice: Play Number Scrolls and Next Number activities. Have students record their findings.

Connections: Counting, sets, money, size, operating.

Reflection

Validation: Students go outside and count collections of things by ones or in groups e.g. ants, cars, leaves etc.

Applications/Problem solving. Have students step along a path. Ask them how many steps they took? How did you count your steps?

Extension:

- *Reversing:* Ask students to count backwards using the calculator as collections decrease.
- *Generalising/Changing parameters:* Play Calculator Nim activity. Discover and explore negative numbers by counting backwards from a given number.

Module Review

This section reviews the units in this module. It looks across the units and identifies outcomes that go beyond the particularities of the units. The first of these is general **teaching approaches**, that is, ways of teaching ideas common within the units and across most of mathematics. The second is **models and representations**, that is, common ways of providing students with thinking images that support learning and applying mathematics. The third is **critical teaching points** of **competencies**, that is, abilities that are important across the units and into the future. The final is **later calculating**, information on the mathematics that grows out of this module and provides the reason for its importance.

Teaching approaches

Things we need to consider:

- While classroom approaches to young children's thinking skills aim to develop reasoning, enquiry and creativity, most recent psychological research has focused on children's powers of reasoning and enquiry exclusively (and not **creativity**).
- Current approaches to teaching thinking skills do not draw upon **explicit strategies** to help very young pupils develop their emergent theory of mind or their skills in counterfactual reasoning.
- While the psychological literature reveals that children find some kinds of question more difficult or confusing than others, few studies relating to pedagogical approaches focus on **questioning** specifically.

Encourage teachers to explore the extent to which:

- their questions can focus specifically on stimulating children's thinking;
- they can create timetabled opportunities for "thinking times" which signal to the children that a non-ordinary (and possibly counterfactual) kind of thinking is being encouraged;
- more opportunities can be created in the classroom for structured dialogue;
- children can be invited to construct written opinions and arguments;
- "story-time" can become an opportunity to develop children's thinking;
- traditional sorting and sequencing tasks can be an opportunity for children to verbalise their thinking;
- play equipment can present children with possibilities for developing their imagination;
- children can be given opportunities for solitary as well as social play;
- children can be asked to evaluate their work critically;
- additional adults in the classroom can be used to develop children's thinking; and
- creative activities can encourage creative "possibility thinking", as well as creative skills.

Finally, because algebra is the generalisation of arithmetic, it will be necessary to focus on the development of the new concept of variable as standing for any number and on the big ideas from arithmetic that carry through into algebra (e.g. concepts of operations and equals, principles associated with operations and equals).

Models and representations

It is important that symbols are not introduced as a replacement for models but as an adjunct (another way to represent the number or operation). The end product of operations is a **symbol language** (e.g. 2 + 3 by the end of Module O2). The symbols are then calculated to come to a symbol answer (e.g. 5 by the end of Module O3). The symbol language must be understood and the best way is to maintain contact with other representations. For example, the focus on answers and not relationships leads many students to believe that 2 + 3 = 5 is acceptable and 5 = 2 + 3 is not; for many students the equals sign has become a symbol for "put the answer here” or “do something” when its real meaning is “same value as”. Mathematically, *7 subtract 3 equals 4* is represented with symbols as 7 – 3 = 4; it must be seen as **7 – 3 is the same value as 4**. This means that it is possible and equally correct to show 7 – 3 = 4 as 4 = 7 – 3. Similarly, students also need to be familiar with relationships that are not closed to a single answer, such as 2 + 3 = 4 + 1. This symbol language has to relate to everyday language and real instances. Many forms of symbols are possible and all relate to stories as the addition stories below show. The interesting point is that treating the symbols like a concise language (so 7 + 4 is 7 things joining 4 things), enables the symbols to tell stories and to describe the world, an outcome more powerful than answers. This is a major part of building the concepts of the operations (see Module O1, subsections 2.1 and 3.1).

One way to represent this symbol language is to use **models**. Interestingly, models unify mathematics (where symbols tend to emphasise difference). For operations, there are three models: **set** (e.g. Unifix and counters), **length** (e.g. Unifix stuck together or number lines), and **array or area** (e.g. counters, Unifix, dot paper or graph paper), which is for multiplication and division (e.g. 3 × 4 is 3 rows of 4 or a 3 by 4 rectangle). These models do much more than just help with meaning – they show structure and apply across many topics. For example, the array model can help with the following:

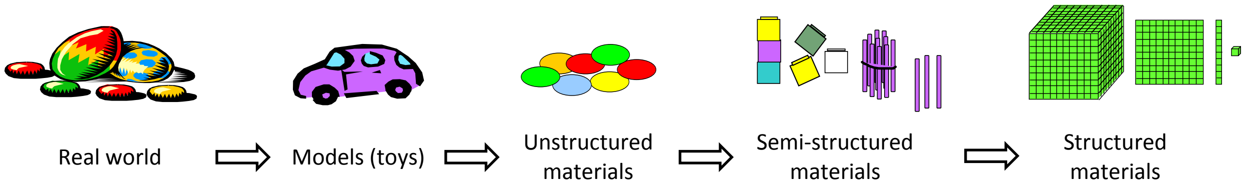
(a) basic facts (e.g. 4 × 7 is 4 rows of 7 which is 4 rows of 5 plus 4 rows of 2);

(b) algorithms (e.g. 24 × 7 is 24 rows of 7 so it is 20 rows of 7 plus 4 rows of 7); and

(c) fractions/decimals and percent (e.g. 0.2 × 0.4 is a rectangle 2 tenths by 4 tenths which gives 8 squares in 100 or 0.08).

