Accelerated Inclusive Mathematics Early Understandings Project

YuMi Deadly Maths

AIM EU

Module A2

Algebra: Functions and Equations

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The YuMi Deadly Centre acknowledges the traditional owners and custodians of the lands in which the mathematics ideas for this resource were developed, written and refined and presented in professional development sessions.

YUMI DEADLY CENTRE

The YuMi Deadly Centre is a research centre within the Faculty of Education at the Queensland University of Technology which is dedicated to enhancing the learning of Indigenous and non-Indigenous children, young people and adults to improve their opportunities for further education, training and employment, and to equip them for lifelong learning. The YuMi Deadly Centre (YDC) can be contacted at ydc@qut.edu.au. Its website is http://ydc.qut.edu.au.

"YuMi" is a Torres Strait Islander Creole word meaning "you and me" but is used here with permission from the Torres Strait Islanders' Regional Education Council to mean working together as a community for the betterment of education for all. "Deadly" is an Aboriginal word used widely across Australia to mean smart in terms of being the best one can be in learning and life. YuMi Deadly Centre's motif was developed by Blacklines to depict learning, empowerment, and growth within country/community. The three key elements are the individual (represented by the inner seed), the community (represented by the leaf), and the journey/pathway of learning (represented by the curved line which winds around and up through the leaf). As such, the motif illustrates the YuMi Deadly Centre's vision: *Growing community through education*.

DEVELOPMENT OF THE AIM EARLY UNDERSTANDINGS MODULES

In 2009, the YuMi Deadly Centre (YDC) was funded by the Commonwealth Government's *Closing the Gap: Expansion of Intensive Literacy and Numeracy* program for Indigenous students. This resulted in a Years 7 to 9 program of 24 half-term mathematics modules designed to accelerate learning of very underperforming Indigenous students to enable access to mathematics subjects in the senior-secondary years and therefore enhance employment and life chances. This program was called Accelerated Indigenous Mathematics or AIM and was based on YDC's pedagogy for teaching mathematics titled YuMi Deadly Maths (YDM). As low-income schools became interested in using the program, it was modified to be suitable for all students and its title was changed to Accelerated Inclusive Mathematics (leaving the acronym unchanged as AIM).

In response to a request for AIM-type materials for early childhood years, YDC decided to develop an Early Understandings version of AIM for underperforming Years F to 2 students titled Accelerated Inclusive Mathematics Early Understandings or AIM EU. This module is part of this new program. It uses the original AIM acceleration pedagogy developed for Years 7 to 9 students and focuses on developing teaching and learning modules which show the vertical sequence for developing key Years F to 2 mathematics ideas in a manner that enables students to accelerate learning from their ability level to their age level if they fall behind in mathematics.

YDC acknowledges the role of the Federal Department of Education in the development of the original AIM modules and sees AIM EY as a continuation of, and a statement of respect for, the *Closing the Gap* funding.

Contents

Module Overview

This algebra module A2 *Functions and Equations* is the second Accelerated Inclusive Mathematics Early Understandings (AIM EU) module on algebra. Like the first algebra module, A1 *Patterning,* its focus is not on variables, x's and y's ("algebra early"), but on **general relationships** ("early algebra") in operations that also hold in algebra. As an example, knowing that 3 + 4 = 7 is arithmetic, if it is understood as 3 objects joining 4 objects makes 7 objects, but knowing that 3 + 4 = 4 + 3 is algebra if the students know that it is true because, for any numbers, first number + second number is always the same as second number + first number. In other words, arithmetic is knowledge of addition computation, namely, that 3 + 4 = 7, while algebra is knowledge of the general rule of addition that changing the order does not change the answer, namely, that 3 + 4 = 4 + 3 for any 4 and 3.

This module A2 is based on two major general ideas in algebra. The first, called *functions*, is that operations in arithmetic change numbers, that is, for example, 3 + 4 = 7 can be thought of as 3 changing to 7 by adding 4. Thought of in this way, arithmetic leads to the algebraic ideas of **inverse** (every operation can be undone or can be backtracked), that if 3 changes to 7 by +4, the inverse is that 7 changes to 3 by the inverse −4. The second, called *equations*, is that operations in arithmetic can be thought of as **equivalent**, that is, for example, 3 + 4 adding to 7 means that 3 + 4 = 7, 7 = 3 + 4 and 3 + 4 = 8 − 1 as both sides are 7, the same value. Thought of in this way, arithmetic leads to the algebraic idea of **balance rule** (whatever happens to one side of an equation happens to the other side to remain equal), that is, for example, 3 + 4 + 11 = 8− 1 + 11.

These algebra modules form part of a set of nine (see **Appendix A**) designed to provide support in Years F to 2 to improve Year 3 mathematics performance. Similar to the Years 7–9 AIM modules, the Years F–2 modules provide teaching and learning information in terms of a vertical sequence of units. Each unit is a step in the sequence and provides information and teaching ideas for that step. This section, Module Overview, precedes the units and looks at the content (connections, big ideas and pedagogy) covered by the module.

Function and equation ideas

This module looks at the major mathematical ideas that underlie functions and equations in school mathematics. It also introduces functions and equations as two perspectives that can be applied to any mathematical idea. It relates them to the informal words for them (change and relationship) and it also relates them to other ideas such as identity, inverse, arrowmath, and backtracking for functions; and equivalence, expressions, and balance rule for equations. It lists the major ideas to be developed and describes the two main models (i.e. function machine for functions and number line for equations). It shows how both lead to variables and graphs.

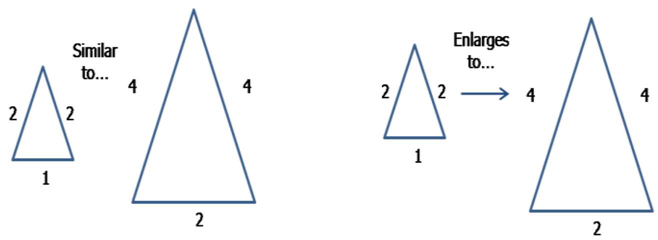
Only some of these are early childhood ideas, but it is important that the early childhood ideas in functions and equations are taught by teachers who know where they will be used in later years so that appropriate emphasis can be given to crucial parts.

Differentiating between function and equation

$$3 \xrightarrow{+2} 5$$

Real-world situations can be translated into changes or **relationships**. For example, "3 joining 2 to make 5" is a **relationship** if considered as 3 + 2 = 5 and is a **change** if considered as 3 changing to 5 by +2 (as on right). Change is the basis of functions and relationships are the basis of equations. Change is often informally used for function (as relationship is used informally for equation).

These real-world situations do not have to involve operations. For example, two triangles being similar can be considered as a **relationship** of angles being equal and sides in ratio, or can be considered as a **change** due to a projection that enlarges the first shape to the second shape, as on right.



The important point is that change and relationship are two perspectives of the same mathematics. If we have both, we will have two ways we can use to solve mathematics problems.

Function and change ideas

The major ideas to be covered in functions deal with mathematical forms that describe change and with the major ideas that emerge from changes, namely: (a) changes that do not change anything (e.g. +0, ×1, a 360° turn) – the **identity principle**; and (b) changes that reverse other changes (e.g. −6 reversing +6, ÷8 reversing ×8) – the **inverse principle** and **backtracking**. The major ideas in equations deal with how the two sides of the relationship are kept equal (or in balance) leading to the **balance rule**.

Changes can also be represented as equations but it is easier to understand them if they are represented by **arrowmath notation**. For example, the situation *I bought some $3 pies and a $5 chocolate, how much did I spend?* cannot be calculated because there is not enough information given. However, if we knew the number of pies, we could calculate the answer by multiplying this number by 3 and adding 5. Thus, the notation can be thought of as the equation $n \times 3 + 5$ (or $3n + 5$), or in arrows:

$$n \xrightarrow{\times 3} \xrightarrow{+5}$$

The arrowmath notation makes studying the above change easy. First, if we change forward, it is easy to work out what money will be paid for different numbers of pies; for example, if the number of pies is 7, then the answer is $26.

$$7 \xrightarrow{\times 3} \xrightarrow{+5} 26$$

Second, by reversing the change, it is possible to find the number of pies; for example, if I paid $38 for the pies and chocolate (see below). We use the inverses of the operations and backtrack (as shown in the bottom arrows) to the answer of 11 pies.

$$n \xrightarrow{\times 3} \xrightarrow{+5} 38$$

$$11 \xleftarrow{\div 3} 33 \xleftarrow{-5} 38$$

Sequencing of function ideas

Thus, the major ideas that can be developed in this section relate to inverse but include:

(a) developing the notion of change and inverse (backtracking) in unnumbered situations;

(b) extending the notions of change and inverse to numbers and operations (first with addition and subtraction, second with multiplication and division, and third with more than one operation);

(c) introducing drawings (number lines and function machines), tables and arrowmath notation to describe changes and inverses;

(d) relating change and inverse (backtracking) to real-world situations and vice versa;

(e) generalising change and inverse and using this to introduce variables and algebraic expressions and equations (including conversions between arrowmath and equation notations);

(f) interpreting real-world problems in terms of change and using backtracking to solve for unknowns; and

(g) representing generalised change with graphs and relating real-world situations, arrowmath and equation notation, and graphs and change to graphs in all directions.

Equation and equivalence ideas

Relationships are most often represented as equations and this form of notation is good for seeing equals or equivalence as balance and for applying the balance principle (that there is a left-hand side and a right-hand side and they have to stay in balance). However, mathematically, 2 + 3 does not equal 5, only 5 equals 5. The correct interpretation is that 2 + 3 is **equivalent** to 5. Thus, although we use an equals sign in our equations, they represent equivalence.

Equivalence and equations cover understanding of number sentences involving numbers, operations, variables, and equals, greater than and less than signs. These are called *equations*, *inequations* and *expressions*, and are defined as follows: (a) an *equation* is a sentence, usually involving numbers, operations and variables, that has an equals (=) to show a relationship between two things (e.g. $2 + 3 = 5$, $2x + y = 16 + y^2$); (b) an *inequation* is an equation with greater than or less than symbols (< or >) showing an order relationship (e.g. $2 + 3 > 4$, $2x + y < 16 + y^2$); and (c) an *expression* is one side of an equation; it has numbers, operators and variables but no equals or greater than or less than symbols (e.g. $2 + 3$, $16 + y^2$). Thus an equation is the equivalence of two expressions and an inequation is an order relationship between two expressions. To study equations and inequations is also to study expressions.

Sequencing of equation ideas

The major ideas to be covered in equivalence and equations deal with using equations to model real life and manipulating and solving the equations to solve these real-life problems. The sequence of activities designed to build this understanding and proficiency is as follows:

(a) introducing the notion of same and different and relating this to introduce equal, unequal, greater than and less than in length and mass (balance) situations;

(b) using mass and length in unnumbered situations to build understanding of equals and equations and to develop the equivalence and order principles;

(c) using mass and length in numbered situations to build understanding of arithmetic equations and to reinforce the equivalence and order principles in numbered situations;

(d) relating arithmetic equations to real-life situations and vice versa (e.g. telling stories about the world);

(e) using mass and length models to introduce the balance principle that equations stay equal if the same thing is done to both sides of the equation;

(f) using mass and length models to introduce unknowns, and relate equations with unknowns to real-world situations and vice versa;

(g) extending mass and length models to mathematical versions (in picture form) where all operations are possible;

(h) using the balance principle to find solutions to equations with unknowns;

(i) developing rules/principles that enable expressions to be manipulated (including simplification and substitution); and

(j) introducing graphical representations of equations and showing how graphs, equations and everyday life relate.

Connections and big ideas

YuMi Deadly Mathematics (YDM) is based around (a) connections, big ideas and sequencing; and (b) the Reality–Abstraction–Mathematics–Reflection (RAMR) model (**see Appendix B**). It endeavours to achieve three goals: (a) reveal the structure of mathematics, (b) show how the symbols of mathematics tell stories about our everyday world, and (c) provide students with knowledge they can access in real-world situations to help solve problems. YDM argues that the power of mathematics is based on **connections and big ideas**. For arithmetic (number and operations), these come from algebra and lead to algebra. The best way to learn mathematics is by using these connections and big ideas; that is, using algebraic thinking to generalise arithmetic. This needs to be developed early so that students can better understand number and operations and be better prepared for formal algebra.

Connections

We believe that mathematics is best understood and applied in a connected manner (a rich schematic structured form) which contains knowledge of **when and why as well as how**. Connected nodes facilitate recall and problem-solving. Knowledge of the structure of mathematics can assist teachers to be effective and efficient in teaching mathematics, and enable students to accelerate their learning. It enables teachers to:

(a) *determine what mathematics is important to teach* – mathematics with many connections or based on big ideas is more important than mathematics with few connections or little use beyond the present;

(b) *link new mathematics ideas to existing known mathematics* – mathematics that is connected to other mathematics or based on the one big idea is easier to recall and provides options in problem solving;

(c) *choose effective instructional materials, models and strategies* – mathematics that is connected to other ideas or based around a big idea can be taught with similar materials, models and strategies; and

(d) *teach mathematics in a manner that enables later teachers to teach more advanced mathematics* – by preparing linkages to other ideas and foundations for big ideas later teachers will use.

