



**YuMi Deadly Maths**

**AIM EU**

**Module 02**

# **Operations: Meaning and Operating**

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**Prepared by the YuMi Deadly Centre  
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## ACKNOWLEDGEMENT

The YuMi Deadly Centre acknowledges the traditional owners and custodians of the lands in which the mathematics ideas for this resource were developed, written and refined and presented in professional development sessions.

## YUMI DEADLY CENTRE

The YuMi Deadly Centre is a research centre within the Faculty of Education at the Queensland University of Technology which is dedicated to enhancing the learning of Indigenous and non-Indigenous children, young people and adults to improve their opportunities for further education, training and employment, and to equip them for lifelong learning. The YuMi Deadly Centre (YDC) can be contacted at [ydc@qut.edu.au](mailto:ydc@qut.edu.au). Its website is <http://ydc.qut.edu.au>.

“YuMi” is a Torres Strait Islander Creole word meaning “you and me” but is used here with permission from the Torres Strait Islanders’ Regional Education Council to mean working together as a community for the betterment of education for all. “Deadly” is an Aboriginal word used widely across Australia to mean smart in terms of being the best one can be in learning and life. YuMi Deadly Centre’s motif was developed by Blacklines to depict learning, empowerment, and growth within country/community. The three key elements are the individual (represented by the inner seed), the community (represented by the leaf), and the journey/pathway of learning (represented by the curved line which winds around and up through the leaf). As such, the motif illustrates the YuMi Deadly Centre’s vision: *Growing community through education*.

## DEVELOPMENT OF THE AIM EARLY UNDERSTANDINGS MODULES

In 2009, the YuMi Deadly Centre (YDC) was funded by the Commonwealth Government’s *Closing the Gap: Expansion of Intensive Literacy and Numeracy* program for Indigenous students. This resulted in a Year 7 to 9 program of 24 half-term mathematics modules designed to accelerate the learning of very underperforming Indigenous students to enable access to mathematics subjects in the senior secondary years and therefore enhance employment and life chances. This program was called Accelerated Indigenous Mathematics or AIM and was based on YDC’s pedagogy for teaching mathematics titled YuMi Deadly Maths (YDM). As low-income schools became interested in using the program, it was modified to be suitable for all students and its title was changed to Accelerated Inclusive Mathematics (leaving the acronym unchanged as AIM).

In response to a request for AIM-type materials for early childhood years, YDC decide to develop an Early Understandings version of AIM for underperforming Years F to 2 students titled Accelerated Inclusive Mathematics Early Understandings or AIM EU. This module is part of this new program. It uses the original AIM acceleration pedagogy developed for Years 7 to 9 students and focuses on developing teaching and learning modules which show the vertical sequence for developing key Years F to 2 mathematics ideas in a manner that enables students to accelerate learning from their ability level to their age level if they fall behind in mathematics.

YDC acknowledges the role of the Federal Department of Education in the development of the original AIM modules and sees AIM EU as a continuation of, and a statement of respect for, the *Closing the Gap* funding.

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## *List of Abbreviations*

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AIM EU	Accelerated Inclusive Mathematics Early Understandings
BAE	begin–act–end
RAMR	Reality–Abstraction–Mathematics–Reflection
SPDC	see–plan–do–check

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## Module Overview

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This module, *O2 Meaning and Operating*, is the sixth of the nine Accelerated Inclusive Mathematics Early Understandings (AIM EU) modules. These modules are designed to provide support in Years F to 2 to improve Year 3 mathematics performance. The AIM EU modules are based on the AIM Years 7 to 9 modules, which are designed to accelerate mathematics teaching and learning to where underperforming mathematics students (at around Year 3–4 level in Year 7) can learn six years of mathematics in three years and thus access Year 10 mathematics as mainstream students. The AIM EU modules are designed to accelerate learning in the early years so that students with little schooling cultural capital at the start of their Foundation year can learn the school mathematics understandings normally taught in home, plus those taught in Years F–2, in three years and reach Year 3 with strong Year 3 mathematics knowledge.

This module is the second in a sequence of three on operations, *O1 Thinking and Solving*, *O2 Meaning and Operating* and *O3 Calculating*, which cover Operations from before Year F to Year 2. This module is designed to ensure that teachers build meaning before symbols with operations. It is also written to ensure that the module covers some prior to Year F work in operations as well as the normal F–2 work. The nine AIM EU modules covering Number and Algebra Years F to 2, plus background on the modules, are shown in sequence in **Appendix A**.

AIM EU uses the YuMi Deadly Mathematics (YDM) pedagogy ideas, which is based around the structure of mathematics (sequencing, connections and big ideas) and the Reality–Abstraction–Mathematics–Reflection (RAMR) teaching cycle that is described in **Appendix B**. The YDM pedagogy endeavours to achieve three goals: (a) to reveal the structure of mathematics, (b) to show how the symbols of mathematics have meaning in that they tell stories about our everyday world, and (c) to provide students with knowledge they can access in real-world situations to help solve problems. The YuMi Deadly Centre (YDC) argues that the power of mathematics is based on how the structure of connections, big ideas and sequences relates descriptively (with language) and logically (through problem-solving) to the world in which we live.

This chapter introduces and overviews the module by discussing: (a) connections and big ideas with respect to meaning of operations; (b) sequencing of operations ideas (across and within modules); (c) teaching and cultural implications; and (d) summarising the structure of the module.

### Connections and big ideas

The starting point for all YDC AIM modules is the connections between mathematics topics and using these connections to accelerate learning, in particular in the formation of big ideas whose learning will provide understanding across mathematics topics and across year levels.

### Importance of connections and big ideas in early years

YDC believes mathematics is best understood and applied in a schematic structured form which contains knowledge of when and why as well as how. Schema has knowledge as connected nodes, which facilitates recall and problem-solving. As O1 argues, understanding schematic structure enables teachers to:

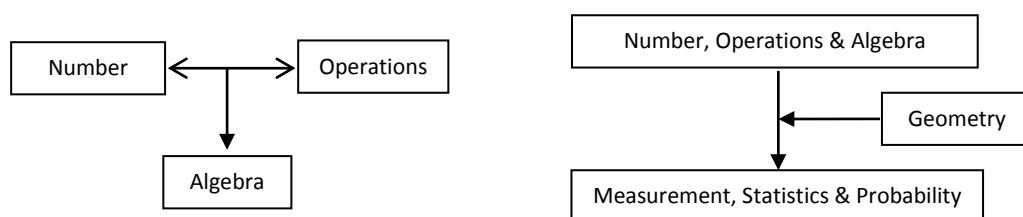
- (a) *determine what mathematics is important to teach* – mathematics with many connections or based on big ideas is more important than mathematics with few connections or little use beyond the present;
- (b) *link new mathematics ideas to existing known mathematics* – mathematics that is connected to other mathematics or based on the one big idea is easier to recall and provides options in problem-solving;
- (c) *choose effective instructional materials, models and strategies* – mathematics that is connected to other ideas or based around a big idea can be taught with similar materials, models and strategies; and

- (d) *teach mathematics in a manner that enables later teachers to more easily teach more advanced mathematics* – by pre-empting the knowledge that will be needed later, preparing linkages to other ideas, and building foundations for big ideas the later teachers will use.

The ideas of schema are as crucial in the early years as later, maybe even more so, because the early years lay the foundations for later understandings. Strong foundations build acceleration in learning and powerful mathematics ideas. This is particularly so when big ideas are identified that cover a variety of topics and are useful across more than one year level, and also cover both pre-operational and operational thinking.

## Connections with respect to operations

Operations concepts are used with Number and lead into Algebra. Finally, through the inclusion of Geometry, Number, Operations and Algebra lead onto Measurement and, more directly, to Statistics and Probability. This can be diagrammatically represented as below:



The major connections between Operations and the other topics are to topics that use number and/or operations as the basis of their mathematics (e.g. Number, Algebra, Measurement, Statistics and Probability). Major connections are as follows.

- *Operations and Number* – an obvious connection as operations need numbers to act on. In particular, the strategies for computation relate to the numeration concepts, that is: (a) separation strategy relies on a place value understanding of 2- to 4-digit numeration; and (b) sequencing and compensation strategies rely on a rank understanding of numeration.
- *Operations and Algebra* – again an obvious relationship as algebra is generalisation of arithmetic activities. In particular,  $2x + 3$  relates to an example like  $2 \times 5 + 3$ . The difference is that 5 is an actual number while  $x$  is a variable.
- *Operations and Measurement* – measurement involves a lot of operations particularly with respect to formulae (e.g. perimeter, area).
- *Operations and Statistics and Probability* – both of these involve operations (e.g. in calculating mean and chance).

## Big ideas underlying operations

Operations have many big ideas. They can be global or based on concepts, properties, and strategies. The following two lists cover global big ideas because these apply to all mathematics, and operation big ideas as these apply to the operation work in both O2 and O3. The others will be listed in module O3 Calculating as they lead to ideas that are the basis of calculating and strategies for calculation.

The ideas of schema are as crucial in the early years as later, maybe even more so, because the early years lay the foundations for later understandings. Strong foundations build acceleration in learning and powerful mathematics ideas. This is particularly so when big ideas are identified that cover a variety of topics and are useful across more than one year level, and also cover both pre-operational and operational thinking.

For YDM, **curricula should be taught so that big ideas and connections are emphasised**. Thus, our aim is to construct teaching frameworks that specify and sequence topics and draw attention to connections, particularly those that result in big ideas. The ideas of schema are as crucial in the early years as later, maybe even more so, because the early years lay the foundations for later understandings. Strong foundations build acceleration in learning and powerful mathematics ideas.

### Global big ideas

1. **Symbols tell stories.** The symbols of mathematics enable the world to be described succinctly and in a generalised way (e.g.  $2 + 3 = 5$  means caught 2 fish and then caught another 3 fish, or bought a \$2 chocolate and \$3 drink, or joined a 2m length of wood to a 3m length ... and so on).
2. **Relationship vs change.** Mathematics has three components – objects, relationships between objects, and changes/transformations between objects. All relationships can be perceived as changes and vice versa. This is particularly applicable to operations; 2 plus 3 can be perceived as relationship  $2 + 3 = 5$  or change  $2 \xrightarrow{+3} 5$ .
3. **Interpretation vs construction.** Things can either be interpreted (e.g. what operation for this problem, what properties for this shape) or constructed (write a problem for  $2 + 3 = 5$ ; construct a shape of 4 sides with 2 sides parallel).
4. **Accuracy vs exactness.** Problems can be solved accurately (e.g. find  $5\,275 + 3\,873$  to the nearest 100 – rounding and estimation) or exactly (e.g.  $5\,275 + 3\,873 = 9\,148$  – basic facts and algorithms).
5. **Part-part-total/whole.** Two parts make a total or whole, and a total or whole can be separated to form two parts – this is the basis of numbers and operations (e.g. fraction is part-whole, ratio is part to part; addition is knowing parts, wanting total).
6. **Triad:** Any relationship which has three components (like BAE) has three types of problem. For example,  $3+5=8$  can be: (a) *There were some children, 5 joined them, this made 8, how many children to start with?* (3 unknown); (b) *There were 3 children, some joined them, this made 8, how many children joined?* (5 unknown); and (c) *There were 3 children, 5 joined them, how many children?* (8 unknown). Triads also apply to many other topics such as measures (object, unit, number and percent (amount, percent, percentage)).

### Basic operation big ideas

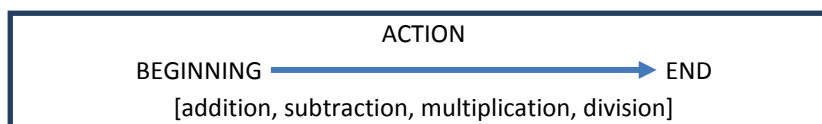
1. **Concepts of the operations.** This covers the meanings of addition, subtraction, multiplication and division (the focus of this module). It also covers the knowledge that only addition and multiplication are true operations with subtraction and division being their inverses.
2. **Concepts of equals and order.** This covers the meanings of equations and inequations (and equals and order). It covers the three principles of equals and order, that is: (a) reflexivity/non-reflexivity –  $A=A$  but  $A$  is not  $> A$ ; (b) symmetry/antisymmetry –  $A=B \rightarrow B=A$  while  $A>B \rightarrow B<A$  and  $A<B \rightarrow B>A$ ; and (c) transitivity –  $A=B$  and  $B=C \rightarrow A=C$  and  $A>B$  and  $B>C \rightarrow A>C$ . It also covers that the true meaning of equals is equivalence (LHS is same value as RHS).
3. **The operation principles.** This covers the six properties of the operations: (a) closure – operations with numbers always gives a number; (b) identity – addition/subtraction of 0 and multiplication/division by 1 do not change things; (c) inverse – changes that undo other changes (such as  $+2$  and  $-2$ , and  $\times 3$  and  $\div 3$ ); (d) commutativity – order does not matter for  $+$  and  $\times$  but does for  $-$  and  $\div$  (e.g.  $8+6 = 6+8$ ;  $4\times 3 = 3\times 4$ ); (e) associativity – what is done first does not matter for  $+$  and  $\times$  but does for  $-$  and  $\div$  (e.g.  $(8+4)+2 = 8+(4+2)$ , but  $(8\div 4)\div 2 \neq 8\div (4\div 2)$ ); and (f) distributivity –  $+$  and  $-$  act on like things while  $\times$  and  $\div$  act on everything (e.g.  $2\times(3+4) = 6+8$ ;  $(6+8)\div 2 = 3+4$ ).
4. **Extensions of the equals and operation principles.** (a) balance – whatever is done to one side of the equation is done to the other for the equation to stay true; (b) compensation – ensuring that a change is compensated for so answer remains the same – related to inverse (e.g.  $5+5 = 7+3$ ;  $48+25 = 50+23$ ;  $61-29 = 62-30$ ); (c) equivalence – two expressions are equivalent if they relate by  $+0$  and  $\times 1$  – also related to inverse, number, fractions, proportion and algebra (e.g.  $48+25 = 48+2+25-2 = 73$ ;  $50+23 = 73$ ;  $\frac{2}{3} = \frac{2}{3} \times \frac{2}{2} = \frac{4}{6}$ ); (d) inverse relation for  $-$ ,  $\div$  and direct relationship  $+$ ,  $\times$  – a higher the number can result in a smaller answer for  $-$ ,  $\div$  (e.g.  $12\div 2 = 6 > 12\div 3 = 4$ ;  $\frac{1}{2} > \frac{1}{3}$ ) but a higher number always results in a higher answer for  $+$ ,  $\times$  (e.g.  $4+3 < 4+7$ ), and vice versa for lower numbers; and (e) backtracking – using inverse to reverse and solve problems (e.g.  $2y+3 = 11$  means  $y\times 2+3$ , so answer is  $11-3\div 2 = 4$ ).

5. **Strategies for calculation.** This covers (a) basic fact strategies – counting, doubles, near 10, patterns, connections, think addition, think multiplication; (b) algorithm strategies – separation, sequencing and compensation; and (c) estimation strategies – front end, rounding, straddling and getting closer.

## The two important big ideas from O1

As the middle module in a sequence of modules on operations, O2 covers some prior to school knowledge, but much less than Module O1. The focus in O2 is on the actions that lead to the four operations. Thus, it focuses on the two teaching big ideas that were described in Module O1 and which are particularly important for AIM EU's operation modules, namely, begin-act-end (BAE) and see-plan-do-check (SPDC).

1. **Begin act-end.** The basis to understanding operations is BAE because all operations have a beginning situation, an action that is the operation, and an end situation, as below.



In Module O1, it was argued that, to understand operations students have to be able to: (a) **recognise** when there is a BAE activity – e.g. *I started with a stack of blocks, John knocked them down and I ended with a pile of blocks*; (b) **identify** differences in beginning and ending – the way things looked, the number of things, and so on; and (c) **discriminate** between different kinds of actions – things that make things messy, things that are tidy, things that remove, things that make larger, and so on. We also looked at reversing the situations – that is knowing the end, what is the beginning for a given action.

In Module O2, we apply BAE to operations by making the beginning and end both numbers and the action a way of changing numbers. We continue reversing and this leads to deeper understanding of operations.

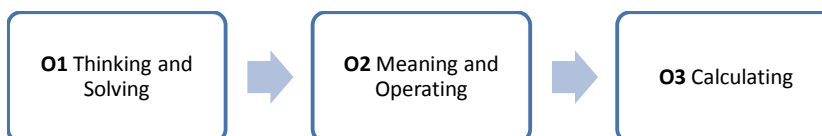
2. **See-plan-do-check.** In Module O1, it was also argued that the basis of thinking about and solving problems is the Polya's four stages (SPDC) big idea, that powerful thinking when faced with a problem to solve is to work out what the problem is saying (**see**), make a plan to solve it (**plan**), apply that plan to get a solution (**do**) and then check the answer (**check**), as below:



SPDC is an approach that can work for any problem but again, for operations, the problem has to act on numbers and the types of things that can be covered have to relate to the actions that give rise to operations. This does not give the direct link to concepts of the four operations as BAE does but it is strong in its application to word problems.

## Sequencing

As shown in O1, pre-operations precedes operations so that the three modules follow the sequence below:



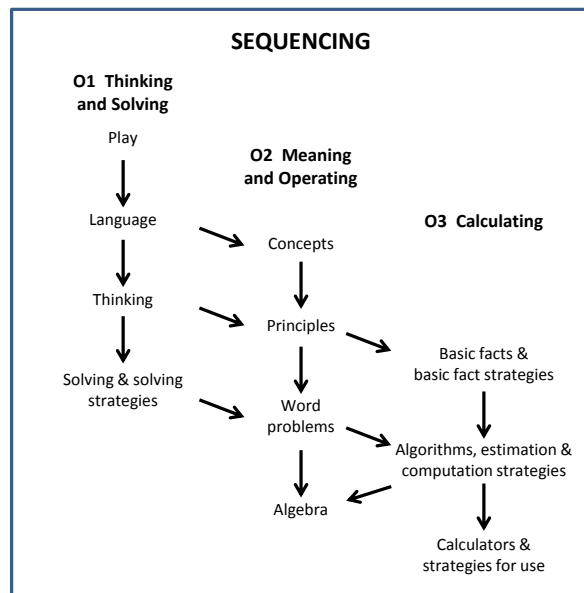
To teach for rich schema, it is essential for teachers to know the mathematics that precedes, relates to and follows what they are teaching, because they are then able to build on the past, relate to the present, and prepare for the future. Thus, YDM presents information in this module as sequences of ideas that relate to and

connect with each other. This is particularly important for early understandings as this module is designed to cover some early work although its emphasis is on the Year 2 sequence.

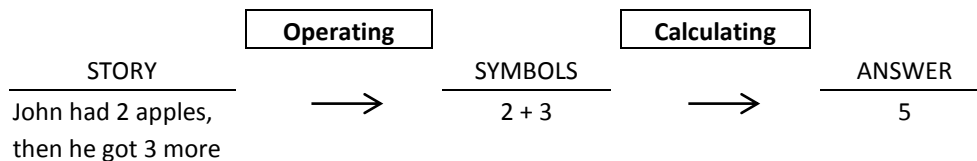
## Sequence across modules

In general, YDM advocates that operations be divided into **two columns or sides**, meanings-operating and computation-calculating. Meanings-operating covers concepts, principles, word problems and extension to algebra, while computation-calculating covers basic facts, algorithms and estimation. This is done to highlight that problem-solving and algebra are based on the meanings-operating side (i.e. concepts and principles), not computation. Of course, expertise with respect to operations is a combination of both sides.

The three AIM EU modules on operations follow this separation in Modules O2 and O3. However, to cover pre-operational thinking and prior-to-school experiences), Module O1 (*Thinking and Solving*) is placed into the start of the sequence as on right.



The move across O1, O2 and O3 was one from story to answer as is seen below and discussed in Unit 4:



The problem of this sequence is if the move loses the story, we end up with operations and calculations where the symbols are decontextualised from other representations. This can lead to problems with understanding as number activity as modelling the world changes to number activity as solving calculations through rote processes. More on this in Unit4.

## Sequence within module

To ensure that this module passes through prior-to-school experiences and F-2 experiences, this module will be broken into five sections as follows.

1. **Early ideas.** Exploring the ideas that underlie BAE for operations and ensuring students understand the prerequisites. This includes same/different and more/less, and discussing what actions bring these situations about (beginning the process to addition/subtraction). They also involve relationships between stories, material use, drawings of materials (models), language and BAE frameworks.
2. **Basic meanings.** Focusing on the basic actions that lead to addition, subtraction, multiplication and division. This includes joining separating, combining, partitioning (sharing), and equal/unequal groups, It also includes developing the relationships between the four operation, when are they similar and dissimilar. This will use the set, array and number line models. This will involve relationships between stories material use, drawings of materials (models), language and symbols.
3. **Other meanings.** Focusing on other actions that also lead to some or all of the four operations. These include inverse, comparison and, for multiplication and division only, combinations. These actions will use set, array, number-line and combinations models and relate these models to stories, materials, language and symbols.

