Accelerated Inclusive Mathematics Early Understandings Project

YuMi Deadly Maths

AIM EU
Module N3
Number: Quantity

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ACKNOWLEDGEMENT

The YuMi Deadly Centre acknowledges the traditional owners and custodians of the lands in which the mathematics ideas for this resource were developed, refined and presented in professional development sessions.

YUMI DEADLY CENTRE

The YuMi Deadly Centre is a research centre within the Faculty of Education at the Queensland University of Technology which is dedicated to enhancing the learning of Indigenous and non-Indigenous children, young people and adults to improve their opportunities for further education, training and employment, and to equip them for lifelong learning. The YuMi Deadly Centre (YDC) can be contacted at ydc@qut.edu.au. Its website is http://ydc.qut.edu.au.

"YuMi" is a Torres Strait Islander Creole word meaning "you and me" but is used here with permission from the Torres Strait Islanders' Regional Education Council to mean working together as a community for the betterment of education for all. "Deadly" is an Aboriginal word used widely across Australia to mean smart in terms of being the best one can be in learning and life. YuMi Deadly Centre's motif was developed by Blacklines to depict learning, empowerment, and growth within country/community. The three key elements are the individual (represented by the inner seed), the community (represented by the leaf), and the journey/pathway of learning (represented by the curved line which winds around and up through the leaf). As such, the motif illustrates the YuMi Deadly Centre's vision: *Growing community through education*.

DEVELOPMENT OF THE AIM EARLY UNDERSTANDINGS MODULES

In 2009, the YuMi Deadly Centre (YDC) was funded by the Commonwealth Government's *Closing the Gap: Expansion of Intensive Literacy and Numeracy* program for Indigenous students. This resulted in a Year 7 to 9 program of 24 half-term mathematics modules designed to accelerate learning of very underperforming Indigenous students to enable access to mathematics subjects in the senior secondary years and therefore enhance employment and life chances. This program was called Accelerated Indigenous Mathematics or AIM and was based on YDC's pedagogy for teaching mathematics titled YuMi Deadly Maths (YDM). As low income schools became interested in using the program, it was modified to be suitable for all students and its title was changed to Accelerated Inclusive Mathematics (leaving the acronym unchanged as AIM).

In response to a request for AIM-type materials for Early Childhood years, YDC is developing an Early Understandings version of AIM for underperforming Years F to 2 students titled Accelerated Inclusive Mathematics Early Understandings or AIM EU. This module is part of this new program. It uses the original AIM acceleration pedagogy developed for Years 7 to 9 students and focuses on developing teaching and learning modules which show the vertical sequence for developing key Years F to 2 mathematics ideas in a manner that enables students to accelerate learning from their ability level to their age level if they fall behind in mathematics.

YDC acknowledges the role of the Federal Department of Education in the development of the original AIM modules and sees AIM EU as a continuation of, and a statement of respect for, the *Closing the Gap* funding.

Contents

Module Overview

This module, N3 *Quantity*, is the fifth of the nine Accelerated Inclusive Mathematics Early Understandings (AIM EU) modules. These modules are designed to provide support in Years F to 2 to improve Year 3 mathematics performance. The nine modules covering Number and Algebra Years F to 2 are shown in **Appendix A**.

This module is the third in the sequence looking at number. It follows on from N1 *Counting* and N2 *Place Value*. It focuses on quantity, initially from a number-line perspective and then from a perspective that integrates quantity and place value.

The reason for this third module is twofold: (a) to ensure that the number line receives a lot of attention, and (b) to ensure students understand the difference between discrete things and continuous things with respect to number (see section on continuous vs discrete below).

This overview will cover quantity in early childhood and primary, connections and big ideas in quantity, sequencing, teaching and cultural implications, and the structure of this module.

Quantity in early childhood and primary

Initially we will look at quantity in terms of position on a number line. To understand why this is important, we will look at continuous vs discrete and the effect that applying number to continuous entities has had on our view of numbers, particularly zero, and where numbers begin and end, which is important later for understanding parts of the whole.

Continuous vs discrete

Number was developed for discrete entities or objects, from the set model. This is because discrete objects enable one number name to be matched to one object so that the last number name tells how many. This is not possible in length, area, volume/capacity, mass, angle or even time. These are continuous; they do not have discrete parts to match with number names.

However, Western cultures were developing commerce and wanted to buy and sell continuous things (e.g. land area). To allow this, they developed the notion of **unit**, an amount of the continuous entity that could be used to determine the amount of the continuous part (e.g. metres to measure across the front of a house block). This gave a number to, for instance, a length of wood which could now be sold per length. This “discretifying” of the continuous by the unit broke the length being measured into unit length pieces which were discrete and could be counted (because it was now possible to match a discrete unit length part with number names).

This had two effects: the first was a philosophical effect – it changed our **view of reality**. We started to look at the land and its trees, water and other resources in terms of numbers, which started to prevent holistic understandings and reinforced the view that we had to increase the land’s output in terms of those numbers: Western society’s view that the land is for us to use and dominate.

The second effect was mathematical – it changed our **view of number**, particularly 0. When we divide a length into units as below, we can number the unit spaces (A) or number the ends of the length (B).

A		1	2	3	4	5	6	7	8
B		1	2	3	4	5	6	7	8
C	0	1	2	3	4	5	6	7	8

With measures, you are deciding how much from the start to where you are and so B is the long-term more appropriate way to consider counting (and also measuring devices). This leads to the question, what do we call the starting point? The only answer is 0 (see C above). This changes the meaning of 0 from nothing to something, the starting point. It also changes the meanings of numbers 1 to 8 (in the above example) from counting objects to denoting the end of units.

Note: To teach this new understanding of number, start with **number tracks** where you label each gap between lines, move on to **number ladders** which label each rung (0 is the ground or bottom rung), and finally move on to **number lines** (like rulers).

Deeper meanings

Quantity is the amount of something. First, when that something is continuous, in a line, its value is determined from 0 to the amount. This means that the number is not just what is there but the result of **comparing it to an amount in relation to 0**. In terms of determining number, this is not just noticing how many there are but comparing to a measurement device which comprehends where the zero is (and may need calibrating like a beam balance, or placing the zero at the start like has to be done with a ruler). Also, if we get the zero mixed up, we end up with, for example, the year 2001 (rather than the year 2000) being 2000 years after the birth of Christ. This is because Christ's birthday was considered to be 1 and not 0 (there was no year 0).

Second, because of the nature of a number line, it is sometimes **difficult to determine** number. For example, *how many years between 2004 and 2007 or from 2004 to 2007?* If this is October 2004 to October 2007, it is 3 years. But otherwise, we tend to count it as 4 years (i.e. 2004, 2005, 2006 and 2007). Another example is the problem about posts being put in. *If posts are to be placed every 5 m, how many posts for 40 m?* If we divide 40 by 5, we get 8, but the number of posts needed is 9. This is because there has to be a post at 0.

Third, because of what they are, **negatives** have to be discretified continuous measures for which going below 0 has meaning. They cannot be discrete because we cannot get a negative chair or person. They have to be measures that go below zero, such as temperature. In these cases, the 0 is determined by people. We could say that height is 0 when it is 75 cm. Then a 100 cm person is 25 and a 50 cm person is −25.

Connections and big ideas

We advocate that mathematics is best understood and applied in a schematic structured form that contains knowledge of **when and why as well as how**. Schema has knowledge as **connected nodes**, which facilitates recall and problem-solving. We argue that knowledge of the structure of mathematics, particularly of connections and big ideas, can assist teachers to be effective and efficient in teaching mathematics, and enable students to accelerate their learning. It enables teachers to: (a) *determine what mathematics is important to teach* – mathematics with many connections or based on big ideas is more important than mathematics with few connections or little use beyond the present; (b) *link new mathematics ideas to existing known mathematics* – mathematics that is connected to other mathematics or based on the one big idea is easier to recall and provides options in problem-solving; (c) *choose effective instructional materials, models and strategies* – mathematics that is connected to other ideas or based around a big idea can be taught with similar materials, models and strategies; and (d) *teach mathematics in a manner that enables later teachers to teach more advanced mathematics* – by preparing linkages to other ideas and foundations for big ideas later teachers will use.

Number has a structure that is amenable to connections and big ideas as we show with some examples below.

Connections

The quantity component of number from ones to hundreds–tens–ones deals with number lines and with equivalence (and the role of ones and zeros). It is connected to:

(a) **all measures from the measurement strand** (i.e. length, mass, capacity, perimeter, area, volume, time, angle, money and temperature); in particular, most measures directly use number lines (e.g. length –

rulers, capacity – measuring cylinders, temperature – thermometer), some of which are circular (mass – scales, time – clock), and this is the reason for their importance;

(b) **directed numbers**, that is, positive and negative numbers (e.g. temperature, above and below sea level, lift in a tall building);

(c) **decimal numbers and fractions** through the application of number lines – putting whole numbers on a number line gives space between, say, 2 and 3 which can be filled with fractions (e.g. $2\frac{1}{3}$) and decimals (e.g. 2.87);

(d) **operations** through using number lines to show addition and subtraction by jumps along the line (e.g. 46 + 23 is starting on the line at 46, jumping forward 2 tens and then 3 ones) and multiplication and division by repeated jumps; and

(e) **equations and percent, rate and ratio** through the application of the double number line (e.g. an equation such as 6 – 1 = 2 + 3 is starting at 0, one side jumping forward 6 and back 1, while the other side jumps 2 and then 3 to end at the same point, while ratio is 2 on one side of the line being the same as 3 on the other side which leads to 4 being opposite 6, 6 opposite 9 and so on).

Big ideas

There are five big ideas in number. These are given in Module N2. We repeat them below, with the first three in summarised form.

1. **Part-whole (includes notion of one)**. The basis of number is the unit which is grouped to make large numbers and partitioned to make fractions. Thus everything can be seen as part and whole – a ten is a tenth of a hundred and a whole of 10 ones. There is also a sub-big idea in that the pattern for the groupings and partitioning in numbers is a **pattern of threes**, that is, ones-tens-hundreds ones, ones-tens-hundreds thousands, ones-tens-hundreds millions and so on. The concept for this big idea is **place value**, and the processes are **reading and writing**.

2. **Additive structure**. A number's value is the sum of its parts, e.g. 234 = 200 + 30 + 4. Thus, place values increase and decrease by adding and subtracting. The consequence of this is that each place-value position counts forwards and backwards and values of place-value positions increase and decrease.

   There is also a sub-big idea in that counting follows a pattern, the **odometer pattern**. The concept for this idea is **counting** and the processes are **seriation** and **counting patterns**.

3. **Multiplicative structure***.* A number's value is a sum of the products of value and place, e.g. 234 = 2 × 100 + 3 × 10 + 4 × 1. This means that there is a multiplicative relationship between adjacent place-value positions – values increase ×10 when moving to the left and decrease ÷10 when moving to the right. Thus, the value of a number is found by adding the multiples of digit × place value (e.g. 204 = 2 × 100 + 0 × 10 + 4 × 1).

   The concept for this idea is **multiplicativity** and processes are **renaming** and **flexibility**. Flexibility is thinking of things more than one way by changing the unit (e.g. 300 is 30 tens or 0.3 thousands and so on).

4. **Continuous vs discrete/Number line**. Regardless of place values and digits, each number is a single quantity represented by a point on a number line, has rank and can be compared to other numbers. The number-line representation changes perspective of number – for example, the 0 is no longer nothing, it is the starting position of positive whole numbers (and of rulers and other measuring devices). A **quantity** is one position on the number line.

   The concepts for this idea are **comparison/order**, **rank** and **density** and the processes are **comparing/ ordering**, **rounding**, and **estimating**. Order just means to work out the larger/largest, but rank means to place on a line where they should be proportionally. For example, 91 being larger than 32 can be shown by students in a row with 91 after 32, but rank is shown by the 32 being on a line one third of the way between 0 and 100 and the 91 being near the 100. Density is how many numbers can be placed between two other numbers adjacent to each other.

5. **Equivalence**. Sometimes a single quantity can be represented by more than one number. For example, 04 is the same as 4, 2.40 is the same as 2.4, $\frac{2}{3}$ is the same as $\frac{4}{6}$, and 3:5 is the same as 6:10. Equivalence often reflects adding zero (the additive identity) or multiplying by one (the multiplicative identity). It also reflects where zeros can be placed to change and not to change the number. We will use this big idea to cover the role of zero in numbers.

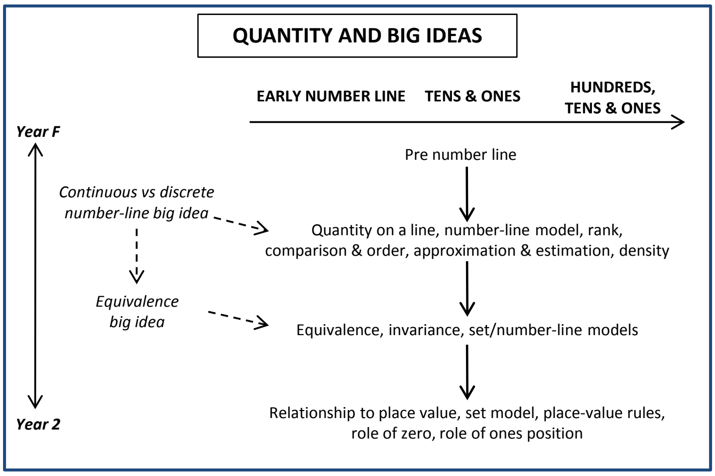
Big ideas 4 and 5 are the basis for this module.

Sequencing

Similar to Module N2, *Place Value*, this module has the problem of two sequences integrated:

(a) the size of numbers Os → T-Os → H-T-Os (left to right in diagram on right); and

(b) the complexity of the activities in the big ideas (top to bottom in diagram on right).



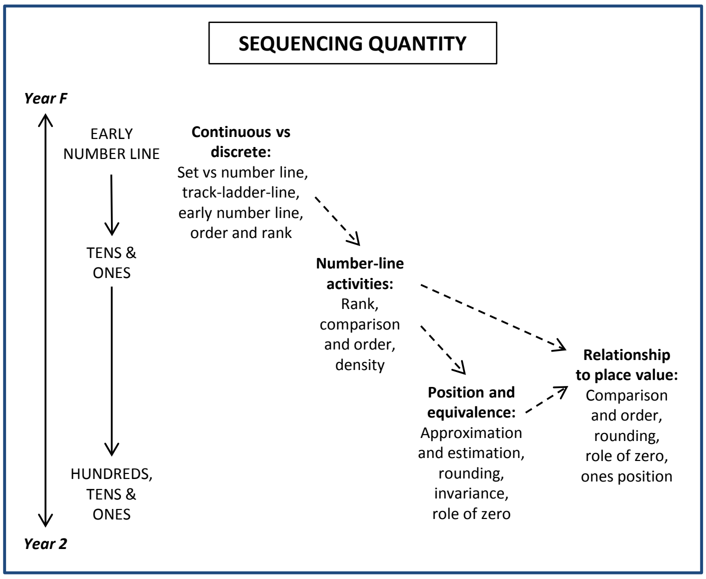
It also has two other issues:

(a) unlike N2, this module does not integrate into the ideas from N1 in that it does not follow on from the set model ideas of N1 but introduces **a new model** (in fact, a new perspective on number) of the number line which requires an early look at what this is (the pre number-line work in the diagram above); and

(b) the requirement to integrate the number-line work back in with the **place-value/set** work of N2 (as in the "relationship" work in the diagram above).