Connections in early algebra

Because it is the generalisation of arithmetic ideas, early algebra aims to ensure that arithmetic is seen as a structure of connected ideas, not a series of algorithms and rules to be learnt. That is, early algebra is not about x's and y's; it is about doing and understanding arithmetic in a deeper way that builds arithmetic structure and prepares students for algebra. This is achieved if teachers:

(a) know the mathematics that precedes, relates to and follows what they are teaching, because they are then able to build on the past, relate to the present, and prepare for the future and show the connections between these three levels;

(b) develop the underlying connections informally and relate them to the world of the student and, in particular, teach from unnumbered to numbered activities (and later to unknowns and variables);

(c) connect arithmetic to students' reality so that this can be extended to algebra (e.g. "bought a pie for $5 and a drink for $4" is 5 + 4 = 9, while "bought a pie and a drink for $4" is x + 4) – this means a heavy emphasis on symbols → reality and reality → symbols so that abstract symbols like $2 \times 7 + 3$ and, later, $2y + 3$ are given meaning in reality; and

(d) ensure students spend time finding and expressing generalisations (which is why Module A1 precedes this Module A2) and knowing and understanding some of the common generalisations, such as "turnarounds" (or, more formally, the commutative law) like $6 + 5 = 5 + 6$.

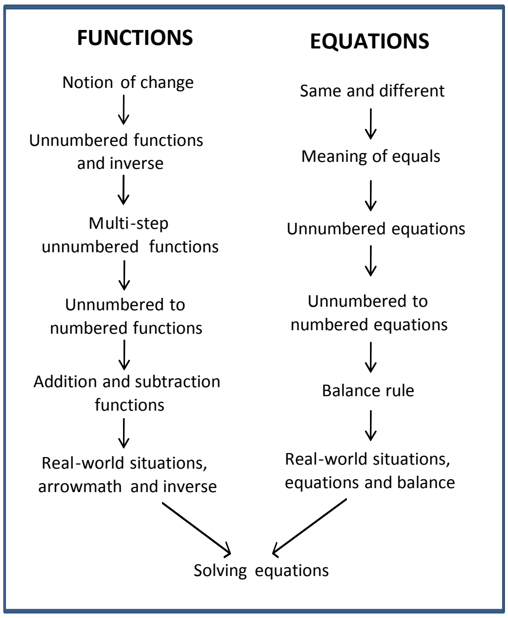
Big ideas

Big ideas are mathematical ideas which recur and are useful in many strands/topics of mathematics and across many year levels. There are five types of big ideas: global, concept, principle, strategy and teaching. The **major big ideas for algebra** are listed under these headings as follows.

1. **Global**. These are big ideas that apply very widely (some apply to all mathematics). An example is "symbols tell stories", that is, symbols are a concise shorthand language for describing the world. This concept applies to arithmetic, algebra, geometry and all of mathematics. The algebra global big ideas are those from operations: symbols tell stories, relationship vs change, interpretation vs construction, accuracy vs exactness, and part-part-total, plus extra emphasis on unnumbered to numbered.
2. **Concept**. These are meanings of central ideas, such as the concept of place value (also applies to mixed numbers and to measures) or the concept of subtraction (applies to all types of numbers and to algebra). Often there is more than one concept for each term (e.g. the part-of-a-whole concept of fraction and the division concept of fraction). Algebra concept big ideas are similar to those from operations but with some extra that are particular to algebra, as follows:
    - operation concepts – four operations, equals, order (> and <); and
    - new algebra concepts – generalisation, expression, equation, unknown, variable, linear, and function.
3. **Principle**. These are relationships whose meaning is encapsulated in the relationship between the components of the idea not in the actual content focus of the idea, such as the formulae for the volume of a cylinder or the distributive principle (e.g. 24 × 3 = 20 × 3 + 4 × 3; 6 × 7 = 6 × 2 + 6 × 5). Again, algebra principle big ideas are similar to those from operations but with some extra that are particular to algebra, as follows:
    - operation principles (field principles) – closure, identity, inverse, commutativity, associativity, distributivity, compensation, equivalence, inverse relation, and triadic relationships;
    - equals/order principles – reflexivity/nonreflexivity, symmetry/antisymmetry, and transitivity; and
    - new principles – balance rule, backtracking, expansion, factorisation, changing subject of a formula.
4. **Strategy**. These are general "rules of thumb" that point towards a solution of a problem or procedure. For example, the separation strategy for adding numbers of two or more digits (applies to subtracting mixed numbers, measures and variables as well as whole numbers), or the problem-solving strategy of make a drawing, diagram or graph (which is universally applicable to problems of any type). The strategies are similar to operations – separation, sequencing, and compensation.
5. **Teaching**. These are big ideas for teaching mathematics (not for mathematics itself) that apply to a variety of, if not all, teaching situations. An example of a teaching big idea is the RAMR reflection strategy of "reversing". In fact all of RAMR is a big teaching idea. Algebra will have all or most of the teaching big ideas that other topics have. However, particularly important in algebra is generalisation.

Sequencing

The sequencing of the teaching of early functions and early equations is shown on right. It is based on the sequences given earlier under the heading *Function and equation ideas*.



The major sequencing idea is to move from unnumbered to numbered with the unnumbered being taught first. Thus, in this module, instruction begins with unnumbered activities. This is because students appear more easily able to look for patterns and generalisations in unnumbered activities than in numbered situations, where they tend to look for answers. The activities in this book, as far as possible, start with unnumbered activities, then move to numbered activities and then to variable activities. This involves generalisation in which students progress from working with small numbers, to large numbers (this is called quasi-generalisation), everyday language, and finally, formally with variables. The figure below illustrates this diagrammatically.

Unnumbered activities
↓
Numbered activities
↓
Variable activities

GENERALISATION

↓ Small to large numbers
Everyday language
Variables

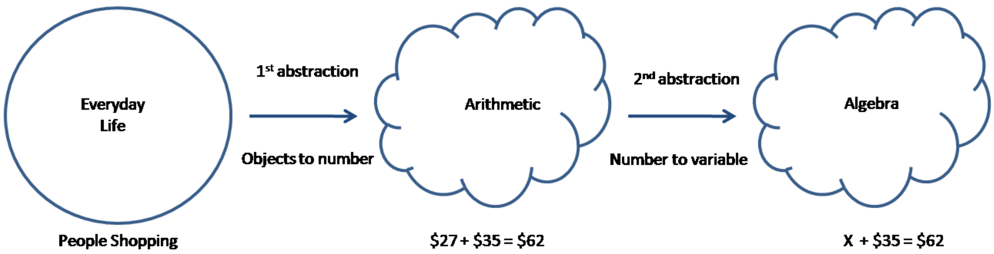
Teaching and cultural implications

This section looks at teaching and cultural implications, including the Reality–Abstraction–Mathematics–Reflection (RAMR) framework and the impact of Western number teaching on Indigenous and low-SES students.

Teaching implications

Many of the teaching implications have already been covered. First, the RAMR framework (Appendix B) should be used. Second, teaching should start at unnumbered activities. However, there are four other points to be made, as follows.

1. **Connecting symbols and reality**. As also discussed earlier, the relationship between everyday life and algebra is a two-step abstraction that goes through arithmetic (see below). This means, first, that the act of generalising is at the core of algebra and proficiency must be built in both the act of generalising (how to generalise) and the products of generalisation (the mathematical ideas/principles that result from generalising). Second, it means that the symbols of algebra, notably the letters, are far removed from everyday life and their meaning must be built with care through: (a) continuous connections being made between symbols and real-world stories, and (b) using sequences of materials and activities that become progressively more abstract.



2. **Pre-empting and prerequisites**. The figure above also explains why algebra is difficult for many students. It shows that the step from everyday life (reality) to arithmetic must be well built because the step from arithmetic to algebra is built upon it. It is very difficult for students to invent stories for $2x - 1$, if they cannot invent stories for $2 \times 5 - 1$.

3. **Generalising and reversing**. As algebra is generalisation of arithmetic, the reflection step of generalising is crucial. All activities should be discussed in terms of what they mean in general – what would "any number" do? As well, to reach the generalisation often involves many steps – always **reverse** these steps in the next activity.

4. **Dual perspectives**. Functions and equations are dual perspectives of the same thing. We look at 5 × 7 = 35, and we can think of it as a change – that 5 has been changed to 35 by multiplying by 7 and that this change can be reversed by dividing by 7. Or we can think of it as relationship, that 5 × 7 balances/equals 35 and, if we add 2 to 35, we have to add 2 to 5 × 7 to main the balance/equals, that is, 5 × 7 + 2 = 35 + 2 = 37. In the long run, although arrowmath is really useful in understanding functions and solving for unknowns, formal equations are the language/writing of algebra, and so we have to bring the two perspectives together.

Cultural implications

There are two implications for algebra from the above: (a) what is the best way to teach it? and (b) what is the best way to teach it to Aboriginal and Torres Strait Islander and low-SES students?

Teaching algebra. The power of mathematics lies in the structured way it relates to everyday life. Knowledge of these structures gives learners the ability to apply mathematics to a wide range of issues and problems. This is best achieved if the knowledge is in its most generalised form, which is algebraic form. Thus, the most effective way to present mathematical knowledge is through algebra. However, any topic of mathematics can be presented instrumentally (as a set of rules). Although algebra is the direction for power in mathematics, it has to be presented structurally, showing the generalisations that can be used in many examples. Powerful algebra teaching focuses on extending arithmetic to generalisations that can apply across all arithmetic; that is, teaching that builds holistic understandings of structure that can then be applied to particular instances (from the whole to the part). If students are fortunate enough to gain this structured understanding of mathematics, the subject becomes easy. This is because it is no longer seen as a never-ending collection of rules and procedures but rather as the reapplication of a few big ideas.

Teaching Indigenous students. Aboriginal and Torres Strait Islander students tend to be high context – their mathematics has always been built around pattern and relationships. Their learning style is best met by teaching that presents mathematics structurally as relationships, without the trappings of Western culture. Powerful Indigenous teaching is therefore holistic, from the whole to the part. As Ezeife (2002) and Grant (1998) argued, Indigenous students should flourish in situations where teaching is holistic (from the whole to the parts). Thus, holistic algebra teaching has two positive outcomes for Indigenous students: (a) it teaches a powerful form of mathematics, and (b) it teaches it in a way that is in harmony with Indigenous learning styles. Algebra taught structurally, then, is something in which Indigenous students should excel. However, this is just a general finding. What does this mean in practice for the teaching of algebra? It means that we will not be teaching rules for manipulating letters. Letters and algebraic expressions and equations will be understood in terms of everyday life and algebraic ideas will be generalised from arithmetic. This will mean a lesser focus on algorithms and rules, and a greater focus on generalisations and applications to everyday life.

Teaching low-SES students. Interestingly, holistic teaching is also positive for low-SES students. Three reasons are worth noting. First, low-SES students tend to have strengths with intuitive-holistic and visual-spatial teaching approaches rather than verbal-logical approaches. Thus, an algebraic focus on teaching mathematics should also be positive for low-SES students. Second, many low-SES students in Australia are immigrants and refugees from cultures not dissimilar to Aboriginal or Torres Strait Islander cultures. They are also advantaged by holistic algebraic approaches to teaching mathematics. Third, many low-SES students have themselves experienced failure in traditional mathematics teaching, and so have members of their families. This results in learned helplessness with regard to mathematics and what is called *mathaphobia*, where students believe that no effort on their part will enable them to learn mathematics. Holistic-based algebraically oriented teaching of mathematics is sufficiently different that students may not apply their phobia to it – particularly if taught actively and from reality as in the RAMR model.

Thus, for the Indigenous and low-SES students for whom YDM was developed, **algebra is the key for mathematics success** – not x's and y's but the generalised holistic thinking that is the basis of it.

Structure of module

Components

Based on the ideas above, this module is divided into this overview section, four units, a review section, test item types, and appendices as follows.

Overview: This section covers sequencing, connections and big ideas, teaching and culture, and a summary of the module structure.

Units: Each unit includes examples of teaching ideas that could be provided to the students, some in the form of RAMR lessons, and all as complete and well sequenced as is possible within this structure.

Unit 1: Early functions. This unit covers the major understandings of the functions part of this module, change and inverse. It focuses on unnumbered changes and all the various types of activities that can be done in that topic. It introduces function machine activities with their focus on Input and Output.

Unit 2: Early equations. This unit covers the major understandings of the equations part of this module – same, different, equals, not equals – and introduces the mass and length models. Like Unit 1, it focuses on unnumbered activities and concludes with equations and the equals properties.

Unit 3: Early numbered functions. This unit covers the extension of Unit 1 to numbers, looking at one- and multi-step function machine activities that lead to inverses and backtracking of changes.

Unit 4: Early numbered equations. This unit covers the extension of Unit 2 to numbers, developing numbered equations, relating these to real-world situations and the balance rule. It also shows how models develop from physical to more abstract pictorial models.

Module review. This section reviews the module, looking at important components across units. This includes the way that two different ideas (change and balance) can be related and modelled to enable the balance rule to be implemented to solve problems.

Test item types: This section provides examples of items that could be used in pre- and post-tests.

Appendices: This comprises three appendices covering the AIM EU modules, the RAMR pedagogy, and proposed teaching frameworks for functions and equations.