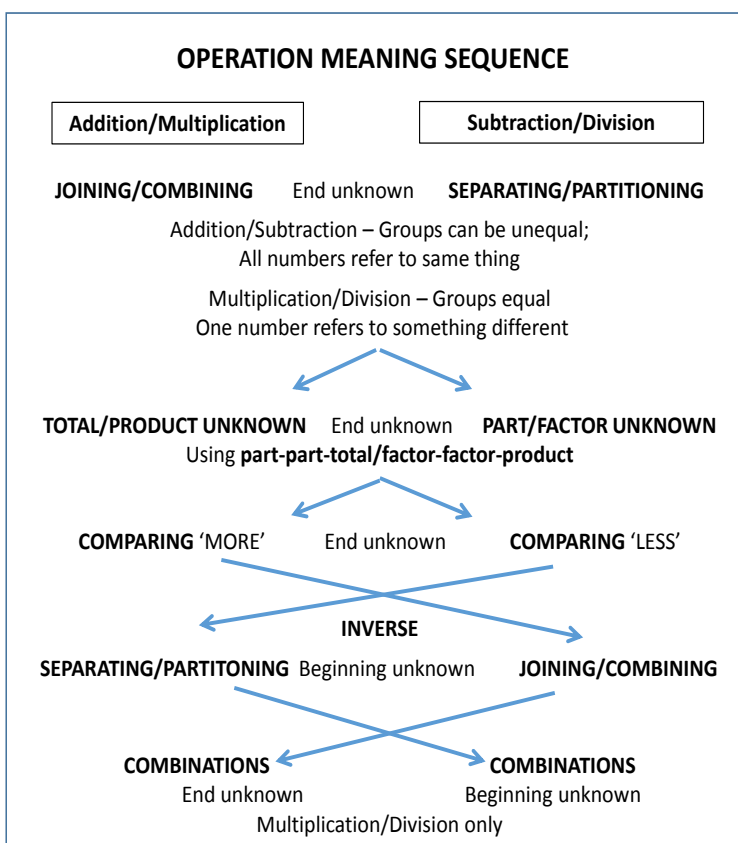
4. **Word problem-solving.** Using the above information to solve word problems. Taking stories and relating them to BAE so can determine what meaning, and therefore what operation, is being used in the action, taking into account the beginning, the end, and what is unknown. This unit will also reprise SPDC to help determine what beginning, action and end is being talked about.
5. **Principles and decontextualisation.** Teaching the properties of operations that relate to other operations without calculation and are true regardless of the numbers and the calculation (the operation principles, number 3 of the basic operation big ideas of page 3). They include: (a) “the operations that do not change value” (i.e.  $+0$  and  $\times 1$  – identity); (b) “undoing the change” (e.g.  $-3$  undoes  $+3$  – inverse); (c) “turn around to the same thing” (e.g.  $2+5 = 5+2$  – commutative law); (d) “it does not matter how I add them up” (e.g.  $2+3 + 5 + 3 = 5+2+3+3 = 7+7 = 13$  – associative law); and (e) “multiplication and division act on everything in additions and subtractions” ( $4 \times (3+5) = 4 \times 3 + 4 \times 5$  – distributive law). As AIM EU is Years F to 2, the distributive law is not covered in this module (in Unit 5).

**Appendix C** shows the teaching scope and sequence for O1, O2 and O3 as a table, and provides a proposed year-level teaching framework.

### Sequencing the meanings and models:

There are four meanings to addition and subtraction: (a) joining and separating (or take-away); (b) part-part-total; (c) comparison (e.g. ‘difference’, ‘4 more’); and (d) inverse. The first three have the End of BAE unknown while the last (inverse) has Beginning unknown). Inverse means that joining can be subtraction and take-away can be addition if the Beginning is the unknown. For example, *Jim took away 3 books – this left 5 books. How many books were there to start with?* The answer is 3+5.

There are 5 meanings of multiplication and division: (a) combining and partitioning (sharing-grouping); (b) factor-factor-product; (c) multiplicative comparison (‘four times more’); (d) inverse – with Beginning of BAE unknown; and (e) combinations. The first four of the meanings are number  $\times$  rate while the last is number  $\times$  number and is most important for area.



As diagram (above right) shows, the overall sequence is:

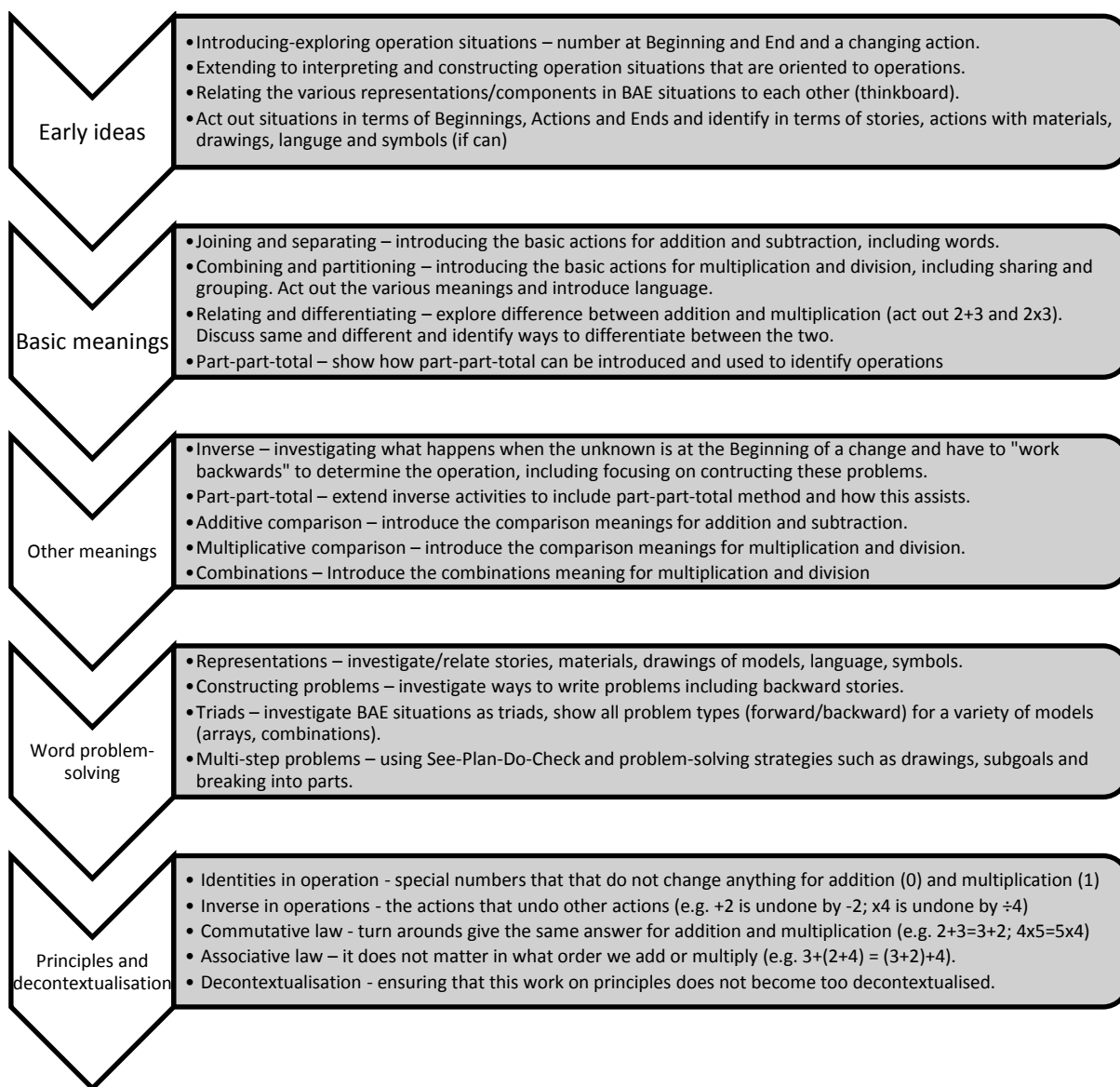
Joining-separating    $\Rightarrow$    Part-part-total    $\Rightarrow$    Comparison    $\Rightarrow$    Inverse    $\Rightarrow$    Combinations  
 Combining- Partitioning    $\Rightarrow$    Factor-factor-product    $\Rightarrow$    add/mult

By focusing on Beginning instead of End, all problems have an inverse where the opposite operation describes the action. For example, 4 joined 5 to give total means  $4+5=9$ , but some joined 5 to give 9 is  $9-5=4$ . Part-part-total enables the correct operation to be determined and is the only method that is unaffected by whether Beginning is unknown of End is unknown. It works regardless of where unknown is.

These meanings are commonly first investigated with set models and then number line models. Multiplication and division are also investigated with array and combinations models (e.g. tree diagrams).

## Sequences within units

The following summarises the activities with each of the five units in this module *O2 Meanings and Solving*



## Teaching and culture

This section looks at teaching and cultural implications, including the Reality–Abstraction–Mathematics–Reflection (RAMR) framework and the impact of Western number teaching on Indigenous and low-SES students.

### Teaching implications

Because the teaching in this module moves from before-school knowledge through to Year 2 knowledge, the teaching implications are as follows.

1. **Language teaching ideas.** The module will require you to do play and language activities that will build operations. Some ideas for this are: (a) use storytelling time to develop thinking and to model verbalised thinking; (b) create ideas for dialogue using thinking language, particularly opportunities for structured

dialogue; (c) develop questioning, focusing on students' ability to ask questions; and (d) wonder together with students.

2. **Thinking–solving teaching ideas.** There will also be a large focus on starting students on thinking and solving. Some ideas for this are: (a) develop spatial, logical, creative and flexible thinking skills and skills in decision-making, plus metacognition; (b) provide opportunities to explore identifying and describing attributes, matching and sorting, and comparing and ordering; (c) create thinking times and develop the ability to ask questions that stimulate children's thinking and encourage children to elaborate on their ideas; (d) recognise creativity in approach by students and need for solitary as well as social play; and (e) provide opportunities to plan and reflect/evaluate thinking with students and to solve problems.
3. **RAMR cycle.** The module will be based on teaching following the RAMR cycle. Each unit will have at least one RAMR exemplar lesson. The lessons will: (a) start with something students know and in which they are interested; (b) move on to creatively representing the new knowledge through the sequence body → hand → mind; (c) develop language and symbols, and practice and connect components; and (d) finally, reflect the new knowledge back into the lives of the children using problems and applications, and focus on ensuring flexibility, reversing and generalising (by changing parameters if needed). The RAMR cycle is in **Appendix B**.
4. **Models.** It is important that students connect symbols, language, real world situations and models in many and varied contexts and forms. For models, it is important that there is a balance of set and number line (or length) models. Set models are discrete items like money, fingers, counters and other objects, while the number line model is a ruler or steps or jumps along a line (this means that  $2+3$  can be two books joining 3 books, set model, or two steps and then three more steps, number line model). With new meanings, it is also important to begin to use array models and combinations models.

### Cultural implications

In this section, we move on from just looking at teaching to the cultural implications in this teaching, because students who need AIM EU modules include Indigenous and low-SES students.

**Teaching Indigenous students.** Aboriginal and Torres Strait Islander students tend to be high context – their mathematics has always been built around pattern and relationships. Their learning style is best met by teaching patterning that presents mathematics structurally as relationships, without the trappings of Western culture. As Ezeife (2002) and Grant (1998) argued, Indigenous students should flourish in situations where teaching is holistic (from the whole to the parts). Thus, problems and investigations where there is opportunity for creativity and patterning should have positive outcomes for Indigenous students as long as the problems are realistic, make sense within the Indigenous students' context and matter to the students. In general, this means a lesser focus on algorithms and rules, and a greater focus on patterning, generalisations and applications to everyday life. It also means a strong language focus to translate the students' abilities to the world of standard English.

**Teaching low-SES students.** Interestingly, holistic teaching is also positive for low-SES students. Three reasons are worth noting. First, low-SES students tend to have strengths with intuitive–holistic and visual–spatial teaching approaches rather than verbal–logical approaches. Thus, a focus on solving problems that make contextual sense and for which the answers matter and with a strong language component should be positive for low-SES students. Second, many low-SES students in Australia are immigrants and refugees from cultures not dissimilar to Aboriginal or Torres Strait Islander cultures. They are also advantaged by holistic algebraic and patterning approaches to teaching mathematics. Third, many low-SES students and their families have long-term experience of failure in traditional mathematics teaching, resulting in learned helplessness. This can be overcome with a focus on investigations along with a strong language focus. Holistic-based problem-oriented teaching of mathematics through patterns is sufficiently different that students can escape their helplessness – particularly if taught actively and from reality as in the RAMR model.

**Prior-to-school knowledge.** Both Indigenous and low SES students can come to school with lots of knowledge from their culture and background, but little knowledge that helps with school work. The modules in these books are designed to provide this ‘prior-to-school’ school knowledge, but it is important that their cultural and context knowledge is also equally appreciated and maintained. Pride in heritage and connection to heritage is important in learning ‘school’ knowledge.

## Structure of module

Based on the ideas above, this section describes the components of the module, including four units, and prepares the reader for tests and test item types, appendices, plus RAMR lessons.

### Components

Based on the ideas above, this module is divided into this overview section, five units, a review section, test item types, and appendices, as follows.

**Overview:** This section covers a description the module’s focus, connections and big ideas, sequencing, teaching and culture, and summary of the module structure.

**Units:** Each unit includes examples of teaching ideas that could be provided to the students, some in the form of RAMR lessons, and all as complete and well sequenced as is possible within this structure.

*Unit 1: Early ideas* – operation situations, interpreting-constructing, relating and acting out.

*Unit 2: Basic meanings* – joining-separating, combining-partitioning, relating and parts and totals.

*Unit 3: Other meanings* – inverse, extending to part-part-total, comparison and combinations.

*Unit 4: Word problem-solving* – representations, constructions, triads and multi-step problems.

*Unit 5: Principles and decontextualisation* – identity, inverse, commutative and associative, plus discussion of role of context.

**Module review:** This section reviews the module, looking at important components across units. This includes the teaching approaches, models and representations, competencies and later activity (where the activity in this module leads to in Years 3 to 9).

**Test item types:** This section provides examples of items that could be used in unit pre- and post-tests.

**Appendices:** This comprises three appendices covering the AIM EU modules, the RAMR pedagogy, and proposed teaching frameworks for operations.

### Further information

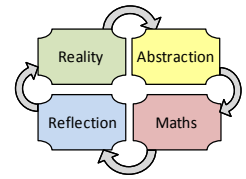
**Sequencing the teaching of the units.** The four units are in sequence and could be completed one at a time. However, each of the units is divided into sub-ideas (concepts and processes) that are also in sequence within the unit. Therefore, schools may find it advantageous to: (a) teach earlier sub-ideas in a later unit before completing all later sub-ideas in an earlier unit; (b) teach sub-ideas across units, teaching a sub-idea in a way that covers that sub-idea in all the units together; or (c) a combination of the above.

The AIM EU modules are designed to show sequences within and across units. However, it is always YDC’s policy that schools should be **free to adapt the modules to suit the needs of the school** and the students. This should also be true of the materials for teaching provided in the units in the modules. These are exemplars of lessons and test items and schools should feel free to use them as they are or to modify them. The RAMR framework itself (see Appendix B) is also flexible and should be used that way.

Together, the units and the RAMR framework are designed to ensure that all important information is covered in teaching. Therefore, if changing and modifying the order, try to ensure the modification does not miss something important (see **Appendix C** for detailed teaching frameworks).

**RAMR lessons.** We have included RAMR lessons/activities as exemplars wherever possible in the units of the module. These lessons provide teaching ideas for Reality, Abstracting, Mathematics and Reflection activities. They are provided as examples – all teachers should modify them for their students' needs. The module identifies when RAMR lessons are given by the symbol on the right.

**Suggestions for improvement.** We are always open to suggestions for improvement and modification of our resources. If you have any suggestions for this module, please contact YDC.



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# Unit 1: Early Ideas

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Unit 1 bridges from the exploration of situations related to operations in Module O1 to the meanings of the operations in Module O2. Module O1 introduces the notion that operations are special cases of situations where there are three parts: a beginning, an action and an end (called BAE), where the special cases has **beginnings and ends that involve number and actions that effect number**. Module O2 takes up these special cases and identifies the different meanings of the operations that can be represented by BAEs.

This unit is therefore, focusing on early ideas. It begins by identifying and exploring situations that lead to operations. It follows this by both interpreting and constructing these situations. Then the various representations of the situations are related, the stories, the manipulation of materials, the drawing of the action, and any language and symbols, are related. Finally, the situations are acted out and components identified.

## Background information

BAEs are situations which have a beginning, an action, and an end. These can be unnumbered or numbered, that is, refer to situations where numbers are not used (like blowing up a balloon) or refer to situations where there are numbers and actions that change numbers (like finding 3 more pencils). YuMi Deadly Maths follows the mathematics education researcher Davidov and recommends exploring unnumbered situations before focusing down onto numbered situations. We believe that teaching enables the big ideas behind operating with numbers to be better understood if learning begins with work on unnumbered situations.

As stated above, O1 contains much of the unnumbered work at the start of operations. This it does by exploring situations which involve a beginning, an action and an end (e.g. a balloon → blowing → blown up balloon), and seeing when these relate to mathematical operations (e.g. the beginning and the end have number and the actions operate in the numbers). However, this unit (O2) can still explore ideas at the cusp of the shift from unnumbered to numbered situations that underlie BAE for operations. This consists of:

- (a) Identifying that number is involved in the three parts of BAE (beginning, action and end);
- (b) Determining what are the consequences of numbers being involved in the BAE (how do the number/numbers change?);
- (c) Using the identifications and determinations to allow students to interpret and to construct BAEs; and
- (d) Relating the component parts of the BAE situation: determining the story, acting out the change, drawing a picture of the change, and providing a description of the change in words and symbols (when students are ready for this).

The above will be done for the operations of addition and subtraction. In the next unit multiplication and division will be integrated into

## 1.1 Introducing operation situations

1. Look at same and different and more and less:
  - (a) Ensure students can determine same and different in everyday situations – look at attributes that are the same and are different (colour, size, and so on).
  - (b) Focus on same and different using the attribute of number – ensure students can see the number-attribute difference in terms of more and less (and can also see sameness in terms of not more or less)

- (c) Be flexible in this – for instance, six blocks and seven blocks are different and the 7 blocks represents more in terms of number, but it is straightforward – what about one car, one dinosaur and three bears being different to one car, one dinosaur and four bears and the one car, one dinosaur and four bears being more in terms of bears.
  - (d) Use a variety of representations and situations (e.g. materials, pictures, stories) to explore and discuss same and different and more and less, generally and in terms of number.
2. Look at same and different and more and less using the BAE big idea::
    - (a) Discuss situations where the beginning is same as the end (e.g. balloon resting, person squishes balloon, balloon returns to shape it was at the beginning) – discuss situations where the beginning is not the same as the end or the end is different to the beginning (e.g. balloon resting, person squishes balloon, balloon bursts).
    - (b) Discuss situations where beginning and end have number and there is same, different, more and less in the numbers at the beginnings and ends (e.g., what is the significance of number increasing at the end from what it was at the beginning?).
  3. Ensure students can see which is more or less and how much more or less. Begin with stories but also display objects and use drawings. Discuss situations when this is important – counting students in a class on an excursion
  4. Activities to include:
    - (a) Hidden collections (students have a small quantity of the collection covered by both their hands. Ask them to make two groups, one under each hand, without looking. Ask which hand covers more? Which hand covers less? Then ask them to reveal the ‘less’ group. Follow by the ‘more’ group. Ask: how can you be sure? Say: cover less. Cover more. Reverse order asked to request ‘more’ group first then less group. Repeat several times with varying collections.
    - (b) Domino sort/Domino more than, less than – e.g. sort Dominoes with more dots/those with less dots/those with equal dots, those where one more dot, those with two less dots, and so on.
    - (c) Dice and counters – Students work in pairs taking turns to roll the dice reading the number out loud and putting the number of counters in a group. They then decide who has more in their group. On another occasion play who has less and the child with fewer counters wins.
    - (d) Use cards to play ‘winner takes all’ – each child takes a card, reads the number and place the number of counters/buttons/bears/etc. in a group, after an agreed number of turns the student with ‘more’ wins, or reverse and have the student with less win.
    - (e) Use number balance scale and a 1, 2, 3 dice Roll and place that many on your arm of the balance. Object is to identify who has more. Who has left? Can we ever have the same?
  5. Ensure language is understood – same, different equal, more, less, not equal, and especially “fewer”.

## 1.2 Extending to interpretation and construction

1. Consider BAE situations where the beginning is a set of objects and the end is another set of objects and the action is one which will changes the number of objects, e.g. removing one object.
2. Encourage students to find and describe these situations – use reversing: situation → description/story, AND description/story → situation. Encourage students to act out the situation using other children, physical materials and pictures.
3. Change the situation and discuss how the description/story changes. In particular, look at situations where a number of things increases and then modify the situation to where the number of things decreases. Explore these two situations (number increases/number decreases) – start to act out the differences

between these two without necessarily formally introducing these two ideas and formal names of addition and subtraction. If possible, show that extra things have to be brought in for increase (more) and some have to be removed (for decrease (less) – however, don't prevent students from discussing increase and decrease as reverses (inverses) of each other (e.g. I brought in two extra blocks – this made the end more but also the beginning less).

4. Set up activities when students sets out a change and another student describes it by telling a story. Reverse this to where student describes change and another student sets it out
5. Have students visualise the operating situation e.g. Joe has 4 toy cars. Taylor has twice as many cars. Act this out with toy cars showing the cars for each person. Draw/show me how many Joe has? Draw/ show me how many Taylor has.



### 1.3 Being more formal in bringing in BAE

1. Go through with class a situation where there is a change in number (a BAE which gives more) – use stories, acting out, modelling, drawings, language, and visualising. For example,

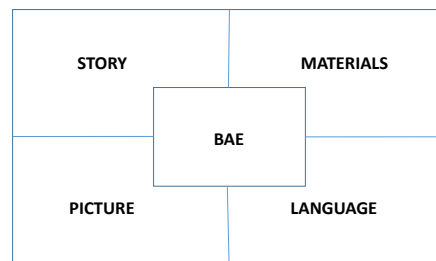
*story* – Jack had 3 pencils, Susie gave Jack two more pencils, how many pencils does Jack have now,  
*acting out* – students act out this scenario showing the giving of the two pencils and the joining of the two and the three;  
*modelling* – use counters to model the pencils  
*drawing* – draw this change as a picture; and language  
*language* – “3 pencils and 2 pencils”,  
*visualising* – seeing the story in the mind

2. Represent the story as a BAE as three sections (as below) and fill in the sections. O1 has ways in which this could be done – state if more or less.

B	A	E
Jack had 3 pencils	Sue gave Jack 2 pencils	Jack now has ___ pencils (more)
Lyn had 5 lollies	She gave her friend ___ lollies	Lyn now has 3 lollies (less)
Bill collected 7 plates	He collected another 3 plates	Bill now has ___ plates (more)
Sue had ___ pencils	She gave 3 to Jack	Sue was left with 5 pencils (less)

3. Reverse this by translating a BAE table to a story (thus we do story → table AND table → story). Bring in the other representations – must relate the BAE to them all – story, acting out/modelling with materials, drawings/pictures, and language/visualising.

4. Repeat above for different stories. If children can do it, use a thinkboard, as on right, broken into sections, for example, story, materials (modelling and acting out), BAE, picture (drawings) and language (visualisation). Start anywhere with respect to these representations give one and ask students to fill in rest of thinkboard. Discuss how relations go in all directions. Discuss all the parts of the thinkboard and how they relate to increasing and decreasing number at the End (addition and subtraction). Note: If wish to retain students materials work, take pictures and display.