Therefore, the sequence that has been adopted for this module attempts to integrate vertical and horizontal sub-sequencing of the diagram above with the pre number-line work and the relationship work at the start and end of the module as in the diagram below.

As can be seen, there are four units:



(a) **continuous vs discrete** that introduces the new number-line model, the sequence track to ladder to line, and looks at early ideas (numbers 0–20);

(b) **number-line activities** that build up the idea of rank, comparison/order and density – numbers 0–99 (T-O) and then 0–999 (H-T-O);

(c) **position and equivalence** that looks at applications of number line to position (approximation, estimation and rounding) and equivalence and invariance (role of zero) – numbers 0–999 (H-T-O); and

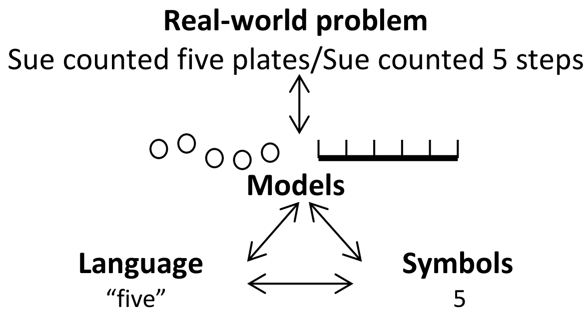
(d) **relationship to place value** that looks at interaction of place value and number line in such activities as comparison and order, rounding, role of zero, and the importance of the ones position (particularly important to enable students to access number-line outcomes from place-value perspectives) – numbers 0–999 (H-T-O).

Teaching and culture

This section looks at teaching and cultural implications, including the Reality–Abstraction–Mathematics–Reflection (RAMR) framework (see **Appendix B**) and the impact of Western number teaching on Indigenous and low socio-economic status (SES) students.

Teaching implications

As in Module N2, the **Payne and Rathmell (1977) triangle** is one of the underpinnings of the Abstraction and Mathematics stages of the RAMR framework. It is important in teaching number because of its focus on the relationship between models, language and symbols. Activities and questions should be constructed that encourage students to connect and move flexibly between models, language and symbols in all directions:



It should be noted that the triangle covers the number line as well as the set model. This means we can have relationships between models as well as between models, language and symbols. This leads to model ←→ model activity being a major part of teaching.

Cultural implications

Aboriginal and Torres Strait Islander students may find the teaching of number confronting because of the differences of the number-oriented culture of the mathematics classroom and their culture, and because many students are from low-SES backgrounds.

Aboriginal and Torres Strait Islander cultures followed a different path from number-oriented cultures (European, Indian-Arabic, and Chinese-Japanese) in the development of mathematics; for Indigenous cultures, people were seen as more important than number so their mathematics specialised in areas other than number. This different focus could be seen as emanating from their cultural beliefs with regard to group rather than individual ownership. Thus, the teaching of number, operations and measurement can bring Australian mainstream Eurocentric school teaching into conflict with Indigenous students; it can be a topic that can, in the terms of Indigenous mathematics and mathematics-education researcher Dr Chris Matthews, designate these cultures as primitive. It must be taught with care as part of a European culture that Indigenous people need to understand. It should not be celebrated as something that raises some people above others.

For low-SES Aboriginal and Torres Strait Islander students in Australia, the outcome is exacerbated. As low-income people, these students are sometimes considered to be unsuccessful. The number systems created as part of Eurocentric mathematics have benefited high-SES people at the expense of low-SES people, and promulgated the idea that bigger numbers (e.g. money, house cost, cars) are better, and mean that the person with the bigger numbers is more successful. The way the numbers function within Eurocentric societies achieves two outcomes simultaneously: (a) it benefits one class of people at the expense of the other, and (b) it puts the blame for their lack of benefit on the actions of the class that is not benefited. The mathematics of number, operations and measurement must be taught with care to low-SES students because its teaching can designate these students as failures. If the students are both Indigenous and low SES, even greater care must be taken.

The number line can be a **more positive teaching approach** than the set model with disengaged students because of its connection to measurement and active pedagogy, and to its use of real-world activities that involve materials and are often outdoor.

Structure of module

Components

Based on the ideas above, this module is divided into this overview section, four units, a review section, test item types, and appendices, as follows.

Overview: This section covers a description of quantity in early childhood and primary, connections and big ideas, sequencing, teaching and culture, and summary of the module structure.

Units: Each unit includes examples of teaching ideas that could be provided to the students, some in the form of RAMR lessons, and all as complete and well sequenced as is possible within this structure

Unit 1: Continuous vs discrete. This unit introduces the number-line model and early ideas (O to T-O).

Unit 2: Number-line activities. This unit builds up ideas of rank, order and density (T-O to H-T-O).

Unit 3: Position and equivalence. This unit covers applications of number line and the role of zero.

Unit 4: Relationship to place value. This unit covers the relationship of place value to length, equivalence, rounding, and comparison and order.

Module review: This section reviews the module, looking at important components across units and at later teaching ideas. For this module, this includes the concept of continuous vs discrete and the notion of the number line.

Test item types: This section provides examples of items that could be used in unit pre- and post-tests.

Appendices: This comprises three appendices covering the AIM EU modules, the RAMR pedagogy, and proposed teaching frameworks for quantity.

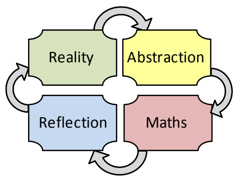
Further information

Sequencing the teaching of the units. The four units cover the teaching of quantity as described in the Sequencing subsection above. The four units are in sequence and could be completed one at a time. However, each of the units is divided into sub-ideas (concepts and processes) that are also in sequence within the unit. Therefore, schools may find it advantageous to: (a) teach earlier sub-ideas in a later unit before completing all later sub-ideas in an earlier unit; (b) teach sub-ideas across units, teaching a sub-idea in a way that covers that sub-idea in all the units together; or (c) a combination of the above.

The AIM EU modules are designed to show sequences within and across units. However, it is always YDC's policy that schools should be free to adapt the modules to suit the needs of the school and the students. This should also be true of the materials for teaching provided in the units in the modules. These are exemplars of lessons and test items and schools should feel free to use them as they are or to modify them. The RAMR framework itself (see Appendix B) is also flexible and should be used that way.

Together, the units and the RAMR framework are designed to ensure that all important information is covered in teaching. Therefore, if modifying the order, try to ensure the modification does not miss something important (see **Appendix C** for detailed teaching frameworks).

RAMR lessons. We have included RAMR lessons as exemplars wherever possible in the units of the module. Activities that are given in RAMR framework form are identified with the symbol on the right.



Suggestions for improvement. We are always open to suggestions for improvement and modification of our resources. If you have any suggestions for this module, please contact YDC.

Unit 1: Continuous vs Discrete

The purpose of this unit is to introduce the idea of number in relating to length (particularly in terms of numbers on a line) and to discuss some of the major applications of length models for number with respect to rank, comparison and order with one-digit numbers and counting numbers up to 20.

1.1 Set vs length (relating number to distance)

In this section, we want to show how counting along a line gets you distance, how this depends on the size of the steps or differences between distances or between objects on a line, and how equal distances between objects relate number to distance. **The focus of this section is that tracks are made so that number gives us an indication of distance**.

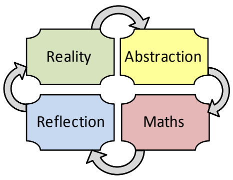
1.1.1 Set vs length activities

1. Act out and discuss how number can be calculated from a set of objects. Recap 1:1 correspondence and matching one and only one object to each number name as the reason for this.
2. Move to objects along a line (like lily pads) – count along the line – discuss how higher counts get you further along the line.
3. Count footsteps again, seeing how more footsteps get you further and fewer footsteps get you less distance. Discuss size of steps and effect of this. Cut out footsteps and place where to step – relate this to objects in a line. Replace footsteps with a rectangle/square in which to place feet.
4. Make a number track, throw dice and move counters along the track – who gets further, why do they get further? Have different students walk a set number of steps. Who gets further and why? Compare different tracks with different spaces to see who gets further – students can choose the track that will get them further.
5. Explore straight and curved tracks – which gets you from A to B in shorter steps even if the squares in the track are the same size? What if the tracks have the same curve?
6. How could we get the objects in line so that the same number of steps is the same distance?

The idea is to get students to realise that the track must be straight or the same curve and the steps the same length for this model to be useful in relating number to distance. It is OK to use the same track for each person but if tracks are different then distances in each step have to be the same.

It is the **experience that is important** – not the correct answer. Treat all students' insights as "correct" at this point. Have discussions and debates – let students construct their own tracks to see what is possible.

1.1.2 RAMR lesson for notion of number line



Learning goal: How number relates to length, i.e. how equal distances between objects relate number to distance.

Big ideas: Continuous vs discrete.

Resources: Blocks, straws, tracks, items you can line up, ladders, stepping stones.

Reality

Local knowledge: Where possible, find real-life contexts to embed the activities in; for example, using relevant objects or situations.

Prior experience: 1:1 correspondence, counting, sorting.

Kinaesthetic: Use materials to construct tracks, e.g. outdoor blocks or stepping stones; act out how number can be calculated by walking/stepping.

Abstraction

Body: Count discrete objects in the environment.

Hand: Use concrete materials such as blocks/straws to "count" length or area of objects.

Mind: Discuss what things can be counted and what cannot. Point out that, normally, items have to be discrete (individual and separated) to be counted. Point out that the world is full of discrete things (chairs, people, animals, days, grains of sand, etc.) but that some things have had to be "changed" to be countable. Discuss length – it is not countable unless a unit of measure is used. Length is continuous (as is area, volume, mass, time, and so on) but units make it discrete. Ask: *Why do we want to turn the continuous into the discrete?* [so we can apply number to it.]

Creativity: Visualise body acting out.

Mathematics

Language/symbols: Number line, points, zero, quantity, equal partitions/lengths, start, end, interval, furthest, closest, larger, smaller.

Practice: Continue to engage in counting and measuring activities that demonstrate the continuous vs discrete big idea along tracks, ladders and with placing number cards.

Connections: With length.

Reflection

Validation: Have students make their own number track games to play with dice.

Applications/Problem-solving: Have students step along a pre-made line. Ask: *How many steps did you take? Did you all take the same number of steps? How could you make it so we all take the same number of steps? Why did the number of steps differ?*

Generalising/Changing parameters: Get students to generalise that discrete can be counted but continuous needs a way (e.g. a unit of measure) to change it into discrete for a number to be used on it. Discuss changes to numbers – count steps from one number to the next and that 0 is now the starting point – not nothing.

1.2 From track to ladder to line

In this section we will work with "steps" being the same distance and lines/tracks being the same curvature at the start. At the end, we will work with straight tracks. **The focus of this section is how tracks are counted (middles or ends)** and the movement from counting the middles to counting the ends.

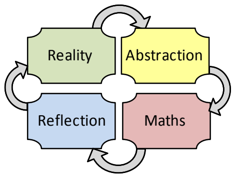
1.2.1 Track to ladder to line activities

1. Student walk – *do we count the steps or the end of the steps?*
2. Number track – *do we count the middle of each track or the end of the track?*
3. Set up a track (all distances the same) and compare starting outside the track and then moving into the spaces at each step with starting on first line and counting along the lines – *what happens to the count?*
4. Start with number tracks and experience these, then go to ladders and experience these (or tracks as ladders) – discuss where the numbers go – make a track and a ladder, make up number cards and place on

track or ladder as student moves along – *what is missing?* [zero for the starting point – need it for ladder but not really for track].

5. Cut down the ladder to one side and short rungs – draw this and call it a number line.
6. Draw number lines for a variety of numbers (0–6, 0–15, etc.) – walk these lines, use them to replace tracks with dice and counters. Make sure students know properties of a track – give them wrong ones to recognise as not proper tracks.

1.2.2 RAMR lesson for number ladder



Learning goal: Relate numerals to positions on the ladder and vice versa.

Big ideas: Continuous vs discrete.

Resources: Number ladder – large on floor, small number ladders with counters (with and without numerals).

Reality

Local knowledge: Look for things that count jumps – walking, house numbers, games, kangaroos, etc.

Prior experience: Check that the students have counting and an understanding of counting and numerals for objects.

Kinaesthetic: Get students to walk the large number ladder – focus on new role for zero and counting movements (e.g. jumps) not stopping points.

Abstraction

Body: Extend ladder actions and activities.

Hand: Repeat counting activities using small number ladders and counters – without numerals and then with numerals – relate real world to model to language and finally to symbols using Rathmell triangle.

Mind: Get students to shut eyes and imagine the number ladder and walk it in their minds.

Creativity: Ask students to make their own number ladder with their own symbols and walk it.

Mathematics

Language/symbols: Model and use formal language and symbols for linear model.

Practice: Relate numerals to positions on the ladder and vice versa. Count out to get positions. Play race games.

Connections: Connect to set model – relate numeral, number name, drawing of a position on a number ladder and set of objects using physical and pictorial materials and games such as bingo and mix-and-match.

Reflection

Validation: Discuss the action of number ladder – students discuss where used (e.g. ruler).

Applications/Problem-solving. Apply to number-ladder problems in world (e.g. house numbers, running blocks).

Extension:

- *Flexibility:* Brainstorm number ladder or length applications (e.g. speedometers).
- *Reversing*: Ensure you go from position on number ladder to language and symbols.
- *Generalising:* Get students to state understanding of zero in number ladder; get them to discuss what the numbers do (tell you jumps), show examples where count ends instead of jumps (e.g. 5 + 3 is 5, 6, 7, 8).

- *Changing parameters:* Set up ladder so that every 2 or 5 steps is bolded or coloured red – count twos and fives. Extend ladder to hundreds and count in tens or fives. Make each step more than one (e.g. to count as two or five); discuss what this means.

1.3 Early rank

For this we will work in correct number lines. We will divide number lines into **regular intervals** (called **cross-marks** or just **marks**) and use this to find numbers. We will focus on rank which is the ability to place numbers correctly in relation to end points.

1. Set up a number line – put numbers on number-line marks from 0 onwards. Ask students to find numbers on the number line – *Where is 7? Point to 9* and so on.
2. Replace this number line with one with no numbers but all cross-marks; put in numbers at regular intervals along marks – say for a 0–12 line, put in numbers at 0, 3, 6, 9, 12. Use these numbers to place and find numbers for other marks – *Where is the 7? Where is the 5?* Repeat this for many examples.
3. Repeat above for examples where numbers are not put in regularly – *Is this more difficult? Why?* Repeat this. Repeat the above with numbers placed but no cross-marks. *Is this more difficult? Why?*
4. Reverse procedure – put out blank number lines and ask students to place some numbers to make it easy to identify where other numbers go. Example 0–16 number line – have to place three numbers to make it easy [4, 8, 12]. Repeat this.
5. Finally, put out 0–20 number line and ask for numbers to be placed to make it easier to find or place other numbers. Do an extension – put out a blank number line with no cross-marks and ask students to place numbers. State that to **rank** numbers is to place them so that they are in the same place they would be if all numbers were on the line.

So we have **four types of number line** – (a) cross-marks and numbers, (b) cross-marks and some regular numbers, (c) no cross-marks but regular numbers, and (d) no cross-marks and no numbers. Students should know that intervals for cross-marks are regular.

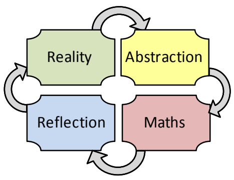
1.4 Early comparison and order

Here we classify a number as larger if it is further along the line away from 0 and smaller if it is nearer the 0. In other words, to compare two numbers, we place them on the line and the one further from 0 is the larger (and the opposite for the smaller). For ordering more than two numbers, we place them on the line and their order is determined by their position on the line relative to each other and to zero.