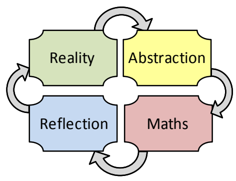
Further information

Sequencing the teaching of the units. The four units are in sequence and could be completed one at a time. However, Units 1 and 3 cover functions and Units 2 and 4 cover equations. Units 1 and 3 are in sequence and so are Units 2 and 4; but such sequencing is not in Units 1 and 2, and 3 and 4. Therefore, schools may find it advantageous to teach across functions and equations with earlier sub-units in Unit 2, for example, being taught before later units in Unit 1. A similar sequencing may also work in Units 3 and 4.

The AIM EU modules are designed to show sequences within and across units. However, it is always YDC's policy that schools should be free to adapt the modules to suit the needs of the school and the students. This should also be true of the materials for teaching provided in the units in the modules. These are exemplars of lessons and test items and schools should feel free to use them as they are or to modify them. The RAMR framework itself (see **Appendix B**) is also flexible and should be used that way.

However, this does not mean that any order will do. Together, the units and the RAMR framework are designed to ensure that all important information is covered in teaching and that later builds on earlier. Therefore, if changing and modifying the order, try to ensure the modification does not, within topics, miss something important or fail to provide a prerequisite for a later unit (see **Appendix C** for proposals for detailed teaching sequences and frameworks).

RAMR lessons. We have included RAMR lessons as exemplars wherever possible in the units of the module. Activities that are given in RAMR framework form are identified with the symbol on the right.



Suggestions for improvement. We are always open to suggestions for improvement and modification of our resources. If you have any suggestions for this module, please contact YDC.

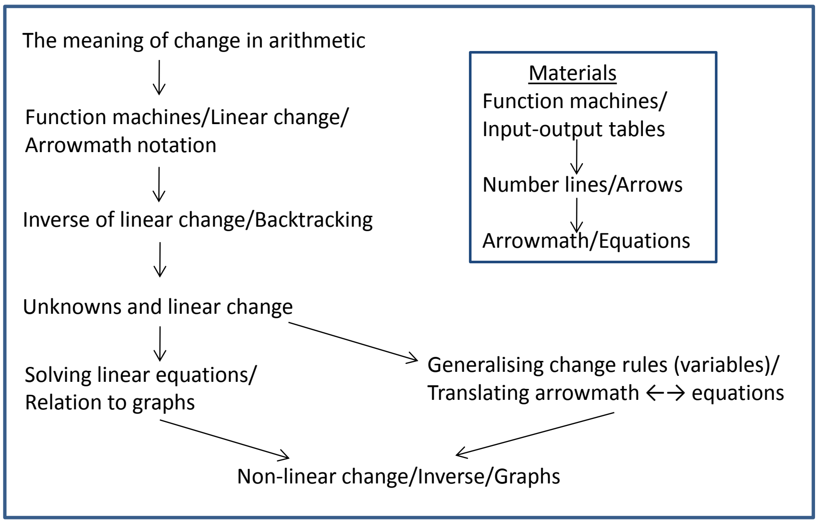
Unit 1: Early Functions

Early functions introduces the idea of change and no change. It is unnumbered in its activities, so it does not cover the four operations (addition, subtraction, multiplication and division). It looks at change and inverse of change in everyday activities. For example, washing your car is changing it from dirty to clean and the inverse of this is clean to dirty. The early years are when unnumbered activities are used to develop the language of change (the words change, input and output) and the notions of change and reversing change. This unit, therefore, explores how to represent everyday life in terms of change or transformation.

The sections in this unit are constructed around activities suitable for early year levels. The unit begins by providing background information and then provides a sequence of four activities (in this module, each activity involves a sequence of numbered steps) – notion of change, unnumbered functions and inverses, “guess my rule”, and extensions (two function machines and early number).

Background information

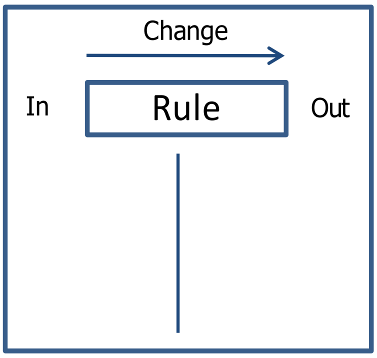
The sequence of activities that makes up functions for Years F–9 covers *change* and *operations as change* and is designed as a precursor to functions. The sequence for teaching is summarised in the figure below. The sequence moves from unnumbered to numbered activities, addition and subtraction to multiplication and division, and one operation to multiple operations. It introduces input–output tables and arrowmath notation. It develops inverse of one operation, inverse of sequence of operations, and backtracking to solve for unknowns. It relates arrowmath notation to equations, and graphs the results, relating all aspects in all directions. It continually relates symbols and models back to everyday situations so that it can model reality. Finally, it allows change to be generalised into a rule using algebraic notation, thus introducing variable and leading to function.



It introduces the symbols, notation and rules for change and functions (including input–output tables and arrowmath symbols and their relation to equations and graphs). In the long run, it returns to representing functions using equation notation. It is important that arrowmath and equations be related at the end so that both relationship and change ideas can be applied to algebraic situations using the same notational forms (the expression and the equation). To get to this point requires studying (a) change in unnumbered situations; (b) linear change in arithmetic situations (numbers and operations but with no squares, cubes, and so on); (c) linear change in algebraic situations (variables and operations); and (d) nonlinear change. Similarly to patterns, most of the change and function work for Years P to 9 is in linear form. However, the groundwork for nonlinear changes should be laid to pre-empt the needs of Year 10.

1.1 Notion of change activity

1. Discuss with students how things change (e.g. we clean things, we grow things, weather gets hotter, things are moved around, we change clothing, change hair colour, and so on).
   (a) Discuss before and after – before washing hair, after washing hair, before putting on a dress, after putting on a dress.
   (b) Take photographs – use these as before and after discussion points (e.g. what happened here?). Look at relationship patterns – what goes with what? For example, shoes with feet, shirts on bodies, hats on heads, and so on. Focus on how before and after have to be related in some way.
2. Play card games like "switch" (or its commercial form, UNO). If a spade is put down, you have to follow it. To change suit you can put the same value on top, and so on. Play Snakes and Ladders – things change if you land on a snake or a ladder.
3. Set up a function machine that will have an input and output and a rule for change; see example on right for whiteboard.
   (a) Give students a small copy of the board to record their changes (doing this with a small chalkboard [slate or whiteboard] was very successful in one school).
   (b) Choose an unnumbered change and put this in the RULE box. Discuss what input and output are – act out some changes.
   (c) Have students walk in on the left with a picture of a thing to be changed and stick this on input. Discuss what it could change to.

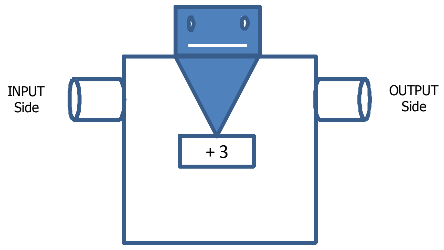


Whiteboard

   (d) Give students a picture of this change to put on the right-hand side (RHS). Students should also record on their recording sheets or chalkboards, the input and output (the In and the Out).
4. Repeat this activity – examples of unnumbered change include:
   (a) lower case to capital letter (e.g. input h and output H);
   (b) "cook it" (e.g. input a picture of a potato and output a picture of chips);
   (c) "wash it" (e.g. dirty car to clean car);
   (d) "wear it" (e.g. hand to glove, foot to shoe);
   (e) add "at" (e.g. b to bat, fl to flat);
   (f) add "ing" (e.g. r to ring , s to sing);
   (g) move first letter of word to end of word; and
   (h) let the students make up their own!
5. Involve students in bringing out picture cards (or potatoes or letters or whatever is relevant for the RULE) and working out what the output will be. Get students to discuss what is happening, and encourage students to think of things to change and to think of changes.
6. Reverse the activity – there are two ways the activity can be reversed: (a) give students the output and the rule, and they have to work out what the input could be; (b) give students the output and the input and have them work out or guess the change or rule. (This is called the "What's my rule?" or "Guess my rule" game and is described further in section 1.3.)

Note: Attribute logic blocks or pattern blocks can also be used – changes can be blue to red or large to small or triangle to square. The thing to remember here is that attributes that are not mentioned should not change.

The function machine can also be a "robot" made out of a large cardboard box with input-output openings, with students inside taking the inputs and sliding out the outputs on the other side. This really raises the engagement of the students – they love being chosen to be in the robot.



One school had such a large box that they put a door on each side and the students could walk through the "robot", hang up their input card and take the output card. The whole school loved this one.

1.2 Unnumbered functions and inverses activity

Use the function machine from 1.1 above to look at the notion of inverse.

1. Make up a matching set of pictures before and after cooking (e.g. pasta in a packet to spaghetti bolognaise, and so on). Organise students to have slates or whiteboards with an input–output table drawn on them.
2. Organise students to come out front and pick an input card. Get students to show the picture to the class and stick it on the input side of the table. Discuss what would be on the output side. Select the likely picture from output cards.
3. Continue in this way – then switch to other examples of change. Students not involved in front at the function machine are to draw inputs and outputs on their slates.
4. After doing this for some time, ask one of the students to select an output card and stand on the RHS, at output. Discuss what the input card could be. Students draw on input–output table on their slates.
5. Repeat this as often as required. Initially, get students to think what input could give this output, e.g. *We have mashed potato, what could we cook to get this?* But after a while, get them to "**think backwards**", e.g. *What do we get when we "uncook" the mashed potato?*
6. This is easier done with, for example, the "add *at*" rule. Here, input of *s* goes to output of *sat* – we have "added" the "at". When we look at an output of *rat*, it is fairly straightforward to consider removing, or "taking away", the "at" to find an input of *r*. Similarly, input and output that makes *h* a capital *H* can be considered to have "capitalise" as its change rule, and we can think of output to input as "uncapitalising", e.g. *R* to *r*.
7. Repeat and practise with different change rules.

1.3 "Guess my rule" activity

This activity is a reverse of the above two activities. Instead of following a rule, students have to guess the rule.

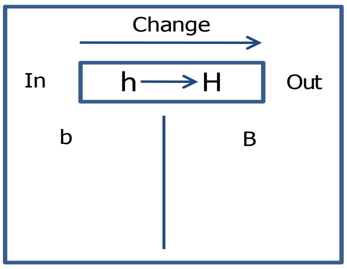
1. The teacher makes up a change rule (e.g. "add *ing*") and prepares input and output cards for this rule.
2. The teacher asks students to come out and pick up an input card and put it on the input side. Students draw this on their input column. Then the teacher places a card as output on the machine and students draw this.
3. Students have to guess the change rule. This is repeated for harder examples.
4. This works really well for logic blocks as the change could be a change of colour, shape, size or thickness, or a combination of these four.
5. You could let students play in groups – a student selects a change but tells no one – other students hand the first student a card and the student changes it. Keep going until someone guesses the change rule.

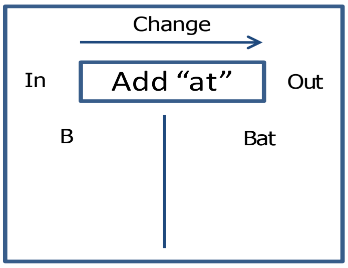
1.4 Multi-change activity

There are two ways we can begin extending what we have done in the above. The first of these is to involve two or more function machines (and two or more changes in sequence). The second is to begin the process of extension from unnumbered to numbered change. We do the first here and leave the second for Unit 3.

1. Think up two changes that could occur one after the other. Construct two function machines side by side. We can then act out and reverse double changes.

2. For example, consider the two changes: “capitalise” followed by “add *at*”. This is two changes that students have to move through with the two function machines (see below) – first the change from b to B and second the change from B to Bat.





   (a) Repeat the activities of 1.1 but with two changes in a row – students pick up input card, go through the first machine to the middle (between the two machines) with the first output card, then take this card through the second machine and pick up final output card. The slates and whiteboards can have three columns – Input, Middle, Output. Students continue to copy what happens on the two machines.

   (b) Try to reverse both changes, e.g. if we ended with *Fat*, then this goes back to *F* and then *f*. (It should be noted that changes like *e* to *Eat* could be challenging as well as *fl* to *Flat*.) Have the students walk left to right to go through the two machines. Have students walk backwards from RHS machine to LHS machine when reversing.

   (c) Start in the middle – what happens at input and output if you have a capital M in the middle?

3. Try to do sufficient activities so students can understand that if they go forward it is:

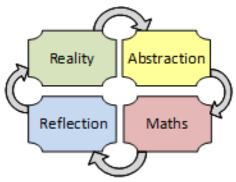
   first change → second change

   but if they reverse, it is:

   reverse of first change ← reverse of second change

   not reverse of first change → reverse of second change.

1.5 RAMR lesson for early functions



Learning goal: Follow a rule to change values; reverse the change by reversing the rule; and identify the function rule when given a set of input values and a set of output values. (Unnumbered)

Big ideas: Language ←→ picture ←→materials ←→ action/function machine; visualising movements (backtracking) through kinaesthetic activity.

Resources: General classroom items, manipulatives, toys that range in size, colour, texture, mass, length.

Reality

Local knowledge: Items that change and events that change that are familiar to children; play and construction.

Prior experience: Sorting, collections and materials, identifying and observing change/no change around us.

Kinaesthetic: **Follow the rule** activity. Large box or puppet theatre to use as function machine. Label input and output slot. Attach a rule to the outside of the function machine. As students place words or values or items in the input slot, another student applies the rule and "posts" the output through the output slot. Students take turns in roles. Students in audience predict and check and/or record on whiteboard what is happening. (Teacher to model this recording in initial stages of the activity.) Rule ideas: change colour; cook; wear; add an item; turnover; turn upside down; etc. Have students reverse the change and identify the inverse rule.