## 1.4 Using BAE to identify operation

1. Start to identify the mathematics operation 'addition' with actions where numbers increase from B to E and to identify the mathematics operation 'subtraction' with actions that decrease number from B to E. Look for commonalities in actions and words in BAEs for addition, commonalities in actions and words for subtraction and differences between the two.
2. Encourage students to look for a collection of words that signify addition, that is, when B to E increases. Encourage students to do the same for subtraction, when B to E decreases. Act this out with materials. What happens when number decreases? When number increases? Sort words/actions into those associated with more and those associated with less.
3. Start to associate addition and subtraction with different actions and words – giving, joining, gathering more, and so on for addition and taking away, removing, losing, and so on for subtraction. Start to use major words like joining and taking with most examples. The RAMR lessons that follow will pick this up for early addition and subtraction
4. Use the following worksheet types with 4 columns as follows

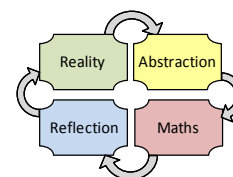
<i>Description of Beginning</i>	<i>Description of Action</i>	<i>Description of End</i>	<i>Underline operation</i>
Peter had 2 cups	Jan gave Peter 3 cups	Peter has 5 cups	<u>Addition</u> Subtraction

The idea is that students can relate BAE situations to operations (not answers). Can be flexible in what provide in columns: (a) can provide B, A and E and ask for operation, (b) can provide operation and ask for B, A and E, (c) can only provide one column and students invent rest, and so on.

## 1.5 RAMR lesson on Early Addition

Learning goal: Identifying and describing **adding** processes

Big ideas: Language  $\leftrightarrow$  picture  $\leftrightarrow$  materials  $\leftrightarrow$  action; visualising through kinaesthetic activity; Beginning-action-end; relationship and change



Resources: A carefully-chosen, inviting set of resources that offer lots of freedom to play, explore, question and try out ideas. Start from something that children might enjoy doing as they play and explore. Such as: construction materials/blocks; sorting/counting materials; pattern making materials/pattern blocks; shape picture materials; puzzles/jigsaws; role playing areas and materials; measuring tools; nesting materials; robots/windup toys for routes.

### Reality

Local knowledge: Base this on play items, construction items, and games that are familiar to the children. Use **everyday experiences such as** food being added to a plate (e.g. placing biscuits and or fruit onto a plate) or Children coming into the classroom.

#### Learning area experiences-

**The Arts**: Filling a space in art, e.g. adding more dots

**English**: Stories, rhymes and songs e.g. this old man, he played one...

**HPE**: students going into room and then students going out of room.

**Science**: Floating and sinking—how many objects float, how many sink

**Technology**: Okta's Rescue; and Five Frame

Prerequisites: Material in modules N1 and O1

Kinaesthetic: Use activities such as acting out the situation to reinforce beginning, action and end

## Abstraction

### *Is there evidence to suggest that the student is 'ready' to move into the abstraction?*

Provide multiple opportunities to experience and engage in activities. Teacher labels and constantly reinforces the target vocabulary, reinforced with nominated symbols/signs. Consider the need to explore one concept (adding to) before introducing the next.

**Body:** Notice and label the effect of people filling a defined space. Acting out a rhyme/ a song/ number story where collection of numbers are altered.

**Hand** Filling up / emptying containers –stacking towers, blocks, joining connectable maths resources – adding one by one.

**Mind** Responds to a request to add more. Predicts and or responds to the effect of adding to a quantity and space. Shut eyes and visualise.

## Mathematics

*Language/symbols:* more/less, how much more/how much less, joining, separating, add, take

### *Is there evidence to suggest that the student is 'ready' to move into the mathematics?*

Provide opportunities for student to request more of an object. Require students to comment using target vocabulary in counting songs—more / less etc. Students role play counting songs with objects. Students respond to requests to add to or increase.

Use BAE to label components of a change situation that increase number. Focus on A (action) and look for common words that help to understand a change that increases number or has more at the end when compared to the beginning.. Encourage students to see that such changes can be described by the word add or adding involve adding. Try to also attach words join, part and total to what is happening in these situations *Practice.* Give a variety of situations, act them out, identify those that involve adding. Have activities that go situation → addition and addition → situation.

*Connections:* Ensure that students see that adding is based on sorting, identifying attributes, patterns, matching, ordering, counting. In particular that it requires counting and ordering to restrict the name 'add' to those where end is more than beginning. Also try to encourage students to see that addition is reverse to situations where you get less.

## Reflection

*Validation.* Look for adding experiences in everyday world of students. Chooses quantities of collections linked to personal preference, e.g. take a bigger pile of lollies

As part of this can developing understanding of quantity in relation to space: (a) area – e.g. gather more dots to use in a dot painting as the space is large; and (b) capacity – e.g. gather more objects to fill a space. Expresses relative amount of collection through appropriate gesture, hand sign, language, ALS.

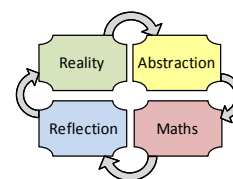
*Applications/PS:* Students respond to requests to add to in a range of learning areas and everyday experiences.

*Extension: Flexibility* – try to find many situations that add; *Reversing* – effect a change to a collection by using appropriate actions (e.g. change an action from adding to taking away); *Generalising* – look at situations which add and not add, and discuss the general situations which lead to addition.

## 1.6 RAMR lesson on Early Subtraction

Learning goal: Identifying and describing **subtraction** processes

Big ideas: Language ↔ picture ↔ materials ↔ action; visualising through kinaesthetic activity; Beginning-action-end; relationship and change



Resources: A carefully-chosen, inviting set of resources that offer lots of freedom to play, explore, question and try out ideas. Start from something that children might enjoy doing as they play and explore. Such as: construction materials/blocks; sorting/counting materials; pattern making materials/pattern blocks; shape picture materials; puzzles/jigsaws; role playing areas and materials; measuring tools; nesting materials; robots/windup toys for routes.

## Reality

*Local knowledge:* Play items, construction items, and games that are familiar to the children. Use **everyday experiences** such as food being taken or added to a plate, e.g. biscuits disappearing from a packet, spoons full of food being added or taken away at meal time. Tissues disappearing from a box. Packing away or adding classroom resources to a collection

### Learning area experiences

**The Arts:** Filling a space in art, e.g. adding more dots

**English:** Stories, rhymes and songs e.g. 10 green bottles; five little ducks.

**HPE:** students going into room and then students going out of room.

**Technology:** [Grouping and grazing](#)

*Prerequisites:* Material in modules N1 and O1

*Kinaesthetic:* Use activities such as acting out the situation to reinforce beginning, action and end

## Abstraction

### *Is there evidence to suggest that the student is 'ready' to move into the abstraction?*

Provide multiple opportunities to experience and engage in activities. Teacher labels and constantly reinforces the target vocabulary, reinforced with nominated symbols/signs. Consider the need to explore one concept (adding to) before introducing the next.

*Body:* Act out BAE situations where E has less than B. For example, notice and label the effect of people filling a defined space; Acting out a rhyme/ a song/ number story where collection of numbers are altered

*Hand:* Use materials, pictures, etc. to represent BAE where E is less than B. For example, filling up – use materials / emptying containers –stacking towers, blocks, joining connectable maths resources – adding one by one/ removing one by one

*Mind:* Responds to a request to take away; Predicts and or responds to the effect of taking away on quantity and space. Visualises situations where number of objects reduces.

## Mathematics

*Language/symbols:* more/less, how much more/how much less, joining, separating, take away

### *Is there evidence to suggest that the student is 'ready' to move into the mathematics?*

Use BAE to label components of a change situation that decrease number. Focus on A (action) and look for common words that help to understand a change that decreases number or has less at the end when compared to the beginning.. Encourage students to see that such changes can be described by the word subtract or subtracting. Try to also attach words take, part and total to what is happening in these situations

*Practice.* Give a variety of situations, act them out, identify those that involve subtracting. Have activities that go situation → subtraction and subtraction → situation. Provide opportunities for student to request less in terms of number of objects. Require students to comment using target vocabulary in counting songs—fewer / less etc. Students role play counting songs with objects. Students respond to requests to take away or remove.

*Connections:* Ensure that students see that subtracting is based on sorting, identifying attributes, patterns, matching, ordering, counting. In particular that it requires counting and ordering to restrict the name 'subtract' to those where end is less than beginning. Also try to encourage students to see that subtraction is reverse to situations where you have addition.

## Reflection

*Validation.* Look for subtracting experiences in everyday world of students. Chooses quantities of collections linked to personal preference, e.g. take less sandwiches. Students respond to requests to add to or take away in a range of learning areas and everyday experiences. As part of this can developing understanding of quantity in relation to space: (a) area – e.g. gather less photographs for the presentation as the space is small; and (b) capacity – e.g. gather less objects to fill a space. Expresses relative amount of collection through appropriate gesture, hand sign, language, ALS.

*Applications/PS:* Students respond to requests to take away in a range of learning areas and everyday experiences. Look at situations to determine whether addition or subtraction.

*Extension: Flexibility* – try to find many situations that subtract; *Reversing* – effect a change to a collection by using appropriate actions (e.g. change an action from taking away to adding); *Generalising* – look at situations which subtract and, in general, what conditions lead to subtraction.



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## Unit 2: Basic Meanings

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In this unit, we begin the process of extending the ideas from O1 and Unit 1 to new meanings for the operations. Because we are still in the early years, we will look at two basic meanings for the operations addition and subtraction and one basic meaning for the operations multiplication and division. The two meanings for addition and subtraction are:

- (a) **The form of the 'action' in the BAE big idea** – by seeing addition as being an action to increase number in E in relation to B, and by seeing subtraction as being an action to decrease number in E in relation to B, with **joining** and **taking away** being the main form of action; and
- (b) **the part-part-total components in the beginning and end in the BAE big idea** – by seeing addition as when a **part is unknown** and wanted and seeing subtraction as when a **total is unknown** and wanted.

The one basic meaning for multiplication and division is the same as the first for addition and subtraction, it is based on the **form of the 'action' in the BAE big idea** – by seeing multiplication as being an action to increase number in E in relation to B by **combining equal groups** and by seeing division as being an action to decrease number by **partitioning into equal groups**.

(Note: There is a factor-factor-product understanding of multiplication and division that is similar to part-part-total for addition and subtraction. Because of the conceptual complexity in understanding what factors are, and the fact that this is a Year F-2 module, factor-factor-product will only be covered briefly in Unit 3.

### Background information

Officially, in arithmetic, there are only two operations – addition and multiplication. These two follow the commutative and associative law where subtraction and division do not. For example,  $5+2 = 2+5 = 7$  but  $5-2$  does not equal  $2-5$ ; similarly  $3 \times 4 = 4 \times 3 = 12$  but  $15 \div 3$  does not equal  $3 \div 15$ . The operations subtraction and division are adding a negative and multiplying by reciprocal respectively, that is,  $5-3 = 5+(-3)$  and  $8 \div 2 = 8 \times \frac{1}{2}$ .

However, for the early years, we consider subtraction and division as operations. They are widely used as operations in the everyday world of school and business. However, we are aware that subtraction and division have this inverse relationship with addition and multiplication respectively and use it when we can.

### Meanings of addition and subtraction

There are four meanings for addition and subtraction as follows:

MEANING	ADDITION [Unknown]	SUBTRACTION [Unknown]
1. Action	Joining, more, increasing [End] 3 children playing, 4 joined them, how many playing now?	Take-away, separating, removing [End] 6 children playing, 5 went home, how many playing now?
2. Components	Part-part-total [Total] 4 boys and 5 girls, how many children?	Part-part-total [Part] 3 boys, 8 children playing, how many girls?
3. Comparison	Y has more than X [Y] John has 3 pens, Jack has 4 more pens than John. How many pens does Jack have?	Y has less/fewer than X [Y] John has 5 pens, Jack has 3 pens fewer/less than John. How many pens does Jack have?
4. Inverse to normal action	Take-away, separating, removing [Beginning] Children were playing, 5 left. This made 4 playing. How many children at the start?	Joining, more, increasing [Beginning] Children were playing. 5 joined them. This made 11 playing. How many children at the start?

The two interesting meanings are components and inverse. Components can work for situations where there is no action and they help overcome confusion that can occur with inverse – this is because addition and

subtraction situations involve two parts and a total and, regardless of the action, **addition** is a situation when parts are known and total is wanted and **subtraction** is a situation when total and one part is known and the other part is wanted. Inverse does the opposite of the original meaning by simply having the unknown at the start. This is because, for inverse, we are working backwards – looking for the beginning not the end.

Because this module focuses on Years F to 2, we focus mainly on addition and subtraction and, although we introduce all meanings in this module, we begin with what we call the “basic meanings” – **action** and **components**. Thus, this unit, Unit 2, focuses on action and components, with Unit 3 focusing on the other meanings

## Meanings of multiplication and division

There are five meanings for multiplication and division as follows:

MEANING	MULTIPLICATION [Unknown]	DIVISION [Unknown]
1. Action	Combining equal groups [End] There were 4 bags each containing 3 lollies. The lollies were all poured into a bowl. How many lollies in all.	Partitioning into equal groups [End] There were 15 lollies. They were put into packs of 5. How many packs (Grouping) There were 15 lollies. They were shared amongst 5 children. How many did each child get? (Sharing)
2. Components	Factor-factor-product [Total] There were There were 3 teams of 5 children. How many children?	Part-part-total [Part] There were 4 equal teams and 20 children. How many children in each team? (Sharing) There were teams of 4 and 20 children. How many teams? (Grouping)
3. Comparison	Y has times more than X [Y] John had 4 lollies. Jack had 3 times as many lollies as John. How many lollies did Jack have?	Y has times less than X [Y] John has 15 lollies. Jack has 3 times less. How many lollies does Jack have?
4. Inverse to normal action	Partitioning into equal groups [Beginning] There was a pile of lollies. They were shared amongst 4 children. Each child got 3. How many lollies at start?	Combining equal groups [Beginning] There were 4 bags of lollies with the same number of lollies in each bag. They were all poured into a bowl. There were 20 lollies. How many lollies in each bag at the start?
5. Combinations-Area	Options to combinations [End] There were 3 shirts and 4 pants, how many outfits? The rectangle was 4m by 3m in length, what is area in m <sup>2</sup> ?	Combinations to options [End] There were 24 outcomes from 2 spinners. One spinner had 3 options, how many did the other have? Rectangle area 24 m <sup>2</sup> & length 8m. Width in m?

As for addition and subtraction, the interesting meanings are components and inverse. The components meaning provides a strategy that can encompass all other meanings. It can work for situations where there appears to be no action and it helps overcome confusion that can occur with inverse. This is because multiplication and division situations involve two factors and a product and, regardless of the action, **multiplication** is a situation where factors are known and product is wanted and **division** is a situation when product and one factor is known and the other factor is wanted. The inverse does the opposite of the original meaning by simply having the unknown at the start. This is because, for inverse, we are working backwards – looking for the beginning not the end.

Because this module focuses on Years F to 2, our main focus is on action. Thus, this unit, Unit 2 Basic Meanings, covers only the first one of the five multiplication-division meanings – the **actions** of combining and partitioning.

## Teaching meanings

To teach the various meanings of the operations, it is useful to do the following:

1. **Model with BAE.** Model situations using the BAE big idea and determine what actions relate to what operations. Begin with the simpler ones – joining for addition, take-away for subtraction, combining for multiplication and partitioning for division.
2. **Reinforce useful language.** Use the language of the actions when interpreting the actions and relating them to operation names – we are “joining”, we are “taking-away”, and so on. In particular, talking about joining in terms of “parts” and “totals” – joining as two parts making a total and separating as a total becoming two parts.
3. **Interpret and construct.** Do activities where you interpret situations as joining or separating and then do activities where you construct situations that are joining and separating.
4. **Use a variety of models.** Addition and subtraction can be acted out by moving sets of objects of same and different size and with same and different hops along a number line or a 99 board (set model and number line model), while multiplication and division can be acted out with sets of objects of same size in groups or rows and multiple hops along number lines or number boards (set model, array model and number line model).
5. **Sequence and relate.** Connect new ideas to old – for instance, look at addition as joining and then multiplication as an extension of this, as joining 2 or more equal groups.
6. **Distinguish and show difference:** As well as sequencing and looking for similarities (as in 4 above), need to distinguish new ideas from previously known ideas. As we will see in this unit, addition and multiplication both involve joining, but they are different in that:
  - (c) addition may have different sized groups, while multiplication must have equal groups; and
  - (d) addition’s three numbers all refer to the same thing, that is, the number of objects, while one of multiplication’s three numbers refers to the number of groups leaving only two numbers to refer to the number of objects.

Note: There is a similar difference between subtraction and division to that between addition and multiplication. The separation in division is with equal groups and one of division’s numbers refers to the number of groups.
7. **Introduce and connect language and symbols.** Introduce formal language (add, multiply, subtract, divide) and symbols (+, ×, −, and ÷) for the various means when feel appropriate. Reinforce by relating language ↔ symbols ↔ stories ↔ models, where models can be BAE diagrams but also set, array and number line models using materials and pictures.
8. **Identifying when there is no operation.** Also be able to identify that not all situations are addition or subtraction and not all situations are even operations

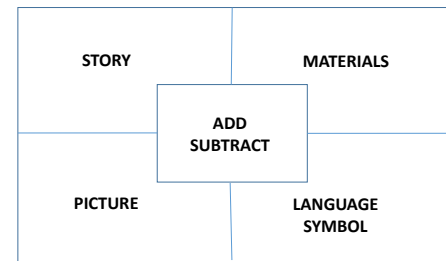
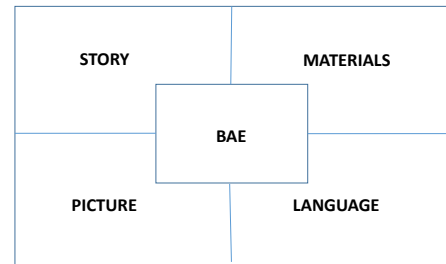
Note: Free resources are available on the YDC website giving activities for introducing part-part-total and factor-factor-product – these are the result of an Australian Research Council Linkage funded project to train Indigenous teacher aides. Five booklets for addition and subtraction and five for multiplication and division are available under Professional Learning resources.

## 2.1 Joining and separating as meanings for addition and subtraction

1. Consider situations in which two sets are joining – act out with students and discuss what is happening and what results. Consider situations in which a set is separating into two sets (and one set is being taken away) – act out with students and discuss what is happening. Discuss the above two situations in terms of the actions – which one increases size of group, which one decreases?

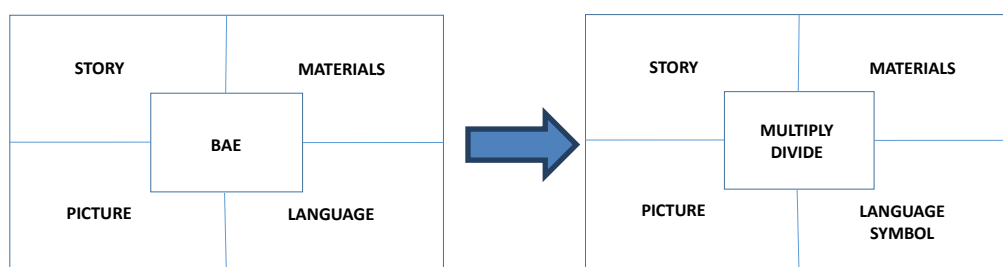
Reverse what is being done and ask for actions that increase/decrease size of group. Act out and discuss these. It is best if we look at real world/everyday situations before we model with materials and we model with materials before formal language (and symbols).

2. Introduce 'taking away from' and 'adding to' – act this out with students (best if action before words. Discuss which one of joining and separating are these action part? Relate in terms of actions as well as descriptions of actions – visualise the actions, the activities and the implications of these activities. Reverse and relate joining and separation to adding and taking away. Discuss if everything is covered, whether there are some actions that don't fit in as well as others? For example separating and take away need experiencing – separating is forming two sets from one – where is the taking away? However, taking away needs separating.
3. Practice the relationships in 1 and 2 above with thinkboard as in Unit 1 – story, materials, drawing, language (increases-decreases), and BAE – provide students with one of the 5 parts of the thinkboard and students fill in all the other sections.
4. When explored all that can or you feel students are ready – introduce add for when group increases and subtract for when group decreases, in place of BAE –and introduce language and symbols, informally and formally, if think students are ready. Please note that, at this point in the sequence, we are not considering answers – we are relating situations (e.g. BAEs) to the operation.
5. Repeat the above for adding and subtracting more than one group. Use this to connect 2.2 to 2.1. Also spend time ensuring children understand when something is not an operation (here addition and subtraction) or not an operation at all.



## 2.2 Combining and sharing as meanings for multiplication and division

1. Repeat 2.1 for more than two groups with the groups equal in number of objects. Introduce “combining” and “partitioning” after actions
2. Repeat 2.1 for special situation of more than two equal groups. Explore this and then talk about it. Relate partitioning to sharing in action and then words.
3. Explore the above for a time and then introduce “multiplication” and “division” for these actions. Explore grouping as well as sharing in division. Sharing is where divide equally amongst a number of people – check that this understood by all children – this is difficult for Indigenous students where “sharing equally” does not mean the same number to each student.
4. Use thinkboards – story, materials, pictures, language, BAE – start from anywhere and finish all other things (that is, as teacher, complete one of the 5 parts of the thinkboard and ask for the others to be completed). Construct and interpret, that is, complete thinkboards for situations related to combining equal groups, sharing equally, and partitioning into equal groups (i.e. division) AND interpret completed thinkboards in terms of multiplication and division situations (i.e. combining equal groups, sharing and grouping). Use the two thinkboards in the sequence below when you think students can go on to symbols.



5. Use three models when drawing multiplication and division pictures – sets and equal subsets, arrays, and repeated same size hops along a number line. For the thinkboard activities, repeat the activities/ideas from 4 above. However, when moving to symbols, discuss how for multiplication/division that one of the numbers refers to the number of groups/sets, rows/columns or hops.
6. Also remember to ensure students can also identify non operations.