Once again, we go through the four types of number line, placing numbers as we go and then writing them in order small to large (away from 0) and large to small (near to 0). Work with rank helping and then without rank.

Activity for Preps – line up on ages, line up on birthdays – let students determine how to do this.

1.4.1 RAMR lesson for early rank, comparison and order



Learning goal: Using regular intervals; exploring that a number is larger if it is further along a line away from 0; and order is determined by position on the line.

Big ideas: Continuous vs discrete; quantity on a number line; order.

Resources: Tape, rope, pegs, number cards, whiteboards, rank markers, rope ladder.

Reality

Local knowledge: Where possible, find real-life contexts to embed the activities in; for example, using relevant objects or situations. For example, taking a number when queuing at the deli counter at the supermarket (order); things in the environment that occur at regular intervals such as fence posts or dotted lines on pavement, distance thrown or other relevant length situations (rank).

Prior experience: Counting numbers, number to space connection, place numbers on lines with regular intervals.

Kinaesthetic: Play numbers around the room. Each student takes a number card (0–10, 0–40 or numbers appropriate to the group). Ask students to arrange themselves from least to most. Ask students to justify why they have chosen to stand where they are. Do they use language of greater, less, larger number, smaller number, next to, between? Do they identify the "spaces" or missing numbers? Do they line up by "order" or by "rank"? What do they do if two students have the same numbers? Encourage discussion about this. When is order sufficient? When do we need rank? How do they decide to create equal intervals?

Abstraction

Body: Track to ladder to line activity from section 1.2.

Hand: Have pegs with numbers written on them and a rope. Ask students to construct a number line 0–20. Provide blank lines and various activities where students place numbers on lines; have students write them in order small to large (away from zero) and large to small (near to zero); work with rank helping, then without rank. Use Unifix cubes, bead strings with markers, etc.

Mind: Play forwards and reverse to develop an imagined number line. Have a student count forwards by ones/clap slowly while the other students imagine counting forwards along a number line each time the leader claps. After several claps, the leader stops the count so the students can say what number they are at. Vary the game by using signals to change to counting backwards. The leader can decide when to change direction of the counting. Later the leader can direct the students to change from counting by ones to skip counting (e.g. by threes, fives, tens) by giving a unit of say three claps, then repeating it for each jump.

Creativity: Ask students to create their own number sequence in "order" and then in rank. Ask: *What is the same? What is different?*

Mathematics

Language/symbols: Number line, points, zero, quantity, equal partitions/lengths, start, end, interval, furthest, closest, larger, smaller, rank, comparison, order, midpoint, cross-marks.

Practice: Give the students a number line with 0–20 marked divisions and the numbers 0 and 10 on it. Ask them to fill in/complete the numbers. Have them circle the beginning-point number and the end-point number. With a blue pencil have them circle the midpoint number. With a yellow pencil have them circle the number between 10 and 12; the number after 16; the number before 7. With a green pencil ask them to circle two numbers greater than 10; with a pink pencil circle two numbers smaller than 10.

Next have them look at an open (unmarked) number line. Ask them to make a number line for the numbers 0–10. Do they identify the beginning point, end point, and midpoint? What intervals do they use?

Connections: Length, measure, patterns, operating and problem-solving.

Reflection

Validation: Ask students: *What is a number line? Why would we write numbers on a number line? What can you learn about numbers from a number line?*

Applications/Problem-solving: Ask students: *What numbers are related to each other and how? Can you think of a problem you might be able to solve using a number line?*

Extension:

- *Flexibility*: Ask: *What other numbers could you place on a number line?*
- *Reversing*: Give students numbers 6, 8, 10, 12 and ask them to draw and place these numbers on a number line.
- *Generalising/Changing parameters*: Ask students: *Can you see some interesting number patterns? Can you make a number pattern on a number line?*

Unit 2: Number-Line Activities

This unit looks at the central ideas that emerge from number lines with and without cross-marks and with and without numbers. This is done for 0–99 (T-O) and then for 0–999 (H-T-O).

2.1 Rank

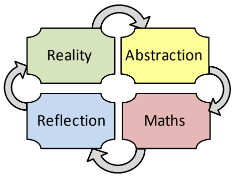
This is the ability of students to place numbers on a number line in relation to start number and finish number so that they are close to where they would be if all numbers were on the line evenly spaced.

To do rank appropriately requires a lot of knowledge at once – to know proportionally where different numbers are, to have in the mind a breakdown of the distance between start and finish that clumps of number would be around. Most usefully, place-value positions become the better way to do this, but in H-T-O it may be easier to see the line in blocks of fifties and twenties not tens. This of course can lead to errors with students thinking that, for example, 21 on a 0–100 number line with marks/numbers every two is 20 and one more step (instead of half a step).

2.1.1 Rank activities

1. Do rank for numbers to 100 with the **four types of lines**:
    (a) cross-marks and numbers,
    (b) cross-marks and some regular numbers,
    (c) no cross-marks but regular numbers, and
    (d) no cross-marks and no numbers.
2. There are **four types of activity** to do this with:
    (a) a masking tape line with/without marks, and number cards to place where card should be placed;
    (b) a rope with start and finish numbers on students holding the rope, with/without marks, number cards and pegs so that students can peg their numbers on the rope/cord;
    (c) a large ship's rope with/without marks and numbers that students can walk or run alongside (or throw bean bags at); and
    (d) using a computer (virtual lines and placement).
3. There are **four different lesson ways** that the lesson could go:
    (a) structured – put out line with start and finish known and with regular intervals marked and fit the numbers in between (using Tens positions and Ones rank knowledge of placing numbers 0–10), then slowly remove the guides;
    (b) learn as go – put out line with start and finish known and few/no marks and let students explore where to put the numbers;
    (c) using bodies – could be lining up themselves on characteristics or numbers given; and
    (d) open inquiry method – put out line with nothing on it and give students numbers and they have to work out what the start and finish could be and where numbers best go.

2.1.2 RAMR lesson for rank, comparison and order



Learning goal: To place numbers on a number line in relation to start number and finish number so that they are close to where they would be if all numbers were evenly spaced on the line.

Big idea: Quantity on a number line – rank.

Resources: Tape, rope, number cards, pens, whiteboards, pegs, markers, rank markers.

Reality

Local knowledge: Where possible, find real-life ordering and ranking contexts to embed the activities in, using relevant and familiar objects or situations.

Prior experience: Begin with 0–10; then move to 0–100; then 0–1000.

Kinaesthetic: Peg Up. Hang a piece of string across the room. Ask each student to write any number on a card. Select one student to peg their card on the line. Then, invite other students to determine whether their number is larger or smaller and, in turn, add their number to the line. As more numbers are added to the line, students will need to decide whether the position of some cards has to be changed in order to get the sequence right. Over time students can add more numbers to the line as they arise. Do they use language of greater than, less than, larger number, smaller number, next to, between? Do they identify the "spaces" or missing numbers? Do they peg up by "order" or by order and rank? What do they do if two students have the same numbers? Encourage discussion about this. When is order sufficient? When do we need rank? How do they decide to create equal intervals?

Abstraction

Body: A large ship's rope with/without marks and numbers that students can walk or run alongside (or throw bean bags at).

Hand: A masking tape line with/without marks, and number cards to place where card should be placed.

Mind: Visualising and checking using a computer (virtual lines and placement).

Creativity: Drawing their own number line.

Mathematics

Language/symbols: Number line, points, zero, quantity, equal partitions/lengths, start, finish, interval, furthest, closest, larger, smaller, rank, comparison, order.

Practice: Using bodies – could be lining up themselves on characteristics or numbers given.

Put out line with start finish known and with regular intervals marked and have students fit the numbers in between; use individual whiteboards for students to record what they do.

Provide lines with and without marks and with and without numbers and ask students to complete them.

Connections: Measurement – length; operations – count on.

Reflection

Validation: Ask students to construct a number line using numbers 0–100. Have them count forwards and backwards by tens. Ask: *At what number does your number line need to start? Does it need to show all the numbers in between the counted numbers? Why? Why not?*

Play the "number line" game. Pack of cards with numbers on them, a number line with numbers for checking. Students in pairs take turns to turn over a card. They must say the number, the number one greater than it, and the number one smaller than it. Then check the number line to see if they had the correct order. If so they keep the card and it counts for one point.

Applications/Problem-solving. Have students do some problem-solving where they need to order measurements of length (say using straws). Ask them to record their work on a number line.

Extension:

- *Flexibility*: Have students create a number line with their own numbers and rankings.
- *Reversing*: Have students consider some number lines with arrows indicating points on the line. Ask students to decide what number is indicated by each arrow.
- *Generalising/Changing parameters*: Ask students to choose two numbers. Have them show the quantities on a number line. Ask: *Which number is larger? How do you know? Which is smaller? How do you know?*

2.1.3 Tens and Ones

The activities here are to pick a way and just repeat different situations – could do 0–20 in fives, 0–100 in tens, 0–100 in twenties, 0–100 with only 50 shown, or only the end points shown.

Recommendations:

- Get students to work together – students taking out 27 to place are helped by class.
- Justify – when placed, ask students to justify their placement (have string same length as line to be folded for halves, quarters, fifths, etc.)
- Focus on benchmarking – encourage students to pick something they can relate the number to – e.g. 45 is near halfway (50) but less than 50, but close to 50 when looking at 0–100 line.

2.1.4 Hundreds, Tens and Ones

This is the same as above but more complex – can use hundreds to place numbers, can then break 100 into fifties or twenties or tens for more accuracy. Try to get students to use experience and learn from what they do – e.g. in 0–1000, the numbers 9 and 23 are both very near the 0 while 990 is very near the 1000.

May need to spend time looking at similarity between 0–100 broken into tens and 0–1000 broken into hundreds.

2.2 Comparison/Order

Once we have rank, it seems easy to do comparison and order – just rank the numbers and we have the order. However, in the longer run we need to do more as will be shown in Unit 4. The focus of order has to be that in 0–100 it is the tens that matter, and in 0–1000 it is the hundreds that matter. This is so we can do order with place value.

The rule of comparison/order is that the larger/largest number is the one with the biggest digit in the biggest place value. DO NOT say that the largest number is the longest as this does not work for decimals.

2.2.1 Tens and Ones

It is necessary to teach students that the number with the most tens is longer – it is the tens that count (unless the tens are the same).

A good way to do this is to cut 10 cm straws (two colours), and join with string through them with alternate colours – label the tens if needed – then measure so students have to say lengths are 4 tens and a little more, 3 tens with nearly all for the next 10 – they can record the longer and look for the pattern.

An activity that works well is to play front-row forward on a rugby oval and get students to run, for example, 53 m and 38 m – they'll soon learn that the more tens is longer!

Also, dice order or chance order are good games here.

2.2.2 Hundreds, Tens and Ones

It is necessary to teach students that the number with the most hundreds is longer – it is the hundreds that count (unless the hundreds are the same – then the tens count). To support this, it is important that activities and discussion in the above subsection and this one should begin to develop the generalisation/rule for order.

Do similar activities with 0–1000 as you did with 0–100. The large rope is good – make it 0–1000, mark hundreds – and then walk distance to place numbers – or line whole class up at right angles to the rope and all walk together to called-out numbers. This works better if the 100 is a long jump – students like jumping the hundreds and adding on the rest. Or do basketball games of running to numbers and back again or throw bean bags – closest wins a point for their side.

2.3 Density

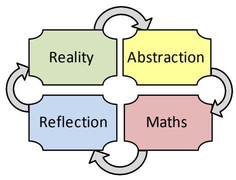
Whole numbers are not dense but fractions and decimals are – there are no whole numbers between 2 and 3 but an infinite number for fractions and decimals (e.g. $2\frac{1}{2}$, $2\frac{1}{3}$, and so on and 2.5, 2.55, and so on). Ideas:

(a) Line up students on a track marked 1, 2, 3 and so on – fill all the squares – can't fit anyone in – musical chairs – adding and removing chairs.

(b) Move to number line – can fit students in between 2 and 3 so number line has potential to be dense.

(c) Density activity with Preps – line up on ages so they can see clumps.

2.3.1 RAMR lesson for density



Learning goal: Whole numbers are not dense but parts of wholes are; there are numbers between whole numbers.

Big ideas: Continuous vs discrete; comparison and order.

Resources: Calculators, number-line materials, pens, drawing paper, cards to write on, Blu Tack.

Reality

Local knowledge: Look for things that move in order or steps or sequence.

Prior experience: Check that the students have counting and an understanding of counting and numerals for objects.

Kinaesthetic: Age groups: Have students enter their age into their calculators, then organise themselves into groups according to the number shown on their calculators. Ask: *What does your 5 (6, 7, and 8) mean? Who is exactly 5 (6, 7, 8) years old? Who is more than 6 years old, but not 7 years old? Can you show this on the calculator?* Look for a response that can be developed into placing their ages in order: "I am six and a little bit, a part of a year" and so on. Record on a birthday-cake number line. A cake for 5, a cake for 6, a cake for 7, and have children place their birthdates in appropriate sequence. Ask: *Can birthdays be on the same spot on the birthday-cake number line?*

Abstraction

Body: Explore birthdates in the current month by creating a calendar using the grid mat.

Hand: Have students construct (use Unifix cubes or similar materials) their own number lines to show the movements for when they were younger and when they will be older. Where are the whole years? Where are the parts of the year? Ask: *At what number does your number line need to start? Does it need to show all of the numbers in between the birthday numbers? Why? Why not?*

Mind: Have students visualise their birthday number line. *Can you see where on the number line you will be in two years? When you are a teenager? An adult? A grandparent?*

Creativity: Ask students to create a birthday number line for someone special in their life.

Mathematics

Language/symbols: Interval, line model, spaces and lines, zero.

Practice: Use bead strings and number cards for students to practise creating and reading number lines. Have cards with stories and situations or problems that could be represented on the number line. Ask students to do this (can be done individually or working in groups). Can students place interval markers? Do they use zero as the beginning reference point? Do they order and compare their numbers appropriately? Do they recognise there are numbers between whole numbers? Have students/student groups compare their solutions with others. Ask each student to record their number-line situation and actions.

Connections: Measuring time – calendars; months as parts of a whole year. Fraction parts.

Reflection

Validation: Ask students: *Where else can you think of where numbers come in clumps?*

Applications/Problem-solving: Ask students: *What is a number line? Why would we write numbers on a number line? What can you learn about numbers from a number line?*

Flexibility. Ask students: *What numbers are related to each other and how? Can you think of a problem you might be able to solve using a number line?*

2.4 Construction of number lines

This is reversing the activities in the previous sections so that students construct the number lines. To do this, the students need some parameters for the line such as end points and purpose. Then the students have to construct a line that meets that purpose. Mostly the purpose is for the line to act as a simple quick visual method for determining amount. However, it can be a line for accurate measurement (e.g. injection needle or micrometer gauge). For example, what is the best line for a speedometer – do we have a mark every km/hr or every 5 km/hr?

Activity can be based on the three components needed for ranking on lines:

A. a line with a start and a finish;

B. intervals marked on the line; and

C. numbers to be put on the line.

Students can be given one of these three and asked to construct the other. This means that we **reverse** what we have done in earlier sections – from line → placement of numbers **TO** numbers to be placed → line. As an added activity, students can be asked to construct **easy and hard** lines on which to place the given numbers.