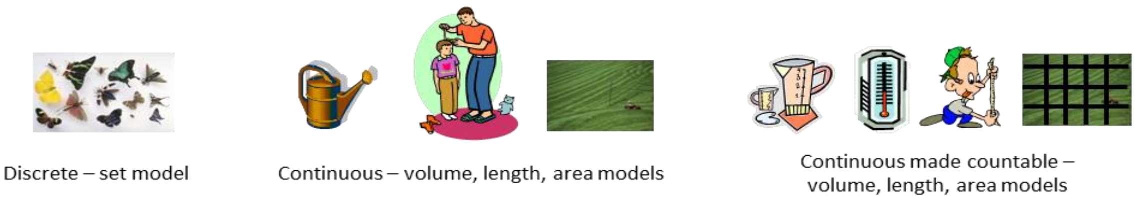
Multiplication is the inverse of division and ×1 or $\times \frac{2}{2}$ or $\times \frac{3}{3}$ and so on leaves everything unchanged (e.g. $\frac{2}{3} = \frac{2}{3} \times \frac{2}{2} = \frac{4}{6}$, and so on).

YDM focuses on knowing connections between real-world situations, models, language and symbols, and strategies that lead to meaning and generalisation rather than rote definitions and procedures (or algorithms) that lead to particular answers for particular numbers. In this approach, models can be used in the same sequence as for number and can be tailored to suit the level of number representation students are working at. Kinaesthetic acting out of operations should be completed first, which can then be transferred to tangible concrete models and finally abstract number sentences (see figure below).

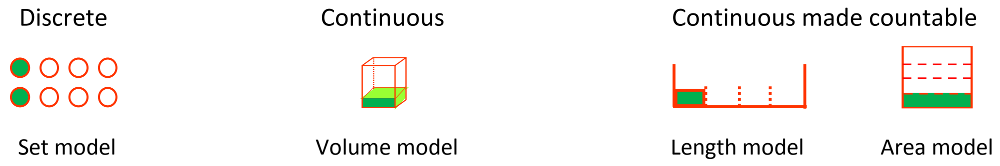


Sequence of physical material representations

Models also assist us to link **discrete and continuous**. There are a range of models that cross that divide, as the examples below show. The continuous made countable models are particularly important.



Real-world models for discrete, continuous, and continuous made countable



Examples of discrete and continuous models to support maths learning

As number is taken from discrete objects and applied to continuous attributes, its nature changes. Numbers designate ends of units and zero represents the start of units and not nothing (although it represents no units).

Critical teaching points

The critical teaching points for calculating are that in computational situations students can do the following:

1. Use one-to-one counting, combine collections, skip count, and identify beginning–action–end (BAE).
2. Know strategies to derive basic number facts, use flexible mental strategies to calculate to 100, and use written approaches as back up.
3. Use calculator as a recording device, to explain thinking to others, and as a computational and solving tool.
4. Consider accuracy in terms of whether it is appropriate, necessary, or better to estimate or approximate.
5. Check whether the calculation makes sense in terms of the numbers, operations, and context.

Later calculating

As students move through the years of schooling, the following calculating understandings are important.

Basic number facts

It is recommended to build basic facts through the use of strategies because: (a) strategies can help develop students' understanding of the operation concepts as well as the basic facts themselves; and (b) automaticity in basic facts is best achieved by speeding up strategy steps rather than making direct connections. After introduction through strategies, repeated practice can lead to automaticity. Therefore, continued and ongoing practice and diagnosis of basic fact strategies for addition, subtraction and multiples of ten is recommended through the later years.

Using facts to calculate to 100

Addition and subtraction algorithms for whole numbers are accurate (not approximate) addition and subtraction calculations for numbers of two or more digits. Today, they can be completed mentally, with pen and paper, and with calculators. This module looks at them through the three strategies that can be used, namely *separation*, *sequencing* and *compensation*. We recommend that these strategies be reserved for computations of up to three digits – over this size we recommend estimation and calculators.

Calculation using written approaches

Once students are able to confidently carry out an operation using mental and informal pen-and-paper strategies it may be appropriate to help them develop standard written methods that will be efficient for more difficult calculations.

Schools need to plan what they want from algorithms. The position of YDM is that there should be opportunities for students to create their own ways of recording, and that they should be able to show their working in some way. This requires students to examine their own thinking and, thus, develop metacognition. These creations need not be the same for each learner.

In general, the position of YDM is that all methods should be taught to all students as the methods are more important than getting answers, which can be done with a calculator or, close enough, by estimation. However, in a remedial situation, as in AIM, only one method is needed. Our recommendation would be to ask the students how they would do a sum – determine which method it is and, if they are happy with it, support the student to be accurate with that method. If students have no method, choose one which is common across your school and teach that.

With regard to recording, there are three ways it could be done: (a) answer only – when full mental methods or calculators are used; (b) informal writing or doodling – numbers and drawings that assist the mental processes, mostly idiosyncratic to the learner; and (c) pen/paper recordings that imitate the material manipulation – ways of recording that lead on from materials and can replace the material thinking.