Abstraction

Body: Have small groups of students enact rules and changes with function machine. Provide rules and materials. Ask: *Can you reverse the change? What is the rule for this? Can you act out the change without the rule displayed and can your partners identify the rule?*

Hand: Have students record the action from the function machine with drawings of a function machine. Ask: *Where would you put the rule? Where is the input slot? Where is the output slot? Can you draw the reverse change? What is the rule? Can you draw a machine with the input and output values but no rule?* Ask a partner to identify the rule.

Mind: Visualise a fantastic function machine. Describe what it looks like to a friend. Think of a rule and see in your mind's eye the fantastic function machine working as you input items.

Creativity: Have students draw/make their own fantastic function machine with their own rule example. Swap with another student to test the machines.

Mathematics

Language/symbols: Rule, input, output, function machine, value, change, change process, reverse, inverse, identify, table, backtracking.

Practice: 1. Have students construct individual function machines from small boxes (milk carton is ideal). Give them rules to practice. Consolidate the notion of change. 2. Model how to record this change on the Input/output table. 3. Have students make up their own change and record it on an input/output table. 4. Ask: *Can we undo the change?* Consolidate the notion of reversing the change. Have students practise finding the unknown by backtracking. It is important to show the students that they are working backwards or backtracking by changing the direction of the arrow and reversing the rule. 5. Identify the inverse rule. 6. Have the students record this on an input/output table. 7. Play guess my rule: Give the students an input/output table with values but without the rule. Can the students identify the rule?

Connections: Encourage students to describe and discuss the change processes. Where might you see/might we use change processes like this? Where do we see/use change processes in other areas of maths?

Reflection

Validation: Consolidate the notion of change rule, reversing and identifying the rule by having students create these for their partner to use with various function machines.

Extension: Create situations with two changes. Use two function machines beside each other. Provide two input/output tables beside each other for students to record the two changes. Have the students reverse the change and record this process.

Reversing: Give students input /output tables with values and no identified rule. Ask students to tell/write rule.

This RAMR could be used in a sequence of lesson topics:

- Lesson 1: A change occurs based on a simple rule.
- Lesson 2: Reverse the function/rule.
- Lesson 3: Identify the rule.
- Lesson 4: Backwards and forwards.
- Lesson 5: More than one step.

Unit 2: Early Equations

This unit looks at early equations. Following the ideas from Unit 1, this means:

(a) getting to know the ideas behind equations – the meanings of same and different, the meaning of equals, and the meaning of equation from an unnumbered perspective;

(b) starting with unnumbered relationships or relationships in unnumbered situations, and then moving on to numbered;

(c) exploring how to represent everyday life in terms of relationships and relating this to equals and equations; and

(d) studying the symbols, notation and rules for equations (number sentences with equal signs) and relating them to real-life situations.

The unit consists of background ideas, followed by activities (in this module, each activity involves a sequence of numbered steps) relating to same and different, the meaning of equals and what this means for equations, unnumbered equations, and finally properties of equals.

Background ideas

In the long run, equations are used with numbers, operations, letters, unknowns and variables. To get to this point requires the study of the following: (a) equations in unnumbered situations and in arithmetic (no variables or letters); (b) equations with unknowns for which calculation is in arithmetic form (called pre-algebra by some curricula); and (c) equations with variables where calculation is in algebraic form (considered to be full algebra – these are normally where there are variables on both sides of the equation). It is imperative that students learn **symbols as ways of telling stories** about everyday life. Initially, the symbols will be as in arithmetic (e.g. numbers and operation symbols). However, as problems move to where not all numbers are provided (e.g. *I bought a hat for $88 and a coat and spent $227 altogether*), the symbols will include unknowns (pre-algebra) and variables (algebra).

This means that we look first at the meaning of equals, the meanings of greater than and less than, at number sentences with these symbols (equations and inequations), and finally at their principles (properties of equals, greater than and less than). Then the balance rule is introduced (first for unnumbered situations and then for numbered), and used to find unknowns. This leads to the method of solving equations which has the widest application because it can be used with variables on both sides of the equations. Finally in junior secondary, we begin to look at nonlinear equations (equations that are not a straight line such as those with variables as squares or cubics), and set the scene for modelling algebraic situations. This sequence is shown below.

Equals, equation and inequation (relation to reality)
↓
Principles (equivalence and order)
↓
Balance rule (unnumbered → numbered)
↓
Unknown/Solving linear equations
↓
Nonlinear equations/Formulae
↓
Modelling algebraically

Materials
Physical balance/Lengths
↓
Pictures and counters/
Number lines
↓
Abstract 'Math' balance/
Drawings/Equations

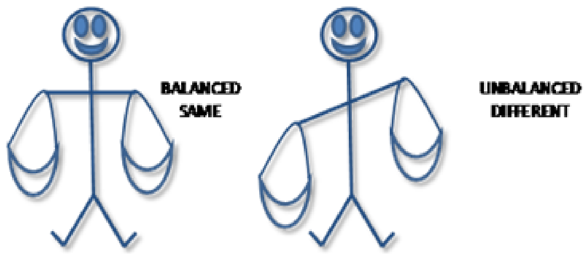
There are two models used to introduce equations – mass (the beam balance) and length (using strips and number lines). In the long run, the physical balance and length materials restrict the operations that can be used to addition, subtraction and simple multiplication. They are also kinaesthetic and time consuming – excellent for beginning the teaching but inefficient in later work. The diagrams of balances and number lines enable more activities to be completed but still have restrictions on operations. Thus, it is important that students understand that the balance and the lengths/lines must become **abstract mathematical balances and lines** and able to represent any operation (including division). This means combining the abstract balances/drawings with equations. This leads to the final step which is to dispense with materials and just use equations.

It is also crucial to teach the **three properties of equals**: (a) *reflexivity* – same things are equal, that is A = A for any A; (b) *symmetry* – the first thing equalling the second means that the second equals the first, that is, A = B means B = A for any A and B; and (c) *transitivity* – the first equalling the second and the second equalling the third means that the first equals the third, that is, A = B and B = C means A = C, for any A, B and C. The crucial one is symmetry. Many primary students believe that 2 + 3 = 5 but that 5 ≠ 2 + 3 because you are only allowed one thing, the answer, on the RHS. This misunderstanding is one of the major reasons for student difficulties in algebra in junior secondary years.

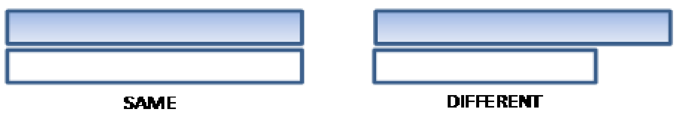
2.1 Same and different

This is the basis of equations – understanding what equals and equivalence mean. To first raise these ideas also means that we have to spend time with unequals.

1. Get students to identify objects which are the same and which are different. Encourage them to learn to describe what is the same and what is different about two objects. Get them to sort objects into those that are the same and to describe the resulting sorted groups.
2. Use physical mass balances and lines/strips to relate different situations and look at what is the same and what is different (as on right and below). The idea here is to use mass and length to be major vehicles for same and different. In doing this, we are considering same and different in terms of real physical mass and length models.



3. For mass, the teacher takes two plastic bags and places on students' arms, puts things in the bags and allows students to feel when things are the same and when they are different. Once this has been repeated enough, you can move to a classical beam balance with trays.



   - The masses to be put in the bags/on the trays can be from the classroom but it is also useful to buy a collection of items from the supermarket.
   - It is important to have things that weigh the same so students experience same in a variety of ways – two 125 g cans feels the same as one 250 g packet.
   - You can also blindfold students as this makes the focus on mass as a heft, as a feeling, not a visual.
4. For length, look at objects (e.g. paper strips) and see if they are the same or different lengths when aligned at one end. These can initially be objects from the room, and can extend to height, strips of paper or pieces of ribbon.
5. Once same and different are understood, the material can be used to introduce the formal language for same and different, namely, equals, not equals, greater than, less than. After the formal language the symbols are introduced, namely =, ≠, <, and >. Discussion is a good way to do this – ask: *What are the mathematics words for same and different?*

2.2 Meaning of equals activity

This activity follows directly on from 2.1. It should flow seamlessly from same and different. It can be done with the "human balance" (plastic bags, blindfolds, groups of three – one acting as balance, one watching the actor so they do not walk into something, and one getting material to put in plastic bags).

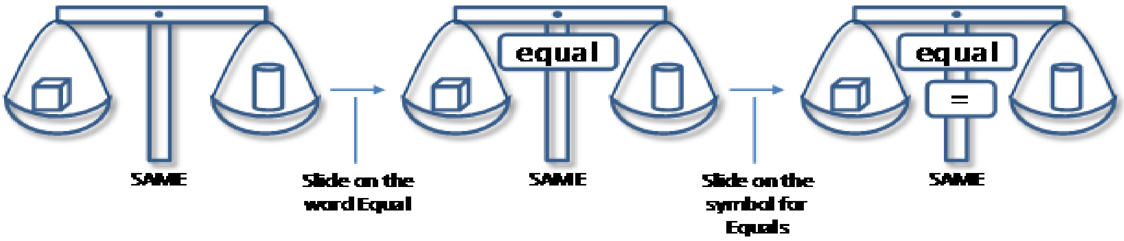
The technique is to discuss what is happening with respect to the balance [it is balanced] and introduce words and symbols by sticking the language and symbols on the balance. Relate the notion of balance to *equals* and imbalance to *not equals*. Early on in the primary years, similar techniques could be used to introduce *greater than* (>) and *less than* (<).

1. The first step is to formalise same and different to equal and not equal. At this point, you may want to switch to the classical beam balance. It is a three-part process as below – using labels *same*, *different*, *equals*, *not equals*, = and ≠.

   (a) set up an equal beam balance;

   (b) discuss with students that they are the same, ask what is the word for same and place the label with this word on the balance; and

   (c) ask students for the symbol and place the label with this symbol below the word.



2. Repeat the above until students get it. Then set up a different balance, where the two things do not weigh the same. Use the *not equals* and ≠ labels to label the balance. Repeat this.

3. It should be noted that some teachers like to also have labels for *same* and *different* and to initially have these labels stuck on and, as part of the process, remove these labels and replace them with the *equals*, *not equals*, = and ≠ labels as appropriate.

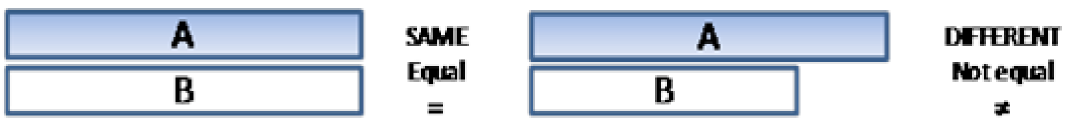
4. When repeating the labelling, stop at the end of the steps and display the balance. Place your hand on balance where it is LHS for the students and, as you move your hand to the students' RHS of the balance, say: "[Whatever is in the LHS tray] equals [whatever is in the RHS tray]". (e.g. "The box equals the cylinder" for the above diagram).

5. Length can also be used in this way (see below), but perhaps not as strongly. For example:

   (a) make labels *same*, *different*, *equals*, *not equals*, = and ≠;

   (b) put out two strips A and B that are the same (see below) and compare, saying "A is the same as B", then "A is equal to B", and write A = B (use labels as appropriate); and

   (c) put out two strips A and B that are different (see below) and compare, saying "A is different from B", then "A is not equal to B", and write A ≠ B (use labels as appropriate).



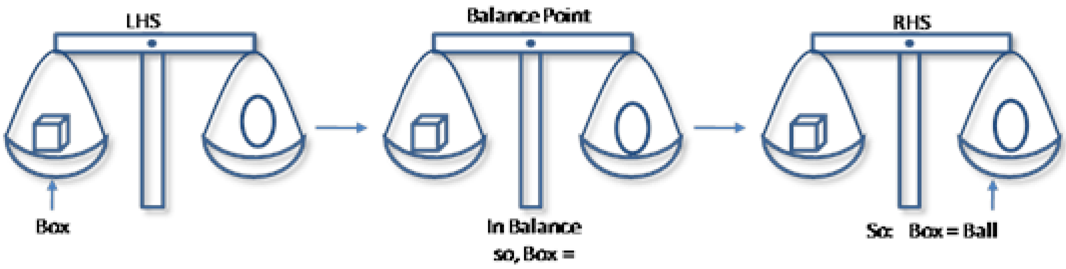
6. The above can be done for *greater than* and *less than* as well as equals and not equals, that is, can extend not equals to considering less than and more/greater than.

2.3 Unnumbered equations activity

As pioneered by Davidov in Russia, unnumbered activities are an excellent way to begin work in a mathematics area. The lack of numbers appears to allow students the freedom to explore structures and principles, and to focus on relationships not answers. Thus, the first formal activities should not use number, just different objects and different lengths. These are explored for equals and not equals and, later, greater than and less than. Students find different things to balance and not balance and record these as "equations" (not in the strictest sense).

1. Take the balance and labels that go with 2.2, and set up an equals balance as on LHS below. Move hand along the balance left → right as the students see it. As you move along, say what it is you are pointing to: on LHS and RHS it will be what is in the trays and at the middle it is to signify that the balance is equals.