So, for this section, the activities devised reverse previous activities where possible. However, there are three types of activities A → B/C, B → A/C, and C → A/B, as the following examples show:

1. Starting from A, provide students with the start and finish and a purpose and ask them to make a number line with that start and finish and numbers to be placed (and possibly some marks with some numbers marked in). The students have to justify what they have constructed in terms of the purpose of the line which could be ease of reading the numbers. Other purposes could be accuracy or simplicity. You could ask the students to construct a number line with marks and a set of numbers that lead to error in many readers. You could further modify by asking for a circular number line.
2. Starting from B, provide the line with marks and ask students to provide a start and finish and a set of numbers to place. For variation, you could ask for a set of number cards and start and finish that are easy to place. Another variation is to ask for a set of numbers and start and finish that are hard to place – test most people's ability to place the numbers correctly.
3. Starting from C, provide the students with the numbers and they have to construct a number line and start and finish that is easy and hard to place the numbers. This could be achieved through the marks or the start and finish.

Overall, ask students to see the patterns in the three types of activities. Encourage them to see that:

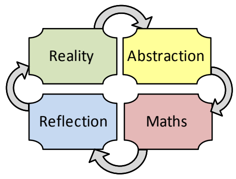
(a) providing A start-finish and B intervals marked means they have to find C cards as the answer;

(b) providing A start-finish and C cards means they have to find B intervals/marks; and

(c) providing B intervals/marks and C cards means they have to find start-finish.

This is another example of the triad big idea.

2.4.1 RAMR lesson for construction of number lines



Learning goal: Construct a number line with regard to the start and finish point.

Big ideas: Continuous vs discrete; comparison and order.

Resources: Calculators, number-line materials, pens, drawing paper, cards to write on, Blu Tack, bead strings, pegs, rope.

Reality

Local knowledge: Where possible, find real-life contexts to embed the activities in, using relevant objects or familiar situations.

Prior experience: Counting, numerals sets, ladders and tracks.

Kinaesthetic: Have students key a number into their calculator. Ask students to order themselves from least to greatest number. Do they use language of greater than, less than, larger number, smaller number, next to, between? Do they identify the "spaces" or missing numbers? Do they line up by "order" or "rank"? What do they do if two students have the same numbers? Encourage discussion about this: *When is order sufficient? When do we need rank? How do you decide to create equal intervals?*

Abstraction

Body: Track to ladder to line activity from section 1.2.

Hand: Have pegs with numbers written on them and a rope. Ask students to construct a number line 0–20. Provide blank lines and various activities where students place numbers on lines, writing them in order small to large (away from zero) and large to small (near to zero); work with rank helping, then without rank. Use Unifix cubes, bead strings with markers, etc.

Mind: Play forwards and reverse to develop an imagined number line. Have a student count forwards by ones/clap slowly while the other students imagine counting forwards along a number line each time the leader claps. After several claps, the leader stops the count so the students can say what number they are at. Vary the game by using signals to change to counting backwards. The leader can decide when to change direction of the counting. Later the leader can direct the students to change from counting by ones to skip counting (e.g. by threes, fives, tens) by giving a unit of say three claps, then repeating it for each jump.

Creativity: Ask students to create their own number sequence in order and then in rank. Ask: *What is the same? What is different?*

Mathematics

Language/symbols: Number line, points, zero, quantity, equal partitions/lengths, start, finish, interval, furthest, closest, larger, smaller, rank, comparison, order.

Practice: "Start and Finish". Hang a piece of line and peg up two cards: one labelled 0 on the left and one labelled 5 on the right. Invite students to say where the cards labelled 2, 3, 4 would go and explain why they think this.

Remove the label 5 and peg the label 10 in its place. Invite the students to say where the cards labelled 2, 3, 4 would go. Encourage students to explain why they have placed their cards in a particular place. Ask another student to place the card 5. Ask: *How does this card help?*

Invite students to record these number lines then create one with their own chosen end point. Have them exchange these with a partner for reading and sharing to the group.

Connections: Length, intervals, comparisons.

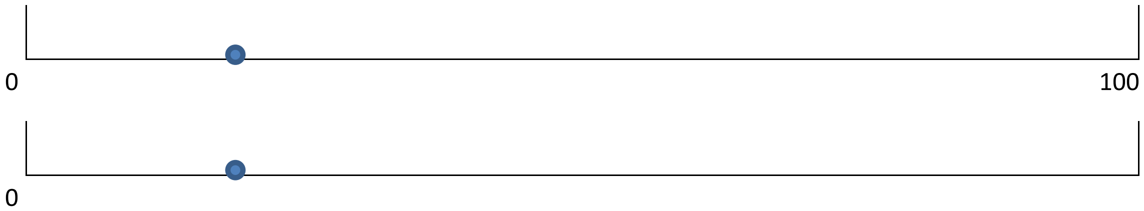
Reflection

Validation: Ask students to construct a number line using numbers 0–100. Have them count forwards and backwards by tens. Ask: *At what number does your number line need to start? Does it need to show all the numbers in between the counted numbers? Why? Why not?*

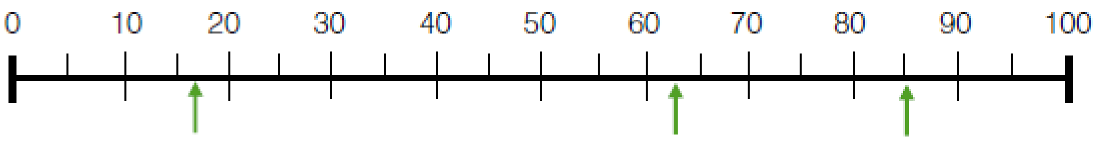
Applications/Problem-solving: Have students do some problem-solving where they need to order measurements of length (say using straws). Ask them to record their work on a number line.

Extension:

- *Flexibility*: Have students create a number line with their own numbers and rankings. Show the number lines below and ask: *What number on the number line might the dots represent?*



- *Reversing*: Have students consider some number lines with arrows indicating points on the line. Ask students to decide what number is indicated by each arrow.



- *Generalising/Changing parameters*: Ask students to choose two numbers. Have them show the quantities on a number line. Ask: *Which number is larger? How do you know? Which is smaller? How do you know?*

Unit 3: Position and Equivalence

This unit continues from Unit 2 in applying number lines to develop approximation and estimation, and rounding, with an added section on equivalence/invariance.

3.1 Approximation and estimation

Many situations exist where exact numbers and calculations are unnecessary. If we have a group at a party, we do not count them, we simply say there were about 20. This means in reality there were probably between 15 and 25 at the party, but for some situations "about 20" could even refer to 27 people or 12 people.

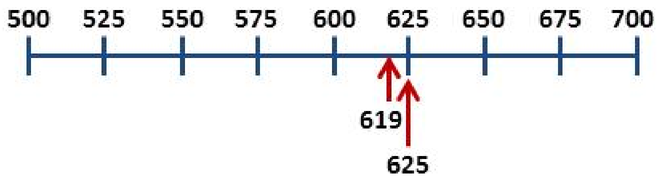
Approximation is used in these situations because it is not necessary to be accurate and we may be using the information in estimations not calculations. However, it works best if the approximation is close to the real number, say within 3 of 20; it can be a problem if the approximation is too far from the actual number. Approximately 20 may mean the actual number is 18, but approximately 200 is not a very good approximation if the actual number is 90. The closeness of the approximation depends on the number size and the needs of the approximater.

Approximations are chosen on the basis of being easy numbers to work with as well as being close. For example, it would be surprising if someone used 253 as an approximation for 262, but acceptable and normal practice to use 250 as the approximation.

3.1.1 Using marks on the number line

Number lines assist approximation and estimation because they allow visually for an estimation to be picked up quickly. For instance, a number line with start–finish of 300–500 with every 10 marked would be good for estimations to the nearest 10. On this number line, 426 would be estimated as 430 because that is the closest 10. If only 50s were marked then 426 would be approximated as 450.

The process is to look at the number to be approximated, say 619, and decide on a suitable length/range of a number line to use, say 500 to 700. Next, mark in certain intervals, say every 25, and place the number on the line. It can quickly be seen that an approximation of 619 to the nearest 25 is 625:



625 approximates 619 with accuracy 25

So the complete process is:

(a) choose the level of accuracy that is suitable for what you want (e.g. within 10);

(b) decide on the range for the number line in which to find the approximation (e.g. 300–500);

(c) mark in the cross-marks on the number line (e.g. every 10);

(d) place the number on the line; and

(e) approximate to the nearest cross-mark.

Activities

1. Make a line 0 to 500 marked every 25. Choose a suitable accuracy. Give an approximation for:

   (a) 368 (b) 37 (c) 409

2. Make a line for steps walked from 0 to 10 000. Choose a suitable accuracy and give approximations for:

   (a) 969 (b) 255 (c) 6 873

3.1.2 Reversing

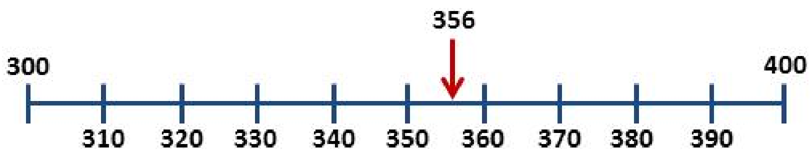
In the work so far we have gone from number → number line → approximation. We can reverse this to go from approximation → number line → number. For example, take a number line from 200 to 400 with cross-marks every 20. Ask students to state numbers that could have been approximated to (a) 340 and (b) 280.

Thus, in Unit 2, we put out a structure of marks on the number line and used these to place another number correctly. Now we have the number and we look at the marks on the number line and find the closest mark. So, before, we had marks and a number → we placed the number between marks; here, we have a number and marks → we look for the closest mark to the number.

3.2 Rounding

Rounding is similar to approximation and estimation in which we place a number and look for the closest mark on the number line. However, for rounding, the mark is given in the task and is usually a place value. There are rules so that the place value alone decides the rounding.

This means that the problem will provide a number and what it is to be rounded to, for example: *What is 356 to the nearest 10?* Now we need a number line that has a start–finish that encloses the number (in this case, 300–400 would be suitable), with marks at every 10:



We now choose the nearest 10, which is 360, so 356 rounded to the nearest 10 is 360.

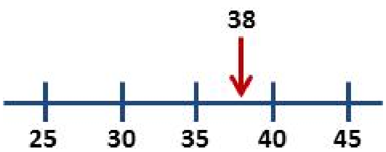
So the process is that we have a regular arrangement of marks on a number line, we place the number and then we round to the nearest mark. For the example: *What is 27 to the nearest 10?* – 27 is placed between 20 and 30 and is nearest to 30, so we would round to 30. However, what happens for a number such as 25 which is halfway between 20 and 30? In these cases there is a **convention** that the halfway goes up, so 25 to the nearest 10 is 30.

3.2.1 Tens and Ones

Here the rounding does not have as many options as for larger numbers. There is a limit on how many different amounts one can round to. If rounding to the nearest 10, then:

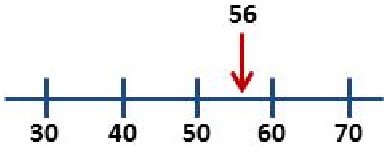
32 → 30	as 32 is nearer to 30 than to 40
56 → 60	as 56 is nearer to 60 than to 50
75 → 80	as 75 is halfway between 70 and 80 but the convention says to round up to the higher number

If rounding to the nearest multiple of 5 then it is a little more difficult. For the example, *round 38 to the nearest 5*, 38 has to be placed on a number line marked in steps of 5, as shown on right; 38 is nearer to 40 than 35, so 38 rounded to the nearest multiple of 5 is 40.



Activity

Use masking tape or a rope on the floor marked from 0 to 100. Give students a number to be rounded, say 56. Find the intervals/marks for rounding to the nearest 10. Place markers on the tape/rope, place number and then discuss rounding.



56 rounded to nearest 10 is 60

3.2.2 Hundreds, Tens and Ones

The addition of the hundreds increases the options for rounding. The marks on a number line to which numbers are rounded are usually the nearest 10 or nearest 100, as these are place values. However, it is possible to have to round to the nearest 20 or 50.

If rounding to the nearest 10, then:

236 → 240 as 36 is nearer to 40 than to 30

455 → 460 as 5 is halfway but the convention is that we round up to the 10 above the number

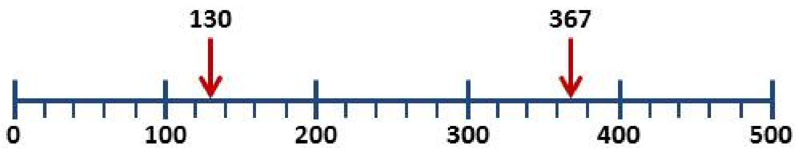
If rounding to the nearest 100, then:

236 → 200 as 236 is nearer to 200 than to 300

455 → 500 as 455 is nearer to 500 than to 400

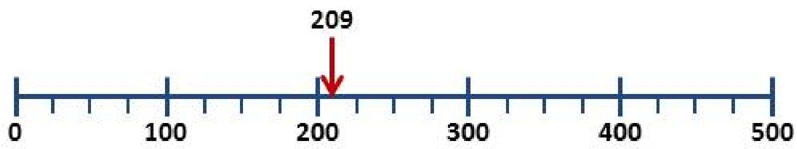
Activity

1. Use masking tape or a rope on the floor marked with 0 and 500 on either end. Give students numbers, say 367 and 130, to be rounded to the nearest 20. Put marks on the number line at intervals of 20 and ask students to place their numbers:



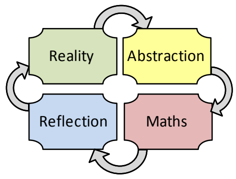
2. Discuss rounding to the nearest 20:
   (a) 130 is between 120 and 140 and is halfway between, so following the convention that numbers halfway in between round up, 130 rounds up to 140.
   (b) 367 is between 360 and 380 and is nearer 360, so rounds to 360.

The difficult rounding is to the nearest 25:



As an example we see that 209 is between 200 and 225 and is closer to 200, so 209 rounded to the nearest 25 is 200.

3.2.3 RAMR lesson for rounding



Learning goal: Approximation and estimation – recognising the marks on the number line are usually place value and thus can round to 10 and 100; also use the convention of halfway goes up on the number line.

Big idea: Continuous vs discrete; comparison and order.

Resources: Rope, cards numbered 1–99 (more than one of some numbers), packs of smaller cards numbered 1–99, printed number lines, counters, board games, dice, laminated number-line strips, paperclips.

Reality

Local knowledge: Where possible, find real-life contexts to embed the activities in; for example, using relevant objects or situations.

Prior experience: Continuous/discrete; comparison and order.

Kinaesthetic: Play "Have the power". Have a rope laid out. Ask students to take a card with a number on it (1–99 with tens numbers ending in zero removed). Have them order themselves from least value (on the left) to most value (on the right) along the rope. Do they use language of greater than, less than, larger number, smaller number, next to, between? Do they identify the "spaces" or missing numbers? Do they line up by "order" or "rank"? What do they do if two students have the same numbers? Encourage discussion about this. When is order sufficient? When do we need rank? How do they decide to create equal intervals? Say: *Today all the numbers with zeros in them have the power,* as you walk along the rope and lay them out from 0–100. *If your number value is close to one of the zero numbers you can go and stand with that zero number and "have the power" too.* Encourage and invite discussion about each number and its distance from a "power" number. What do the students decide to do with the "5" numbers? Do they recognise them as a midpoint? Introduce name "rounding" and the convention of halfway goes up.