Introducing and using calculators

Following this students need to understand the limitations of calculators. Most basic calculators have an eight-digit display. The largest number that may be entered into an eight-digit calculator is 99 999 999. As you enter digits they start on the right and move to the left. Each time you press nine and the digit moves one place to the left it increases by a power of ten.

Later, in order to make use of a calculator properly, students need to have a good understanding of place value. Overall, they need to eventually gain an understanding of:

- when it is sensible to use a calculator, for example, to monitor calculations to assist accuracy, estimating and validation when solving problems;
- how to mentally estimate the results of a calculation to check the reasonableness of calculator results; and
- how to use a variety of checking techniques, one of which is estimation.

Strategies instead of calculation

It is recommended that the focus on calculation be in terms of strategies to enable answers rather than calculation for answer getting. The reason for this is that strategies exemplify big ideas that can be used later in schools, while answer getting from calculation is more efficient with a calculator or a spreadsheet. Two calculation strategies that can be useful for later mathematical understanding are *separation* and *compensation*.

1. **Separation**. This strategy involves breaking a difficult problem into parts where the parts can be independently solved and the solutions combined for the total solution. This is the strategy of the standard algorithm and it can be extended to more advanced calculations as shown:

$$\begin{array}{r} 2\,3 \\ +\ \underline{4\,1} \\ 6\,4 \end{array} \qquad \begin{array}{r} 2\text{m} + 61\text{cm} \\ +\ \underline{4\text{m} + 27\text{cm}} \\ 6\text{m} + 88\text{cm} \end{array} \qquad \begin{array}{r} 3\text{h } 31\text{min} \\ \times\ \underline{5\text{h } 22\text{min}} \\ 8\text{h } 53\text{min} \end{array} \qquad \begin{array}{r} 3a + 2b \\ \times\ \underline{2a + 6b} \\ 5a + 8b \end{array}$$

2. **Compensation**. When faced with a difficult problem, this strategy involves looking for a simpler but related problem that can be completed and then compensating for any changes. Some examples are:

   $8 + 5 \rightarrow 10 + 5 = 15 \rightarrow 15 - 2 = 13$ (add 2 and subtract 2 to compensate)

   $38 + 26 \rightarrow 40 + 26 = 66 \rightarrow 66 - 2 = 64$ (add 2 and subtract 2 to compensate)

   $x^2 + 6x + 4 = 0 \rightarrow x^2 + 6x + 9 - 5 = 0$ (add 5 and subtract 5 to compensate)
   $\rightarrow (x + 3)^2 - 5 = 0$ [Note: $x^2 + 6x + 9 = (x + 3)^2$]
   $\rightarrow (x + 3)^2 = 5$
   $\rightarrow x + 3 = \sqrt{5}$
   $\rightarrow x = \sqrt{5} - 3$

Problem-solving strategies

Problem-solving strategies are also a useful focus for calculation. For example, the part-part-total strategy enables solvers to understand that the story: *John withdrew $1106 from the bank. This left $789 in the bank. How much was in the bank to start with?* is addition not subtraction because $1106 and $789 are parts while the unknown (*how much was in the bank to start with?*) is the total, and parts known and total unknown is always addition regardless of the story.

Building big ideas

Finally, it is always useful to ensure big ideas become a major focus of calculation. This can be seen in powerful but simple ideas such "**add like things**" which is the basis of the separation strategy above. It also applies to the "**multiplication is area**" idea that is built from arrays. It is easy to show that 24 × 3 is 3 lots of 24 which is 3 lots of 20 and 3 lots of 4 which is 60 + 12 = 72. This leads on to algebra where $x(x+1)$ is seen as x lots of $x+1$, which is x lots of x (i.e. x^2) and x lots of 1 (i.e. x). Thus, $x(x+1) = x^2 + x$.

The most powerful big idea groups for operations are the **equals or equivalence properties** (reflexivity, symmetry and transitivity) and the **operation or field properties** (identity and inverse and the associative, commutative and distributive laws). The distributive law is the basis of the area model.

Test Item Types

This section presents instructions and the test item types for the subtests associated with the units. These will form the bases of the pre-test and post-test for this module.

Instructions

Selecting the items and administering the pre-post tests

This section provides an item bank of test item types, constructed around the units in the module. From this bank, items should be selected for the pre-test and post-test; these selected items need to suit the students and may need to be modified, particularly to make post-test items different from pre-test items. The purpose of the tests is to measure students' performance before and after the module is taught. The questions should be selected so that the level of difficulty progresses from easier items to more difficult items. In some modules this will follow the order of the units and subtests, and in other modules it will not, depending on the sequencing across the module and within units. The pre-test items need to allow for students' existing knowledge to be shown but without continual failure, and the post-test items need to cover all the sections in a manner that maximises students' effort to show what they can do.

In administering the pre-test, the students should be told that the test is not related to grades, but is to find out what they know before the topic is taught. They should be told that they are not expected to know the work as they have not been taught it. They should show what they know and, if they cannot do a question, they should skip it, or put "don't know" beside questions. They will be taught the work in the next few weeks and will then be able to show what they know. Stress to students that **any pre-test is a series of questions to find out what they know** before the knowledge is taught. They should do their best but the important questions come at the end of the module. For the **post-test**, the students should be told that **this is their opportunity to show how they have improved**.