## 2.3 Relating adding/subtracting to multiplying/dividing

1. Get the students to act out addition and multiplication situations that involve the same numbers and compare what happens, that is, compare  $2+3$  and  $2 \times 3$ ,  $5 \times 2$  and  $5+2$  and so on (simple examples like this which have the same numbers for what is being added and multiplied. Repeat this acting out with materials, e.g. unifix cubes, asking students what is the same and what is different?
2. Discuss how addition and multiplication are similar– encourage/facilitate the understanding that “both are joining”? Discuss how they are different – encourage/facilitate the understandings that “addition can have unequal groups” (though can be equal) while multiplication “must have equal groups”. Keep discussing – try to draw out the important one difference that all numbers relate to objects, counters or students while multiplication has one number which refers to the number of groups. For example, display and join objects as below:

$3+4$      $\text{OOO} \rightarrow \leftarrow \text{OOOOO}$

joining different groups

3 objects and 4 objects

$3 \times 4$      $\text{OOOO} \rightarrow \text{OOOO} \leftarrow \text{OOOO}$

joining same groups (4 objects)

3 is the number of groups

3. Extend above to other models. Discuss this way of representing addition and multiplication using a number line and how addition and multiplication are different [addition – numbers are the lengths of the hops; multiplication – one number is length of all hops while the other number is number of hops]. Discuss how arrays work for multiplication – how one number gives the number of objects in each row of the array while the other number gives the number of rows.
4. Focus on practicing the relationships for all situations for all operations. Relate story, BAE, models and acting out with materials, language and symbols to each other – use thinkboard as in 2.3 above, activity 4 – but focus on similarities and differences. Extend discussion to similarities and differences between subtraction and division.

## 2.4 Part-part-total

1. The secret to part-part-total is to use the words part and total from the start – even when working with BAE always describe a joining or separating in terms of parts and totals – ask all the time “what is the part?” “what is the total?” – construct parts and totals as well as interpret what is going on in terms of them
2. Move backwards and forwards – “here are two parts” – “I will join them to make a total” – “I’ll break the total up into two different parts” – “which were the parts which were the total?” – ask the children to form totals from parts and parts from totals – “now you gather two parts, now form a total?” and so on.

Get students to see pattern – when add I am finding the total – when I subtract I am finding a part?” So what is addition and what is subtraction?

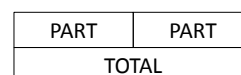
Note: We have two things to reverse here: (a) going from parts  $\rightarrow$  totals, and going from totals  $\rightarrow$  parts; and (b) getting students to follow the teachers directions and answering questions (interpreting; teacher  $\rightarrow$  student) and allowing the students to make up their own parts and totals (constructing; student  $\rightarrow$  teacher).

3. However, the power of part-part-total is its use to solve problems, particularly complex problems (and inverse problems which we do not cover until Unit 3). Regardless of the actions and language of the problem, children need to look at the problem in terms of part-part-total and make decisions on what is wanted. Again students should be allowed to analyse problems themselves. To enable this, provide two types of problems (they are the reverse of each other): (a) students given problem/situation, students determine parts and totals and use this to determine the operation; and (b) students given parts and totals and have to make up their own problems to match the parts and totals they are given.
4. Encourage children to use the type of thinking below – useful to act out or model the stories:

OPERATION	MEANING	STORY	THINKING
Addition	Know parts – want total	Straightforward story: At the party 9 kids were eating ice creams in the pool and 7 were eating ice creams on the verandah. Altogether, how many children were eating ice creams?	“The 9 and 7 are parts. The wanted amount is the total. So, the operation is addition.”
		Complex story: At a party, I saw 9 children take icecreams from a tray, to eat. This left 7 icecreams on the tray. How many icecreams were there to start with?	“The 9 and 7 are parts. The wanted amount is the total. So, the operation is addition the same as above.”
Subtraction	Know total – want a part	Straightforward story: At the party there were 16 icecreams, 9 were eaten how many are left?	“The 16 is the total. The 9 is a part. The wanted amount is a part. So, the operation is subtraction.”
		Complex story: At the party there were 16 icecreams, some chocolate and some banana. The 9 chocolate icecreams were eaten, how many were banana?	“The 9 is a part. The 16 is the total. The wanted amount is a part. So, the operation is subtraction.”

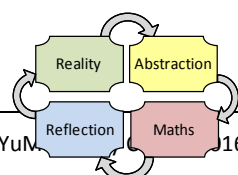
Note: In the long run, it is important to ensure that students have a good understanding of part-part-total because this understanding allows a variety of different stories to be interpreted in terms of addition and subtraction. Activities like those below can help, so spend time on the following:

- act out addition and subtraction situations, like  $4 + 5 = 9$ , with materials, identifying the group of 9 by the name “total”, and the groups of 4 and 5 by the name “part”;
- show how the stories of addition and subtraction are the reverse of each other (addition as  $P + P = T$ ; and subtraction as  $T - P = P$ );
- generally do the same activities as for teaching joining and taking away but continuously use the terms part and total instead of other language or symbols; and
- use a part-part-total diagram like that on the right – place information from the stories onto it – the numbers and a question mark for the unknown. Note – this diagram will also help recognise non-operation situations.



## 2.5 RAMR lesson for addition/subtraction meanings

Meanings for each operation Addition and Subtraction: joining /separating and part-part-total



Learning goal: How number relates to length, i.e. how equal distances between objects relates number to distance.

Big idea: Symbols tell stories.

Resources: Everyday objects; Sorting and counting materials; dice; unifix cubes

## Reality

*Local knowledge:* Where possible, find real-life contexts to embed the activities in; for example, using relevant objects or situations. Identify everyday situations where there is addition and subtraction – people leaving the classroom, people arriving for a party, and so on.

*Prior experience:* 1:1, counting, sorting plus understandings from O1, particularly re BAE.

*Kinaesthetic:* Use materials to construct stories and problem situations

Ask students to think about their favourite number. Is it a number you can count? Is it a number you can make? Is it a number you can write? What is your favourite number between 0-10. Make me a story with your number – then make me a story where the number gets larger/smaller. Or Read a story with numbers in it. Is a story that numbers get larger/smaller includes some addition/subtraction actions?

Play 'make 10' with using groups of students. Have students sit in two teams. First team rolls the dice. They make a group with that number of students. Second team rolls the dice and makes a group with that number of students. Ask: How many in each group? Have students take turns to count and confirm. Who has more? Who has less? Have students compare their groups to confirm. Roll one die, make a group of that number of students/counters – roll another die and make a second group and join it to the first. What has happened to the group? Continue to act out **joining** the groups. State that this is adding.

How many more do you need to 'make 10'? Have students show how they know this by acting out the group change. Is it possible to roll that number with the dice? Why? Why not? Have each group roll the dice a second time, make the second group and. Ask: Did your group make 10? Why? Why not? You may need a third roll of the dice for both teams. Each time a team makes ten they score a point recorded as a tally mark. When they get 5 tally marks they are the overall winner.

Reverse the above process- play 'Make zero' where you begin with 10 students in each group and separate the group each time you roll the die. The die gives the number to be removed – larger the number, the more removed – what remains is less than the start – so state this is subtraction. Compare the team situations referring to total and parts. Act out making bigger by joining on a new group ("adding it in) and smaller by removing a group ("taking-away) and ensure that adding and take-away relate to joining parts and taking a part away respectively.

## Abstraction

*Body:* Play tell a story and act it out. Students tell the story while other students act it out Lots of turns (all is good). Count and group discrete objects in the environment. Join and separate the groups. Arrive at totals. Compare. Enact on a human size PPT board.

*Hand:* Other students can make the story with materials e.g. unifix cubes or paddle pop sticks or counters or blocks or etc. or draw it; Play 'make 10' with sorting sets; real objects and unifix cubes. Draw P-P-T pictures. Also can use ten frames, the number ladder, a bead string

*Mind:* Discuss situations where things are joined and separated. Have students tell a joining/separating story and the others visualise what is happening.

*Creativity:* Have students create their own joining and separating stories using materials and mediums of their choice and share them with each other. Can also have the students invent own symbols for joining and separating before show formal symbols.

## Mathematics

*Language/symbols:* introducing addition and associated symbols and words, subtraction and associated symbols and words, joining, separating, total, part, thinkboard, action, change, opposite actions, problem, model, representation,

*Practice:* Focus on practicing the relationship between formal words and symbols and stories, drawing, actions and models (BAE and part-part-total where useful). Look to ideas in 2.1 and 2.2. **Note.** We can find answers but the focus of the practice should be on:

- (a) identifying a story/BAE situation in terms of the operations addition or subtraction – that is identifying meaning (story  $\rightarrow$  operation);
- (b) constructing a story/BAE situation when given addition or subtraction (operation  $\rightarrow$  story, i.e. reversing (a)); and
- (c) identifying difference between stories/situation which are not operations and those that are.

This means that there are three outcomes that need to be identified – addition, subtraction and no operation.

Make ten frames; Model on a PPT board. Have students Model on a group Thinkboard; Show addition and subtraction reverse/inverse. Tell one more one less stories; model what was the first part? What was the other part? What was the total? Have students tell oral problems. Publish and share.

*Connections:* Counting, sets, and real world applications (e.g. money) plus ideas in BAE in AIM EU module O1. But spend particular effort on seeing addition and subtraction as inverses.

## Reflection

*Validate:* Students find situations in their world where they add and subtraction (join and take-away). Students prepare 'addition in my world' or 'subtraction in my world' type posters

*Applications/Problem-solving.* Set problems that require understanding of add and/or subtract. For example, an application is to find length by counting steps. There are two parts so have to add steps. This application can lead to wider applications that have to do with length. For example if have students step along a premade line. Ask them how many steps they took? Ask: did you all take the same number of steps? How could you make it so we all take the same number of steps? Why did the number differ? Get students to note that still adding steps but measurement needs the steps to be the same size.

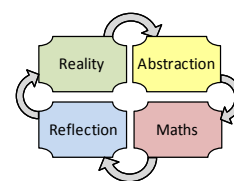
*Extension. Flexibility* – How many different ways can we add to 10. How many different things do we add in our daily life. Show 7-2 in as many different ways as you can. *Reversing* – changing direction of stories, for example, given 7 kangaroos, make up a subtraction; given a subtraction story about kangaroos, put number 7 in the story. Important here that ensure addition and subtraction are reversing – ensure know that  $7-3=4$  is opposite to  $3+4=7$  – join and separate to show inverse relation. *Generalising/Changing parameters:* If I gave you two numbers, how would you add them? How would you subtract them? Whatif they were two digit numbers?

## 2.6 RAMR lesson outline for teaching addition/subtraction models

Learning goal: to teach addition as joining, and subtraction as separating or taking away, using the set and number-line models.

Big Ideas: Symbols tell stories

Resources: Thinkboards; sorting materials; counters; pens, pencils; paper;



## Reality

**Prerequisites.** Knowledge in modules N1 and O1 and in Unit1. Ability to act out joining and take away situations.

**Reality.** Discuss where students see things being joined and things being separated (taken away) in their world.

**Kinaesthetic.** Act out problems that students talk about with the students. For example, act out (a) Addition – *there were 3 students sitting at a table and 2 more students joined them. How many students altogether?*; and (b) Subtraction – *7 students were eating their lunch together. 3 students had to go to a meeting and walked away. How many students were left?*

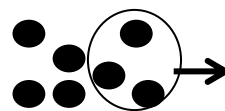
## Abstraction

### Set model

**Body.** Start with acting out a problem with the body – identify parts and whole.

Encourage students to model the acting out with counters:

John has 7 pencils, he gives  
3 to Jane, he has 4 left.



**Hand.** Have students draw a picture with their own representation for the story (can include own images for standard symbols for later discussion):

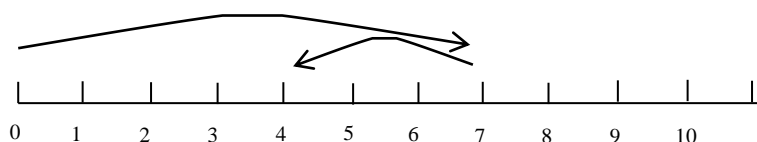
John has 7 pencils, he gives  
3 to Jane, he has 4 left.



### Number-line model

**Body.** Start with acting out a problem (and finding the answer if important). This time students model the acting out with a number line. Encourage students to walk a number line.

John walks 7 blocks and then walks 3 back, how far is he from where he started?



**Hand.** Have students draw a picture with their own representation of the story (including symbols if this is where students are).

**Mind.** Imagine the movements with the objects (set model) and walking (number line model). Visualise what is happening.

**Creativity.** All students to use their own drawings and the make up their own symbols –get different students to use other students' symbols.

**[Overall note.** Do both addition and subtraction and discuss actions and drawings. Label the drawings with the story but also the operation.]

## Mathematics

**Introduce language and formal symbols.** For example: How many pencils did John have? [7], how many did he give Jane? [3], and how many were left? [4]. This would be “seven subtract three is four” and  $7 - 3 = 4$ . Use the students' own symbolism and forms of recording as a halfway house to this end.

*Practice.* Relate symbols to stories – e.g. thinkboard, mix and match cards, cover the board game, bingo. See booklets of materials in YDC website (not the blackboard website). Begin with any of the representations and complete the others, that is, relate all five representations: story/act out  $\leftrightarrow$  material/models  $\leftrightarrow$  pictures  $\leftrightarrow$  language  $\leftrightarrow$  symbols. Using the thinkboard, fill in one area (any of the areas), and ask for the other areas to be completed. For example, see thinkboards in 2.1 and 2.2 below.

*Connections.* Ensure that addition and subtraction are related and these related to Modules N1, N2 and O1.

## Reflection

*Validation.* Prepare a collage – addition and subtraction with sets and lines in my world.

*Applications/problem-solving.* Set up stories to solve that involve sets (e.g. money) and number line (e.g. riding bike).

*Extension.* Flexibility – discuss all ways can add or subtract to something; Reversing – ensure know how all representations relate two-way (e.g. symbol to drawing and drawing to symbol) and how addition and subtraction relate two way. Focus on undoing an operation. Generalising/Changing parameters. Say to student – if I gave you two numbers, in what situations would you add and in what situation would you subtract – what if the numbers were two-digit?

## 2.7 RAMR Structure for Part-Part-Total (and word problem-solving)

Reality	Abstraction
<p>Ask students to think their favourite number. Is it a number you can count? Is it a number you can make? Is it a number you can write? Make me a story with your number. Or Read a story with numbers in it.</p> <p>Make up a story that increases your number. Make another story which decreases your number. Make sure the stories have a beginning, action and end (BAE). Which story is addition; which is subtraction?</p> <p>Make me two stories with your number as a total; one story using addition and one using subtraction. Make me two stories with your number as a part; one story using addition and one using subtraction.</p> <p>Get students to talk to others in their class. Do they see any patterns? Which stories are hard to make?</p>	<p>Organise students to tell a story that involves BAE and increasing/decreasing numbers and to act this out. Teacher repeats story but using <b>part-part-total words</b>. Start with unknown only at the end of the action</p> <p>Ensure students get many opportunities to act out a variety of part-part-total situations. Other students not involved in acting can make the story with materials (e.g. unifix cubes, paddlepop sticks, counters or blocks or etc.) or draw the story.</p> <p>Ensure that the part-part-total stories cover a variety of addition and subtraction situations and that these are acted out, modelled with materials, drawn, spoken about and described with symbols as appropriate. Ensure all directions are covered in activities that relate stories to meanings.</p> <p>Discuss what happens if the unknown is at the beginning of the action.</p>
Reflection	Mathematics
<p>Students construct problems from their world – other students determine parts and wholes and work out the operation and thus the answer.</p> <p>One and more than one step problems involving addition and subtraction and part-part-whole can be set for students to solve from identifying parts and wholes and using rules from 2.4 act 2(a).</p> <p>Reverse the above and focus on students constructing problems. Give numbers to the students for which students have to write a story.</p>	<p>Model increasing/decreasing BAE stories with ten frames or on PPT boards. Also model on a group Thinkboard. Practice determining the operation meaning for stories – large LH column for story – small RH column with operations – circle the correct one. Then reverse – give operation and numbers and students write story.</p> <p>Using parts and totals, show addition and subtraction as reverse/inverse of each other. Tell one more one less stories; model the stories.</p>

Take numbers and write all possible addition and subtraction stories for those numbers. Use the triad big idea and write a completed story with the numbers and then rewrite the stories with one of the numbers unknown, and the next one unknown and so on. Look at what would happen if numbers were larger (1 digit to 2 digits).

Identify parts and totals. Provide stories and ask what was the first part? What was the other part? What was the total? Provide word problems – determine what given numbers are (part or whole). Determine operation from rules (see Discuss what this means for solving the problem. See 2.4 act. 2(a)



## 2.8 RAMR lesson outline for multiplication and division

### Operating –Meanings for each operation multiplication and division

Learning goal: writing number sentences to show arrays, groups, sets, number lines

Big Ideas: Symbols tell stories

Resources: Everyday objects; Sorting and counting materials; dice; unifix cubes

### Reality

*Local knowledge*: Scenarios of multiplication in everyday situations: birthdays-cupcakes arranged in arrays, lolly bags for guests with same number of lollies in each; rows of seats at movie/concert etc.; shopping 1 bag of carrots cost 5\$ how much for 5 bags? Include division situations (e.g. sharing, grouping) which have same number of objects.

*Prior experience*: with general operations are aware that there are 4 operations; have worked with addition and subtraction – now extending to multiplication and division; have been exposed to BAEs, and symbols telling stories especially the operation symbols – they know N1, O1 and start of O2.

*Kinaesthetic*: Use materials to construct stories and problem situations involving forming equal groups and the how many altogether question? Act out equal groups, arrays (use mat) and division situations (grouping and sharing). Have students form equal groups and determining the result of combining, and have students share themselves equally or break into equal groups.

### Abstraction

*Body*: Act out and tell a story of combining equal groups for students in a class (e.g.  $3 \times 8$ ;  $2 \times 12$ ;  $4 \times 6$ ;  $1 \times 24$  for 24 students in a class). Note that it is a BAE and the action is the change the 6 lots of 4, for example, into one group of 24. Also act out arrays (form students into rows of 6) and on number lines (students take 6 jumps of 4). Stress the need for equal groups and to do addition a form of addition with these groups (repeated addition). Relate the multiplication and division back to the addition and subtraction – check differences and similarities (see 2.3).

*Hand*: Other students can make the story with materials e.g. unifix cubes or paddle pop sticks or counters or blocks or etc. or draw it; Play ‘make groups of’ with sorting sets; real objects and unifix cubes. Also can use the number ladder, a bead string

*Mind*: Discuss situations where things are grouped. Have students tell a grouping story and the others visualise what is happening- along a line or an array.

*Creativity*: Have students create their own grouping stories using materials and mediums of their choice and share them with each other. Let them choose/construct their own symbols.

## Mathematics

*Language/symbols:* Reveal all multiplication and division associated symbols and words, e.g. grouping, sharing, total, part, thinkboard, action, change, opposite actions, problem, model, representation, factor and product,

*Practice:* Practice relationship between stories, modelling with materials, drawings, language and symbols. Have students model on a group Thinkboard and relate what is acted out to equations to represent sets, arrays and jumps along a number line; Show multiplication and division are inverses of each other – form equal groups, combine, form equal groups, combine, and so on. Note that the numbers being used are such that one number is the number of groups.

Have students tell oral problems. Publish and share.

*Connections:* Counting, sets, money, and lengths. Showing that multiplication and division are inverses of each other and consolidating relationships between multiplication and addition and between division and subtraction.

## Reflection

*Validate:* Students solve each others problem; Play make groups with numbers of their choice. Gather information on multiplication and division in their world.

*Applications/Problem-solving.* Make problems that require combining/partitioning into equal groups. For example, relate back the birthday party scenario, e.g. 8 invites, lolly bags have 7 lollies in each- how many lollies for each guest/ how many lollies altogether? 25 balloons to put into 5 party bags, how many balloons in a bag?

*Extension. Reversing* – The answer is 12, what might the parts of the equation be? Tell me a story about these 15 kangaroos. Note show that multiplication and division are inverses. *Flexibility* – Try to find as many different ways to show 3, 5 and 15 in word problems. Remember that the unknown can be in the End and in the Beginning and these two scenarios give different answers. *Generalising/Changing parameters* – Say that there are three numbers related by multiplication – ask for all the ways that they could be in multiplication or division? This can also be done for the following: How many ways can you make 16? What makes 5 a special number? Show 6 in as many different ways as you can.

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## Unit 3: Other Meanings

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This unit looks at the more difficult meanings for the four operations. In Unit 2, we covered three meanings:

- (a) addition and subtraction as **joining and separating** (taking away);
- (b) addition and subtraction as **part-part-total** (components); and
- (c) multiplication and division as **combining and partitioning**.

In this unit, we cover more of these meanings notably:

- (a) addition, subtraction, multiplication and division as **inverse** of normal action;
- (b) addition and subtraction as **additive comparison** (having more and having less/fewer); and
- (c) multiplication and division as **multiplicative comparison** (having times more and having times less).

As well, the unit will cover, in a limited way, two other more complex meanings, namely:

- (a) multiplication and division as **factor-factor-product**; and
- (b) multiplication and division as **combinations**.

### Background information

In this section, we look deeper into the other meanings that also lead to some or all of the four operations. These include inverse and comparison, and, for multiplication and division only, combinations. We will use set, and number line models for all operations and array and combinations as extra models for multiplication and division. As for Unit 2, the representations used are stories/situations, acting out, modelling with materials, drawings, language and symbols.

### Inverse

If we have a story such as *Fred had two books and Jan gives him 3 more, how many books does Fred have?*, we can see that BAE is as follow and has part-part-total as follows:

BEGINNING Fred with 2 books	ACTION get 3 more books	END How many books does Fred have now
PART	PART	TOTAL?

Thus we have an action that increases the number of books and the operation is addition  $2+3=5$ . However, if we move the unknown from End to Beginning, we have the following story *Fred has some books and Jan gives him 3 more, Fred now has 5 books, how many did he have at the start?*, and BAE/part-part-total as follows:

BEGINNING Fred has some books	ACTION gets 3 more books	END Fred has 5 books, How many at start?
PART	PART	TOTAL

We now have an action that increases but the unknown at start means that it decreases End to Beginning. Thus it is subtraction  $5-3=2$ . Thus for this new meaning, we enlarge the focus of the BAE from what is the action to also include where is the unknown. We can see that with unknown at start, addition becomes subtraction and multiplication becomes division.