Abstraction

Body: Provide packs of number cards for small groups to play "Have the power".

Hand: Use printed number lines and cards for students to create number lines and model "have the power" with counters. Play board games that rely on rounding with each dice throw.

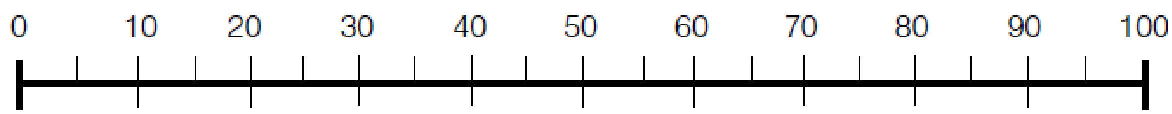
Mind: Invite students to visualise rounding numbers on the number line. Perhaps recall earlier introduction activity and see with the mind's eye where the groups of students ended up.

Creativity: Have each student create a rounding game to share with the class.

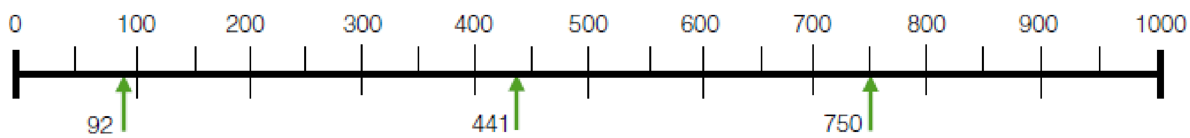
Mathematics

Language/symbols: Round, nearest ten, nearest 100, place value.

Practice: Students mark the numbers 17, 71, 38, 89 on the number line below, and round to the nearest ten.



Look at numbers, say 92, 441, and 750 on the number line below and round to the nearest 100.



Connections: Measurement – length and calibrated scale; early fraction understanding.

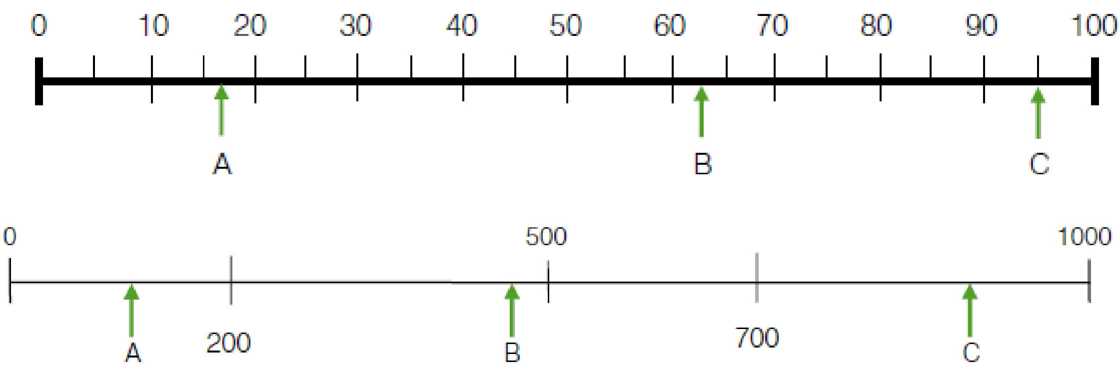
Reflection

Validation: Make laminated number-line strips for each student to use with a paperclip slider. In pairs play "roll to ten" where students take turns to roll a die. They move their clip from zero to the number then have to state if they can round to 10 or not. The first person to reach 100 wins.

Applications/Problem-solving: Ask students: *Can you find a problem to solve using the number line and rounding?*

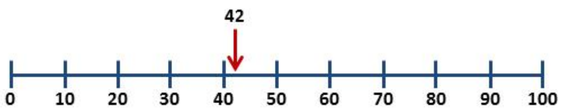
Extension:

- *Flexibility:* Invent variations of the game above, e.g. reverse and round down from 100.
- *Reversing*: Show number lines below and ask students: What numbers could be at A, B and C?



3.3 Equivalence (role of zero)

Equivalence is not a major concept for whole numbers and decimal numbers – it is most important for fractions and ratio (proportion). However, it is effective for students' learning to present equivalent numbers as two numbers occupying the same point on a number line.



1. Draw a number line and pick a point, say 42.
2. What other numbers can be on the same point?
3. To do this, look at zero. When does a zero change the number? A rule for whole numbers is that if the zero is to the left of all the digits in the number and not to the right or in between digits, the number does not change. For example:

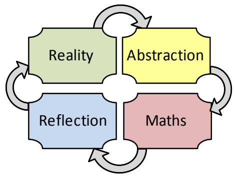
   42 = 042 = 0042 and so on
   42 ≠ 420 42 ≠ 402 and so on

4. If the number is a decimal, then the rule is that a zero before all digits or after all digits does not change the value, but a zero between digits, or between the decimal point and digits, does change the value. For example:

   14.2 = 014.2 = 0014.2 = 14.20 = 14.200 and so on
   14.2 ≠ 140.2 14.2 ≠ 14.02 14.2 ≠ 104.2 and so on

Note. The advent of calculators and computers has meant equivalence has changed. For computers, zeros before digits can mean a different number in situations where the number of digits is important for identification. For example, the month of July is expressed as 07 rather than 7.

3.3.1 RAMR lesson for equivalence



Learning goal: Role of zero; equivalent numbers can occupy the same spot on the number line.

Big idea: Equivalence.

Resources: Calculators, number-line construction materials, number cards, zero cards.

Reality

Local knowledge: Where possible, find real-life contexts to embed the activities in; for example, using relevant objects or situations, such as ordering measures.

Prior experience: Counting; early rank and number-line activities.

Kinaesthetic: Ask students to enter a number between 1 and 30 in their calculators. Organise students into small groups. Ask each group to order themselves, from the greatest number displayed to the lowest number. Then, have two small groups combine and repeat the process. Combine groups again and repeat. When two large groups remain, ask one group to read its sequence. Ask students in the second group to think about where their numbers will fit in the first sequence, before combining the groups to form one sequence. Discuss with students the need to deal with the situation where numbers are doubled up or tripled up. Equivalent numbers occupy the same spot. Have the students change their number sequence to a number line. Ask: *What changes? What stays the same?*

Abstraction

Body: Have students enact placing various numbers on an open number line. Do they use zero as their anchor point? Do they place equivalent numbers on the same spot? Do they rank and order numbers accurately?

Hand: Introduce number-line activities with different representations of the same number. Have activities with numbers with zero in them.

Mind: Have students visualise moving backwards and forwards on number lines; choose from those created in the earlier activities.

Creativity: Have students create their own number lines with as many different number representations as they can.

Mathematics

Language/symbols: Make connections between different presentations of the same number and place them in the same spot on the number line in relation to distance from zero; greater/smaller is further/closer; numbers with zero.

Practice: Have students create a number line. In pairs have them ask their partners to place number cards alternately on each other's line. Have students ask: *Why did you place your number there? How do you know?* Repeat with sets of number cards with numbers that contain zero.

Connections: Measurement – length and calibrated scale; beginning of rulers; early fraction understanding.

Reflection

Validation: Have students investigate where they find numbers with zero in them. Create a collection and order and compare them. Can students place them on the number line?

Applications/problems. Ask students to think of problems where zero is important to consider when solving.

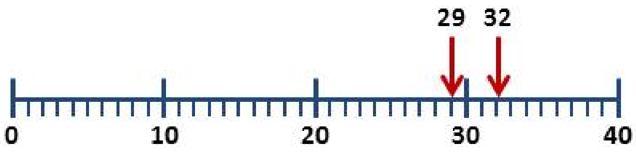
Extension:

- *Flexibility:* Have students create a number line with hundreds and consider what happens with zero now. *Where does it occur on the number line?*
- *Reversing*: Ask students what is the same and what is different about these two numbers – 10, 01.

Unit 4: Relationship to Place Value

In this unit we integrate number line with place value – that is, we relate number as a quantity on a number line with number as place value.

This integration is important because some properties of numbers are best experienced by number lines but should be understood in terms of place value. One of the simplest examples of this is ordering 2-digit numbers, say 32 and 29. If we use a number line we can easily see that 32 is larger than 29 because it is further away from the zero.



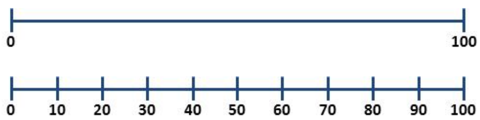
The integration of quantity and place value in this unit will cover distance or length, equivalence, rounding and estimation, and comparison and order.

Background information

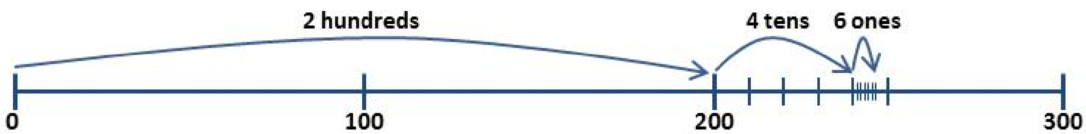
Numbers can be represented in two different but powerful ways: (a) place value, and (b) position on a line. Place value is the important representation because it is how we read and write numbers. However, position on a line is useful in the way it represents certain characteristics of numbers, such as length and order. Therefore, we need to integrate place value with the number line so that number-line properties can be translated to place-value properties. For example, if we make a large number line from 0 to 100 we can easily experience, through walking, that the more tens you have the longer the distance. This immediately shows, for instance, that 62 is longer than 47 because 62 has more tens.

4.1 Length and place value

If we take something like a number line, we can relate it to 2-digit place value by saying the line is a hundred long and then partitioning it into 10 equal parts (see diagram on right).



If we think of numbers as hundreds, tens and ones, we can see that a hundred is the whole length, the tens are $\frac{1}{10}$ of the hundred, and the ones are $\frac{1}{10}$ of the tens. This means that a number such as 246 is 2 hundred-lengths, 4 ten-lengths and 6 one-lengths:



Similarly, 563 would be 5 hundred-lengths, 6 ten-lengths, and 3 one-lengths. This gives an image in terms of the number line of what place value is in terms of length, for solving problems such as: *Which has more length, 349 or 612?* This image is also accurate in terms of the relationships between adjacent place-value positions, e.g. 342 is 3 long, 4 medium and 2 short lengths, with the mediums being $\frac{1}{10}$ of the longs and the shorts being $\frac{1}{10}$ of the mediums.

Activity

Develop a number line long enough to show hundreds, tens and ones. Place numbers on the line. Discuss the relationship between the place-value digits and the length of the line.

4.2 Equivalence and place value

Section 3.3 looked at equivalence and the role of zero. In this section we focus on equivalence and place value relating to length (considering place value positions as a length)

We know from section 3.3 that 042 = 42 but 402 ≠ 42. Let's explore why this is so:

- 42 is 2 one-lengths and 4 ten-lengths; and
- 402 is 2 one-lengths, 0 ten-lengths, and 4 hundred-lengths.

Obviously, 402 is much longer than 42 and is not equivalent.

Looking at place value, anything that changes the place-lengths is not equivalent, i.e. 420 ≠ 42 because there are 4 hundred-lengths on the left-hand number. Similarly, for decimal numbers where we add in tenth-lengths and hundredth-lengths, 4.02 ≠ 4.2 because both have 4 one-lengths but 4.2 has two tenth-lengths while 4.02 only has two hundredth-lengths, making 4.02 shorter than 4.2.

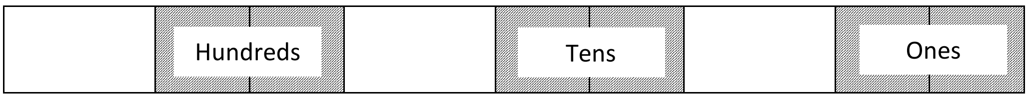
4.3 Rounding, estimation and place value

This section focuses on rounding, as estimation can be considered as a form of rounding in which the estimator makes a decision about what the number is rounded to. For example, 782 rounded to the nearest 10 is 780, but for estimation the rounding is not specified. The estimator could decide to round to the nearest 100 which would make the estimation 800.

Consider again a number line where there are ones, tens (10 × longer) and hundreds (10 × longer again). Since place value is hundreds, tens and ones, then choosing a place value to round to is choosing a place-value position. For example, rounding 27 to the nearest 10 means rounding it to the tens place-value position, which in length terms means that 27 is between 20 and 30 and rounds up to 30 because this is closer.

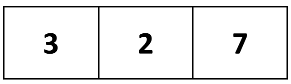
Activity

1. Obtain a number expander for Hundreds–Tens–Ones:

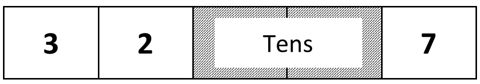


2. Fold this to 3 digits:

3. Place in a number to be rounded:



4. Expand the expander at the rounding number (tens); this shows that you have 32 tens and 7 left over:



5. Since 7 is closer to 10 than to 0, the rounding is to 33 tens or 330.
6. Use a number expander to round:

   (a) 457 to nearest 10 (b) 355 to nearest 10 (c) 2 368 to nearest 100

With activity and discussion, students will soon not need the length idea or the number expander for rounding; e.g.

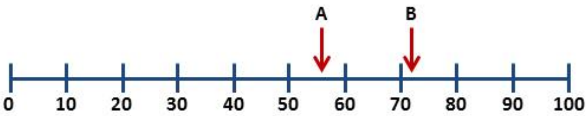
- 674 to nearest 10 is 6 7 4 67 tens, or 670
  ↑ ten ↑ less than 5
- 876 to nearest 10 is 8 7 6 88 tens, or 880
  ↑ ten ↑ more than 5

4.4 Comparison, order and place value

The focus here is to get across that for 2-digit numbers, the number of tens is important and ones only count in comparison if the tens are the same (and similarly for hundreds or 3-digit numbers). An activity for developing this idea is as follows.

Activity

1. Construct a number line:

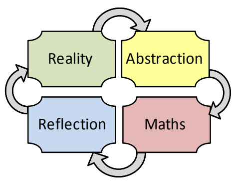


2. Use the number line to measure objects as tens and left-over ones. Ensure that you give students objects like A and B above to measure – that is objects with high tens and low ones and objects with low tens and high ones. This is so that students see that the tens are more important even though there is a large number like 9 in the ones (e.g., 21 is larger than 19).
3. Record objects, measures and which one is longer on a table:

Object	Measure	Longer
Red stick	3 tens and a lot	Blue stick
Blue stick	5 tens and a little	

4. Look for the pattern (longer has more tens or more ones if the tens are the same).
5. Transfer this to place value, for example:
   - 56 is greater than 49 because 56 has more tens.
   - 47 is greater than 43 because they have the same tens and 47 has more ones.

4.5 RAMR lesson for number lines and relationship to place value



Learning goal: Using place-value concepts on the number line.

Big ideas: Continuous vs discrete; comparison and order; equivalence.

Resources: Sets of digit cards (large and small), wall number line, dice, counters, small number lines.

Reality

Local knowledge: Where possible, find real-life contexts to embed the activities in; for example, using relevant objects or situations.

Prior experience: Continuous/discrete; comparison and order; equivalence.

Kinaesthetic: Play “Number line up”. Use one set of demonstration digit cards and small digit cards for class use, one set for each student. Select 2–6 students and give each student one of the demonstration digit cards. Each student selected moves to the front of the classroom and holds the digit card in front of them so that classmates can easily see the numeral. Students at their desks should select the same digit cards from their decks and put the other cards aside. Say a 2–6 digit number aloud, depending on the level of your students. Students with the demo digit cards must arrange themselves to create that number. Students at desks must use their small digit cards to create the dictated number. Ask students to say the number in words. Ask students to point to the digit in the hundreds place, the tens place, etc. Students holding cards in the front of the classroom should raise the correct card. Have a student record the number on a wall number line.