2. This means that the three movements for LHS to balance point to RHS will be stated as "*box equals ball*". This is an unnumbered equation.



3. Move on to more complicated examples and so more complicated equations – see example on right. This is LHS = RHS which is:

   box + ball = jar + scissors

   Ensure students understand how this is done by "reading" the equation from the balance. Encourage students to see that when there are two items, they "join" together to balance what is on the other side. Introduce the mathematical language for joining (addition) by using the words "add" or "plus".

4. Do many examples:

   (a) one object LHS and one object RHS → more than one object on one of the sides → two objects on both sides → to more than two objects, and so on;

   (b) spend time on examples with one object on LHS balanced with more than one on RHS and one object on RHS balanced with more than one on LHS (examples such as "*can = box + scissors*" have been a problem for some classes); and

   (c) **reverse the activity** – we have been doing balance → equation, now do equation → balance – for example, ask students to make *spaghetti + baked beans = peas* on a balance (supermarkets have many cans and other packets at 125 g, 250 g, 500 g and so on that make balancing easy).

2.4 Properties of equals activity

There are three properties of equals (or, more correctly, equivalence): *reflexivity* (A = A), *symmetry* (A = B means B = A), and *transitivity* (A = B and B = C means A = C). These properties can be explored in the early years with the balances and paper strips.

1. **Reflexivity using mass**

   (a) Obtain a beam balance and groups of two examples of items with the same mass. Place the matching items on LHS and RHS of balance and note that they balance. Also note the equation this produces.

(b) Let students explore to see if they can find items that do not do this.

(c) Ask students if this is always the case – that the same things always balance and are equal. Discuss with students situations where this might not happen.

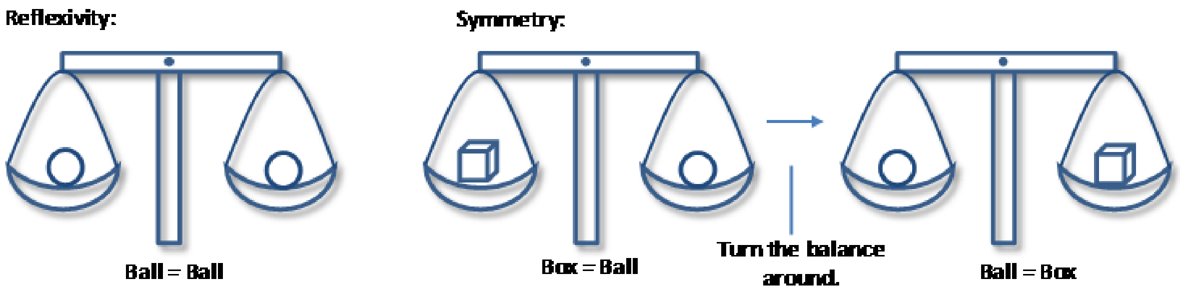
(d) State that this is a rule of equals which is true no matter what the situation. As long as the two things are identical in all ways. See example below.

2. **Symmetry using mass**

(a) Make up a beam balance where A = B (two different items of same mass). Then rotate the beam balance 180 degrees and students will see that B = A. If concerned with rotation, change positions in trays.

(b) Repeat this with more than one item in one or both of the trays, again turning the balance showing that LHS = RHS also means that RHS = LHS. This is where difficulties may occur because students sometimes believe that RHS can only have one object (e.g. they believe *box = ball + cylinder* is not allowed).

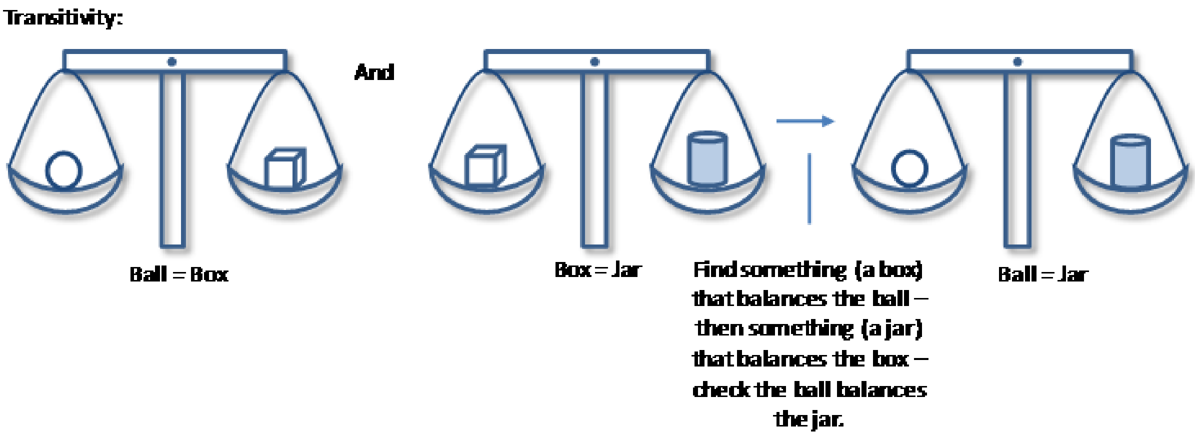
(c) Discuss the result with students – ask what it means for exercises or "sums".



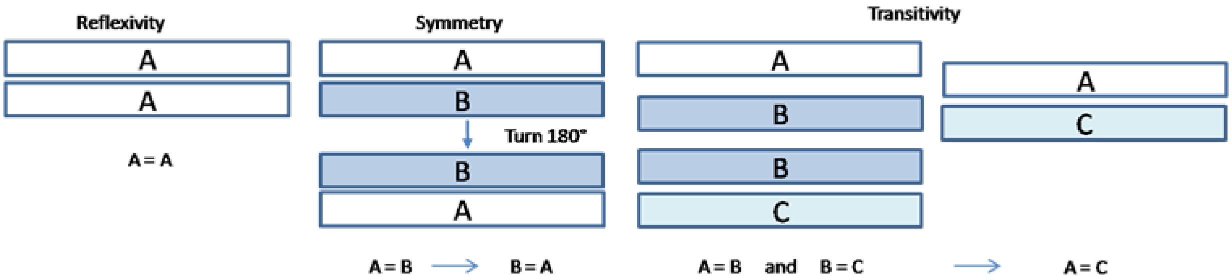
3. **Transitivity using mass**

(a) Find three items of the same mass. Show students that the first = second, and then, that second = third. Ask if first = third? Check this with balance. Ask if this is always the case?

(b) Repeat with more than one item in each tray. Let students explore possibilities, as in diagram below.



4. **Length model**. Similar work can be done with length and, because of ability to put things side by side, the length model can represent some things strongly. For example:



5. **Transfer**. It is important that the three rules of equals are known and understood because they become the basis of equivalence (e.g. fractions, $\frac{2}{5} = \frac{6}{15}$) as well as arithmetic and algebra. They enable many of the practices in solving computation in later year levels to be allowed, for example:

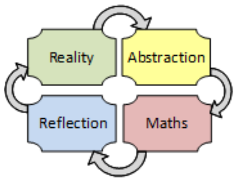
$$x^2 - 5x + 6 = 0$$

$$\text{means} \quad (x-2)(x-3) = 0$$

$$\text{means} \quad x = 2 \text{ or } x = 3$$

Thus, the three rules of equals should be transferred to arithmetic and used in computation.

2.5 RAMR lesson for early equations



Learning goal: Develop comparative language to describe equivalent and non-equivalent situations; use balance to develop an understanding that equals means that the two sides of an equation are equivalent or balanced.

Big ideas: Language ←→ picture ←→materials ←→ action; visualising movements (balance) through kinaesthetic activity.

Resources: General classroom items, manipulatives, balance baskets, bags, beams or similar for mass and length.

Reality

Local knowledge: Local/familiar situations that can be explored as equal/not equal; equivalent or not equivalent.

Prior experience: Sorting, collections and materials, identifying and observing actions around us.

Kinaesthetic: Same and different activity: Read and have students help tell the story through acting it out of "Me and You" by Henry Pluckrose (or similar, another is by Genevieve Cote). Establish the concept of same and different and enable students to use the language: same as, different from, equal to, equivalent, attribute, similar, compare as they tell/retell story using puppets, animals or each other.

Abstraction

Body: Use a full length mirror. Have pairs of students identify what is the same and different about them. How many ways can they come up with to describe this?

Hand: Have students compare towers they build, lengths they cut/make, and hefting objects. Use bags, baskets, balances, blocks. Ask: *What is happening? What do you see? What can you change? What is the same? What is equal? What is not equal?*

Mind: Visualise looking at themselves in the mirror. Ask: *What do you see?* Do the same while using balance baskets.

Creativity: Have students draw/make puppets of themselves. See if other students can identify their puppet.

Mathematics

Language/symbols: Same as, different from, equal to, equivalent, attribute, similar, compare, balance, =.

Practice: 1.Tell and act out Mrs. Oops-a-Daisy story. Mrs. Oops-a-Daisy is an old lady struggling to get home with her shopping over the bridge (balance beam). She struggles because she cannot balance as she always carries her shopping in one hand. Through a series of questions encourage the students to give Mrs. Oops-a-Daisy some suggestions to help her carry her shopping home without falling down. 2. Students each have an opportunity to walk the beam individually with the shopping balanced and unbalanced. 3. Ensure that students work out that the "weight" of Mrs Oops-a-Daisy's shopping must be the "same as"/balanced on the right and the left. If it is not equal, it is not balanced; it is "different from". 4. Students observing draw the event on individual whiteboards. 5. Build a word list as students use the "new" maths words and encourage students to label their drawings. 6. Divide into groups and have students use balance scales and/or baskets/bags to practise equal and not equal with a variety of grocery items. Ask: *Which is equal? Which objects don't balance?* These are unequal. 7. Have students record balanced/equal and unbalanced/unequal examples.

Connections: Encourage students to describe and discuss what is happening with balance situations. Where might you see/might we use this idea? Where do we see/use balance in other areas of maths?

Reflection

Validation: Consolidate the notion of balance and equals. Have students divide themselves/materials into equal groups. Compare them on a balance scale. Ask: *How do they know they are equal?* Have students create equal lengths of wool/string/Unifix cubes, etc. Ask: *How do they know they are equal?*

Extension: Have students make two teams. Consider the attributes and attempt to make the teams as equal as possible. Ask: *Do you think the teams are balanced? Why? Why not?* Discuss predictions for outcomes. What will happen if the teams are equal? Not equal?

Reversing: Provide balanced examples and ask students how they can unbalance the situation and vice versa. Ask: *What aspect is equal? What examples are unequal? Can you change/undo/reverse these?*

This RAMR could be used in a sequence of lesson topics:

- Lesson 1: Attributes and the language of equivalence.
- Lesson 2: Equal and not equal – balance.
- Lesson 3: Equal and not equal – length.
- Lesson 4: More than one attribute.
- Lesson 5: Symbols, pictures and words for equal and unequal.

Unit 3: Early Numbered Functions

This unit looks at early numbered functions. In the early to middle years of primary, change and functions work progresses to include numbered activities – first with the operations of addition and subtraction and then with multiplication and division. The numbered activities begin with one operation and involve presentation of change with formal symbols for the first time. Then the activities move on to two-operation function machines.

At all times, it is important that this work with operations begins from reality (from real-world problems), involves tables, inverse and arrowmath, and reverses everything (problem → rule for change, and rule for change → problem).

The unit again consists of background ideas, followed by activities (in this module, each activity involves a sequence of numbered steps) for unnumbered to numbered functions, one-step operation functions, multi-step operation functions and, finally, introducing arrowmath notation.

Background ideas

Suggested sequences for introducing the operations and for types of activities are provided below.

Order of operations

The sequence in bringing in the operations is as follows:

- addition → subtraction → multiplication → division
- one operation → two operations
- addition/subtraction and multiplication/division → addition/multiplication and subtraction/division.

Note: The reason for the last sequence is that when you have one of addition and subtraction with one of multiplication and division, changes of order change answers. For example, $12 + 4 - 3 = 12 - 3 + 4$ and $12 \times 4 \div 3 = 12 \div 3 \times 4$ but $12 \times 4 + 3 \neq 12 + 3 \times 4$.

Order of activities

The order of activities is as follows.

1. **Early activities (one operation)**. These are explained in detail in 3.2.
   - Give students a real-world problem.
   - Students consider the problem as change and then draw a function machine.
   - Act out change with the function machine.
   - Fill in input–output table.
   - Reverse the change.
   - Develop inverse.
   - Use arrowmath notation (as on right).
   - Do activity without machines – just with input–output table.

$$6 \xrightarrow{+5} 11 \qquad\qquad 8 \xleftarrow{-5} 13$$

Change   Inverse change

2. **Later activities (one operation)**. These are given in more detail here as they are not early childhood and hence have not been included in this unit's activities.