However, there is one more thing – the unknown can also be in the action *Fred has 2 books, Jan gives him some more, Fred now has 5 books, how many did Jan give him?*, again we see this is subtraction  $5-2=3$ . This means

that unknown anywhere but at End means operation is inverse to action. Part-part-total overcomes this problem by the rule that unknown is total means operation is as per action and unknown is part means operation is inverse of action.

Note: Inverse can be understood in terms of change as provided in module A2. We can rethink BAE in terms of arrow math as below:



If the unknown is at the end then the operation is applied to the beginning and is straightforward. If unknown is at the beginning, then to find it requires the arrow to be reversed, that is, the **inverse of the action** has to be followed – this multiplication becomes division and so on.

## Comparison

Starting with module O1, we have tried to relate operations to situations where there is a Beginning, Action and End. However, there are some meanings of operations for which students have to visualise the action as it is not obvious in the story. For example, *There were 3 cats and 4 dogs, how many animals?*. The visualisation that goes with this problem takes the cats and dogs and joins them into one group.

Similar to the above, comparison has no obvious action but again can be easily visualised. An example of comparison more could be *John has 4 dollars more than Frank, Frank has \$7, how much does John have?* A visualisation is that of joining the \$4 to whatever Frank has. Then the \$4 is joined with the \$7.

There are three types of comparison: (a) numerical – 24 is larger than 8; (b) additive – 24 is 16 more than 8; and (c) multiplicative – 24 is 3 times as many as 8. The comparison used for meanings in addition and subtraction is additive while the comparison used in multiplication and division is multiplicative (e.g. *John has 4 times as much money as Frank, Frank has \$7, how much does John have?*).

## Factor-factor-product

This is similar to part-part-total and enables all multiplication and division to be covered by determining whether the unknown is a product or a factor. Factors and products are a little advanced for Year 2 so we will not make it part of what we study in AIM EU. However, this approach can be usable in lower grades if we restrict ourselves to products – we can have the simple rule: product/total is wanted means multiplication and product/total given (not wanted) is division. Examples of this are:

Sue bought 5 bottles, each cost \$4, how much did she have to pay?

The 5 bottles and the \$4 are factors, the total is unknown, so this is multiplication:  
 $5 \times \$4 = \$20$

Sue bought some bottles, each cost \$4, she spent \$16, how many bottles did she buy?

The total is \$16 and is given, \$4 is a factor, so this is division as product is known:  
 $\$16 \div \$4 = 4 \text{ bottles}$

## Combinations

A combinations multiplication example has two sets of things being combined to form something else. For example, 3 shirts and 4 trousers enable 12 outfits, or 4 m by 3 m means 12 m<sup>2</sup>. This is a good activity for young children as it involves dressing, for example a bear, with different outfits as below:

Shirt1/Pants1	Shirt 1/Pants2	Shirt 1/Pants3	Shirt1/Pants4
Shirt2/Pants1	Shirt 2/Pants2	Shirt 2/Pants3	Shirt2/Pants4
Shirt3/Pants1	Shirt 3/Pants2	Shirt 3/Pants3	Shirt3/Pants4

The combinations meaning is important in upper primary and secondary. Normally, multiplication is initially taught as number by rate (e.g.  $4 \times 3$  is 4 rows x 3 objects per row, which gives 12 objects). To multiply number x number is combinations. For example, a spinner with 4 options and a die with 6 options produces  $4 \times 6 = 24$  outcomes – so we are able to have examples that multiply number by number but what the product (number x number) refers to is different to what the numbers refer.

The topic for which this meaning is most important is **area**. Here length x length = area (e.g.  $5\text{m} \times 3\text{m} = 15\text{m}^2$ ). The area model as a way of considering multiplication and division is widely used. It is also used as a teaching method for some quite major areas, including basic facts, the multiplication algorithm, multiplication of fractions and multiplication of algebraic expressions (including factorisation).

Finally, combinations also covers division in terms of being inverse of multiplication. For example, we can handle problems such as *4 shirts and 20 outfits, how many pants?*.

### 3.1 Inverse meaning

1. Recap the part of module A2 on function machines – particularly what happens when know output and want input. Now relook at BAE in terms of input/output as below:



Discuss BAE as an input-action-output – how do we get answer – we go backwards and use inverse. Look at situations, for example, A is joining and unknown is in Beginning. Discuss options and relate to A2. Realise have to go backwards. This means that joining becomes separating and it is possible to have take-away addition as long as unknown is at start. For example, *John has 5 pencils and June gives him another 4. John now has 9 pencils* is joining and addition ( $5+4=9$ ). However, unknown as start changes it as the following example shows, *John has some pencils, June gives him another 4, he now has 9 pencils, how many did he have at the start?* It now is subtraction ( $9-5=4$ ).

2. Look at the above in a different way. BAE for addition is two parts joining to get total. If unknown is at beginning, it means a part is unknown, so by part-part-total the operation is subtraction.

Thus, stories can be addition or subtraction depending on positioning of unknown. Therefore addition can be take-away and subtraction can be joining.

3. Repeat 1 and 2 above for multiplication and division (using factor-factor-product instead of part-part-total). Experience how inverse works. For example partitioning into equal parts is the division action. However, if unknown is at beginning as follows, *Uncle Fred shared some money equally amongst 4 children, each child received \$12, how much money was shared?*, the story is multiplication.
4. Give a variety of stories, some inverse and some not, to students to experience. Include all four operations. Use a thinkboard to relate stories to acting out with materials, pictures, language and symbols
5. If class is up to it, construct problems (numbers and operation  $\rightarrow$  story instead of story  $\rightarrow$  numbers and operation). To do this call the above Backward stories, where we know the end and want the beginning.

Get students to write these backward stories. Provide operation and action and ask students to do a backwards story. For example, begin with the idea of a story joining 2 and 3 backwards Ask this to be backwards and inverse, that is, addition by subtraction backwards. An example of such a story is, *There*

were some cows, the farmer took two away, there were three left, how many to start with?. As can be seen, the problem is solved by addition but the story is subtraction backwards; we can see that addition can be take-away by going backwards. Other examples are:

- (a)  $8-3$  backwards – which is to do addition backwards. *There were 3 cows in the paddock, the farmer put some more cows in with them. This made 8 cows – how many cows were added?* The actions and the words are joining but the result is a story solved by subtraction.
- (b)  $24\div 4$  backwards – which is to do multiplication backwards. *There were 4 paddocks. Each paddock had the same number of cows. There were 24 cows overall. How many in each paddock?* The actions and the words are multiplication but the answer requires division.

**Very special note.** If it can be done, construction is the strongest way to teach inverse.

## 3.2 Comparison meanings

1. Start with additive comparison – look at a variety of stories for addition and subtraction. Here are some examples

- Addition – *Bill has 4 more pencils than Jack, Jack has 3 pencils, how many has Bill?* [7 pencils]
- Subtraction – *Fred has 4 pencils fewer/less) than Anne. Anne has 6 pencils, how many has Fred?* [2 pencils]

Act stories like these out to see how they work – compare with normal BAE operations – look for differences and similarities. Encourage students to see the following difference. For normal joining (say  $3+5$ ), we start with 3 objects and 5 objects and join to make 8 objects; for comparison (say 5 more), we start with 3 objects and another group that has 5 more than 3 (i.e. 8 objects), so we end up with two groups, one of the initial 3 and the other of 3 and 5 joined = 8.

So three differences: (a) the answer is the amount in one group [ $3+5=8$ ], (b) there are two groups not one at the end and the answer lies in the bigger of these, and (c) the number of objects on the table is not 8 but 11 because we are left with the start 3 and the answer ( $3+5$ ) which gives 8.

2. Repeat the above for multiplicative comparison – stories for multiplication and division. Here are some examples.

- Multiplication – *Bill has 4 times as many pencils as Jack. Jack has 3 pencils, how many has Bill?* [12 pencils]
- Division – *Fred has 4 times less pencils than Anne. Anne has 20 pencils, how many has Fred?* [5 pencils]

Look at differences re multiplication and division in action and multiplication and division in comparison. They should be three differences similar to those in 1. Above.

3. Inverse (see 3.1) also affects the additive comparison situation. If we have the story, *Fred has 4 more than Jack*, we expect Fred's amount to be Jack's amount with 4 added (and the story to be about addition). However, this is only true if Jack's amount is known; if Fred's amount is known, say 7 objects, then to find Fred's amount we have to subtract.

Act out various comparisons looking at effect of inverse. This will give rule for *A has X more than B*. The rule is that the story is addition if B is known and subtraction if A is known.

Repeat this for multiplicative comparison examples and encourage students to arrive at a similar rule for stories of the type *A has X times as much as B*.

4. Understanding the effect of inverse on comparison situations is important. Many students come to word problems believing that use of certain words determines meaning. Particularly this is so for “more” and addition. “More means add” is a common refrain but it should not be supported because, using inverse,

we can write problems where more is subtraction, for example, *Ted had 3 more than Sue. Ted had 7. How many did Sue have?*


Likewise, in subtraction, comparison, we talk about Jen having, say, 4 less than Del. So, if Del has 8, Jan has 4. However, we can also say that the **difference** between Del and Jan is 4. Difference is a common way to talk about subtraction, particularly in comparison situations. Time should be given to acting out situations involving difference, exploring the effect of inverse, ensuring difference is understood, and rules such as below are known, that is, if the difference between A and B is X, then A larger than B means that  $A = B + X$ , while A smaller than B means that  $A = B - X$ . It is easy for difference situations to reflect addition or subtraction.

### 3.3 Factor-factor-product

1. Experience multiplication situations and stories – act them out – get students to see that there are always three parts: the number of groups (say 4), the number in each group (say 3) and the multiplication of this two ( $4 \times 3 = 12$ ). The number of groups and the number in each group are called factors while the multiplication is called the product. Look at numbers such as 3, 4 and 12 – 3 and 4 are factors and 12 is the product because 3 and 4 divide into the product and 12 is the result of multiplying.
2. Like 2.4, the secret of factor-factor-product is to use the words as early as possible and keep referring to them. For example, always ask which number is a factor and which is a product whenever an opportunity arises. Start doing this way back when first introduce the idea of BAE – provide experiences that show that 4 bags and 5 lollies per bag means 4 and 5 are factors and 20 is product. Similarly use inverse for division. For example, 15 shared amongst 5 is 3 to each – 5 and 3 are factors and 15 is product.
3. Construct factor-factor-situations. Move backwards and forwards – “4 sets, 6 in each” – “the 4 and the 6 are factors” – “I will join them all to make a product (24)” – “I will now start with 24, make 4 groups of 6” – “the 4 and 6 are factors” – “now you do the same with 20, 4 and 5?”. Ask the children to form products from factors and vice versa – ask other children to identify factors and identify product.

Note: We have two things to reverse here: (a) factors  $\leftrightarrow$  products, and (b) interpretation  $\leftrightarrow$  construction.

4. Use factor-factor-product to solve problems – particularly inverse. Regardless of the actions and language of the problem, children need to look at the problem in terms of factor-factor-product. Regardless of actions and woes, if the product is the unknown, the problem is multiplication; if a factor is unknown, the problem is division. For example:

ANALYSE STORY AND DETERMINE PRODUCT		PRODUCT KNOWN – IT IS DIVISION
Sue bought some bottles, each cost \$4, she spent \$16, how many bottles did she buy?		The product is \$16 and is given, \$4 is a factor, so this is division: $\$16 \div \$4 = 4$ bottles

5. Use the part-part-total diagram from 2.4 but call it a factor-factor-product diagram.

### 3.4 Combinations meaning

1. For multiplication, the most common understanding is “lots of” or “rows of” or “jumps of” where multiply a number by a rate (“three bags at 5 lollies/bag”, 4 jumps at 2m per jump) – however there is number by number multiplication in situations similar to arrays.
2. Experience by acting out one of the earlier number x number multiplications – making outfits. Make 2 different coats and 6 different pants for a doll (or a cardboard cut-out doll). This will mean 12 combinations (2 coats by 6 pants is 12 outfits). This number by number is identifiable because the answer/product is different things to the factors. (Here factors are coats and pants and product is outfits.

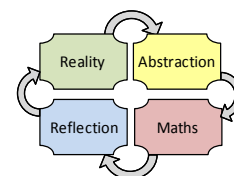
Red coat						
Blue coat						
	Red pants	Blue pants	Green pants	Yellow pants	Orange pants	Pink pants

3. Encourage students to construct combinations – think of two things that interact like colour and rectangles. Have 3 colours and a square and a rectangle – make up different ways they can interact – a colour for a rectangle type.
4. Experience forming rectangles out of square tiles –place in arrays to see how number of rows x number of columns = area. Try to set up the area model for multiplication. Can do this for basic facts – that  $3 \times 4 = 3 \times 2 + 3 \times 2$ .

### 3.5 RAMR Lesson outline on Inverse for addition and subtraction

#### Reality

Make a list of as many words as the students can think of for addition and subtraction that they use every day. Ensure that this list is flexible – sometimes, the same word may be used in different ways for addition and subtraction.



#### Abstraction

1. Take each of these words for addition (e.g. “more”) and act out their normal meaning with two knowns (parts) and one unknown (total).
2. Now give the unknown total a number, and act out the problem with one of the other numbers unknown – students will find that it changes addition to subtraction or vice versa (as in RAMR lesson above). For the example, “more means addition”, the following could eventuate:

Original problem: I had 4 books, I got 3 more, how many books do I have?

This is **addition** because part = 4, part = 3, total = unknown

New problem: I had 4 books, I got some more, now I have seven books, how many more did I get?

This is **subtraction** because part = 4, total = 7, part = unknown

3. Draw diagrams of the two problems.
4. Repeat the above for a subtraction word like “take-away”. Once again students will see that it is possible to have take-away addition, e.g. “I took \$7, this left \$21, how much to start with?”
5. Draw diagrams.
6. Relook at the situation in terms of forward and backward – students should notice that the problems that follow the “normal” use of the words are forward while the problems that give the opposite operation are backward.
7. Look at the drawings – act out the problems – students should see that running addition backward is subtraction and vice versa.

8. Reverse the situation – we start with addition such as  $2 + 3 = ?$  This is problem “2 boys join 3 boys, how many boys playing?” Now think – this is also  $? - 2 = 3$  backwards, that is “2 boys left the group, 3 boys remaining, how many to start with?”.

## Mathematics

Practise backwards problems – give a sum like  $6 + 9$ . Write it as joining forward (e.g. 6 cats on a fence, 9 more join them, how many cats?) and take-away backward (e.g. there were some cats on a fence, 6 jumped off, 9 remained, how many cats to start with?)

Do similar for a subtraction sum. See how this relates to parts and total and which is unknown, and to forward and backward. Write the sums vertically as well as horizontally.

Practise sum  $\rightarrow$  problem and problem  $\rightarrow$  sum for inverse meaning. (Note: Write a problem for three numbers. Then can have three problems, where each of numbers are unknown. This will always give 2 subtractions and one addition).

## Reflection

Try to generalise the process. For example, if addition, how do we change that to subtraction pattern? Also this change from addition to subtraction can be done on symbols – see on right

$\begin{array}{r} 47 \\ + 34 \\ \hline ? ? \end{array}$	$\rightarrow$	$\begin{array}{r} ? 7 \\ + 3 ? \\ \hline 81 \end{array}$
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Look back at module A2 and the function machine – see inverse section of background information for this unit.



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## Unit 4: Word Problem-Solving

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In the preceding units we have explored the various meanings of the four operations. The reason for this is to be able to solve problems by translating the problem to an operation that can be calculated (the focus of *Module O3 Calculation* that follows this module). Thus, the basis of this unit for solving problems is:

story/problem  $\leftrightarrow$  meaning/numbers  $\leftrightarrow$  operation

where  $\leftrightarrow$  represents the ability to move in both directions.

This module will therefore do the following:

- (a) explore how the different ways of modelling a problem/situation enable the appropriate operation to be chosen;
- (b) use construction to deepen knowledge of problem  $\leftrightarrow$  operation;
- (c) use the triad big idea to further deepen knowledge of problem  $\leftrightarrow$  operations;
- (d) extend solving from one-step to multi-step operations; and
- (e) do the above in relation to SEE–PLAN–DO–CHECK.

The focus of this module is very much the problem  $\leftrightarrow$  operation relationship based on knowing the meanings of operations. However, we need to remember that there is an encompassing big idea, Polya's SEE–PLAN–DO–CHECK (SPDC) approach to problems that may assist, plus a collection of strategies, thinking skills and metacognitive ideas that are the basis of problem-solving in general.

### Background information

The focus of this unit is:

problem  $\leftrightarrow$  meaning  $\leftrightarrow$  operation

Thus, simply, the focus of this unit is the ability to **replace a problem with an operation from which a solution can be calculated**. So we use the meanings of the operations to work backwards operation  $\rightarrow$  problem after we know all the ways operation  $\leftrightarrow$  meaning.

### Process for obtaining operation

A complete process for obtaining operation involves three steps:

1. Determine from the problem whether it requires multiplication/division or addition/subtraction: multiplication/division has one of three numbers referring to groups and not objects; addition/subtraction does not – all numbers refer to objects.
2. Having determined it is multiplication/division or addition/subtraction, determine which of the two it is: multiplication is product unknown and addition is total unknown, otherwise it is division or subtraction.
3. Translate problem to operation: operations are  $P + P = T$  or  $T - P = P$  for addition/subtraction; and  $f \times f = P$  or  $P \div f = f$  for multiplication/division.

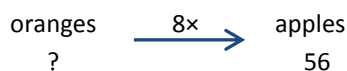
Having determined the correct operation, students can then perform the calculation (see Module O3). An example may help in seeing this three-step process:

**Problem:** *There are 8 times as many apples in a box as oranges. There are 56 apples, how many oranges?*

**Step 1:** Number of apples is 56, unknown number is number of oranges. 8 is number of times/number of groups. So two numbers are objects (fruit) and one is not, so the operation must be multiplication or division.

**Step 2:** In the problem, the 8 is a factor, the 56 is the product and the answer is the number of oranges whose number  $8 \times$  would equal 56. The product is known so the operation is division.

**Step 3:** Looking at the problem in terms of the change idea from Module A2:



The operation is  $56 \div 8$  oranges.

**Note:** Another good way to learn steps 1 to 3 is to **model** them – that is, act out the problem and draw it. So a thinkboard is still useful (see section 4.1 below)

## Essential knowledge

The above three steps can be achieved by three pieces of knowledge:

1. The **difference** between multiplication/division and addition/subtraction: multiplication/division has equal groups and one number focuses on the number of groups.
2. **Part-part-total** for addition/subtraction and that addition is when the total is unknown.
3. **Factor-factor-product** for multiplication/division and that multiplication is when the product is unknown.

However, it helps to know all meanings:

	Addition/Subtraction	Multiplication/Division
Action	join/take away	combine/partition
Comparison	more/less	times more / times less
Inverse (unknown beginning)	take away/join	partition/combine
Components	P-P-T	f-f-P
Combination	—	combinations/options

## Construction and triads

The best way to learn steps 1 to 3 is to **reverse** them – start with an operation and **construct** a problem. This will give experience in the different ways problems/stories can be created. A good way to begin to experience different problems is as follows. It enables three different problems for each situation.

It works best if you are given three numbers, say 12, 3 and 15, and asked to write different problems. You begin by writing a straightforward problem with all the numbers written in. This gives a problem like below:

*Jenny had 12 bowls. She was given 3 more. She now has 15 bowls.*

Then use the **triad big idea**, which is that if a relationship has three numbers, then there are three problems – one for each number as the unknown. This gives three problems based on the initial written problem.

**Problem 1** (12 unknown): *Jenny had bowls, was given 3 more, now she has 15, how many did she have at the start?* [Problem is subtraction  $15 - 3 = 12$ , with meaning inverse of join]

**Problem 2:** (3 unknown) *Jenny had 12 bowls, was given more, now has 15, how many more was she given?* [Problem is subtraction  $15 - 12 = 3$ , with meaning inverse of join]

**Problem 3:** (15 unknown) *Jenny had 12 bowls, was given 3 more, how many does she have now?*

[Problem is addition  $12 + 3 = 15$ , with meaning comparison]

Thus we have constructed three different problems, which together represent all the options..

## Other knowledge

Other problem-solving knowledge can also be used, particularly the SEE–PLAN–DO–CHECK plan of Polya. For the problem “*There are 8 times as many apples in a box as oranges. There are 56 apples, how many oranges?*”, this helps the previous three-step analysis.

<b>SEE</b>	We try to understand what is being asked – use change: <div style="text-align: center;"> <math>\begin{array}{ccc} ? &amp; 8\times &amp; 56 \\ \text{oranges} &amp; \longrightarrow &amp; \text{apples} \end{array}</math> </div>
<b>PLAN</b>	Can use inverse: $56 \div 8$ gives oranges Can see $8\times$ means multiplication/division and the unknown is a factor, so use $\div$
<b>DO</b>	Do the division: $56 \div 8 = 7$
<b>CHECK</b>	Fit the numbers into the problem: <div style="text-align: center;"> <math>\begin{array}{ccc} 7 &amp; 8\times &amp; 56 \\ \text{oranges} &amp; \longrightarrow &amp; \text{apples} \end{array}</math> </div> <i>Yes! We get right number of apples for 7 oranges.</i>

SEE–PLAN–DO–CHECK is a plan of attack for problem-solving that supports problem-solving thinking for problems other than those that come from operating and calculating. It is based on metacognition and supported by thinking skills, strategies, strong affective traits and structural domain knowledge.

**Note:** Download a copy of the *YDM Problem-Solving* book from the Blackboard site that supports YDM training and make yourself aware of the strategies and their role in problem-solving.

## 4.1 Representations in word problem-solving

1. As stated in the background information, the steps to solving word problems are:
  - (a) Step 1: determine if the problem requires multiplication/division or addition/subtraction using the difference between how numbers are used in addition/subtraction and multiplication/division;
  - (b) Step 2: determine which of multiplication/division or addition/subtraction is required using the meanings of each; and
  - (c) Step 3: translate the problem to an operation and then to a calculation.