Repeat this activity a few times to assess student proficiency with basic place-value concepts, changing the digits and selecting different students to hold cards at the front of the classroom. Invite students to discuss the number line that has been created. Ask: *What are the beginning and end points? Is there a midpoint? What intervals might be useful?* Ask students to identify the place-value intervals. Can they see the tens/hundreds on the number line? Consider the role of zero as a beginning point and within numbers.

Abstraction

Body: Encourage small groups to play the above game. Use dice (some with zero) to identify the numbers to make and place on a number line.

Hand: Provide place-value games using dice, counters, number lines for moving in ones, tens and hundreds along and back (increase/decrease) on the number line.

Mind: Have students visualise the tens intervals and place the numbers in between in their mind's eye. *What happens when you get to 100? What do you see?*

Creativity: Encourage students to explore different numbers they find. *Will they go on the number line? Where?*

Mathematics

Language/symbols: Place value, ones, tens, hundreds, intervals, zero, distance, rounding, structures, models.

Practice: Use numbers with zero in them as well as discussing the role of zero as the beginning point on the number line.

Connections: Measurement – length and calibrated scale; beginning of rulers; early fraction understanding.

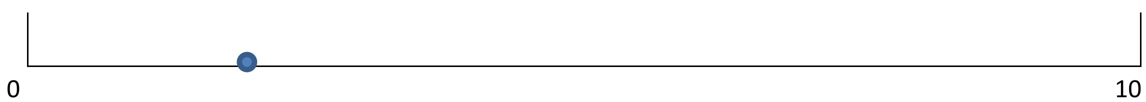
Reflection

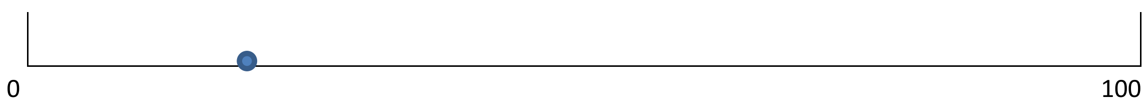
Validation: Compare on a number line how 150 and 105 are alike and how they are different?

Applications/Problem-solving: Have students construct a number line to show the following numbers: (a) 6 tens and 8 ones; (b) 10 tens and 8 ones; (c) 2 tens, 1 hundred and 2 ones.

Extension:

- *Flexibility*: Ask: *What number on the number line does the dot represent?*





- *Generalising/Changing parameters*: Present students with the following problem: *The answer is 110. What is the question? Use a number line to show your answer.*

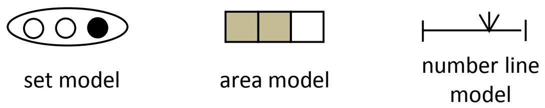
Module Review

This section reviews the module and provides some extra information on number line as a model and representation, critical teaching points, and later number-line ideas.

Continuous vs discrete and notion of number line

We take up the ideas from counting and apply them to the number-line model for grouping. The big idea of continuous or discrete focuses on one of the fundamental differences in the world around us. Things are either: (a) discrete, that is, can be identified as separate entities and counted (e.g. fingers, people, chairs, lollies); or (b) continuous, that is, flowing forever without being naturally broken into pieces that can be counted (e.g. line or length, coverage or area, heft or mass).

Because of the importance placed on determining number and quantity, Western culture found a way of "discretifying" the continuous by inventing **units**. A unit is a small amount of the continuous attribute used to break up the attribute into equal pieces that can be counted so the attribute can be quantified (e.g. metres for length, hectares for area, and kilograms for mass). This allows number to be applied to both discrete and continuous attributes, and is essential for all number types. Whole numbers, decimal numbers, common fractions, rates and ratios must be understood in terms of discrete and continuous – that is, seen in relation to separate objects and seen in relation to continuous entities. In practice this means that number should be represented in terms of **sets**, **area** and **number lines**:



Set, area, and number-line models for two thirds

For continuous entities, the idea is to count "jumps", using number lines and number boards (e.g. 100 boards). One way to begin this is to count steps and this leads to games and activities that use number tracks. However, the number line places numbers at the ends of steps while the number track has numbers in the spaces. Therefore, a number ladder (like a track but numbering the lines) can be a useful intermediary.

A number line is a continuous line that is not naturally divided into discrete sections that can be counted. An interval of length one is used as a unit to divide the line into discrete parts. The placement of the number is based on this equal partitioning of the line. It is excellent for representing the following processes: quantity, rank (each number a single point on the number line), comparison and order, density, and rounding and estimating.

Critical teaching points

Students demonstrate their understanding of quantity using a number line by understanding:

- counting sequences;
- number tracks;
- moving on number lines;
- representing numbers as related to positions on lines as well as sets of objects (Module N2);
- the role of zero/meaning of zero when representing quantity on a number line;
- placing numbers on a number line in order and making comparisons;

- each number is a single point on the number line;
- the placement of the number is based on equal partitioning of the line: midpoints;
- constructing number lines: empty, adding anchor points;
- rank to 10, 100 then greater;
- there are numbers between whole numbers on the number line;
- number lines are useful for relating two-digit numbers and estimating and rounding;
- if the number is closer to 5 or closer to zero; closer to 5 or closer to 10, etc;
- whether zero one is the same as one zero (students need to be able to recognise when the inclusion of zero changes a number and when it does not change a number);
- using the number-line model;
- you can identify, order, and compare two-digit numbers on a number line using: most and least; larger and smaller; largest and smallest; before and after; odd and even;
- there are many ways to represent numbers; numbers are related to positions on lines as well as sets of objects (Module N2); making connections between models, multiple representations of number (Unifix cubes, MAB, ten frames, PV charts, numeral cards, abacus) and the number line;
- making links between visual and non-visual systems of place value.

Later number-line ideas

As larger and smaller numbers begin to populate the students' world, physical models such as MAB become problematic, and the more abstract place-value chart (PVC), on its own, takes over from materials on a PVC as one of the major models for larger and smaller numbers. With powers of 10 and mathematical notation, the PVC can represent most numbers up to complex numbers (which include the square root of −1). However, the place value representation, although providing the basis of counting and number names, has difficulty representing comparison and order, density, rounding and approximation, and multiplicative relationships.

The number line is also an abstract model, particularly when it does not have a scale (the empty number line), and can represent large and small numbers and also fractions, decimals and proportion relationships, along with measures. The following are some examples, sequenced across the primary and junior secondary years of the mathematics curriculum.

1. **Lines to represent whole numbers (to infinity)**. Numbers can easily be represented on lines and students can count along the numbers and get a vision of infinity.

2. **Simple lines for early operations**. Addition 2 + 3 can be seen as two jumps (a jump of 2 and a jump of 3) along a number line; subtraction 5 − 2 can be seen as a jump forward of 5 followed by a jump backward of 2; multiplication 3 × 4 can be seen as 3 jumps of 4, while division 15 ÷ 3 can be seen as 5 jumps of 3 backwards from 15.

   In particular, this leads to strategies for addition and subtraction basic facts: 7 + 2 is 7 plus 2 more jumps (7, 8, 9), while 8 + 5 can be seen as 2 jumps to the 10 and another 3 jumps which gives 13.

3. **Reconstructed lines for early operations and measures (99 boards and clocks)**. A line from 0 to 100 can be cut into 10 strips of 10 steps and placed in rows to make a 99 board – this can be used for adding and subtracting following the sequencing strategy. For example, 35 + 23 = 35, 45, 55 (counting on 2 tens) and then 55, 56, 57, 58 (counting on 3 ones), while 74 − 52 is 74, 64, 54, 44, 34, 24 (counting back 5 tens) and 24, 23, 22 (counting back 2 ones). As well, this 99 board can be used for the compensation strategy: 74 + 19 = 73 + 20 = 93 and 46 − 18 = 48 − 20 = 28.

   It can also be used for multiplication facts – the 3× pattern is as follows and can be used to remember the multiplication of 3× tables:

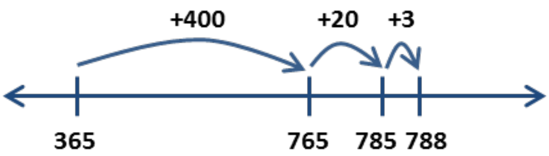
3	6	9
12	15	18
21	24	27
30	and so on	

   As well as all this, straight lines can be bent into circles. This is an excellent way to go from bar or bar-line graphs to circle graphs and to develop the round clock face.

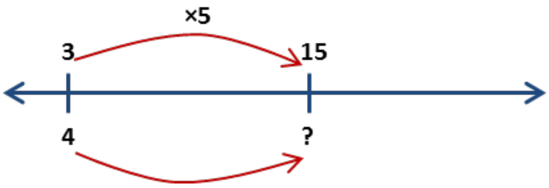
4. **More complex lines for fractions and decimals**. Obviously as the line spreads out, it shows that there is space between the whole numbers for rational numbers (fractions and decimals).

5. **Lines with zero for positive and negative numbers**. The line goes in both directions from zero. Vertical lines can make the representation easier. This line can be used for operations on negative numbers. The "Pirate and Shark" activity is particularly useful here (a video of this activity is available through the YDM Online Blackboard site, in the Operations module, Session One).

6. **Empty lines for algorithms and strategies**. The line can be used to show the sequencing and compensation strategies for three and more digit numbers. For example, 365 + 423:

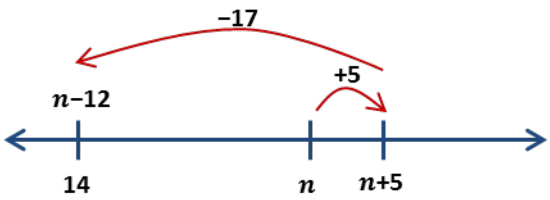


7. **Double number lines for proportion**. This involves using both sides of the line and keeping relationships as multiplication.

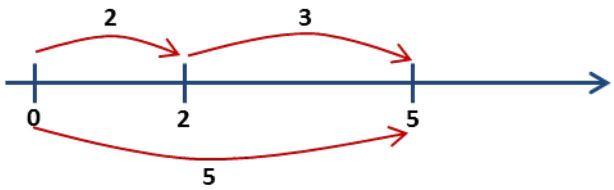


   For the example above, 3 × 5 = 15, therefore ? = 4 × 5 = 20. This model is used in activities with percent, rate and ratio.

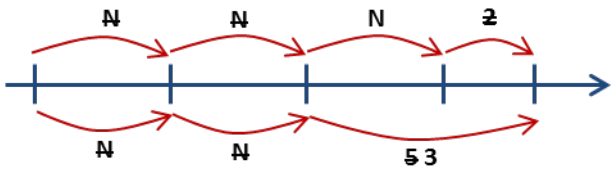
8. **Empty lines for algebraic manipulations and finding unknowns**. Open or empty number lines can also be used for early operations in algebra. For example, *John gets some money from his Dad and another $5 from his mum. He spends $17 and has $14 left, how much did he get from his Dad?* Using the diagram below, a student can act out what is happening on a number line and then backtrack to work out the unknown, n, either by using n – 12 = 14 or by reversing: 14 + 17 − 5 = 26.



9. **Double number lines for equations**. Double number lines can model equations, as below, and then can be used to work out unknowns, as shown. First, 2 + 3 = 5 is modelled as two sides of a line. In the first example, we can see that 2 + 3 = 5 because the jumps of 2 and 3 end where the jump of 5 ends.



In the second example we have modelled 3N + 2 = 2N + 5 by jumps N, N, N and 2 equalling jumps N, N, and 5. Cancelling sees the 2N cancel out 2 of the Ns from 3N and the 2 reduce the 5 to 3, so N = 3.



10. **Two or more lines for two and three dimensional graphing (x, y and z axes)**. Graphs can have an x, y and/or z axis. These axes are just number lines. So the basis of graphs is number lines (including the circular or circle graphs). These axes also grow like number lines. Bar graphs are like number tracks, while line graphs are like number lines. In bar graphs, the squares are filled, but in line graphs the points are on the intersections of the lines.

Test Item Types

This section presents instructions and the test item types for the subtests associated with the units. These will form the bases of the pre-test and post-test for this module.

Instructions

Selecting the items and administering the pre-post tests

This section provides an item bank of test item types, constructed around the units in the module. From this bank, items should be selected for the pre-test and post-test; these selected items need to suit the students and may need to be modified, particularly to make post-test items different from pre-test items. The purpose of the tests is to measure students' performance before and after the module is taught. The questions should be selected so that the level of difficulty progresses from easier items to more difficult items. In some modules this will follow the order of the units and subtests, and in other modules it will not, depending on the sequencing across the module and within units. The pre-test items need to allow for students' existing knowledge to be shown but without continual failure, and the post-test items need to cover all the sections in a manner that maximises students' effort to show what they can do.

In administering the pre-test, the students should be told that the test is not related to grades, but is to find out what they know before the topic is taught. They should be told that they are not expected to know the work as they have not been taught it. They should show what they know and, if they cannot do a question, they should skip it, or put "not known" beside questions. They will be taught the work in the next few weeks and will then be able to show what they know. Stress to students that **any pre-test is a series of questions to find out what they know** before the knowledge is taught. They should do their best but the important questions come at the end of the module. For the **post-test**, the students should be told that **this is their opportunity to show how they have improved**.

For all tests, **teachers should continually check to see how the students are going.** Items in later subtests, or more difficult items within a particular subtest, should not be attempted if previous similar items in earlier subtests show strong weaknesses. Students should be allowed to skip that part of the test, or the test should be finished. Students can be marked zero for these parts.

Information on the Module N3: Quantity test item types

This section includes:

1. Pre-test instructions;
2. Observation Checklist and Teacher Recording Instrument; and
3. Test item types.

Pre-test instructions

When preparing for assessment ensure the following:

- Students have a strong sense of identity; feel safe, secure and supported; develop their emerging autonomy, interdependence, resilience and sense of agency; and develop knowledgeable and confident identities.
- Students are confident and involved learners, and develop dispositions for learning such as curiosity, cooperation, confidence, creativity, commitment, enthusiasm, persistence, imagination and reflexivity.

When conducting assessment, take the following into consideration:

- Student interview for diagnostic assessment in the early learning stages is of paramount importance.
- Use materials and graphics familiar to students' context in and out of school.
- Use manipulatives rather than pictures wherever possible.
- Acknowledge the role of using stories in this early number learning, enabling students to tell stories and act out understandings to illustrate what they know.
- Playdough and sand trays are useful for early interview assessment situations.

Ways to prepare students for assessment processes include the following:

- In **individual teaching times**, challenge students' thinking. "Challenging my thinking helps me to learn by encouraging me to ask questions about what I do and learn. I learn and am encouraged to take risks, try new things and explore my ideas."
- In **group time**, model and scaffold question-and-answer skills by using sentence stems to clarify understandings and think about actions. Encourage students to think of answers to questions where there is no one correct answer, and to understand that there can be more than one correct answer (e.g. How can we sort the objects?).
- In **active learning centres**, use activities such as imaginative play, sand play, playdough, painting, ICTs and construction to think and talk about different ways of using materials, technologies or toys. Ask questions and take risks with new ideas.

Other considerations:

- Preferred/most productive assessment techniques for early understandings are observations, interviews, checklists, diary entries, and folios of student work.
- Diagnostic assessment items can be used as both pre-test and post-test instruments.