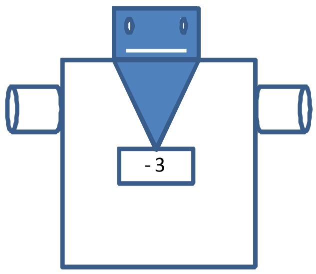
- **Generalise the change and its reverse**. Choose a student and ask to go to input on function machine. Tell other students that this student has a number to input but does not know what it is. Get class to discuss what the output would be. Do the same for any output number – what would be the input? Give students a variety of large numbers to say what the change would be. Get students to write their rules in language.
- **Move on to symbols** (but do not push for accuracy or for everyone getting the answer). Ask what the output would be if input was n. Ask what the input would be if output was k. See if students can write $n + 5$ and $k - 5$.
- **Reverse everything**. Give students a generalisation of a change (as language or as examples of numbers). Ask the students to represent the change and its inverse, with examples, using arrowmath notation. Fill in an input–output table for some values, draw the change as a function machine, and create a problem for it. For example:

Input $n \rightarrow n - 3$ Output

$6 \xrightarrow{-3} 3$ Change

$11 \xleftarrow{+3} 8$ Inverse change

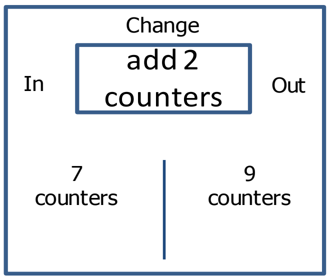
Output $k \rightarrow k + 3$ Input



Input	Output
5	2
10	7
18	15
	84
112	

3.1 Unnumbered to numbered functions activity

1. Begin with an unnumbered activity such as "add *in*" (i.e. add the letters "in"). Have letters as input and words like "bin" and "tin" as output. Act out change using these materials. Also act out reverse of change.
2. Discuss with students what would happen if we were adding counters, for example: If we started with a collection/set of counters, say 7 counters, what would we do if the function machine rule was "add 2 counters"? In the discussion, encourage students to see it is the same as "add *in*", which is to put the extra letters "in" on the end of our starting letter(s). To see the change as adding two counters to a set, we think of adding an extra two counters to our starting number of counters.
3. Next, have the students act out the activity with the 7 counters as the input – add 2 more – this makes 9 counters. Thus, as shown on right, we consider the change to be "adding 2 counters", resulting in an input of 7 and an output of 9.
4. We can now begin to use more numbered activities. However, these first activities should be changing numbers in terms of materials; for example, "remove 3 counters" is like changing from 9 to 6 by removing counters, while the reverse is "add 3 counters" which is adding an extra 3 counters.



5. Students should also experience moving from **actual counters to pictures of counters to number cards** (numerals). The input and output cards can be pictures of counters. Students moving along the function machine take an input card that is a picture of counters and then add or subtract counters as per rules and find the output picture card. This can be extended to input and output cards which just have the numeral. Students record results (input and output) with drawings of counters or with numbers/numerals. Students having difficulties could revert to actual counters. At the end, encourage students to work with numbers and not with materials.
6. Students should act out change both forward and in reverse. After they have recorded a few inputs and outputs, ask the students: *What happens when we reverse adding or reverse subtracting?* Some may be able to notice a pattern (reverse of addition is subtraction and vice versa).

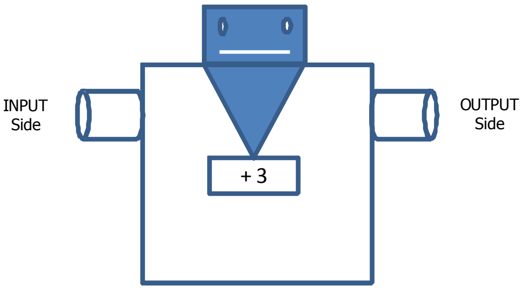
7. As in activity 1.4 in Unit 1, two function machines can be put together. We could add 5 and subtract 2, and discuss what happens and why the overall change is add 3. With experience and discussion, the students could understand that the two operations combined is the overall change. You could give some investigations like what two changes leave things unchanged (e.g. add 3 and subtract 3), and what happens when we reverse order (it reverses order of steps as well the amount added and subtracted in each step – add 5 and subtract 2 in reverse is add 2 and subtract 5, not subtract 5 add 2). These ideas are further explored in section 3.3 on multi-step operation functions.

3.2 One-step operation functions activity

This continues on from 3.1 but with two differences –we only work with numbers (not objects) and we bring in real-world examples. Follow the sequence below.

1. Do this activity for addition first and then for subtraction. When these are done, move on to multiplication and division activities. Discuss with students how addition and subtraction could be thought of as change. For example: *What happens if you have 4 toys and you get 2 more toys? What does 7 toys become if you give 3 toys to a friend?* It may be necessary to relate this to computation – that is relate computation to change, or think of it in a change way.

2. Set up a function machine that adds or subtracts small numbers. The whiteboard function machine is still excellent but, here, we move on to the "robot" function machine. Basically it is a large box (in which students can stand) with a small "head", an opening on each side, and a rule hung around the neck or from the top (if there is no "head"), as in example on right.

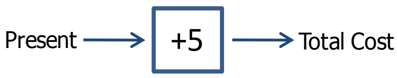


3. Two card sets are made – numbers 1 to 20 for input cards and 1 to 30 for output cards. Students, in twos, bring an input card to LHS of robot and place it in the opening. Students inside add 3 (for the pictured example) and push output card out RHS opening. Remaining students have a calculator to check that correct change has been made (e.g. 6 to 9) and a worksheet on which to record input and output numbers.

4. The following is a sequence of activities found useful.

   (a) Give students a **real-world problem** that adds/subtracts a small number. For example, *It costs $5 to have a present wrapped. What is the total cost of present and wrapping?* Discuss what we can do with this. [We cannot get answer as is but there are two things that can be done: (a) if given the present's cost, we can work out the total cost, e.g. present is $36, total cost is $41; and (b) if given the total cost, we can work out the present's cost, e.g. total cost is $24, present is $19.]

   (b) Students consider problem as a **change** – ask: *What is the operation?* and then draw a function machine as on right.

   (c) **Act out change with the function machine**. Organise a student to go into robot with output cards. Give other students input card numbers and ask them to bring them out front, in turn, to input and then collect a changed card at output.

   (d) **Fill in input–output table**. Students should follow the function machine activity with a calculator, checking calculations and filling in input–output tables on slates/whiteboards. Ask students to complete tables without watching a student at the front use the function machine.

   (e) **Reverse the change**. Teacher directs a student to collect an output card without showing input. Ask class what was the input card. Walk the student backwards from output (act out backtracking) to input as you are doing this. Discuss options and how to find this inverse number. Teacher provides a series of input and output numbers for students to fill in on their input–output tables. Have large numbers as part of this.

(f) **Develop inverse**. Teacher leads discussion on quick ways to find the inverses and encourages students to see that –5 gives inverse of +5.

(g) **Do the activities without the robots** – just working on the input–output table.

(h) **Use arrowmath notation**. Students are directed to write both changes as arrowmath diagrams using examples (as on right). See 3.3 for information on this.

$6 \xrightarrow{+5} 11$ Change

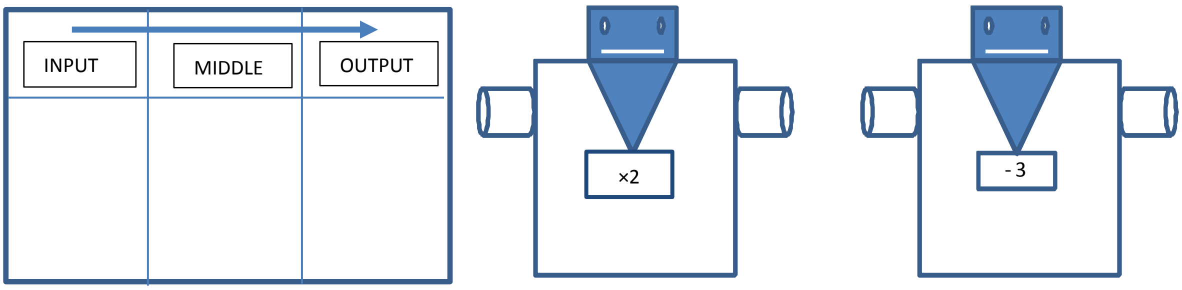
$8 \xleftarrow{-5} 13$ Inverse change

5. Repeat the above for other operations.

Note: This sequence may go too far for P–2. Only go as far as students can or need. See *Background ideas* for this unit.

3.3 Multi-step operation functions activity

This continues on from 3.2 but with one difference –we work with two function machines and a 3-column input–middle–output recording table, as shown below. It is great if students can be inside the machines.



1. The sequence is the same as for 3.2.

   (a) Create a real-world problem: The cost of Seaworld was two times the cost of Waterworld but Seaworld has a $3 discount.

   (b) Translate this to a change and put on the function machines – the change is Waterworld → Seaworld is ×2 –3 (as for function machines above).

   (c) Act out the change with the function machines – this need three sets of cards, input, output and middle, and two sets of students in the robots. Give input cards to other students; they take to LHS robot, put in (via arm), move forward, take out the middle card and move on to RHS robot where they put in and move forward to get out the output card.

   (d) All students fill in the input–middle–output table.

   (e) Reverse the change and develop inverse – remember to reverse the order as well as invert the operations (for machine above, reverse is +3 ÷2, not ÷2 +3).

   (f) Use arrowmath and undertake activity without robots/machines – just using a table.

2. Reverse the whole activity to find the rule ("What's my rule" game). Students give teacher input cards with numbers on them one at a time. Teacher, in turn, gives back output cards which result from the changes the two machines have done to the input cards. Students try to work out what the changes are on the two machines.

3.4 Arrowmath notation activity

1. It is useful for students to think of operations as change and to learn arrowmath notation. So build the idea of “arithmetic excursions”, travelling from number to number by actions or operators, as below.

$$2 \xrightarrow{+7} 9 \xrightarrow{+26} 35 \xrightarrow{-2} 33 \xrightarrow{+61} 94$$

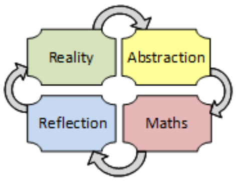
2. Some suggestions:
   - Use calculators to do the computation (unless mental computation is an objective).
   - Give students start and finish and number of changes when this is appropriate – otherwise let them make up their own excursions.
   - Spend time showing them how to do the arrowmath notation.
3. Some ideas for arrowmath excursions include using calculators to:
   - go from 2 to 62 by 4 changes;
   - go from 6 to 101 by 7 changes;
   - go from 7 to 31 going through 209 on the way; and
   - make a long journey from 36 to 97 passing through 2001 on the way.
4. Encourage students to use calculators without first working out in their head where they are going. The aim is for them to understand that making a number larger is achieved through adding, multiplying, or dividing by a fraction; similarly making a number smaller is achieved by subtracting, dividing, or multiplying by a fraction. If they get a decimal, just subtract it on the next move.
5. Use arrowmath notation to study principles for a variety of numbers. For example, does addition followed by subtraction always give the same answer as subtraction followed by addition? Here are some examples to try:
   - Does $\xrightarrow{\times 4} \xrightarrow{\times 3}$ always give the same answer (if starting at same number) as $\xrightarrow{\times 3} \xrightarrow{\times 4}$?
   - Does $\xrightarrow{\times 4} \xrightarrow{+3}$ always give the same answer (if starting at same number) as $\xrightarrow{+3} \xrightarrow{\times 4}$?
6. Reverse arrowmath excursions and use arrowmath to study inverse, as below.

$$3 \xrightarrow{\times 2} 6 \xrightarrow{+8} 14 \xrightarrow{\div 2} 7$$

$$3 \xleftarrow{\div 2} 6 \xleftarrow{-8} 14 \xleftarrow{\times 2} 7$$

7. Importantly, there is a need at the end of middle school to begin relating arrowmath notation to equations, as below.
   - The change $3 \xrightarrow{\times 2} 6 \xrightarrow{+8} 14 \xrightarrow{\div 2} 7$ is $\frac{3\times 2+8}{2} = 7$; with an unknown (say, a) at the start it becomes $\frac{(2a+8)}{2} = 7$ or, cancelling, $a + 4 = 7$.
   - The inverse is $3 \xleftarrow{\div 2} 6 \xleftarrow{-8} 14 \xleftarrow{\times 2} 7$ or $\frac{7\times 2-8}{2} = 3$; with an unknown (say, b) at the end, this becomes $\frac{(2b-8)}{2} = 3$ or, cancelling, $b - 4 = 3$.

3.5 RAMR lesson for early numbered functions



Learning goal: How to show change +/- with materials; how to work the change with numbers; know that there can be two changes; and think of operations as change and use arrowmath notation.

Big ideas: Language ←→ picture ←→materials ←→ action/function machine; visualising movements (backtracking) through kinaesthetic activity.

Resources: Fantastic Function Machine; Small FFM (box); Set of rules and charts (x6); rules and materials from previous lessons; 'In' and 'Out' cards; blank sheets and whiteboard markers; laminated numbers; counters; Unifix cubes.

Reality

Local knowledge: Revisit change situations in your community (adding to collections, saving in a piggy bank) and function machine familiar to students from previous lessons.

Prior experience: Sorting, collections and materials; identifying and observing change/no change around us; function machine input output.

Kinaesthetic: Revisit **Follow the rule** activity. 1. Using the function machine give students only the rule "flip it". They have to choose appropriate materials to act out and demonstrate the rule. Have students reverse the change and identify the inverse rule. Ensure all students have the opportunity to choose and/or participate in solving. Ask: *What goes in? What comes out? What changes? What is the rule? You have gone forwards can you now go backwards? What happens?* 2. Using students as the input and the rule "add one student" have the students build groups using the function machine. Ask same questions as above.