For all tests, **teachers should continually check to see how the students are going**. Items in later subtests, or more difficult items within a particular subtest, should not be attempted if previous similar items in earlier subtests show strong weaknesses. Students should be allowed to skip that part of the test, or the test should be finished. Students can be marked zero for these parts.

Information on the Module O3: Calculating test item types

It is recommended that assessing/testing for this module is best done by interview. Initially we want to find out how they are thinking and operating and build confidence. How well students can read and write mathematics comes later.

This section includes:

1. Pre-test instructions;
2. Diagnostic Mapping Points;
3. Observation Checklist; and
4. Test item types.

Pre-test instructions

When preparing for assessment ensure the following:

- Students have a strong sense of identity; feel safe, secure and supported; develop their emerging autonomy, interdependence, resilience and sense of agency; and develop knowledgeable and confident identities.
- Students are confident and involved learners, and develop dispositions for learning such as curiosity, cooperation, confidence, creativity, commitment, enthusiasm, persistence, imagination and reflexivity.

When conducting assessment, take the following into consideration:

- Student interview for diagnostic assessment in the early learning stages is of paramount importance.
- Use materials and graphics familiar to students' context in and out of school.
- Use manipulatives rather than pictures wherever possible.
- Acknowledge the role of using stories in this early number learning, enabling students to tell stories and act out understandings to illustrate what they know.
- Playdough and sand trays are useful for early interview assessment situations.

Ways to prepare students for assessment processes include the following:

- In **individual teaching times**, challenge students' thinking. "Challenging my thinking helps me to learn by encouraging me to ask questions about what I do and learn. I learn and am encouraged to take risks, try new things and explore my ideas."
- In **group time**, model and scaffold question-and-answer skills by using sentence stems to clarify understandings and think about actions. Encourage students to think of answers to questions where there is no one correct answer, and to understand that there can be more than one correct answer (e.g. *How can we sort the objects?*).
- In **active learning centres**, use activities such as imaginative play, sand play, playdough, painting, ICTs and construction to think and talk about different ways of using materials, technologies or toys. Ask questions and take risks with new ideas.

Other considerations:

- Preferred/most productive assessment techniques for early understandings are observations, interviews, checklists, diary entries, and folios of student work.
- Diagnostic assessment items can be used as both pre-test and post-test instruments.

Remember:

Testing the knowledge can imply memory of stuff; asking the students **what they can do with knowledge** requires construction and demonstration of their understanding at this early understandings level.

O3 Calculating: Diagnostic Mapping Points

1. Basic number fact strategies

Students can use counting and other strategies to mentally solve self-generated or orally presented questions from stories involving small numbers.

1. Use their own mental counting strategies to add and subtract small numbers generated from stories e.g. I had 2 fish, I caught 3 more fish, how many fish do I have?
2. Use imagery to mentally add and subtract small numbers generated from stories, i.e. see 'how many' in their 'mind's eye'. E.g. I have three balls and I pick up two more. How many balls do I have? Ask: How many balls did I start with? If students are having difficulty ask: Show me with counters. How many did I pick up? Show me with counters. Join your counters. How many counters do you now have? How many in total?
3. Counting on
   (a) Play the three dice game. Throw dice. Add, Keep track of the total. Ask: Which two numbers did you add together first? Why? Does the student recognise the larger of the two numbers and subsequently count on from it?
   (b) Have students use a number line to check their counting on and counting back in the three dice game.
   (c) Blocks in a box?
4. Turnarounds
   (a) How many ways can you make a total of 8? Have two plastic bags. Start with 8 marbles in one bag. Ask: how many ways can you put the marbles into two bags? Have them record the combinations and look for the patterns in the number sentences.
   (b) Using dominos ask student to find domino turnarounds.
5. Doubles and near doubles
   (a) Begin with numbers (1–4) of items and ask student to extend the numbers by doubling them.
   (b) Ask which other doubles do you know? (For less confident students -can you make?)
   (c) Use dominos and ask student to find the doubles.
   (d) 6 + 6 = 12 can you use this to help work out 6 + 7, 5 + 6? How?
6. Decompose numbers and represent them in a variety of ways to assist in adding and subtracting, e.g. think of five and three more as the same as two fours. Ask student to act this out with materials.
7. Find hidden numbers mentally with sums to at least ten, e.g. watch as seven dinosaurs are counted into a box and as five are removed one at a time, and say how many are left in the box? Use 10 frames to build number facts to 10.
8. Tens and near tens
   (a) Where would you see 10 things? What are some different ways you could show 10 things? (Ten frames, ten counters, MAB ten, fingers, groups that add to ten etc.) See how many the student can come up with. Write the suggestions as number sentences / equations. e.g. 5+5=10; 2+3+5=10; 12-2=10; Ask students if they can show this in more than one way: 10=9+1=8+2=7+3=6+4=5+5=4+6=3+7=2+8=1+9
   (b) Make 10 sticks with Unifix cubes. Use as a number track.
   (c) Have students complete an addition table to 10

9. Think addition

   (a) For subtraction facts such as 8-3, can also be thought of as “what is added to 3 to make 8” Ask student to make a subtraction fact say 7-3 on a ten frame. Record the number sentence. Ask how else can you think about this fact to solve it? Can the student come up with related facts: 3+4=7 4+3=7 7-4=3

   (b) Repeat above with ‘subtraction’ stories’ to build subtraction facts to ten.