Remember:

Testing the knowledge can imply memory of stuff; asking the students **what they can do with knowledge** requires construction and demonstration of their understanding at this early understandings level.

N3 Quantity: Observation Checklist				
Unit	**Concept**	**Knows**	**Can construct/do/tell/solve**	**Tripping points**
1. Continuous vs discrete	Set vs length (relating number to distance)	How equal distances between objects relates number to distance	Count along a line to get distance; know the track must be straight and the steps the same length	Establishing distances in each step have to be the same
	From track to ladder to line	How tracks are counted by ends	Count along the track from spaces to lines; place number cards for a variety of numbers	Understanding what is not a track
	Early rank	Regular intervals for number lines	Make a number line with regular intervals, cross-marks to show; place numbers	Understanding number to space connection
	Early comparison and order	A number is larger if it is further along a line away from 0; order is determined by position on the line	Compare two numbers on number line to identify the one further away from 0 as larger (and opposite for smaller); place more than two numbers in order on the line	With and without rank helping: to place numbers; to write numbers on number line
2. Number-line activities	Rank	Start numbers and finish numbers; with lines and without lines	Construct a number line with a start number and a finish number; all numbers are evenly spaced	Equal rank markings
	Comparison and order	Ranking the number gives order	The largest number is the one furthest from zero	"Largest number is the longest" furphy
	Density	Whole numbers are not dense, but parts of wholes (fractions) are	Line up class by age to discuss children who are 5, 6, somewhere in between	There are numbers in between whole numbers
	Construction of number lines	Reverse the number line construct	Construct number lines; with start and finish; count number of intervals; place numbers	Zero; forwards and backwards
3. Position and equivalence	Approximation and estimation	To place a number on the number line we look at the marks and find the closest mark	Can choose appropriate interval/ mark for placing the number on the number line	Recognising midpoints; placing numbers close to 0 or end number
	Rounding	Recognises the interval/ mark given in the task is usually a PV	Can round to nearest 10; use convention of halfway goes up on number line	Problem with 5 being midpoint/halfway
	Equivalence (role of zero)	Equivalent numbers can occupy the same spot on the number line	Make connections between different representations of the same number and place them in the same place on number line	Reading/writing/placing zero numbers
4. Relationship to place value	Comparison/ order to place value	Bigger is bigger digit in biggest place value	Groups and ones left over The position of the digit tells us the quantity on number line	Making links between visual/non visual PV systems
	Rounding and estimation to place value	Look at PV before and PV after to determine nearest	Identifies ten-ness Hundreds on number line	Using the number-line model for PV
	Relating equivalence to role of zero in place value	Role of zero as a beginning point and within numbers themselves	Place 0, 10, 01, 100, 101, and so on on number line	Seeing the different ways to show/get to the same number
	Direct relation PV position to distance	Distance from zero greater the number and vice versa	Move along number line to increase and decrease number PVs in ones, tens, hundreds	Moving from unitary to ten structured thinking

N3 Quantity: Observation Checklist Teacher Recording instrument					
Unit	**Concept**	**Knows**	**Can construct/do/tell/solve**	**Tripping points**	**Comments**
1. Continuous vs discrete	Set vs length (relating number to distance)	How equal distances between objects relates number to distance	Count along a line to get distance; know the track must be straight and the steps the same length	Establishing distances in each step have to be the same	
	From track to ladder to line	How tracks are counted by ends	Count along track from spaces to lines; place number cards	Understanding what is not a track	
	Early rank	Regular intervals for number lines	Make a number line with regular intervals, cross-marks to show; place numbers	Understanding number to space connection	
	Early comparison and order	A number is larger if it is further along a line away from 0; order is determined by position on the line	Compare two numbers on number line to identify the one further away from 0 as larger (and opposite for smaller); place more than two numbers in order on the line	With and without rank helping: to place numbers; to write numbers on number line	
2. Number-line activities	Rank	Start numbers and finish numbers; with and without lines	Construct a number line with a start number and a finish number; all numbers are evenly spaced	Equal rank markings	
	Comparison and order	Ranking the number gives order	The largest number is the one furthest from zero	"Largest number is the longest" furphy	
	Density	Whole numbers are not dense but parts of wholes (fractions) are	Line up class by age to discuss children who are 5, 6, somewhere in between	There are numbers in between whole numbers	
	Construction of number lines	Reverse the number line construct	Construct number lines; with start and finish; count number of intervals; place numbers	Zero; forwards and backwards	

Unit	Concept	Knows	Can construct/do/tell/solve	Tripping points	Comments
3. Position and equivalence	Approximation and estimation	To place a number on the number line we look at the marks and find the closest mark	Can choose appropriate interval/ mark for placing the number on the number line	Recognising midpoints; placing numbers close to 0 or end number	
	Rounding	Recognises the interval/ mark given in the task is usually a PV	Can round to nearest 10; use convention of halfway goes up	Problem with 5 being midpoint	
	Equivalence (role of zero)	Equivalent numbers can occupy the same spot on the number line	Make connections between different representations of the same number and place them in the same place on number line	Reading/writing/placing zero numbers	
4. Relationship to place value	Comparison/ order to place value	Bigger is bigger digit in biggest place value	Groups and ones left over; position of the digit tells us the quantity on number line	Making links between visual/non visual PV systems	
	Rounding and estimation to place value	Look at PV before and PV after to determine nearest	Identifies ten-ness; hundreds on number line	Using the number line model for PV	
	Equivalence to role of zero in place value	Role of zero as a beginning point and within numbers	Place 0, 10, 01, 100, 101, and so on on number line	Seeing the different ways to show/get to the same number	
	Direct relation PV position to distance	Distance from zero greater the number and vice versa	Move along number line to increase and decrease number place values in ones, tens, hundreds	Moving from unitary to ten structured thinking	

Subtest item types

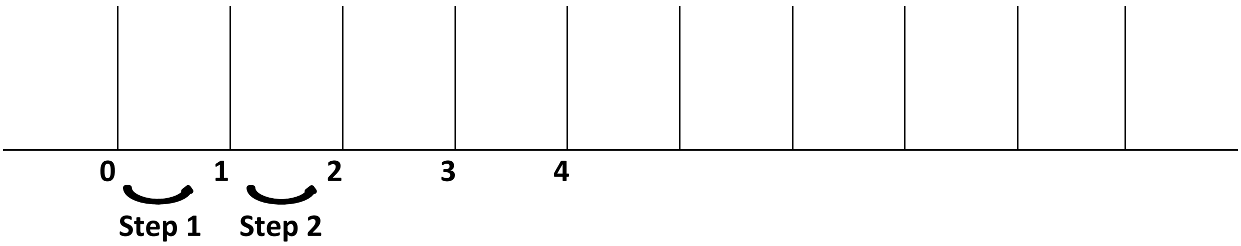
Subtest 1 items (Unit 1: Continuous vs Discrete)

1. Ask student to model 7 steps on stepping stones. Ask student to construct a model of this using Unifix cubes. Join the Unifix cubes. Ask student to step along it with their fingers.

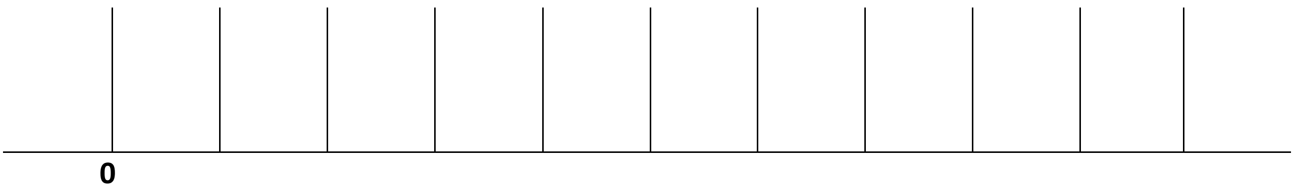
2. Ask student to take 7 steps on a number ladder. What number do you finish on? Ask: *If I take 7 steps along the number ladder will I cover the same distance as you? Why? Why not?* (Same distance from the starting point)

0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	

3. Ask student to step along number line and place number cards.



4. (a) Have student place the numbers 1, 3, 6 on the number line.



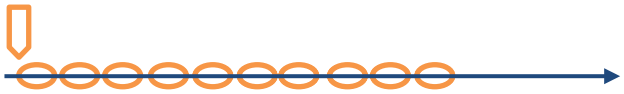
(b) Ask: *Which number is larger? How do you know?* [It is further away from zero on the number line.]

5. (a) Ask student to place 1, 2, 5, 7 on the number line.

(b) Ask: *Which number is smaller? How do you know?* [It is closer to zero on the number line.] Discuss where the student has placed the number rankings relative to zero. Have they used regular intervals to indicate number placement?

Subtest 2 items (Unit 2: Number-line Activities)

1. Using a bead string, ask student to place a start number (0), and a finish number.



Have them mark other numbers on the bead string number line.

2. Give students a number line.

Ask them to place/write the zero at the starting point, and mark 20 for the finishing point, then:

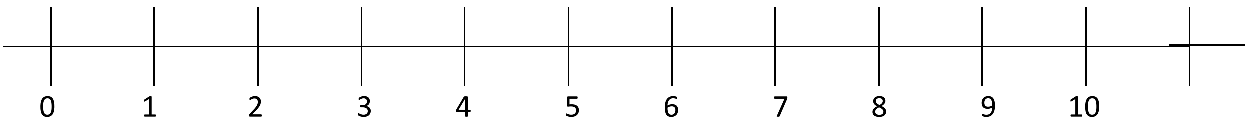
(a) Locate and write 10 on the number line.

(b) Locate and write 1 on the number line.

(c) Locate and write 5, 15, 12 and 19 on the number line.

Has the student attempted to use equal rank markings? Ask: *Which is the larger number? How do you know? Which is the smallest number? How do you know?*

3. Give students a number line and ask them to order students' ages they know on the line.



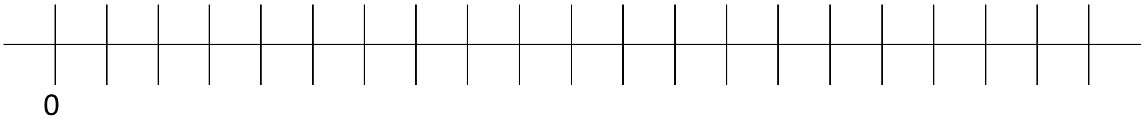
Have cards with students' names on them to place on the number line. Where do they place students who are 6 and a little bit more, 6½? Do they know there are numbers between whole numbers?

4. Ask student to draw a number line to show beginning number 0 and end number 100 (choose number range appropriate to the student). Ask the student to locate and place 15 numbers on their number line.

Can the student count forwards and backwards on the number line they have created?

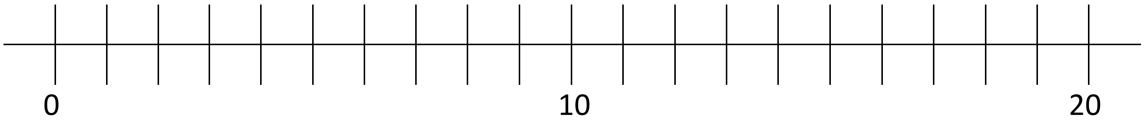
Subtest 3 items (Unit 3: Position and Equivalence)

1. Write the numbers on this number line:

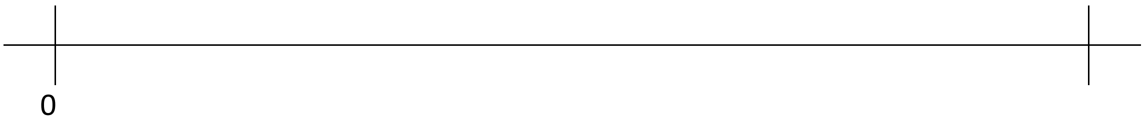


What is the last or end number?

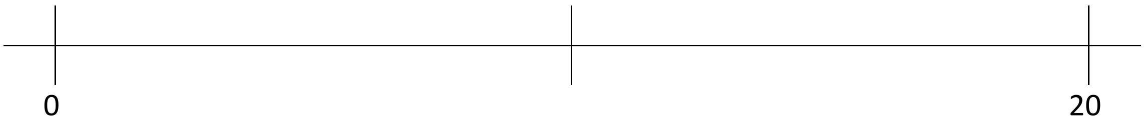
2. Write these numbers: 5, 15, 1, 19 on the number line below:



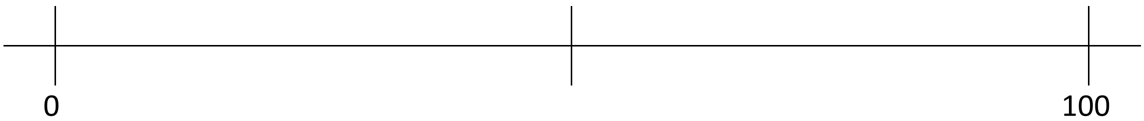
3. Write the numbers 0 to 10 on this number line (0 is done for you):



4. Locate and write the numbers 1, 5, 10, 15 and 19 on this number line:

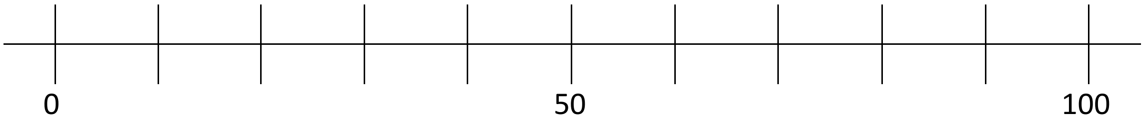


5. Place these numbers on the number line: 50, 10, 90, 40

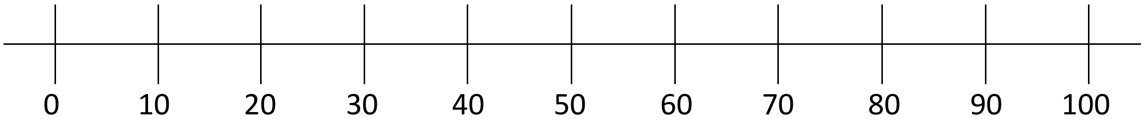


Subtest 4 items (Unit 4: Relationship to Place Value)

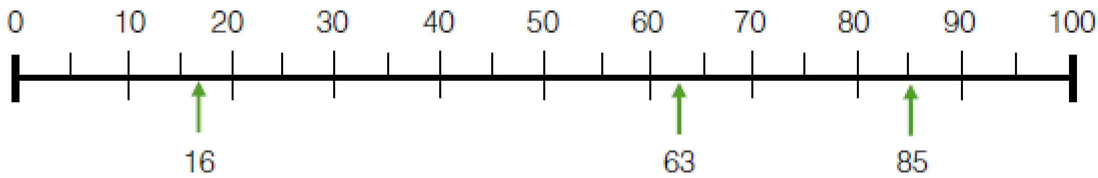
1. Write the following numbers on the number line: 20, 90, 40