Abstraction

Body: Have small groups of students enact rules and changes with function machine. Provide rules and materials. Ask: *Can you reverse the change? What is the rule for this? Can you act out the change without the rule displayed and can your partners identify the rule?*

Hand: Using Unifix cubes and individual function machines, rules add 1, add 2, add 3, on cards, have students enact functions and record the action from the function machine. Ask: *Where do you put the rule? What goes in the input slot? What comes out? Can you reverse the change? What is the rule? How could you record this?* Provide tables and number, word and symbol +/-, → models or cards.

Mind: Visualise function machine. Think of a rule and see in your mind's eye the fantastic function machine working as you input blocks.

Creativity: Have students draw their own fantastic function machine with their own rule example. Have them record what happens in their own way.

Mathematics

Language/symbols: Rule, input, output, function machine, value, change, change process, reverse, inverse, identify, table, backtracking, backwards, forwards, adding, take away, counting, more and less, number, +/- symbols, arrow and arrowmaths.

Practice: Divide students into pairs and each pair will receive a bag which contains a number of marbles (from 1 in one bag through to 11 in one bag). The rule will then be placed on the function machine and the rule is "add 2". Each pair will then be asked to come out and place their bag in the machine, while the other student must follow the rule and add 2 marbles to the bag. The pair will then need to state to the class how many marbles they now have. This will be repeated for each pair. Several pairs however will be sprung with a surprise challenge where the machine will be turned into “reverse” mode and they will have to take away 2 marbles instead of add.

Model tracking these actions on an input/output table. Have students record their change using numbers and symbols +/-, →. Ask: *Can you write down the reverse for your change? What do you notice?* Have them swap rules and repeat the actions.

Connections: Discuss with the students the challenges of putting the machine in reverse and what it actually means (addition and subtraction relationship).

Reflection

Validation: Consolidate the notion of showing change using numbers and symbols. Provide cards with various rules for students to do and record using numbers and symbols.

Extension: Create situations with two changes. Use two function machines beside each other. Provide two input/output tables beside each other for students to record the two changes using numbers and symbols. Have the students reverse the change and record this process.

Reversing: Give students input/output tables with values and no identified rule. Ask students to write the rule. Have students create “identify the rule” scenarios for their partner to try.

This RAMR could also be used in a sequence of lesson topics:

- Lesson 1: How to show change with materials and numbers and symbols.
- Lesson 2: How to work the change with numbers and symbols.
- Lesson 3: Know that there can be two changes.
- Lesson 4: Think of operations as a change.
- Lesson 5: Use numbers and symbols and simple arrowmath notation.

Unit 4: Early Numbered Equations

This unit follows on from Unit 2 in moving the exploration of equations from unnumbered activities to numbered activities. This is simply done by:

(a) replacing supermarket items of differing masses with many masses all of the same amount (e.g. 125 g cans) so we need to count what is on each side and we end up with 2 + 3 = 5 and 5 =2 + 3 rather than can + scissors = jar, and jar = can + scissors; and

(b) replacing strips of paper and ribbon of differing lengths with same-length items such as Unifix or paper/wood cut into equal lengths.

This will enable us to cross over into number and operations with our use of masses/balances and length. However, some operations are difficult to model (e.g. division) and some numbers are impossible to model (e.g. negative numbers). This means that soon we will have to move to more abstract models of mass (drawing of a balance) and length (number lines) that allow all numbers and all operations.

This unit begins with background information and then looks at activities (each activity involves a sequence of numbered steps) for moving from unnumbered to numbered work, relating real-world situations to models and symbols (i.e. equations), and introducing the balance rule. Both mass and length models are used.

Background information

The use of models like balances has assisted in the crucial balance understanding of equations and how to solve them. But, masses and balances are simply a model to assist our image of what has to be done. They are too crude to model all that we need in algebra. This was found in a program based on balance: the students had good basics but when in an exam they were given $y + 5 = 2$ to solve, a problem emerged. The students subtracted the 2 from the RHS and balanced this by subtracting 2 from the LHS. This left the students with $y + 3 = 0$ and they stopped. The reason for this was that the RHS was 0 and when there is nothing on the tray you can only add things. Their physical perception of balance stopped them from seeing that, in mathematics, you can subtract 3 from 0. They should have subtracted 5 from both sides originally to get $y = -3$.

However, this problem does not mean that the good work of balance should be stopped. Rather, it says that we must get students to an abstract mathematical balance on which you can do anything to both sides (negatives, division, and so on). The physical balance and length materials restrict the operations that can be used to addition, subtraction and simple multiplication. They are also kinaesthetic and time consuming – excellent for beginning the teaching but inefficient in later work. The diagrams of balances and number lines enable more activities to be completed but still have restrictions on operations. Thus, it is important that students understand that the balance and the lines must become **abstract mathematical balances and lines** and able to represent any operation (including division). This means combining the abstract balances/drawings with equations. This leads to the final step which is to dispense with materials and just use equations.

The major ideas to be covered in equivalence and equations deal with using equations to model real life and manipulating and solving the equations to solve these real-life problems. The sequence of activities designed to build this understanding and proficiency is as follows:

(a) introducing the notion of same and different and relating this to introduce equal, unequal, greater than and less than in length and mass (balance) situations;

(b) using mass and length in unnumbered situations to build understanding of equals and equations and to develop the equivalence and order principles;

(c) using mass and length in numbered situations to build understanding of arithmetic equations and reinforce the equivalence and order principles in numbered situations;

(d) relating arithmetic equations to real-life situations and vice versa (e.g. telling stories about the world);

(e) using mass and length models to introduce the balance principle that equations stay equal if the same thing is done to both sides of the equation;

(f) using mass and length models to introduce unknowns, and relate equations with unknowns to real-world situations and vice versa;

(g) extending mass and length models to mathematical versions (in picture form) where all operations are possible;

(h) using the balance principle to find solutions to equations with unknowns;

(i) developing rules/principles that enable expressions to be manipulated (including simplification and substitution); and

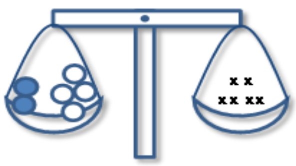
(j) introducing graphical representations of equations and showing how graphs, equations and everyday life relate.

Most of the end of this sequence is not in Years F–2, but it is important for F–2 teachers to know where the knowledge is going so they can prepare the students for it. It is also important that they know that step (g) is the crucial one where the limitations of physical balance and length are replaced by mathematics understandings that allow abstract activity.

4.1 Unnumbered to numbered equations

To move from unnumbered to numbered equations, return to section 2.3 and recap how you can do unnumbered equations: explore getting two sides in balance and writing what is seen as an equation – go in both directions LHS to RHS and RHS to LHS. Remember to stress that addition is joining – 3 objects joining 2 objects is 3 + 2 – thus putting spaghetti with sugar can be considered for the equation as spaghetti + sugar, and putting the red strip with the blue strip is red + blue. Extend this thinking to cans so students can see that 2 cans joined with 3 cans represents 2 + 3. Also stress that addition is turnarounds (commutative law), so 2 + 5 = 5 + 2 and so on.

1. Complete activities like in 2.3. Write the results as equations. Remember to discuss, for example, that putting jam with noodles can be considered as jam + noodles for the equations.
2. Introduce materials (e.g. 125 g cans) that are all the same in mass. It is good to paint them so you have four colours of cans. However, you can also buy four different types of 125 g cans. Introduce Unifix cubes as lengths and stick together the cubes to make towers and rows.
3. Now start to put different types of cans on both sides of the beam balance until you get balance. Start with two types on left and one on right. Note that the drawings below have filled circles, unfilled circles and Xs to represent cans of different colours. Do a similar activity for length. Use three colours of Unifix. Make lengths of the same height with two colours in one length and one colour in the other. Write the equation that they both represent (see below).



2 + 4 = 6

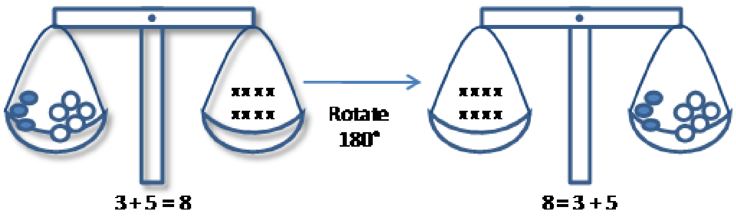
2 + 4 = 6

It is important to read the equations from the materials, e.g. *2 cans plus 4 cans balances 6 cans so 2 + 4 = 6*; and *2 Unifix joined with 4 Unifix is the same length as 6 Unifix, so again 2 + 4 = 6*.

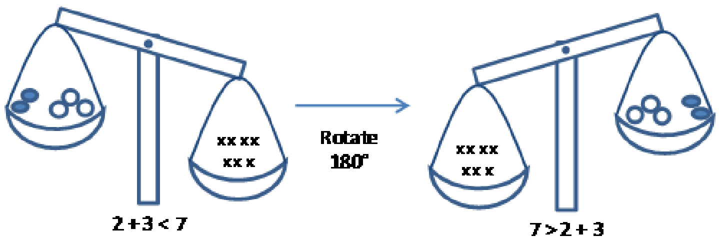
4. Explore the masses and the Unifix to get LHS equal to RHS in masses on a balance and with Unifix in rows and towers. Turn the explorations into equations and deliberately make equations. **Go both ways**: materials → equations and equations → materials. When you have one equation, turn the beam balance 180 degrees and write this new opposite-direction equation. Do the same for Unifix towers – go around and view them from the other side and rewrite them. You will find that 3 + 4 = 7 becomes 7 = 3 + 4.

5. Make the equations more complicated by having more than one type of material on each side. Make sure that there is an addition on both sides, e.g. 2 + 7 = 4 + 4 + 1 and 4 + 4 + 1 = 2 + 7

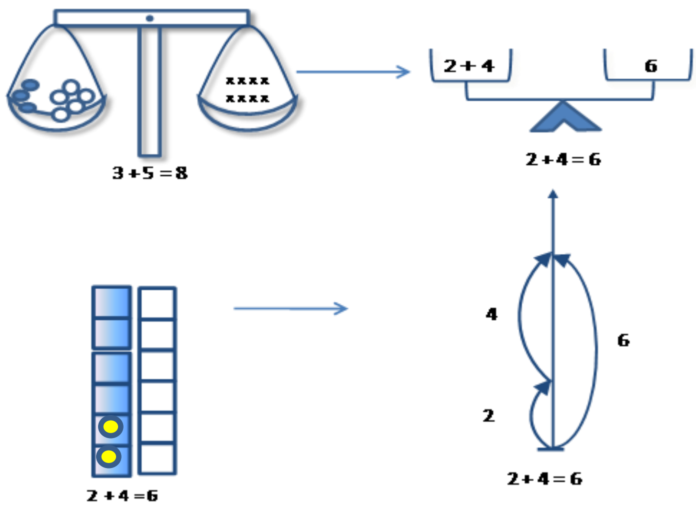
6. Reteach the principles of equals and equivalence (A = A, A = B → B = A, and A = B and B = C → A = C). Spend time particularly on **symmetry** (as below) because of the problem of students thinking only one thing can be on the RHS.



(a) If you wish, extend the mass models to inequations (i.e. greater than and less than). Notice the difference in symmetry.



(b) As students' experience grows, extend models to introduce mathematical balance pictures and double number lines which can handle more operations – more abstract models:

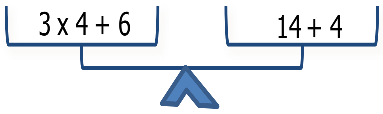


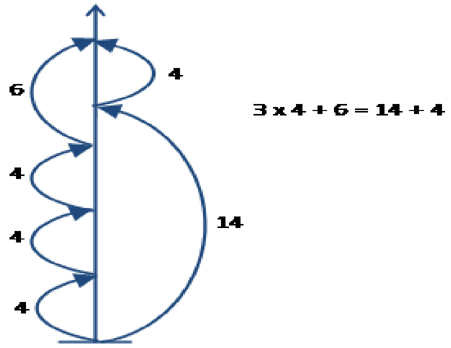
Note: We have turned the cubes and the number line vertical because it shows LHS and RHS. This does not have to be done but it makes the equation easier to relate to the picture. Note also that we have a double number line with jumps on both sides. This is a good way to represent equations. REMEMBER – ALWAYS REVERSE – materials → equation AND equation → materials – so interpret pictures in terms of equations and construct pictures to match equations.

4.2 Relating real-world situations to equations

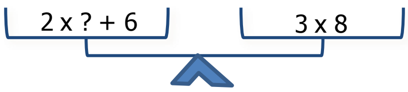
It is important to continue to reinforce the relationship between real-world situations and equations. The relationship is best taught by experience in which equations are deconstructed into parts and related to stories and vice versa (i.e. stories are deconstructed and related to equations). Some examples are as follows.

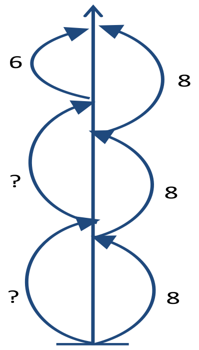
1. **Story to equation**: I bought 3 chocolate bars for $4 each and a pie for $6. I spent the same as June, who bought a meal for $14 and a drink for $4.





2. **Reversing – equation to story**: The equation is 2 × ? + 6 = 3 × 8; what story can this tell?





"My dad and my mum gave me the same amount of money. I already had $6, so I was able to exactly pay for 3 meals at $8 each."