10. Extend facts

   (a) Count on 10 and count back 10 on a number track starting at 0, then 10, then 20, then 30. Ask: what do you notice?

   (b) Play three dice game. In pairs, throw dice. Add. First to reach 50 wins. Ask children to use number sentences to show their working in 2 of their turns. Ask: what do you notice?

   (c) Which of the digits in 23 will change when you add 10? Why? What happens when we add ten again? What if you take away ten? What if you add 9? Take away 9? And so on.

11. Identifies number facts they know.

12. Joins in practice games.

2. Using facts to calculate to 100

Students can count, partition and regroup in order to add and subtract one- and two-digit numbers, using separating, sequencing and compensating (ways of partitioning) to solve two-digit addition and subtraction problems, e.g. write 24 + 37 as 20 + 4 + 30 + 7, which is 50 + 11 = 61; for 34 – 27, think of it as 30 – 27 and add back the four.

1. Number tiles

   (a) Uses set materials to make and break up numbers

   (b) Use ten frames to find ways of breaking up numbers to calculate. E.g. to add 8+5, move two from the five into the frame with eight to make ten and then add the remaining three. Now have students do 9+4 and so on. Ask what is an easy way to work out 7+4? Do they use ‘make 10’?

2. Separating

   (a) Separates numbers into place value parts

3. Sequencing

   (a) Breaks one number up and keeps the other whole

   (b) Break into place value parts

4. Compensating

   (a) change numbers into more manageable parts

   (b) use basic addition facts to work out the others, e.g. *I don’t know 7 + 9 but 7 + 7 is 14 and this will be two more.*

3. Written approaches for calculating

Students can use written approaches to add and subtract whole numbers and amounts of money, to 100.

1. Drawings and jottings

2. Number sentences and equations

3. Making sense of equations

4. Models

4. Introducing and using calculators

Students can use a calculator, know associated language and functions to assist accuracy, comparison and validation when solving problems.

1. Exploring the calculator
   (a) Identifying where each number key is i.e. all calculator keypads show the digits 0 and 1 on the bottom left (most phones show 1 on top left of key pad);
   (b) Function keys Clear Enter = on/off;
2. Recording device
   (a) Enters 0–9 and keys accurately /clears display
   (b) Uses constant function to generate + - sequences
3. Explaining actions to others
   (a) Identifies operations keys + − × ÷ ; uses + − to add and subtract numbers, entering subtractions in correct order
   (b) Reads and interprets display for whole numbers
4. Computation and solving
   (a) Uses calculator for simple checking approaches related to Patterns. Monitors results of simple calculations.

O3 Calculating: Observation Checklist			
Unit	**Concept**	**Knows**	**Can construct/do/tell**
1. Basic Number Facts	Basic strategies addition and subtraction	Counting, counting on Turnarounds Doubles, near doubles Tens and near tens Think addition Extend addition/subtraction facts Multiple-of-ten facts	Using materials such as buttons/bottle tops/gumnuts to model a counting story. For example, mother duck has 4 ducklings with her and 3 more come back. How many has she now? Counts on from 4. Counts back for subtraction. Shows two sets can be joined either way, bigger and smaller or smaller and bigger for turnarounds. Can add using doubles modelled on a tens frame and explain near doubles strategy. Demonstrate tens facts on tens frame and explain near tens thinking: build to ten; add ten; make ten. What is added to 3 to make ten? 3 + 3 = 6 → 13 + 3 = 16 → 23 + 3 = 26 4 + 2 = 6 → 40 + 20 = 60
	Basic strategies multiplication and division	Turnarounds Patterns Connections Think multiplication Families	Knows larger times smaller is the same as smaller times larger: 2 × 5= 5 × 2 = 10 For zero and ones facts, doubles facts, fives/tens facts 4× is double 2× For sharing, thinks how many groups 2 × 3 = 6; 3 × 2 = 6; 6 ÷ 3 = 2; 6 ÷ 2 = 3
	Practice (Automaticity)	Diagnostic strategy Practice games	Identifies number facts and strategies they know Joins in practice games.
2. Using Facts to Calculate to 100	Separation	Breaking into parts: Place value parts Compatible parts Or a combination of both	Using set materials: Make and break bundles of tens. Separate numbers into place-value parts. Breaks both numbers into parts, Identify value of the parts.
	Sequencing	Breaking into parts: Place value parts Compatible parts or a combination of both	Breaks one number up and keeps the other number whole. Does the parts with the whole number in sequence.
	Compensation	Changes numbers into more manageable parts	Adjusts or changes numbers to use numbers that are more manageable.
3. Calculating Written Approaches	Drawing and jottings	What is useful to write down Calculations can be drawn or written in different ways Thinkboard/jottings/journals	Chooses what is useful to record. Draws and jots significant information toward solving problem. Create and link representations on thinkboard – material/drawing/symbols/words/calculator/model.
	Number sentences and equations	Uses operation signs and equals Notices and reads number sentences Equations	Uses operation signs. Models, reads and writes equations. Recognises equivalences: 6 + 3 = 3 + 6 = 9; 8 = 4 + 4; 2 + 2 = 1 + 3
	Making sense of written equations	Interest, involvement, Creates/writes own problems Games and game boards; puzzles Reasonableness of result	Uses real-world examples; explores the environment; uses a story book context to write own problems. Creates and plays games with totals. Consider (a) context, (b) maths strategy used, (c) can this be? (d) rounding (e) role of zero.
	Models	Tens frames Set Number board Number line	Ten frames Arrow maths Number line; set Number board models
4. Introducing and Using Calculators	Exploring the calculator	keys: digits 0–9; operations + − × ÷ Understands clearing the display Uses calculator language	Explains what calculators are used for and who might use them. Uses correct terms to explain calculator use
	Recording device	Enters whole numbers correctly into the calculator Reads the display accurately for whole numbers constant function to generate + and − sequences	Uses 0–9 keys; operations keys; clearing display Enters whole numbers correctly Uses constant function to generate + and − sequences
	Explaining actions to others	Reads and interprets the numbers in the display Makes use of and explains the effect of the operation keys	Reads and interprets display for whole numbers accurately.
	Computation and solving tool	Understands when it is sensible to use a calculator amongst other computational choices.	Uses for simple checking approaches related to patterns. Monitors results of simple calculations.