Students need to practise this three-step process, so provide students with a set of problems and have them apply the three steps.

2. The basis of this three-step process is that it involves stories, acting out, modelling with materials and drawings, as well as language and symbols. To build ability with the three steps, students need to become expert with these 6 representations: (a) stories, (b) acting out, (c) modelling, (d) drawing, (e) language, and (e) symbols. This number can be reduced to 5 representations by considering acting out and modelling as one representation.
3. Experience all these representations as they relate to each other, for example,

take a story (a problem in a context)  $\rightarrow$  act it out or model it with materials  $\rightarrow$  then with diagrams  $\rightarrow$  finally determine the operation.

Repeat this sequence for a variety of problems.

4. **Note:** As will be evident as we complete O2 and move onto O3, the sequence in 3 above is the first half of an overall sequence across O2 and O3: O2 is story → representations → operation; while O3 is operation → calculation → answer.

5. So we have:

**First sequence:** O2 Story → representations → operation/symbols

**Second sequence:** O3 symbols → strategies → calculations

The first sequence is the basis of this Module O2 and the second sequence is the basis of Module O3.

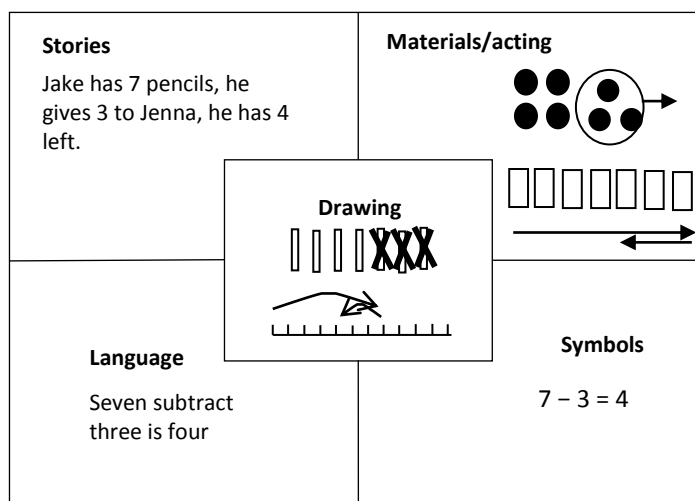
## 4.2 Exploring and extending the thinkboard

We have used a method in this module that covers the five representations in one process by using a thinkboard. In this sub-section we look at how to teach and use thinkboards.

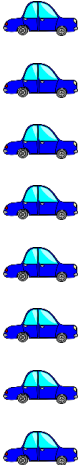
1. The thinkboard needs careful introduction.

- Begin with using four quadrants: language, materials/acting, drawing, symbols/calculator. Folding a piece of paper into quadrants for a group of four to work together makes each problem task less arduous for students. Initially, the teacher provides the “story” problem on a card. Give each student a different role: language, drawing, materials, symbols/calculator, to complete the thinkboard. Ensure every student has the opportunity to try all four roles. At this stage the drawing quadrant is a creative expression of how the student thinks about the maths. Nearly all children connect with this mode and it will frequently engage even the most reticent student.
- Next have students write their own number stories separately on cards. Allow plenty of opportunity for each student to say and compose and write their own word problems. Share these among the groups to show solutions on their thinkboards.
- Introduce drawing models, e.g. tally marks, set, line, to represent drawing word-problem thinking.
- As the students become more confident writing and representing their own word problems, introduce the fifth box on the thinkboard – stories.
- Finally, to encourage students to be able to move from any representation to another, provide cards with different parts of the thinkboard problem-solving illustrated, give each group one and ask them to complete the thinkboard for the rest of the problem.

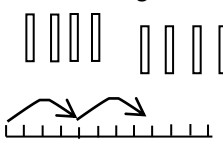
**Note.** Do not assume students can read equations. Students need opportunity and practice to read equations, also writing and reading the different forms of equations:  $7 - 3 = 4$ ;  $4 = 7 - 3$ ;  $7 - 3 = ?$ ;  $7 - ? = 4$ ;  $? - 3 = 4$ .



2. It is important to ensure that:
  - (a) all models are covered – set and number line for addition/subtraction, and set, number line and array for multiplication/division;
  - (b) all connections are both ways – students can write a story for language or symbols, and can interpret a drawing in a story or symbols;
  - (c) stories are used for a variety of situations – shopping, sporting, fishing, driving, TV stories, and so on; and
  - (d) students understand that operations are generic:  $3 + 4 = 7$  means that 3 fish and 4 fish gives 7 fish, \$3 and \$4 gives \$7, 3 m and 4 m gives 7 m and so on. Thus,  $3 + 4 = 7$  holds for every set of objects and every measure in the world.
3. The above steps should be done for addition/subtraction. How far to go with the thinkboard in multiplication/division is up to you the teacher. However, we should include an early multiplication/division example such as: *Joe has 4 toy cars. Taylor has twice as many cars. How many cars does Taylor have?*



<b>Story</b> Joe has 4 toy cars. Taylor has twice as many cars. How many cars does Taylor have?	<b>Materials/acting</b> <div style="display: flex; justify-content: space-around; height: 100px;"> <div style="border: 1px solid black; border-radius: 50%; width: 80px; height: 80px;"></div> <div style="border: 1px solid black; border-radius: 50%; width: 80px; height: 80px;"></div> </div>
<b>Language</b> Twice as many as four is eight Two times four is eight	<b>Symbols</b> $4 \times 2 = 8$

**Drawing**  


### 4.3 Constructing word problems

One of the best ways to understand and interpret word problems is by constructing them. In other words, going from symbols  $\rightarrow$  story, rather than story  $\rightarrow$  symbols. In this section we look at understandings that underlie this idea.

1. **Identifying the parts in a word problem.** The same operation can be said and written in different ways, so it is useful to explore this.
  - (a) Mathematics is written concisely. This makes it different from and in some ways more difficult to read and write than narrative forms of text. Students need to practise moving between the reduced symbolic forms and the various alternative everyday language forms, both oral and written. For example,  $12 - 5$  can be expressed as: *twelve take away five; twelve subtract five; the difference between twelve and five; 5 from 12;* and so on. Or  $6 \times 4$  can be expressed as: *six multiplied by 4; six times 4; the product of 6 and 4; four lots of six; four groups of 6;* and so on.
  - (b) Thus we begin by focusing students' attention on words that describe the actions they use when doing; e.g. putting away materials/toys/equipment: *put with, took out, added, missing, two more,*

another, as well as, separate, subtract, plus, and, the same, make equal, make the same, and so on. However, key words can be deceiving.

2. **What do the words infer?** These two examples infer take away when addition is needed:
  - (a) *Bart ate 6 cookies and David ate 7 cookies. How many cookies did the two boys eat?*
  - (b) *James went to the deli and spent \$2.45. When he got home he had \$2.55 change. How much money did his Mum give him to start with?*
3. So we cannot rely on key words (e.g. “join means add”). **One situation can be represented by different operations.** The “Doorbell Rings” story is an activity that provides an example of this:

[Get doorbell rings story from Robyn]

So is the “How many more” problem:

*Holly said she had 15 marbles and Jesse said he had 11 marbles. Holly said, “I have more than you”. Jesse said, “Not many more”. How many more did Holly have?*

Looking at the “How many more” problem, it can be seen that it can be represented with either an open addition,  $11 + ? = 15$ , or as a subtraction,  $15 - 11 = ?$

4. **One operation can represent apparently different situations.**
  - (a) Here are examples of the subtraction situations that  $9 - 2$  can represent:
    - What’s the difference between 9 and 2?
    - How much more is 9 than 2?
    - What’s left if I take 2 from 9?
    - If I count 2 back from 9 on a number line, what number will I come to?
    - I spent \$2 and I started with \$9. How much do I have left?
    - I have \$2 and I started with \$9. How much did I spend?
  - (b) Ask the students to discuss why all these situations can be represented with subtraction  $9 - 2$ .
  - (c) The various ways  $9 - 2$  can be represented as a story are due to the fact that subtraction has many meanings and two models (set and number line). See if you can connect the stories of  $9 - 2$  to the following meanings and models:
    - taking away
    - changing a quantity so that it is smaller
    - part-part-total: if the quantity is given, then we need to find a part
    - using number lines
    - comparing two quantities
    - separation of a set
    - decreasing a set.
  - (d) It is the same for multiplication and division. Find all the meanings/models of multiplication/division in this module and think of all the ways they could be talked about. Make up a poster.

#### 4.4 Using triad big idea in word problems

1. As we have already stated, the best way to become an expert at interpreting word problems is to learn how to construct them. Constructing word problems so that students experience all types is difficult particularly with inverse. One way to look at doing inverse is to think of them as forward (unknown is End) and backward (unknown is Beginning). We call these forward and backward stories.
2. It is difficult to write stories for a given meaning, and hard at the start to write forward and backward stories. However, it is good practice and sharing ideas is good. So give students an equation such as

$5 + 8 = 13$  and ask them to write forward joining and backward separating stories (and change-comparison and part-part-total problems if appropriate) for set and number-line models and for different day-to-day contexts (e.g. shopping, driving, walking, playing sport, and so on). See 3 below for how to do this.

3. The best way to construct problems and to get some experience of forward and backward stories is to use the **triad big idea**. As all operations have three components, these components form a triadic relationship. If a problem is written with all three numbers showing, it is possible to transform it into three problems by making each of the three numbers the unknown in turn. As we do this, forward and backward stories emerge and it can be seen that these three problems are not all the same operation. See examples in the table below.

PROBLEM (ALL NUMBERS)	THREE PROBLEMS
Sue ran 3 km, then she ran another 4 km, altogether she ran 7 km. [addition, $3+4=7$ , forward]	Sue ran some km, then she ran another 4 km, altogether she ran 7 km, how many km in the first run? [subtraction: $7 - 4 = 3$ , backward]
	Sue ran 3 km, then she ran some more km, altogether she ran 7 km, how many extra km did she run? [subtraction: $7 - 3 = 4$ , backward]
	Sue ran 3 km, then she ran another 4 km, altogether how many km did she run? [addition: $3 + 4 = 7$ , forward]
There were 8 boys playing, 3 ran away, there are now 5 boys playing. [subtraction $8-3=5$ , forward]	There were some boys playing, 3 ran away, there are now 5 boys playing, how many at the start? [addition: $3 + 5 = 8$ , backward]
	There were 8 boys playing, some ran away, there are now 5 boys playing, how many ran away? [subtraction: $8 - 5 = 3$ , backward]
	There were 8 boys playing, 3 ran away, how many are left playing? [subtraction: $8 - 3 = 5$ , forward]

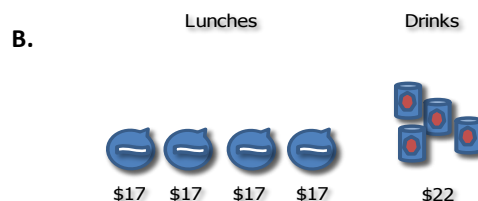
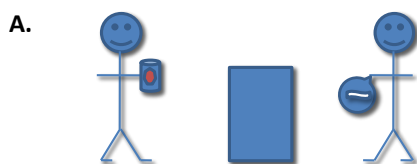
4. Use all methods to build a repertoire of problems. As already stated, the same operation can be said and written in different ways. Have different starts, e.g. write an action story or a comparison story or a part-part-total story. Then go and write triads to get three for every story you thought of. Try to use a variety of words.

## 4.5 Introducing multi-step problems

Of course problems often have more than one step, so this section looks briefly at multi-step problems.

1. Introduce problems with more than one step: e.g. *Julie bought a sandwich for \$8.50 and an orange juice for \$4.20. How much change did she get from \$20?* This has two steps: (a) add the food ( $\$8.50 + \$4.20 = \$12.70$ ), and (b) subtract this total from the money given ( $\$20 - \$12.70 = \$7.30$ ).
2. Multi-step problems need strategies and plans of attack. For simpler two- and three-step problems, there are six particularly useful **strategies**. These are listed below. They are described using the problem: *I bought 4 lunches for \$17 each and drinks for \$22. How much did I spend?*
  - (a) **Break problem into parts.** Do not try to do it all at once; look at the problem as a series of steps – for this problem there would be two steps, the 4 lunches and the drinks.
  - (b) **Be systematic/exhaust all possibilities.** This means use tables or charts to check you have covered everything – list the purchases, 4 lunches and 4 drinks and ensure everything is accounted for (e.g. were the drinks each or a total?) – not complicated in this example but helps in larger problems such as planning a vacation.

- (c) **Make a drawing/act it out.** Get students to make drawings (act it out if this helps) and discuss with them which is the most useful picture and why. For example, B is the most useful drawing below to solve the problem.



- (d) **Given, needed and wanted.** Get students to identify these three things and record:

- Given: 4 lunches, \$17 for each lunch, \$22 for drinks
- Needed:  $4 \times \$17$  to work out lunches
- Wanted: the total amount (food and drinks)

- (e) **Restate the problem/Make it simpler.** Think of it in an easier way and write it down – make the numbers smaller if this helps, work out what to do with these smaller numbers, then replicate this with the larger problem. Think: How could I make the problem easier for a student without giving them the answer? For example, “Work out what 4 lunches at \$17 each is and add that answer to \$22 for drinks to get how much you spent.”

- (f) **Check/learn.** Checking our assumptions: Deciding when using an operation makes sense. Check the answer, reflect on solution and learn from it for later problems.

3. The best **plan of attack** – a metacognitive process for attacking the problem – is still Polya’s four stages (using the SPDC framework):

*See* – spend time simply working out what the problem is – understand the problem.

*Plan* – make up a plan based on known strategies and knowledge to tackle the problem.

*Do* – implement the plan, see if it works and calculate an answer.

*Check* – check the answer for reasonableness and see what you can learn from what you have done.

Other frameworks do exist (though not as general). The following have been found useful.

- (a) **Extension of part-part-total.** The diagram used for one-step problems can be extended. For the example, *I bought 4 lunches at \$17 and drinks for \$22. How much did I spend?* it becomes as on right.

What spent				
4 lunches				\$22
\$17	\$17	\$17	\$17	

- (b) **Working page.** The diagram on right is a working page that YDC staff have found useful. It is based on strategies 2 (c), (d) and (e) above. Use the page to solve the following problems:

- A koala was climbing a 10 m tree. He climbed up 3 m and back 2 m each hour. How many hours to the top of the tree?
- Fred set out on a 90 km walk. On the first day he walked 16 km, the second 32 km and the third 23 km. On the fourth day he finished the 90 km walk and then walked another 11 km to the motel. How far did he walk altogether on the fourth day?

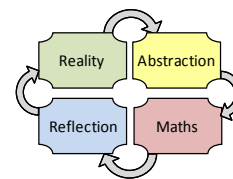
Problem:	
Drawing:	Given: Needed: Wanted:
Restating:	
Working:	

## 4.6 RAMR lesson outline on representing and solving operating situations

**Learning goal:** Students will learn to represent and solve operating /problem-solving situations in multiple ways-Part part whole; number line; tallies; equation; words; calculator; materials

**Big idea:** Symbols tell stories, BAE, Operation meanings (including strategies to assist calculations and solve problems (ACMNA053)

**Resources:** Manipulables, counters, MAB's, blocks, straws, paddle pop sticks, post it's; books and pencils; calculators



### Reality

**Local knowledge:** Choose a number you are comfortable to work with/ is part of your environment/ means something to you. Why are numbers useful?

**Prior experience:** Counting, working with a range of small and large numbers, telling number stories, problem-solving strategies (risk taking, resilience)

**Kinaesthetic:** Post-its/ squares number activity: Write your number on a post it. Find a partner whose number has some relationship to your number. Be prepared to read your number and tell the group how they are related.

### Abstraction

**Body:** With your number choose a partner to make a number story or problem using your two numbers, Do this orally and act out several scenarios.

**Hand:** construct your problem and solve using a range of materials

**Mind:** visualise your number problem. Can you see how big the whole is? Can you see the parts? Can you think of a way to show your solution?

**Creativity:** Freedom to choose and express their own problem with any operation, materials, model, tool

### Mathematics

**Language/symbols:** words for addition; words for subtraction; words for multiplication; words for division; solution; total; is the same as; equal to; missing part; known part; useful information; un-useful; equations information; part-part-whole; number line

**Practice:** Construct a Thinkboard to solve problems: write a problem; draw a solution; use a model; etc. Write a problem for your partner to solve. Compare problem types. You can do this activity with pairs sharing/swapping as well as within pairs.

**Connections:** Use a calculator; basic facts; algorithms; equations; problem-solving; basic facts and computation

### Reflection

**Validation:** Share each other's thinkboards.

**Application/problems:** Solve other students' word problems.

**Extension:**

- **Flexibility:** Play: Make 17; Make 77; Make 177; Play make 204; Take (roll 1,2, or 3 dice) to return to zero; Do with other multiples
- **Reversing:** Here is the whole, what are the parts? E.g. the solution is 78, what is the question?
- **Generalising:** The difference between two numbers is 5. What might the two numbers be? (This focuses on 'difference' as subtraction. Some students will feel confident using larger numbers. Some may record their answers systematically e.g. 6-1, 7-2, 8-3, 9-4 etc.

Make up some different ways to add 9 and 13 in your head. In how many ways can you do it? (E.g. add 7 to 23 to make 30 and then add the other 2 to make 32. Who does it by adding 10 then subtracting 1?)

*I have some marbles. I give some away to my friends and am left with 15. How many marbles might I have started with and how many might I have given away?* (Again look for a range of answers. Note the size of the numbers students are confident to work with)

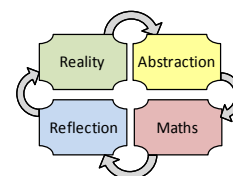
- **Changing parameters:** Make the numbers bigger.

## 4.7 RAMR lesson outline for addition and subtraction construction

**Learning goals:** (1) To construct problems from symbols and to gain a deeper understanding of operations. (2) To get greater insights into interpretation of problems.

**Big idea:** Construction vs Interpretation, Symbols tell stories

**Resources:** A variety of objects for acting out/modelling



### Reality

**Prerequisites:** Knowledge of story ↔ symbols in straightforward situations, knowledge of meaning of operations.

**Give students an equation** such as  $5 + 8 = 13$  and ask them to come up with different problem stories (forward joining, backward separating, change-comparison, and inaction) for set and number-line models and for different everyday contexts (e.g. shopping, driving, walking, playing sport, ...).

**Social interaction roles.** Set students in groups of three to make up and act out stories by assigning roles of director (leader – makes decisions when there is an impasse), continuity (continuously checks for any errors), and script writer (records and reports on the story and how it will be acted).

### Abstraction

**Materials.** Give students materials to work with (e.g. toy models of people, each other, set up a shop) and ask them to make up and act out their story. (One of the best teaching methods for this approach was by a teacher who organised the students to do a claymation of their story.)

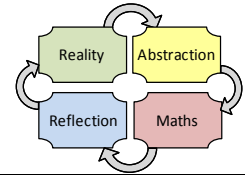
### Mathematics

**Triad approach.** If given  $5 + 8 = 13$ , write a straightforward joining story with all numbers known, then rewrite with one unknown. This will give three stories, one for  $5 + 8 = ?$ , one for  $? + 8 = 13$ , and one for  $5 + ? = 13$ ; the first will be forward and the last two will be backward. After this, write the associated subtraction stories for  $13 - 5 = 8$  and  $13 - 8 = 5$  and then rewrite each of these with one unknown. The students can be encouraged to see that (a) each equation will give three problems, one for each unknown; (b) 13 unknown means addition is the operation to get the answer; and (c) right-hand side unknown means problems forward, otherwise problems backward.

**Use part-part-total.** If given  $5 + 8 = 13$  and asked for a comparison, the 5 and 8 are parts so they have to be the initial number and the increase while the 13 is the large number and is unknown. So have a start of 5 and have an increase of 8 and then find the end. Then write this into context as a story.

## Reflection

**Extend an existing problem.** (1) Give students a problem, then ask the students to add further words to the problem and change the context of the problem to make it harder/easier. (2) Generalise the understandings from mathematics above by examining how changing the component (part, part or total) that is unknown changes the type of problem. For each word problem, act out the normal meaning with two knowns and one unknown, then give the unknown a number, and act out the problem with one of the other numbers unknown – does this change addition to subtraction or vice versa? When is the problem addition/subtraction?



## 4.8 RAMR lesson structure for operations

<b>Reality</b> → <ul style="list-style-type: none"> <li>Acting out problems</li> <li>What problems do we have to solve from day to day?</li> <li>Can you tell me a problem story?</li> <li>What maths tools might help solve problems?</li> </ul>	<b>Abstraction</b> <ul style="list-style-type: none"> <li>Models that help explore operations</li> <li>Set model</li> <li>Draw a picture</li> <li>Length (Number line) model</li> <li>Area model</li> </ul>
↑ <b>Reflection</b> <ul style="list-style-type: none"> <li>Thinkboard to enable students to make their thinking visible</li> <li>Solving problems as teams and/ or individually</li> <li>Change the parameters- make the numbers bigger</li> <li>Reversal- here is the answer what are the questions?</li> <li>Generalise:</li> </ul>	↓ <b>Mathematics</b> <ul style="list-style-type: none"> <li>The maths words and how we say these in sentences or maths problems</li> <li>The maths symbols</li> <li>2 or 3 problems to do</li> <li>What other maths ideas does this new idea or skill connect to</li> </ul>

Ensure: All models experienced; All connections on thinkboard go both ways; Stories can be from a variety of situations e.g. sports, fishing, going out, TV, driving, shopping, books, etc.; Operations are generic  $3+4$  holds for very object and every measure in the world.



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## Unit 5: Principles and Decontextualisation

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Operations are a major part of mathematics. They act on numbers of all types, on algebra and many other mathematical forms (e.g. complex numbers, matrices). Therefore they are part of mathematics across the years, they are a commonality in whole numbers, common fractions, decimal numbers, measures of all types and on to algebra. Because of this they offer a powerful framework for big ideas, that is, ideas that hold across year levels and topics.

This framework we have called the operation principles. It enables arithmetic to have the same structure as algebra and is a powerful way to teach mathematics ideas. The framework (or structure) of operation principles and laws is described on page 3 of this module.