2. Mark these numbers on the number line: 5, 25, 42, 61, 99



3. Use the number line to help you answer the following:

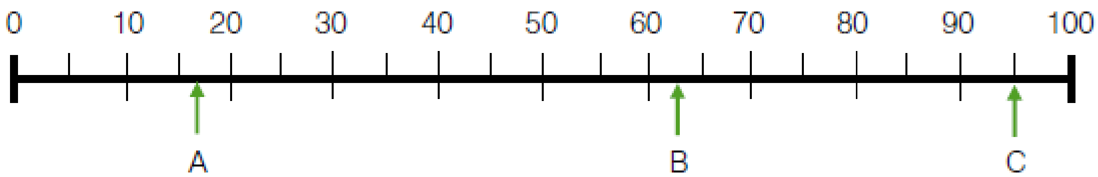


a. Round 16 to the nearest 10. __________

b. Round 63 to the nearest 10. __________

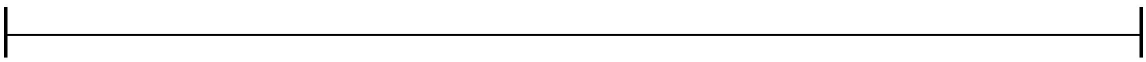
c. Round 85 to the nearest 10. __________

4. What number could be at:

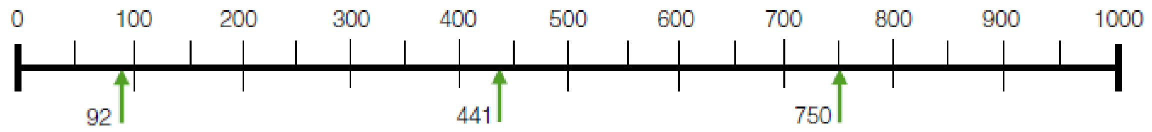


a. A: __________ b. B: __________ c. C: __________

5. Write the following numbers on the number line in order from smallest to largest in value:
10, 0, 01, 100, 101, 50, 30, 1



6. Look at the numbers on this number line and answer the following:

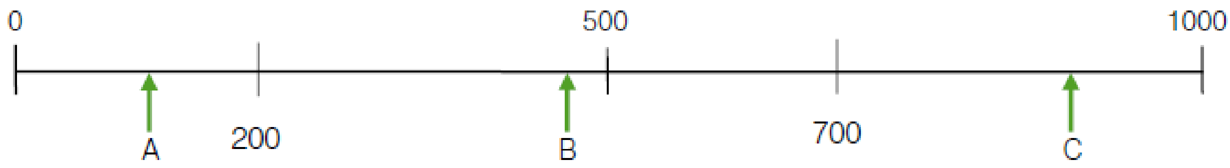


a. Round 92 to the nearest 100 ____________

b. Round 441 to the nearest 100 ____________

c. Round 750 to the nearest 100 ____________

7. What number could be at:



a. A: ____________ b. B: ____________ c. C: ____________

Appendices

Appendix A: AIM Early Understandings Modules

Module content

1st module **Number N1: Counting** *Sorting/correspondence *Subitising *Rote *Rational *Symbol recognition *Models *Counting competencies	**2nd module** **Algebra A1: Patterning** *Repeating *Growing *Visuals/tables *Number patterns	**3rd module** **Algebra A2: Functions and Equations** *Functions* *Change *Function machine *Inverse/backtracking *Equations* *Equals *Balance
4th module **Number N2: Place Value** *Concepts* *Place value *Additive structure, odometer *Multiplicative structure *Equivalence *Processes* *Role of zero *Reading/writing *Counting sequences *Seriation *Renaming	**5th module** **Number N3: Quantity** *Concepts* *Number line *Rank *Processes* *Comparing/ordering *Rounding/estimating *Relationship to place value*	**6th module** **Operations O1: Thinking and Solving** *Early thinking skills *Planning *Strategies *Problem types *Metacognition
7th module **Operations O2: Meaning and Operating** *Addition and subtraction; multiplication and division *Word problems *Models	**8th module** **Operations O3: Calculating** *Computation/calculating *Recording *Estimating	**9th module** **Number N4: Fractions** *Concepts* *Fractions as part of a whole *Fractions as part of a group/set *Fractions as a number or quantity *Fraction as a continuous quantity/number line *Processes* *Representing *Reading and writing *Comparing and ordering *Renaming

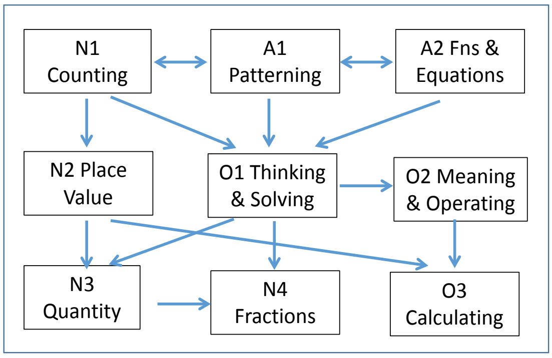
Module background, components and sequence

Background. In many schools, there are students who come to Prep/Foundation with intelligence and local knowledge but little cultural capital to be successful in schooling. In particular, they are missing basic knowledge to do with number that is normally acquired in the years before coming to school. This includes counting and numerals to 10 but also consists of such ideas as attribute recognition, sorting by attributes, making patterns and 1-1 correspondence between objects. Even more difficult, it includes behaviours such as paying attention, listening, completing tasks, not interfering with activity of other students, and so on.

Teachers can sometimes assume this knowledge and teach as if it is known and thus exacerbate this lack of cultural capital. Even when the lack is identified, building this knowledge can be time consuming in classrooms where students are at different levels. It can lead to situations where Prep/Foundation teachers say at the end of the year that some of their students are now just ready to start school and they wish they could have another year with them. These situations can lead to a gap between some students and the rest that is already at least one year by the beginning of Year 1. For many students, this gap becomes at least two years by Year 3 and is not closed and sometimes widens across the primary years unless schools can provide major intervention programs. It also leads to problems with truancy, behaviour and low expectations.

Components. The AIM EU project was developed to provide Years F–2 teachers with a program that can accelerate early understandings and enable children with low cultural capital to be ready for Year 3 at the end of Year 2. It is based on nine modules which are built around three components. The mathematics ideas are designed to be in sequence but also to be connected and related to a common development. The modules are based on the AIM Years 7–9 program where modules are designed to teach six years of mathematics (start of Year 4 to end of Year 9) in three years (start of Year 7 to end of Year 9). The three components are: (a) Basics – A1 *Patterning* and A2 *Functions and Equations*; (b) Number – N1 *Counting* (also a basic), N2 *Place Value*, N3 *Quantity* (number line), and N4 *Fractions*; and (c) Operations – O1 *Thinking and Solving*, O2 *Meaning and Operating*, and O3 *Calculating*. These nine modules cover early Number and Algebra understandings from before school (pre-foundational) to Year 2.

Sequence. Each module is a sequence of ideas from F–2. For some ideas, this means that the module covers activities in Prep/Foundation, Year 1 and Year 2. Other modules are more constrained and may only have activities for one or two year levels. For example, Counting would predominantly be the Prep/Foundation year and Fractions would be Year 2. Thus, the modules overlap across the three years F to 2. For example, Place Value shares ideas with Counting and with Quantity for two-digit numbers in Year1 and three-digit numbers in Year 2. It is therefore difficult, and inexact, to sequence the modules. However, it is worth attempting a sequence because, although inexact, the attempt provides insight into the modules and their teaching. One such attempt is on the right. It shows the following:



1. The foundation ideas are within *Counting*, *Patterning* and *Functions and Equations* – these deal with the manipulation of material for the basis of mathematics, seeing patterns, the start of number, and the idea of inverse (undoing) and the meaning of equals (same and different).

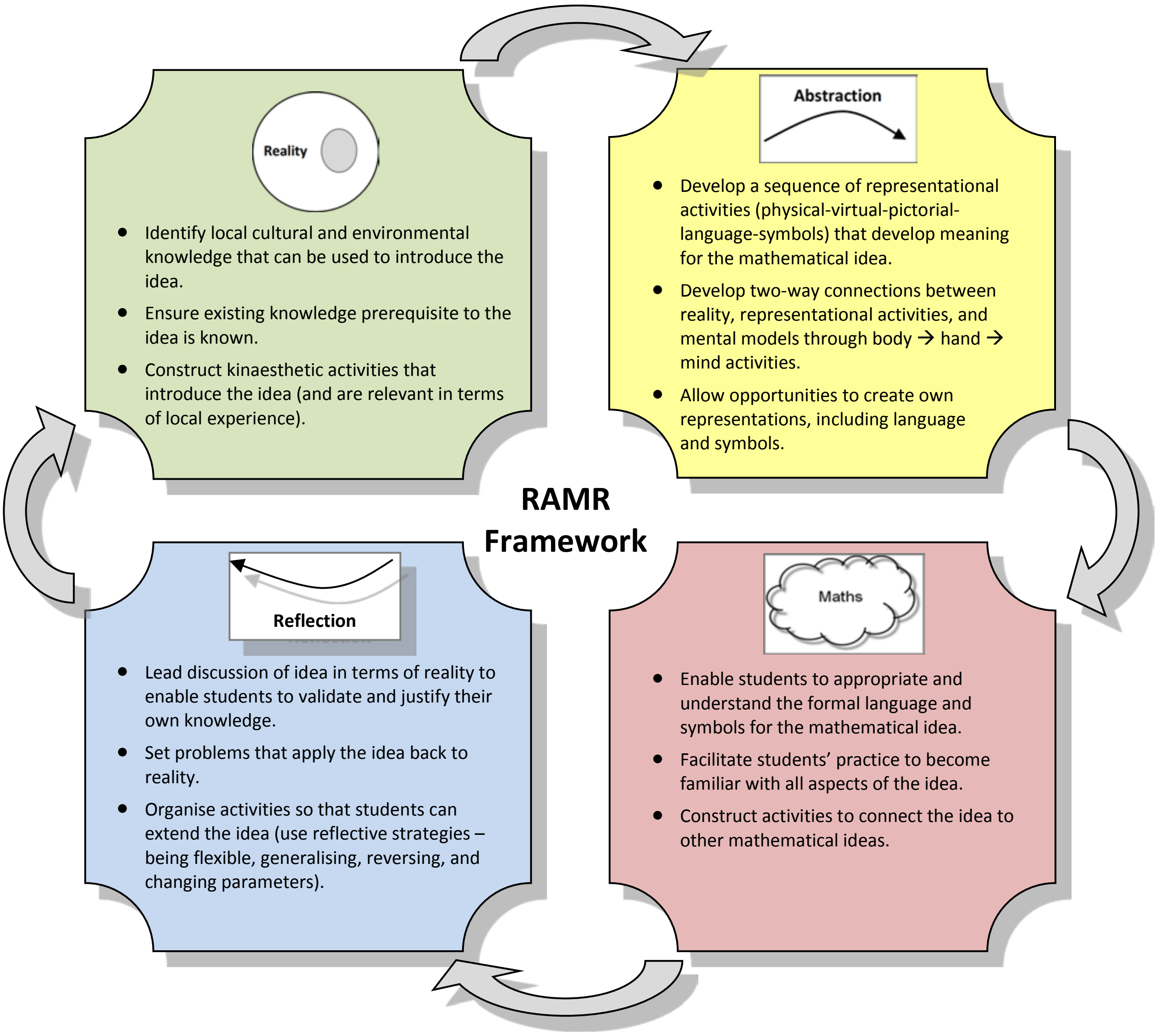
2. The central components of the sequence are *Thinking and Solving* along with *Place Value* and M*eaning and Operating* – these lead into the less important *Calculating* and prepare for *Quantity*, *Fractions* and later general problem-solving and algebra.

3. The *Quantity*, *Fractions* and *Calculating* modules are the end product of the sequence and rely on the earlier ideas, except that *Quantity* restructures the idea of number from discrete to continuous to prepare for measures.

Appendix B: RAMR Cycle

AIM advocates using the four components in the figure below, reality–abstraction–mathematics–reflection (RAMR), as a cycle for planning and teaching mathematics. RAMR proposes: (a) working from reality and local culture (prior experience and everyday kinaesthetic activities); (b) abstracting mathematics ideas from everyday instances to mathematical forms through an active pedagogy (kinaesthetic, physical, virtual, pictorial, language and symbolic representations, i.e. body → hand → mind); (c) consolidating the new ideas as mathematics through practice and connections; and (d) reflecting these ideas back to reality through a focus on applications, problem-solving, flexibility, reversing and generalising. The innovative aspect of RAMR is that Reality to Abstraction to Mathematics develops the mathematics idea while Mathematics to Reflection to Reality reconnects it to the world and extends it.

Planning the teaching of mathematics is based around the four components of the RAMR cycle. They are applied to the mathematical idea to be taught. By breaking instruction down into the four parts, the cycle can lead to a structured instructional sequence for teaching the idea. The figure below shows how this can be done.



The YuMi Deadly Maths RAMR Framework

Appendix C: Teaching Frameworks

Teaching scope and sequence for quantity

TOPICS	SUB-TOPICS	DESCRIPTIONS AND CONCEPTS/STRATEGIES/WAYS
Quantity	Continuous vs discrete	Set vs length (relating number to distance) From track to ladder to line Early rank Early comparison and order
	Number-line activities	Rank Comparison and order Density Construction of number lines
	Position and equivalence	Approximation and estimation Rounding Equivalence (role of zero)
	Relationship to place value	Relating comparison and order to place value Relating rounding and estimation to place value Relating equivalence to role of zero in place value Direct relation of place value position to distance

Proposed year-level framework

YEAR LEVEL	NUMBER – QUANTITY	
	Semester 1	**Semester 2**
Prep	*Symbol recognition* – experience identifying numerals of personal significance (e.g. age); experience real-world, language, set/line models, up to 10 (e.g. storytelling, forming sets of objects, acting out story on an unnumbered number track).	*Rank* – experience number on a number track or ladder to 20. *Comparing/ordering* – numbers to 20 (comparing collections or using a number track, ladder or line); experience terms first and second to indicate order.
1	*Symbol recognition* – reinforce digit recognition to 40 (real world, set, line, language, and symbol). *Rank* – to 40; discuss difference in number on number line to number as applied to a collection. *Comparing/ordering* – up to 40.	*Rote* – forwards/backwards to 100 by 1s, 2s, 5s and 10s, reinforce ordinal numbers to 100; discuss different things that could be counted e.g. objects, groups, steps, position along number line. *Rank* – to 100. *Reading/writing* – numbers to 100. *Seriation* – 1 more/less, 10 more/less to 100. *Comparing/ordering* – up to 100.
2	*Place value* – reinforce teens and zeros to 100, introduce symbol, language (number names) with materials and PVCs to 130. *Rank* – to 130 (placing numbers on number line). *Reading/writing* – to 130. *Seriation* – to 130. *Comparing/ordering* – to 130. *Renaming* – to 100. *Rounding/estimating* – to 100.	*Place value* – connect symbol (teens and 10s), language (number names to 100), models to 1000. *Rank* – reinforce to 1000. *Reading/writing* – to 1000. *Seriation* – experience to 1000 (1, 10 and100 more, 1, 10 and 100 less). *Comparing/ordering* – to 1000. *Renaming* – experience to 1000. *Rounding/estimating* – experience to 1000.
3	*Rank* – reinforce number-line model to 1000. *Reading/writing* – reinforce to 1000. *Seriation* – reinforce to 1000. *Comparing/ordering* – reinforce to 1000. *Renaming* – reinforce to 1000. *Rounding/estimating* – reinforce to 1000.	*Rank* – introduce line model to 10 000. *Reading/writing* – introduce for 5-digits (to 10 000). *Seriation* – connect to odometer to understand one more or less for any PV position up to 100 thousands. *Comparing/ordering* – experience to 10 000. *Renaming* – experience to 10 000. *Rounding/estimating* – introduce for 5 digits.
Focus	Body → Hand → Mind	Construct

The numbers in here will need to be decided by teachers.

Accelerated Inclusive Mathematics Early Understandings Project

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through education

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