Subtest item types

NB. Assessing/testing for this module is best done by interview. Initially we want to find out how they are thinking and operating and build confidence, not how well they can read and write mathematics. This will come later.

Subtest 1 item types (Unit 1: Basic Number Facts)

1. Here are six parrots on a wire:

   (a) If two more have landed on the wire, how many parrots would there be altogether now?

   (b) Show the story using counters.

2. Add the numbers on these tiles:

1	4

   (a) How did you do it?

   (b) Can this be done another way?

3. Add the numbers on these tiles:

2	7	1

   (a) How did you do it?

4. Turnaround dominoes question.

5. Ten-frame question for doubles: ten take away five?

6. Make ten question

7. Think addition question

8. Extension fact question.

9. Multiple-of-ten facts question.

10. Can you show me an easy way to add these numbers?

    6 + 2 + 8 + 4

11. Multiplication fact.

12 Sharing fact.

Subtest 2 item types (Unit 2: Calculation to 100)

Separation

1. Breaking into parts

2. Place-value parts

3. Compatible parts

Sequencing

4. Breaking into parts

5. Place-value parts

6. Compatible parts

Compensation

7. More manageable parts

Subtest 3 item types (Unit 3: Calculation Written Approaches)

Drawing/jottings

1. What is useful to write?

2. Calculations can be written in different ways

3. Thinkboard/journal

Number sentences and equations

4. Uses operation signs and equals

5. Reads a number sentence

6. Equations

7. Writes own problems and discusses results/solutions

Models

8. Ten frames

9. Set

10. Number board

11. Number line

Subtest 4 item types (Unit 4: Introducing and Using Calculators)

Explores

1. Identifies keys.

2. Keys accurate order.

3. Knows language.

Recording device

4. Enters whole numbers correctly.

5. Reads display accurately.

6. Can use constant function to count.

Explaining to others

7. Reads and interprets display to others.

8. Uses operation keys accurately.

Computation and solving tool

9. Knows when calculator is useful.

Appendices

Appendix A: AIM Early Understanding Modules

Module content

1st module **Number N1: Counting** *Sorting/correspondence *Subitising *Rote *Rational *Symbol recognition *Models *Counting competencies	**2nd module** **Algebra A1: Patterning** *Repeating *Growing *Visuals/tables *Number patterns	**3rd module** **Algebra A2: Functions and Equations** *Functions* *Change *Function machine *Inverse/backtracking *Equations* *Equals *Balance
4th module **Number N2: Place Value** *Concepts* *Place value *Additive structure, odometer *Multiplicative structure *Equivalence *Processes* *Role of zero *Reading/writing *Counting sequences *Seriation *Renaming	**5th module** **Number N3: Quantity** *Concepts* *Number line *Rank *Processes* *Comparing/ordering *Rounding/estimating *Relationship to place value*	**6th module** **Operations O1: Thinking and Solving** *Early thinking skills *Planning *Strategies *Problem types *Metacognition
7th module **Operations O2: Meaning and Operating** *Addition and subtraction; multiplication and division *Word problems *Models	**8th module** **Operations O3: Calculating** *Computation/calculating *Recording *Estimating	**9th module** **Number N4: Early Fractions** *Concepts* *Fractions as part of a whole *Fractions as part of a group/set *Fractions as a number or quantity *Fraction as a continuous quantity/number line *Processes* *Representing *Reading and writing *Comparing and ordering *Renaming

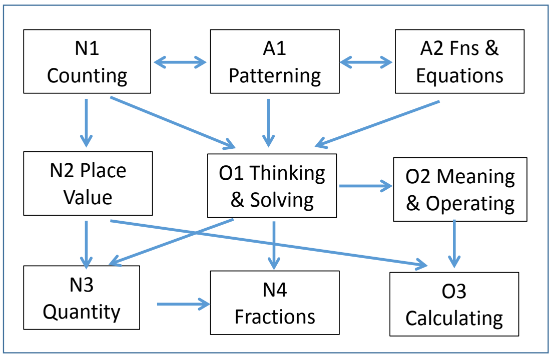
Module background, components and sequence

Background. In many schools, there are students who come to Prep with intelligence and local knowledge but little cultural capital to be successful in schooling. In particular, they are missing basic knowledge to do with number that is normally acquired in the years prior to coming to school. This includes counting and numerals to 10 but also consists of such ideas as attribute recognition, sorting by attributes, making patterns and 1-1 correspondence between objects. Even more difficult, it includes behaviours such as paying attention, listening, completing tasks, not interfering with activity of other students, and so on.