This unit looks at the important principles for Years F to 2 and provides ways of teaching them that are compatible with the year levels. However, the power of the principles lies in their application to many mathematics ideas across many years. This means there is a tendency to see them out of context; for example, commutativity (turnarounds) is most commonly defined as  $a + b = b + a$  for any  $a$  and  $b$ . This can lead to problems in the early years because students are often not ready for decontextualisation like this.

So, we must use the principles within context. For example, we can discuss commutativity by using the term “turnarounds” and by placing it in a context, such as: *3 steps forward followed by 2 more steps is the same as 2 steps forward followed by 3 more steps*; or *A group of 3 children joined with a group of 5 children is the same as a group of 5 children joined with a group of 3 children*.

We must take care with these ideas when working in the early years. Even the term “turnarounds” leads to a decontextual definition, such as  $2 + 3 = 3 + 2$ . It can presuppose that students understand 2 and 3 independent of seeing 2 as 2 objects and 3 as 3 objects. This unit will also look at the problem of decontextualisation.

### Background information

#### Principles

The operation principles are laws that hold for all numbers and operations. They are not based on particular calculations, but on **commonalities** across all calculations.

1. **Identity.** There are two numbers, 0 and 1, which leave things unchanged in addition and multiplication respectively:

e.g.  $7 + 0 = 7$       and       $11 \times 1 = 11$

2. **Inverse.** Every number has an inverse that undoes the action of the operation:

e.g.  $3 + 7 = 10$        $10 - 7 = 3 \rightarrow -7$  undoes  $+7$

$6 \times 3 = 18$        $18 \div 3 = 6 \rightarrow \div 3$  undoes  $\times 3$

3. **Commutativity.** This is best understood as “turnarounds”, and states that for any numbers, addition and multiplication are the same forward as backward:

e.g.  $3 \times 7 = 21$        $7 \times 3 = 21$

$11 + 6 = 17$        $6 + 11 = 17$

4. **Associativity.** This means that the numbers in an addition and multiplication can be added or multiplied in any order:

e.g. adding 3, 2, 8, 5, 6 in any order gives the same answer,

$$3 + 2 + 8 + 5 + 6 = 24 \quad \text{or} \quad 2 + 5 + 8 + 3 + 6 = 24 \quad \text{or} \quad 8 + 6 + 2 + 3 + 5 = 24$$

e.g. multiplying 3, 2, 5, 4 in any order gives the same answer,

$$3 \times 2 \times 5 \times 4 = 120 \quad \text{or} \quad 4 \times 5 \times 2 \times 3 = 120 \quad \text{or} \quad 5 \times 3 \times 4 \times 2 = 120$$

5. **Distributivity.** This means that addition adds like things but multiplication multiplies anything:

$\begin{array}{r} 24 \\ + 3 \\ \hline 7 \\ 20 \\ \hline 27 \end{array}$	$\begin{array}{r} 24 \\ \times 3 \\ \hline 12 \\ 60 \\ \hline 72 \end{array}$
$\begin{array}{r} 7 \quad (4 + 3) \\ 20 \quad (20 + 0) \\ \hline 27 \end{array}$	$\begin{array}{r} 12 \quad (3 \times 4) \\ 60 \quad (3 \times 20) \\ \hline 72 \end{array}$

These principles, or laws, cover operations in everything – whole numbers, fractions, algebra – and provide big ideas for learning mathematics. For Years F to 2, the following principles can be introduced: identity, inverse, commutativity and associativity. The distributive law is introduced in later school years. The method of introduction should meet the level of the students.

Interestingly, the principles mean that subtraction and division are not true operations. This is because subtraction and division do not obey the commutative and associative laws. For example:

$9 - 3 - 2$  could equal 4 or it could equal 8;

$24 \div 6 \div 2$  could equal 2 or it could equal 8; and

$9 - 3$  is not equal to  $3 - 9$  and  $6 \div 2$  is not equal to  $2 \div 6$ .

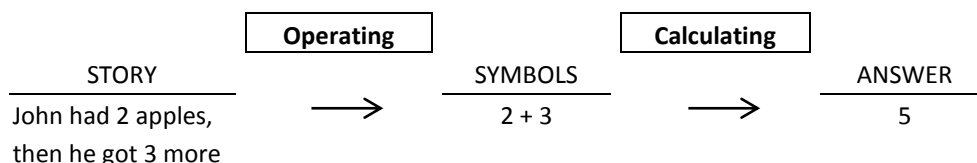
## Decontextualisation

Initially, children learn mathematical concepts using objects. After sorting, patterning and matching objects, children learn that there is a new characteristic that can be applied – number. So the following –  $\square \square \square$  – can be called 3 objects. However, it takes time, activity with many examples, and development for children to see “three-ness” independent of the objects. So, for many students, the number three remains contextualised, requiring three objects, or thinking in terms of three objects, for understanding. This is OK unless the focus of teaching becomes the symbol and there is no contextualisation.

When operations arrive, students still seeing numbers in terms of numbers of objects will see 3 joined with 2 as  $\square \square \square \rightarrow \leftarrow \square \square$  (3 objects joined with 2 objects) and the resulting 5 is seen as the objects together. It takes experience and maturation before 2 and 3 are able to be understood independently of the objects that students count and “ $2 + 3$ ” can stand alone as something not supported by 2 objects joining 3 objects.

This means that we must be aware of ensuring young children understand 3 independently of 3 “somethings” and understand  $2 + 3$  independently of joining 2 “somethings” to 3 “somethings” before reducing the focus of mathematics to numerals and operation symbols or number sentences, and calculating numeral answers.

The implication for O2 is that the two sides of operations – meanings/operating and calculating – represent a change in contextualisation, as below:



Thus, as we move from operating to calculating, we must be sure that:

- (a) students have an understanding of  $2 + 3$  appropriate to their knowledge and year level;

- (b) teaching does not move to decontextualised exercises, e.g.  $3 + 7 = \underline{\quad}$ ,  $2 + 3 = \underline{\quad}$ , without ensuring understanding; and
- (c) teaching includes students being able to reverse:  $7 \rightarrow "4 + 3" \rightarrow \text{story}$   
"4 plus 3"

This is particularly so when there are laws that apply across symbols and year levels. We must maintain contextualisation, and recontextualise when we feel that it is important to check understandings.

## 5.1 Identities in operations

This is better known as the "actions that do not change anything". Identities consist of special numbers, 0 for addition and 1 for multiplication, that do not change anything.

Activities that show the operations that do not change anything (identities) are as follows:

1. Join 0 to 4, take 0 away from 7, what happens? Why?
2. Is this the only thing to do? Could we get 0 some other way than 0? What about John walking 7 steps, he walks back 2 steps and forward 2. Act this out by walking – by hopping along a number line. Where does he end up? Why?
3. What about having \$13, spending \$3 on a chocolate, getting \$7 from your mum and then buying a drink for \$4. What has happened? Act it out with play money. Why?
4. Make up your own magnificently expansive operations that don't change anything.
5. Can you do this with multiplication and division?

It should be noted that anything that gives 0 and 1 can have the same effect as 0 and 1. For example:

$$6 + 4 - 2 + 1 - 3 = 6 \text{ because } + 4 - 2 + 1 - 3 = 0; \quad 3 \times 6 \div 2 \div 3 = 3 \text{ because } 6 \div 2 \div 3 = 1$$

## 5.2 Inverse in operations

These are the actions that undo other actions e.g.  $+2$  is undone by  $-2$ ;  $\times 4$  is undone by  $\div 4$ .

Activities that show the actions that undo other operations (inverse) are as follows:

1. Add 4 to 3 and then subtract 4. What happens? Why?
2. Is this the only thing we can do to get 3 back to 3 by undoing  $-4$ ? Which of the following will work?
  - (a) Walk forwards 3 steps, walk backwards 2, walk forwards 5 steps and walk backwards 3 steps. What do you notice? Why does it work?
  - (b) Use play money. You have \$13 to spend. You spend \$3 on a drink and \$4 on a chocolate, your mum gives you \$7. How much do you have now? Why?
3. Make up your own stories where actions/operations get undone. Make it long and expansive. Act it out with friends and play money.
4. Can you do this for multiplication and division?

## 5.3 Commutative law

This is the "turnarounds", e.g.  $2 + 3 = 3 + 2$  and  $4 \times 5 = 5 \times 4$ .

Activities that show commutative law are:

1. Join 5 children to 3 children, join 3 children to 5 children. Which group is bigger? Why? Does this always work? What if you add 11 to 5 and 5 to 11. Is the turnaround still true?

2. What about 3 rows of 4 children, what about 4 rows of 3 children – what do you notice? Does this always hold? Is it true for all arrays? What about 3 bags of 5 lollies and 5 bags of 3 lollies?
3. Can the above help me add 29 to 4?
4. Can you do this for multiplication and division?

## 5.4 Associative law

It does not matter in what order we add or multiply, e.g.  $3 + (2 + 4) = (3 + 2) + 4$ .

Activities to show associative law are:

1. Add 4 numbers in more than one way.
2. Subtract 3 numbers in more than one way.
3. What do you notice?
4. Can you do the same in multiplication and division? (use a calculator)

## 5.5 Decontextualisation

This looks at the change to understanding where context is not needed and the effect on students when given decontextualised activities when still requiring context. Activities are as follows:

1. Determination of student development:
  - (a) Check if students need to have objects to understand numbers – What is 7? What is 48? What is  $7 + 4$ ? How do they see numbers? How do they think of  $4 + 5$  for instance?
  - (b) Check that students can translate symbols to stories both for numbers and for operations, that is, check that students have the contextualisation to understand number sentences.
2. Undertake number studies:
  - (a) Give students calculations and see how they solve them – do they use doodles, pictures or objects?
  - (b) Share different students' methods of completing calculations and see if there are unusual methods of completing the problems.
3. Get students to describe operations – what does  $3 \times 4$  mean to you?
4. Try always to have contextualised activities (e.g., stories) when moving work across Modules O2 and O3, that is, from story  $\rightarrow$  symbols  $\rightarrow$  calculation.

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## Module Review

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This section reviews the units in this module. It looks across the units and identifies outcomes that go beyond the particularities of the units. The first of these is general **teaching approaches**, that is, ways of teaching ideas common within the units and across most of mathematics. The second is **models and representations**, that is, common ways of providing students with thinking images that support learning and applying mathematics. The third is **competencies**, that is, abilities that are important across the units and into the future. The final is **later thinking and finding**, information on the mathematics that grows out of this module and provides the reason for its importance.

### Teaching approaches

Things we need to consider:

- Whilst classroom approaches to young children's thinking skills aim to develop reasoning, enquiry and creativity, most recent psychological research has focused on children's powers of reasoning and enquiry exclusively (and not **creativity**)
- Current approaches to teaching thinking skills do not draw upon **explicit strategies** to help very young pupils develop their emergent theory of mind or their skills in counterfactual reasoning
- whilst the psychological literature reveals that children find some kinds of question more difficult or confusing than others, few studies relating to pedagogical approaches focus on **questioning** specifically#

Encourage teachers to explore the extent to which:

- their questions can focus specifically on stimulating children's thinking
- they can create time tabled opportunities for 'thinking times' which signal to the children that a non-ordinary (and possibly counterfactual) kind of thinking is being encouraged
- more opportunities can be created in the classroom for structured dialogue
- children can be invited to construct written opinions and arguments
- 'story-time' can become an opportunity to develop children's thinking
- traditional sorting and sequencing tasks can be an opportunity for children to verbalise their thinking
- play equipment can present children with possibilities for developing their imagination
- children can be given opportunities for solitary as well as social play
- children can be asked to evaluate their work critically
- additional adults in the classroom can be used to develop children's thinking
- creative activities can encourage creative 'possibility thinking', as well as creative skills

Finally, because algebra is the generalisation of arithmetic, it will be necessary to focus on the development of the new concept of variable as standing for any number and on the big ideas from arithmetic that carry through into algebra (e.g. concepts of operations and equals, principles associated with operations and equals).

## Models and representations

The end product of operations is a **symbol language** (e.g.  $2 + 3$  and  $2 + 3 = 5$ ). This is developed across O2 and O3 and the activities that follow O3. However, if this is done before students have the a decontextualised understanding of numbers and operations, there will be problems. For example, there are problems in this symbol language because a focus on answers, and not relationships, leads many students to believe that  $2 + 3 = 5$  is acceptable and  $5 = 2 + 3$  is not; for many students the equals sign has become a symbol for “put the answer here” or “do something” when its real meaning is “same value as”. Mathematically, *7 subtract 3 equals 4* is represented with symbols as  $7 - 3 = 4$ ; it must be seen as  **$7 - 3$  is the same value as 4**. This means that it is possible and equally correct to show  $7 - 3 = 4$  as  $4 = 7 - 3$ . Similarly, students also need to be familiar with context and meaning to ensure they do not see  $2 + 3 = 4 + 1$  as incorrect. This requires knowing how symbols relate to models and stories.

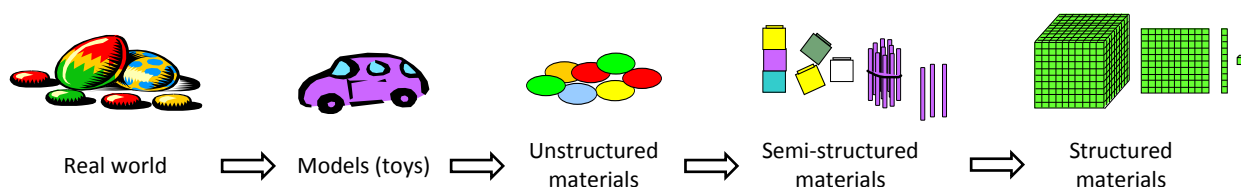
### Connecting symbols, language, models and real-world situations

This symbol language has to relate to everyday language and real instances. Many forms of symbols are possible and all relate to stories as the addition stories below show. The interesting point is that treating the symbols like a concise language (so  $7 + 4$  is 7 things joining 4 things), enables the symbols to tell stories and to describe the world, an outcome more powerful than answers. This is a major part of building the concepts of the operations (see sections 2.1 and 3.1).

One way to represent this symbol language is to use **models**. Models connect and unify mathematics, whereas symbols tend to emphasise difference. For operations, there are three models: **set** (e.g. Unifix and counters), **length** (e.g. Unifix stuck together or number lines), and **array or area** (e.g. counters, Unifix, dot paper or graph paper), which is for multiplication and division (e.g.  $3 \times 4$  is 3 rows of 4 or a 3 by 4 rectangle). These models do much more than just help with meaning – they show structure and apply across many topics. For example, the array model can help with: basic facts (e.g.  $4 \times 7$  is 4 rows of 7 which is 4 rows of 5 plus 4 rows of 2); algorithms (e.g.  $24 \times 7$  is 24 rows of 7 so it is 20 rows of 7 plus 4 rows of 7); fractions/decimals and percent (e.g.  $0.2 \times 0.4$  is a rectangle 2 tenths by 4 tenths which gives 8 squares in 100 or 0.08). Multiplication is the inverse of division and  $\times 1$  or  $\times \frac{2}{2}$  or  $\times \frac{3}{3}$  and so on leaves everything unchanged (e.g.  $\frac{2}{3} = \frac{2}{3} \times \frac{2}{2} = \frac{4}{6}$ , and so on).

### Types of models

YDM focuses on knowing **connections between real-world situations, models, language and symbols**, and strategies that lead to meaning and generalisation rather than rote definitions and procedures (or algorithms) that lead to particular answers for particular numbers. In this approach, models can be used in the same sequence as for number and can be tailored to suit the level of number representation students are working at. Kinaesthetic acting out of operations should be completed first which can then be transferred to tangible concrete models and finally abstract number sentences (see figure below).



Models also assist us to link **discrete and continuous**; there are a range of models that cross that divide as the examples below show. The continuous made countable models are particularly important.



### Discrete



Set model

### Continuous



Volume model

### Continuous made countable



Length model



Area model

As number is taken from discrete objects and applied to continuous attributes, its nature changes. Numbers designate ends of units and zero represents the start of units and not nothing (although it represents no units).

## Competencies

The major competencies for this unit are that students can:

- -understand the meaning and connections between counting, joining and separating (number partitions), and addition and subtraction, using them to represent situations involving all four basic operations
- -connect stories and symbols: stories  $\leftrightarrow$  action; stories  $\leftrightarrow$  operating language; stories  $\leftrightarrow$  operation symbol; stories  $\leftrightarrow$  equation; stories  $\leftrightarrow$  drawings/models
- -decode number stories: use beginning, action, end (BAE) and relate to equation parts. Find the missing or hidden part.
- -experience different problem types: Is it the start the change or the end that is needed?
- -write number stories and solve problems

## Later meaning and operating

Meaning and operating was the time of the introduction of principles. The principles make up a the structure of both arithmetic and algebra. The complete set of principles for primary and junior secondary is as follows. It includes the principles for equals (which are the equivalence class principles).

### Field principles for operations or operation properties

1. **Closure.** Numbers and an operation always give another number (e.g.  $2.17 + 4.34 = 6.51$  – for any numbers  $a$  and  $b$ ,  $a + b = c$  which is another number; and  $2.17 \times 4.3 = 9.331$  – for any numbers  $a$  and  $b$ ,  $a \times b = c$ , where  $c$  is another number).
2. **Identity.** 0 and 1 do not change things (+/- and  $\times/\div$  respectively). Adding/subtracting zero leaves numbers unchanged (e.g.  $9 +/- 0 = 9$ , where 0 can equal  $+1-1$ ,  $+6-3-3$ ,  $+11-14+3$ , and so on). Anything multiplied by 1 = itself (e.g. for any  $a$ ,  $a \times 1 = 1 \times a = a$ ). Anything multiplied by 0 = 0.
3. **Inverse.** A change that undoes another change. Addition is undone by subtraction and vice versa (e.g.  $+5-5 = 0$ , so  $2 + 5 = 7$  means  $7 - 5 = 2$ ). Multiplication is inverse of division and vice versa (e.g.  $\times 5 \div 5 = 0$ , so  $2 \times 5 = 10$  means  $10 \div 5 = 2$ ). This principle holds for fractions and indices (e.g. the inverse of 7 is  $\frac{1}{7}$  because  $7 \times \frac{1}{7} = \frac{1}{7} \times 7 = 1$ ; for any  $a$ , there exists  $a^{-1}$  so that  $a \times a^{-1} = a^{-1} \times a = 1$ ).
4. **Commutativity.** Order does not matter for addition but does for subtraction (e.g.  $3 + 4 = 4 + 3$ , but  $7 - 5$  is not = to  $5 - 7$ ). Order does not matter for multiplication but does for division (e.g.  $12 \times 4 = 4 \times 12$  but  $12 \div 4$  is not =  $4 \div 12$ ; for any  $a$ ,  $b$  and  $c$ ,  $(a \times b) \times c = a \times (b \times c)$ ). Also known as ‘turnarounds’.
5. **Associativity.** What is done first does not matter for addition and multiplication but does matter for subtraction and division (e.g.  $(8+4)+2 = 8+(4+2)$ , and  $(8 \times 4) \times 2 = 32 \times 2 = 64$  and  $8 \times (4 \times 2) = 8 \times 8 = 64$  but  $(8 \div 4) \div 2$  does not =  $8 \div (4 \div 2)$ ).

6. **Distributivity.** Multiplication and division are distributed across addition and subtraction and act on everything (e.g.  $3 \times (4 + 5) = (3 \times 4) + (3 \times 5)$ ;  $(21 - 12) \div 3 = (21 \div 3) - (12 \div 3)$ ). Distributivity does hold for all operations (for example,  $7 \times (8 - 3) = (7 \times 8) - (7 \times 3)$ ,  $(56 + 21) \div 7 = (56 \div 7) + (21 \div 7)$  and  $(56 - 21) \div 7 = (56 \div 7) - (21 \div 7)$ ).

### Extension of Field/operation properties

1. **Compensation.** Ensuring that a change is compensated for so the answer remains the same – related to inverse (e.g.  $5 + 5 = 7 + 3$ ;  $48 + 25 = 50 + 23$ ;  $61 - 29 = 62 - 30$ ).
2. **Equivalence.** Two expressions are equivalent if they relate by adding or subtracting 0 and multiplying or dividing by 1; also related to inverse (e.g.  $48 + 25 = 48 + 2 + 25 - 2 = 73$ ;  $50 + 23 = 73$ ;  $\frac{2}{3} = \frac{2}{3} \times \frac{2}{2} = \frac{4}{6}$ ).
3. **Inverse relation.** The higher the second number in subtraction and division, the smaller the result (e.g.  $12 \div 2 = 6 > 12 \div 3 = 4$ ;  $\frac{1}{2} > \frac{1}{3}$ ). For division, the more you divide by, the less you have (e.g.  $24 \div 8$  is less than  $24 \div 6$ ). This principle does not apply to addition or multiplication.
4. **Triadic relationships.** When three things are related there are three problem types where each of the parts are the unknowns. For example,  $2 + 3 = 5$  can have a problem for:  $? + 3 = 5$ ,  $2 + ? = 5$ ,  $2 + 3 = ?$ . This principle holds for all four operations:  $a + b = c$  ( $a + b = ?$ ,  $? + b = c$ ,  $a + ? = c$ );  $a - b = c$  ( $a - b = ?$ ,  $? - b = c$ ,  $a - ? = c$ );  $a \times b = c$  ( $a \times b = ?$ ,  $? \times b = c$ ,  $a \times ? = c$ );  $a \div b = c$  ( $a \div b = ?$ ,  $? \div b = c$ ,  $a \div ? = c$ ).