Teachers can sometimes assume this knowledge and teach as if it is known and thus exacerbate this lack of cultural capital. Even when it is identified, it can be time consuming to build this knowledge in classrooms where children are at different levels. Thus, it can lead to situations where Prep teachers say at the end of the year that some of their students are now just ready to start Prep and they wish they could have another year with them. These situations lead to a gap between some students and the rest that is already at least one year at beginning of Year 1. For many students, this gap becomes at least two years by Year 3 and is not closed and sometimes widens across the primary years unless schools can provide major intervention programs. It also leads to problems with truancy, behaviour and low expectations.

Components. The AIM EU project was developed to provide Years P-2 teachers with a program that can accelerate early understandings and enable children with low cultural capital to be ready for Year 3 at the end of Year 2. It is based on nine modules which are built around three components the mathematics ideas are designed to be in sequence but also to be connected and related to a common development. The modules are based on the AIM Years 7-9 program where modules are designed to teach six years of mathematics (end Year 3 to end Year 9) in three years (start Year 7 to end Year 9. The three components are: (a) Basics – A1 *Patterning* and A2 *Functions and equations*; (b) Number – N1 *Counting* (also a basic); N2 *Place value*; N3 *Quantity* (number line); and N4 *Fractions*; and (c) Operations – O1 *Thinking and solving*; O2 *Meaning and operating*; and O3 *Calculating*. These nine modules cover early Number and Algebra understandings from before Prep to Year 2.

Sequence. Each module is a sequence of ideas from P-2. For some ideas, this means that the module covers activities in Prep, Year 1 and Year 2. Other modules are more constrained and may only have activities for one Year or for two Years. For example, Counting would predominantly be Year P and Fractions Year 2. Thus, the modules overlap across the three years P to 2. For example, Place value shares ideas with Counting and with Quantity for two digit numbers in Year1 and three digit numbers in Year 2. It is therefore difficult, and inexact to sequence the modules. However, it is worth attempting a sequence because, although inexact, the attempt provides insight into the modules and their teaching. One such attempt is on the right. It shows the following:

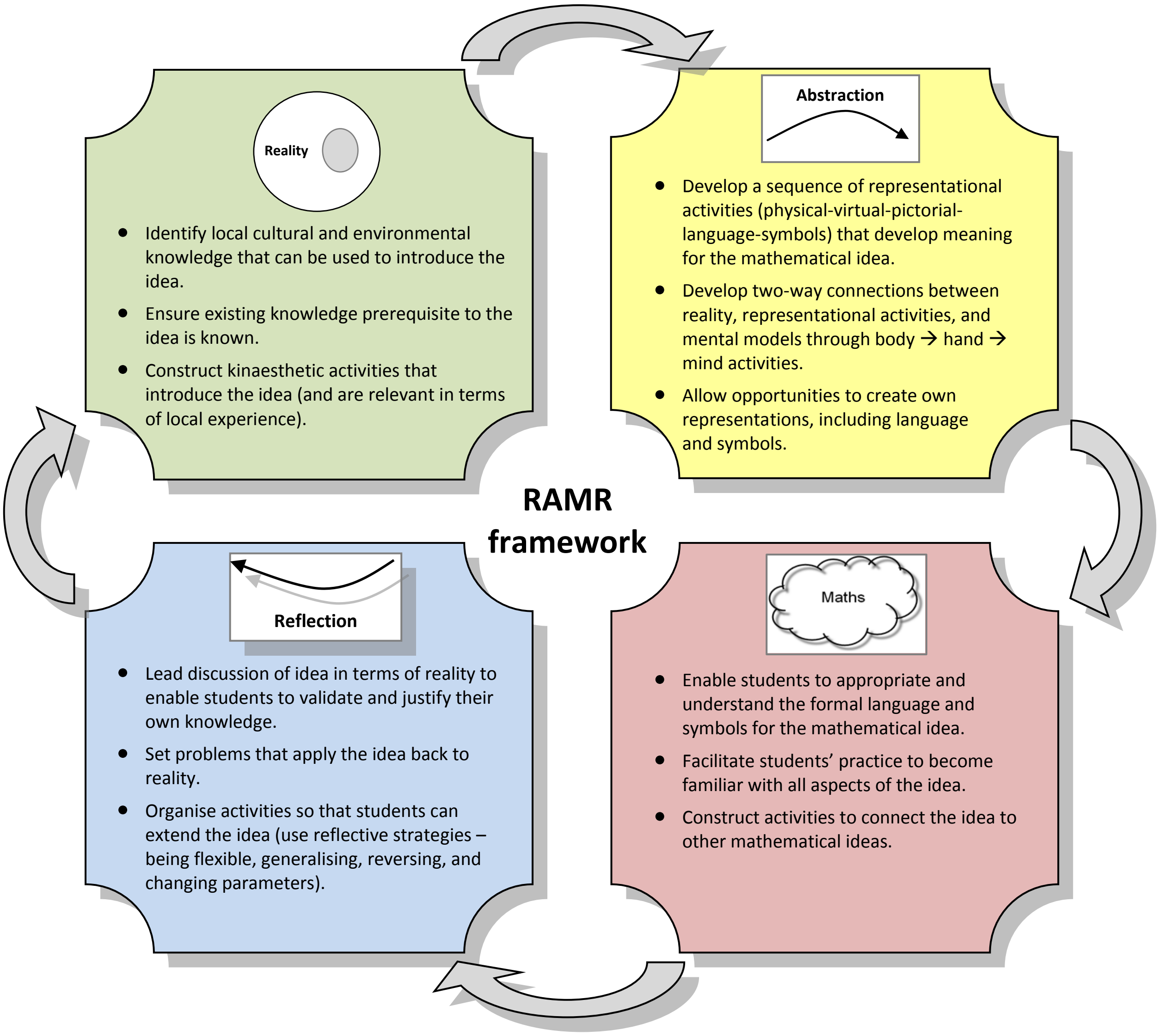


1. that foundation ideas are within *Counting*, *Patterning* and *Functions and Equations* – these deal with the manipulation of material for the basis of mathematics, seeing patterns, the start of number, and the idea of inverse (undoing) and the meaning of equals (same and different);
2. that the central components of the sequence are *Thinking and solving* along with *Place Value* and M*eaning and operating* – these lead into the less important *Calculating* and prepare for *Quantity*, *Fractions* and later general problem-solving and algebra; and
3. that *Quantity*, *Fractions* and *Calculating* are the end product of the sequence and rely on the earlier ideas, except that *Quantity* restructures the idea of number from discrete to continuous to prepare for measures.

Appendix B: RAMR cycle

AIM advocates using the four components in the figure below, reality–abstraction–mathematics–reflection (RAMR), as a cycle for planning and teaching mathematics. RAMR proposes: (a) working from reality and local culture (prior experience and everyday kinaesthetic activities); (b) abstracting mathematics ideas from everyday instances to mathematical forms through an active pedagogy (kinaesthetic, physical, virtual, pictorial, language and symbolic representations, i.e. body → hand → mind); (c) consolidating the new ideas as mathematics through practice and connections; and (d) reflecting these ideas back to reality through a focus on applications, problem-solving, flexibility, reversing and generalising. The innovative aspect of RAMR is that Reality to Abstraction to Mathematics develops the mathematics idea while Mathematics to Reflection to Reality reconnects it to the world and extends it.

Planning the teaching of mathematics is based around the four components of the RAMR cycle. They are applied to the mathematical idea to be taught. By breaking instruction down into the four parts, the cycle can lead to a structured instructional sequence for teaching the idea. The figure below shows how this can be done.



The YuMi Deadly Maths RAMR cycle

Appendix C: Teaching framework

Teaching scope and sequence for O3 Calculating

TOPICS	SUB-TOPICS	DESCRIPTIONS AND CONCEPTS/STRATEGIES/WAYS
Calculating	Basic number facts	+/− counting and counting on, near doubles, near tens, think addition ×/÷ skip counting, patterns, connections, think multiplication =/+/−/×/÷ turnarounds Multiple-of-ten facts, extension facts Practice games → instant recall
	Using facts to calculate to 100	Separation (do parts separately) Sequencing (take a part and bring in rest in sequence – including additive subtraction) Compensation (do an easier but equivalent operation)
	Written approaches for calculating	Drawings and jottings Number sentences and equations Making sense of written calculations Models: arrow maths, number lines and algorithms
	Computation and solving tools – calculator	Exploring the calculator Using as a recording device Explaining results to others Solving tool

Proposed year-level framework

YEAR LEVEL	OPERATIONS – CALCULATING	
	Semester 1	**Semester 2**
Prep	Identifying change – beginning–action–end problems with small numbers 0–10 and counting. Drawing to record. Exploring the calculator.	Modelling actions of adding, taking away, sharing, grouping small numbers. Building number facts 0–10. Drawing to record. Calculator as a recording device.
1	Basic facts strategies – counting on, counting back 0–100 with materials; jotting. Connecting representations with thinkboard. Calculator as a recording device.	Basic facts strategies – partitioning and rearranging parts 0–100 with materials; jotting. Connecting representations with thinkboard. Explaining calculator actions to others.
2	Basic fact strategies – turnarounds, doubles, near doubles, tens and near tens. Fact families/connections between = and +, –, × and ÷; links modelled/mental/written computation. Writing and modelling own story number problems; using thinkboard to link representations. Writing number sentences/stories/equations. Explaining calculator actions to others.	Basic fact strategies – think addition, think multiplication, connections, patterns, extended number facts. Writing and solving problems using mental and written strategies/models; using thinkboard to link representations. Writing number sentences/stories/equations. Calculator as a computation and solving tool.
Focus	Body → Hand → Mind – acting out, constructing, visualising.	Body → Hand → Mind – acting out, constructing, visualising, recording.

Accelerated Inclusive Mathematics Early Understandings Project

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