### Equivalence class principles or equals properties

1. **Reflexivity.** Anything always equals itself (e.g.  $2 \times 4 + 7 = 2 \times 4 + 7$ ) any number  $a$  is equal to itself, that is,  $a = a$ .
2. **Symmetry.** If something equals another thing then the another-thing equals the something (e.g.  $2 + 3 = 5$  means  $5 = 2 + 3$ ). For all numbers  $a$  and  $b$ , if  $a = b$  then  $b = a$  (the order of an equation can be reversed or “turned around”). This is important for equations as it means that  $2 \times 3 = 6$  and  $6 = 2 \times 3$ ,  $2 \times 3 = 12 \div 2$  and  $12 \div 2 = 2 \times 3$  are all correct and true.
3. **Transitivity.** If something equals another thing and the another-thing equals a third thing, then the original something equals the third-thing (e.g.  $2 + 3 = 5$  and  $5 = 9 - 4$  means  $2 + 3 = 9 - 4$ ). For all numbers  $a$ ,  $b$  and  $c$ ,  $a = b$  and  $b = c$  means  $a = c$ . This is also important for equations because it means we can say:  $4 \times 3 \div 2 = 12 \div 2 = 6$  and so  $4 \times 3 \div 2 = 6$

### Problem-solving and algebra

Because the field principles are the same for arithmetic and algebra, it means that arithmetic understanding can be translated to algebra understandings. In particular, the standard algorithm strategies for 2-digit numbers can directly apply to algebra if we use vertical setting out as the start of algebraic computation, as the following show

$\begin{array}{r} 23 \\ + 31 \\ \hline 54 \end{array}$	$\begin{array}{r} 2a + 3b \\ + 3a + b \\ \hline 5a + 4b \end{array}$
<p>4 adding ones</p> <p>5 0 adding 10s</p> <p>5 4</p>	<p>4b adding b's</p> <p>5a adding a's</p> <p>5a + 4b</p>

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# Test Item Types

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This section presents instructions and the test item types for the subtests associated with the units. These will form the bases of the pre-test and post-test for this module.

## Instructions

### Selecting the items and administering the pre-post tests

This section provides an item bank of test item types, constructed around the units in the module. From this bank, items should be selected for the pre-test and post-test; these selected items need to suit the students and may need to be modified, particularly to make post-test items different from pre-test items. The purpose of the tests is to measure students' performance before and after the module is taught. The questions should be selected so that the level of difficulty progresses from easier items to more difficult items. In some modules this will follow the order of the units and subtests, and in other modules it will not, depending on the sequencing across the module and within units. The pre-test items need to allow for students' existing knowledge to be shown but without continual failure, and the post-test items need to cover all the sections in a manner that maximises students' effort to show what they can do.

In administering the pre-test, the students should be told that the test is not related to grades, but is to find out what they know before the topic is taught. They should be told that they are not expected to know the work as they have not been taught it. They should show what they know and, if they cannot do a question, they should skip it, or put "don't know" beside questions. They will be taught the work in the next few weeks and will then be able to show what they know. Stress to students that **any pre-test is a series of questions to find out what they know** before the knowledge is taught. They should do their best but the important questions come at the end of the module. For the **post-test**, the students should be told that **this is their opportunity to show how they have improved**.

For all tests, **teachers should continually check to see how the students are going**. Items in later subtests, or more difficult items within a particular subtest, should not be attempted if previous similar items in earlier subtests show strong weaknesses. Students should be allowed to skip that part of the test, or the test should be finished. Students can be marked zero for these parts.

### Information on the Module O2: Meaning and Operating test item types

This section includes:

1. Pre-test instructions;
2. Diagnostic Mapping Points;
3. Observation Checklist; and
4. Test item types.

## Pre-test instructions

When preparing for assessment ensure the following:

- Students have a strong sense of identity; feel safe, secure and supported; develop their emerging autonomy, interdependence, resilience and sense of agency; and develop knowledgeable and confident identities.
- Students are confident and involved learners, and develop dispositions for learning such as curiosity, cooperation, confidence, creativity, commitment, enthusiasm, persistence, imagination and reflexivity.

When conducting assessment, take the following into consideration:

- Student interview for diagnostic assessment in the early learning stages is of paramount importance.
- Use materials and graphics familiar to students' context in and out of school.
- Use manipulatives rather than pictures wherever possible.
- Acknowledge the role of using stories in this early number learning, enabling students to tell stories and act out understandings to illustrate what they know.
- Playdough and sand trays are useful for early interview assessment situations.

Ways to prepare students for assessment processes include the following:

- In **individual teaching times**, challenge students' thinking. "Challenging my thinking helps me to learn by encouraging me to ask questions about what I do and learn. I learn and am encouraged to take risks, try new things and explore my ideas."
- In **group time**, model and scaffold question-and-answer skills by using sentence stems to clarify understandings and think about actions. Encourage students to think of answers to questions where there is no one correct answer, and to understand that there can be more than one correct answer (e.g. *How can we sort the objects?*).
- In **active learning centres**, use activities such as imaginative play, sand play, playdough, painting, ICTs and construction to think and talk about different ways of using materials, technologies or toys. Ask questions and take risks with new ideas.

Other considerations:

- Preferred/most productive assessment techniques for early understandings are observations, interviews, checklists, diary entries, and folios of student work.
- Diagnostic assessment items can be used as both pre-test and post-test instruments.

Remember:

**Testing the knowledge** can imply memory of stuff; asking the students **what they can do with knowledge** requires construction and demonstration of their understanding at this early understandings level.

## O2 Meaning and Operating: Diagnostic Mapping Points

### Early Ideas

-understand the meaning and connections between counting, number partitions, and addition and subtraction

1. same/different
2. more/less,
3. what actions bring the above situations about (beginning the process to addition/subtraction).
4. relationships between stories, material use, drawings of materials (models), language
5. BAE frameworks.

### Basic Meanings

--understand the meaning and connections between counting, joining and separating (number partitions), and addition and subtraction, using them to represent situations involving all four basic operations

--connect stories and symbols: stories  $\leftrightarrow$  action; stories  $\leftrightarrow$  operating language; stories  $\leftrightarrow$  operation symbol; stories  $\leftrightarrow$  equation; stories  $\leftrightarrow$  drawings/models

1. basic actions that lead to addition, subtraction, multiplication and division.
2. joining-separating,
3. taking away-adding to,
4. combining-partitioning (sharing), and
5. equal/unequal groups,
6. developing the relationships between the four operations, when are they similar and dissimilar. This will use the set, array and number line models. This will involve relationships between stories material use, drawings of materials (models), language and symbols.

### Other Meanings

Looking at more difficult meanings for the four operations

-decode number stories: beginning, action, end and relate to equation parts. Find the missing or hidden part

-experience different problem types: Is it the start the change or the end that is needed?

1. addition, subtraction, multiplication and division as **inverse** of normal action;
2. addition and subtraction as **additive comparison** (having more and having less/fewer); and
3. multiplication and division as **multiplicative comparison** (having times more and having times less).

As well, the unit covers, in a limited way, two other more complex meanings, namely:

4. multiplication and division as **factor-factor-product**; and
5. multiplication and division as **combinations**.

## Word Problem-Solving

-write number story and solve problems using stories  $\leftrightarrow$  action; stories  $\leftrightarrow$  operating language; stories  $\leftrightarrow$  operation symbol; stories  $\leftrightarrow$  equation; stories  $\leftrightarrow$  drawings/models

1. Representation in word problem-solving –Can student move from problem to operation using 1) story  $\rightarrow$  representations  $\rightarrow$  symbols then 2) story  $\rightarrow$  Symbols  $\rightarrow$  Calculations
2. Exploring and extending the think board- Can the student represent the operation in different ways using the think board?
3. Constructing word problems. Can students construct their own word problems? Do they take into account what words may infer but may or may not mean?
4. Using triad big idea in word problems.
5. Multistep problems

## Principles and Decontextualisation

Identities in operations – special numbers that do not change things

Inverse in operations-the actions that undo other actions

Commutative law -turn around to the same thing

Associative law -does it matter in what order I add them up?

Decontextualizing word problems

O2 Meaning and Operating Observation Checklist			
Unit	Concept	Knows	Can construct/do/tell
1. Early Ideas	Introducing operating situations	Same /different More/less	Basic actions that lead to addition, subtraction, multiplication, division
	Extending to interpretation and construction	What actions bring about the above situations	Actions and describe change
	BAE framework	To find Beginning-action-end parts of the story	beginning the process to addition/subtraction
2. Meanings	Joining-separating	Taking away-adding to	Connects stories and symbols: stories $\leftrightarrow$ action; stories $\leftrightarrow$ operating language; stories $\leftrightarrow$ operation symbol; stories $\leftrightarrow$ equation; stories $\leftrightarrow$ drawings/models
	Combining –partitioning (sharing)	Equal/unequal groups Grouping and sharing	Connects stories and symbols: stories $\leftrightarrow$ action; stories $\leftrightarrow$ operating language; stories $\leftrightarrow$ operation symbol; stories $\leftrightarrow$ equation; stories $\leftrightarrow$ drawings/models
	Relating- Developing the relationship between the four operations	actions that also lead to some or all of the four operations	Can tell solution using the set, array and number line models. Relationships between stories, material, drawings of materials (models), language and symbols.
	Part-part-total	Identifies what is the part? What is the total?	Constructs parts and totals as well as interprets what is happening. Moves forwards and backwards
3. Other Meanings	Inverse meaning	Inverse	Find the missing or hidden part uses set, array, number line
	Comparison meanings	Different problem types: Is it the start the change or the end that is needed?	Inclusion (sometimes called inaction) Relate these to stories, materials, language and symbols
	Combinations meanings	For multiplication and division only,	uses set, array, number line combinations models
4. Solving	Getting started and working with models	The same operation can be said and written in different ways	Decoding number stories using Thinkboard parts
	Word problem-solving	What do words infer? One situation can be represented by different operations	To decode number stories: beginning, action, end and relate to equation parts Problem parts and possible operations
	Constructing/writing word problems	Forwards –backwards stories One operation can represent apparently different situations	Construct their own word problems for different operations The triadic relationship
	Multi step problems	That you can repeat process of BAE within a problem to address each step.	Strategies of attack to solve; break into parts/steps; draw; given-needed-wanted; a problem within a problem

<b>5. Principles and Decontextualisation</b>	Operations that do not change anything	Knows e.g. $-2$ is the inverse of $+2$ That some changes leave numbers the same ( $+0$ and $\times 1$ ),	Undoing the change The operations that do not change things
	Noticing that 'turn arounds' give the same answer as the original ( $2+5=5+2$ ).	Turn around to the same thing	
	Does it matter how I add them up?	Add/subtract numbers in more than one way	Add 4 numbers in more than way Subtract 3 numbers in more than one way.
	Decontextualizing word problems		

## **Subtest 1 item types (Unit 1: Early ideas)**

1.



# Appendices

## Appendix A: AIM Early Understanding Modules

### Module content

<b>1<sup>st</sup> module</b> <b>Number N1: Counting</b> *Sorting/correspondence *Subitising *Rote *Rational *Symbol recognition *Models *Counting competencies	<b>2<sup>nd</sup> module</b> <b>Algebra A1: Patterning</b> *Repeating *Growing *Visuals/tables *Number patterns	<b>3<sup>rd</sup> module</b> <b>Algebra A2: Functions and Equations</b> <i>Functions</i> *Change *Function machine *Inverse/backtracking <i>Equations</i> *Equals *Balance
<b>4<sup>th</sup> module</b> <b>Number N2: Place Value</b> <i>Concepts</i> *Place value *Additive structure, odometer *Multiplicative structure *Equivalence <i>Processes</i> *Role of zero *Reading/writing *Counting sequences *Seriation *Renaming	<b>5<sup>th</sup> module</b> <b>Number N3: Quantity</b> <i>Concepts</i> *Number line *Rank <i>Processes</i> *Comparing/ordering *Rounding/estimating <i>Relationship to place value</i>	<b>6<sup>th</sup> module</b> <b>Operations O1: Thinking and Solving</b> *Early thinking skills *Planning *Strategies *Problem types *Metacognition
<b>7<sup>th</sup> module</b> <b>Operations O2: Meaning and Operating</b> *Addition and subtraction; multiplication and division *Word problems *Models	<b>8<sup>th</sup> module</b> <b>Operations O3: Calculating</b> *Computation/calculating *Recording *Estimating	<b>9<sup>th</sup> module</b> <b>Number N4: Early Fractions</b> <i>Concepts</i> *Fractions as part of a whole *Fractions as part of a group/set *Fractions as a number or quantity *Fraction as a continuous quantity/number line <i>Processes</i> *Representing *Reading and writing *Comparing and ordering *Renaming

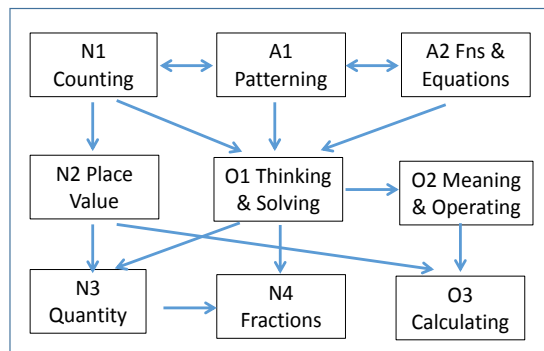
## Module background, components and sequence

**Background.** In many schools, there are students who come to Prep with intelligence and local knowledge but little cultural capital to be successful in schooling. In particular, they are missing basic knowledge to do with number that is normally acquired in the years prior to coming to school. This includes counting and numerals to 10 but also consists of such ideas as attribute recognition, sorting by attributes, making patterns and 1-1 correspondence between objects. Even more difficult, it includes behaviours such as paying attention, listening, completing tasks, not interfering with activity of other students, and so on.

Teachers can sometimes assume this knowledge and teach as if it is known and thus exacerbate this lack of cultural capital. Even when it is identified, it can be time consuming to build this knowledge in classrooms where children are at different levels. Thus, it can lead to situations where Prep teachers say at the end of the year that some of their students are now just ready to start Prep and they wish they could have another year with them. These situations lead to a gap between some students and the rest that is already at least one year at beginning of Year 1. For many students, this gap becomes at least two years by Year 3 and is not closed and sometimes widens across the primary years unless schools can provide major intervention programs. It also leads to problems with truancy, behaviour and low expectations.

**Components.** The AIM EU project was developed to provide Years P-2 teachers with a program that can accelerate early understandings and enable children with low cultural capital to be ready for Year 3 at the end of Year 2. It is based on nine modules which are built around three components the mathematics ideas are designed to be in sequence but also to be connected and related to a common development. The modules are based on the AIM Years 7-9 program where modules are designed to teach six years of mathematics (end Year 3 to end Year 9) in three years (start Year 7 to end Year 9). The three components are: (a) Basics – A1 *Patterning* and A2 *Functions and equations*; (b) Number – N1 *Counting* (also a basic); N2 *Place value*; N3 *Quantity* (number line); and N4 *Fractions*; and (c) Operations – O1 *Thinking and solving*; O2 *Meaning and operating*; and O3 *Calculating*. These nine modules cover early Number and Algebra understandings from before Prep to Year 2.

**Sequence.** Each module is a sequence of ideas from P-2. For some ideas, this means that the module covers activities in Prep, Year 1 and Year 2. Other modules are more constrained and may only have activities for one Year or for two Years. For example, Counting would predominantly be Year P and Fractions Year 2. Thus, the modules overlap across the three years P to 2. For example, Place value shares ideas with Counting and with Quantity for two digit numbers in Year1 and three digit numbers in Year 2. It is therefore difficult, and inexact to sequence the modules. However, it is worth attempting a sequence because, although inexact, the attempt provides insight into the modules and their teaching. One such attempt is on the right. It shows the following:

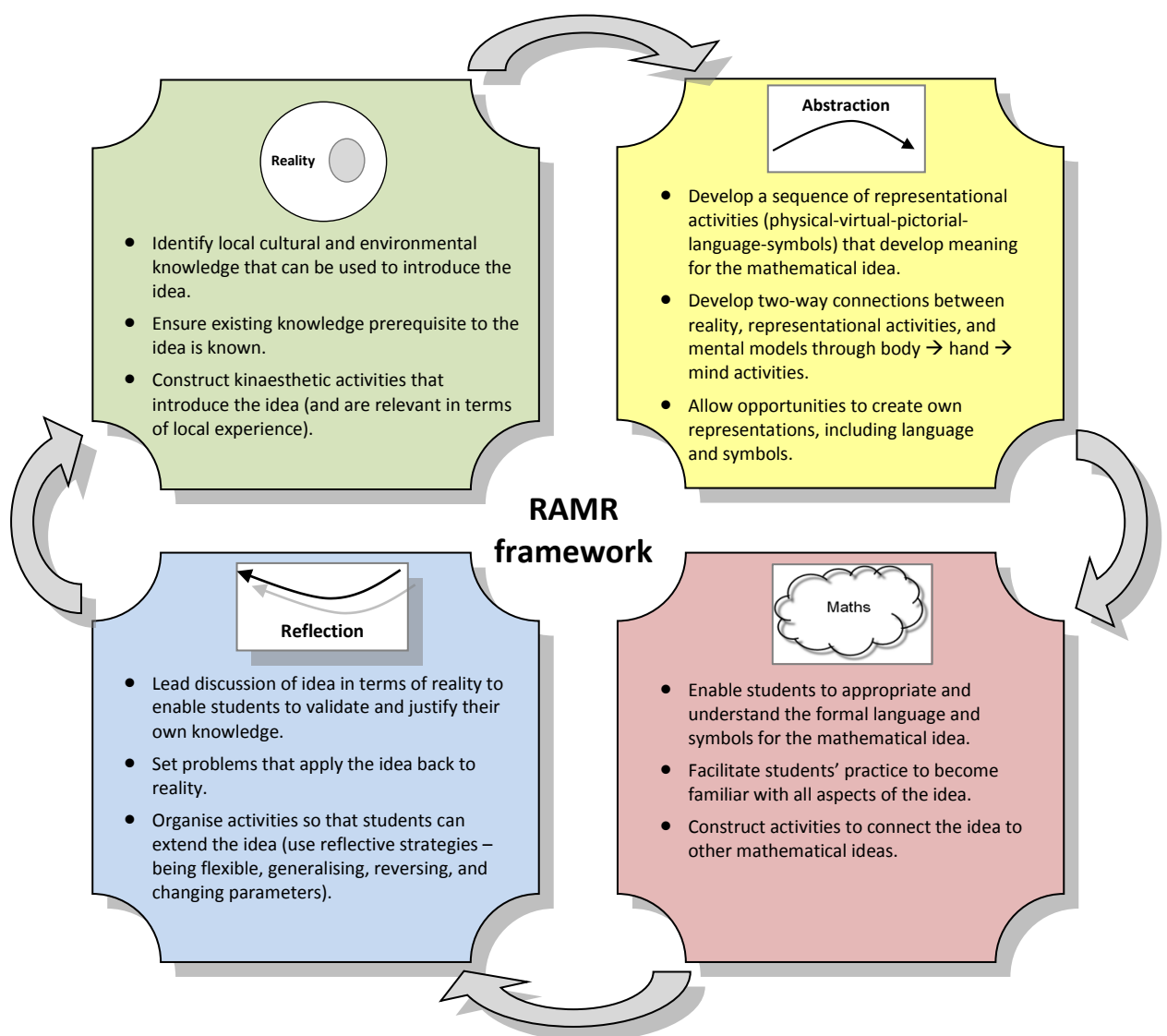


1. that foundation ideas are within *Counting*, *Patterning* and *Functions and Equations* – these deal with the manipulation of material for the basis of mathematics, seeing patterns, the start of number, and the idea of inverse (undoing) and the meaning of equals (same and different);
2. that the central components of the sequence are *Thinking and solving* along with *Place Value* and *Meaning and operating* – these lead into the less important *Calculating* and prepare for *Quantity*, *Fractions* and later general problem-solving and algebra; and
3. that *Quantity*, *Fractions* and *Calculating* are the end product of the sequence and rely on the earlier ideas, except that *Quantity* restructures the idea of number from discrete to continuous to prepare for measures.

## Appendix B: RAMR cycle

AIM advocates using the four components in the figure below, reality–abstraction–mathematics–reflection (RAMR), as a cycle for planning and teaching mathematics. RAMR proposes: (a) working from reality and local culture (prior experience and everyday kinaesthetic activities); (b) abstracting mathematics ideas from everyday instances to mathematical forms through an active pedagogy (kinaesthetic, physical, virtual, pictorial, language and symbolic representations, i.e. body → hand → mind); (c) consolidating the new ideas as mathematics through practice and connections; and (d) reflecting these ideas back to reality through a focus on applications, problem-solving, flexibility, reversing and generalising. The innovative aspect of RAMR is that Reality to Abstraction to Mathematics develops the mathematics idea while Mathematics to Reflection to Reality reconnects it to the world and extends it.

Planning the teaching of mathematics is based around the four components of the RAMR cycle. They are applied to the mathematical idea to be taught. By breaking instruction down into the four parts, the cycle can lead to a structured instructional sequence for teaching the idea. The figure below shows how this can be done.



The YuMi Deadly Maths RAMR cycle

## Appendix C: Teaching framework

### Teaching scope and sequence for Meaning and Operating

TOPIC	Sub-topics	DESCRIPTIONS AND CONCEPTS/STRATEGIES/WAYS
<b>Meaning Operating</b>	Early Ideas	Same/different More/less BAE
	Meanings	joining-separating, taking away-adding to, combining-partitioning (sharing), and equal/unequal groups, part-part-total Connections: real world $\leftrightarrow$ actions $\leftrightarrow$ models $\leftrightarrow$ language $\leftrightarrow$ symbols).
	More Meanings	Inverse Comparisons Combinations
	Solving	Moving from problem to operation Word problem-solving Forwards-backwards Multi step problems
	Principles	Global properties Number size Field & extension of Field Equals (Equivalent class)

## Proposed year-level framework

YEAR LEVEL	OPERATIONS – Meaning and Operating	
	Semester 1	Semester 2
Prep	Early ideas	Early Ideas; joining /separating; taking away-adding in; combining-sharing
1	joining-separating, taking away-adding to, combining-partitioning(sharing),and equal/unequal groups,  Modelled Connections: real world $\leftrightarrow$ actions $\leftrightarrow$ models $\leftrightarrow$ language $\leftrightarrow$ symbols).	Moving from problem to operation, BAE Thinkboard as a group problem-solving strategy Matching Connections: real world $\leftrightarrow$ actions $\leftrightarrow$ models $\leftrightarrow$ language $\leftrightarrow$ symbols).
2	joining-separating, taking away-adding to, combining-partitioning(sharing), and equal/unequal groups, part-part-total  Drawing Connections: real world $\leftrightarrow$ actions $\leftrightarrow$ models $\leftrightarrow$ language $\leftrightarrow$ symbols).  Operations that do not change anything  Noticing that ‘turn arounds’ give the same answer as the original (2+5=5+2).  Does it matter how I add them up?	Inverse, Comparisons, Combinations Moving from problem to operation Word problem-solving Forwards-backwards Multi step problems Decontextualizing word problems



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