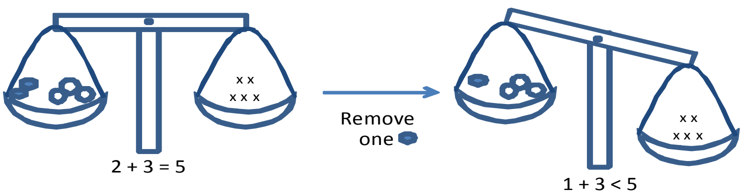
3. Teach this by **building it into all the work in Units 2 and 4**. Every time you construct a model, make up a story for the model. Every time you make an equation, make up a story for the equation. Then try to make up other stories using different contexts.

4. Divide paper into four headings as below – fill one of the columns and ask students to draw and write in the other columns so that models and stories are connected to equations.

STORY	BALANCE DIAGRAM	LINE DIAGRAM	EQUATION

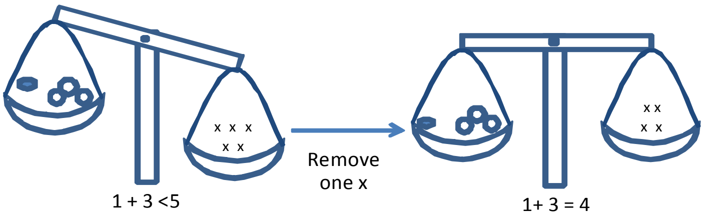
4.3 Balance rule

1. Balance can be used to explore what happens when extra weights (e.g. one weight) are added or removed from a balanced equation (as below).

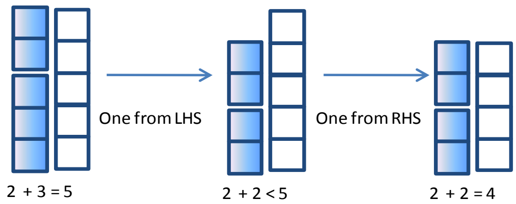


2. Students can be asked how to balance the equation again. There are three possibilities for the example above: (a) put the weight back again (this returns the equation to 2 + 3 = 5); (b) add another weight to the

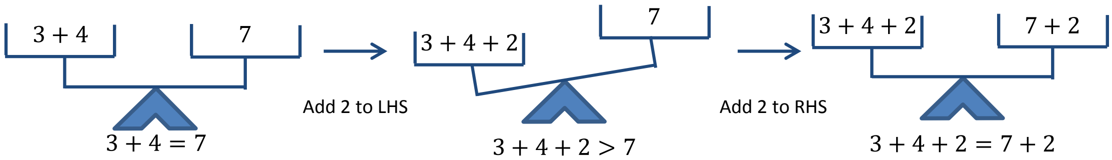
3 on the LHS (this makes the equation 1 + 4 = 5); or (c) remove a weight from the RHS (this makes the equation 1 + 3 = 4). The third possibility is the beginning of the balance principle and should be the focus of questioning (see diagram).

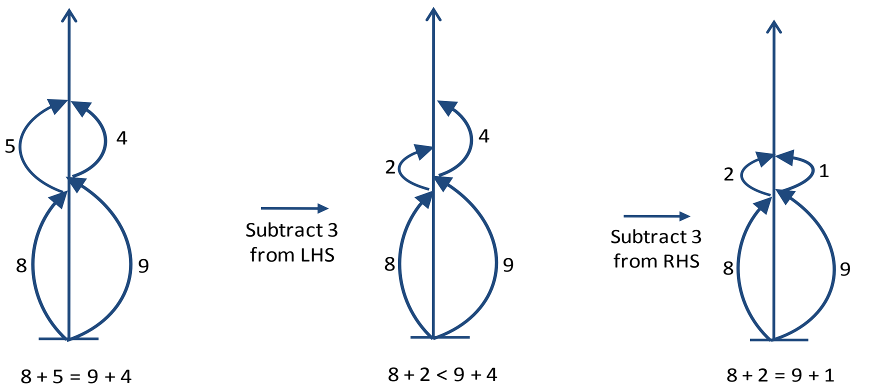


3. Direct the students to add and remove different weights and to rebalance. With questioning, try to get students to generalise this process to the full balance principle (i.e. “whatever you do to one side you do to the other”). The balance principle can also be introduced and demonstrated with length models (as shown below).



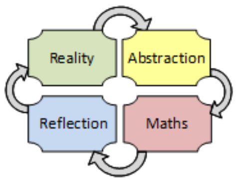
4. The balance principle can be reinforced with mathematical balance pictures and double number lines as follows.





This balance rule is very important and becomes the means for major work on solving equations for unknowns.

4.4 RAMR lesson for early numbered equations



<u>Learning goal</u>: Move from unnumbered to numbered equations and getting the two sides in balance; real-world situations to equations; introduce symbols; and different ways to balance the equation.

<u>Big ideas</u>: Language ←→ picture ←→materials ←→ action; visualising movements (balance) through kinaesthetic activity.

<u>Resources</u>: General classroom items, manipulatives, balance baskets, bags, beams or similar for mass and length.

Reality

Local knowledge: Local/familiar situations that can be explored as equal/not equal; ask the students to consider if they can think of a time when equal groups are important? (Team games)

Prior experience: Sorting, collections and materials; identifying and observing actions around us; previous lessons on concept of equivalence at this stage: comparison, attribute, same as, different from, equal to, not equal to, balance.

Kinaesthetic: Use a story to engage students in acting out various equivalence situations e.g. “Mice Mischief” by Caroline Stills (use a simpler make 5 story for earlier understandings).

Abstraction

Body: Use a see-saw for students to explore who balances.

Hand: Have students represent various equivalence situation stories (prepared on picture cards) with play dough/blocks/marbles, etc. and balance scales. Ask: *How could you write these?* Provide cards with = ≠+ - and numbers on them for students to manipulate. Have students do the same with magnetic picture boards, online interactives, etc.

Mind: Have students visualise in their mind’s eye different equivalent stories you tell. Ask: *Can you see balance? If not, what do you need to change?*

Creativity: Encourage the students to make up an equivalence story that is reflected in their model, e.g. on this leaf there are 5 caterpillars munching away, 5 more caterpillars are munching away on the other leaf; there are an equal number of caterpillars on each leaf. Students share equivalence stories about their models, e.g. caterpillars on a leaf, flowers in a vase, fish in a fishbowl, spiders in a web, bird cage and birds, apple tree and apples, faces getting the measles, washing on the clothes line, fruit into a fruit bowl, people on a bus, sheep in paddock, coins in a piggy bank.

Mathematics

Language/symbols: Same as, different from, equal to, equivalent, attribute, similar, compare, balance, = ≠, number, total, equation.

Practice: Ask pairs of students to represent each of the various combinations of mice in the “Mice Mischief” story using counters. Ask them to record each scenario with numbers and symbols. Ask: *Do your equations balance? Can you explain the two sides?* For example, 9 + 1 = 10; 8 + 2 = 10; 7+ 3 = 10; 6 + 4 = 10; 5 + 5 = 10; also ask is 6 + 4 = 4 + 6 and so on; and further, 6 + 4 = 5 + 5 and so on.

Ask: *What happens when something is added to or removed from one side of the equation?*

Connections: Encourage students to describe and discuss what happens with balance situations. Ask: *Where might you see/might we use this idea of equations?*

Reflection

Validation: 1. Show 2 trees. One tree has four blue birds and one red bird in it. The other tree has three blue birds and two red birds in it. Discuss how the number of birds in tree one is the same as the number of birds in tree two. Record the sum using number symbols and addition sign to show the joining to the two coloured groups of birds in tree one. Repeat this process for the second tree. E.g. 4+1 = 3+2.

2. Have students use a magnetic board to pose their own problems for partner to solve, e.g. using magnetic board resources use examples to explain the rule of equivalence, e.g. pretend to eat a ‘Smartie’ from one biscuit. Ask: *Are they equal now?* Record the equation. *How can they be made equal again?* Does the student know that what we do to one side of an equation must be done to the other to maintain balance?

Extension: Have students make their own equal story. Have them record them on cards and share with other students. Enable students to discuss what they do; sort and compare the various cards.

Reversing: Provide a total and ask students to write an equation e.g. I have seven fish swimming in two schools. How many in each school? Write the equation.

This RAMR could be used in a sequence of lesson topics:

- Lesson 1: Unnumbered to numbered; getting two sides to balance.
- Lesson 2: Addition is joining, also addition is turnarounds.
- Lesson 3: Real-world situations to equation; symbols, numbers, equal and unequal.
- Lesson 4: What happens when something is added or removed? Different ways to balance the equation.

Module Review

This section reviews the units and look for interesting commonalities, differences, and sequences. We have been looking at functions and equations in different ways – functions by function machines and input–output tables, and equations by balance and length (number lines). Therefore we will begin the review by looking at models for functions and equations; in particular, how they relate to each other. This is because, by junior high school, the two perspectives of function and equation will be working together.

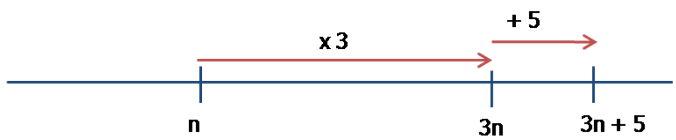
This section looks first at models, seeing how one model can work for more than one approach and one approach can have more than one model. Then it looks at how models sequence for different algebra situations, through to more advanced models. Finally it briefly summarises critical teaching points.

Versatility of models

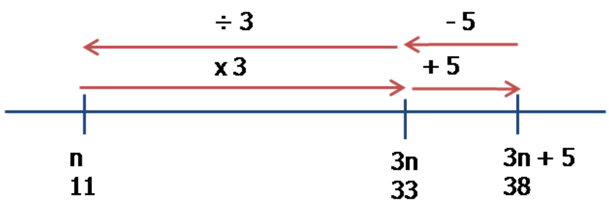
We can see the versatility of models in the use of number-line models in algebra which can apply to both function and equation. We will look at this by considering the following **pie problem**: *I buy 3 pies and a $5 chocolate and spend $38, how much does each pie cost?*

Number lines for the application of functions to the pie problem

Function is based on change and inverse of change. We present the pie problem on a number line as change and then we can reverse the change for the solution. To do this, we start with the cost of the pie, which we call n. The operations to get to $38 can be shown as changes on a line as below.

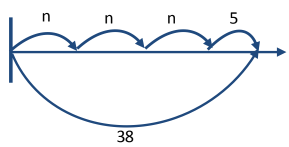


Using the inverse of the changes, we can backtrack from where the problem ended up, at $38. **Backtracking** is going back along the line and undoing what has been changed as seen below. The ×3 and +5 are reversed and show that $n = 11$.

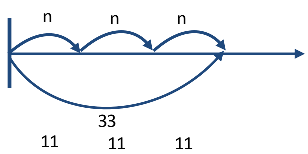


Number lines for the application of equations to the pie problem

Equations focus on setting up a balance using equals. This means a double number line with both sides starting at the same point (0) and ending at the same point ($38). Operations are represented on double number lines as jumps drawn as arrows. For the pie problem, if n is the cost of the pie, we have the following equation: $n + n + n + 5 = 3n + 5 = 38$. The two sides of this can be represented on a double number line as in the top diagram on the right.

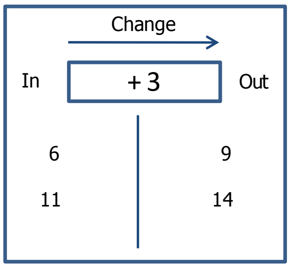


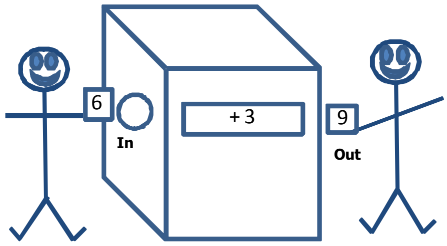
The number of pies (n) can be worked out by first crossing out the 5 on each side, which gives $3n = n + n + n = 33$ and then sharing the 33 evenly among the three n's as in the bottom diagram on the right. This gives the answer of $11.



Modelling functions with machines

Functions describe change; we can set up change by using input–output tables and by constructing function machines. A “machine” can be a whiteboard or blackboard or a box with holes in it and takeout cards, for example:

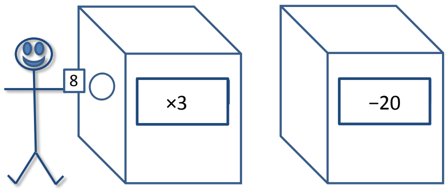




Function machines operate as follows.

1. Situations are described: I sold ice-creams for $3 each. I paid $20 for the ice-cream, how much money can I make?
2. Situations are translated to change activities – what are the operations in action here? [multiply by 3 and subtract 20] These operations (see below) are put on the two function machines (as on right):

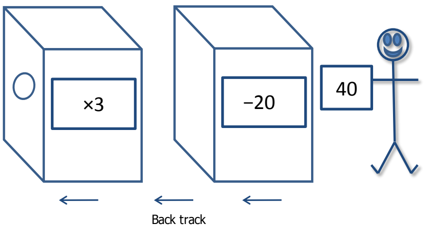
$$\xrightarrow{\times 3}\ \xrightarrow{-20}$$



3. The changes are acted out by students with numbers on cards and examples are put on an input–output table (shown on right) starting with input numbers.

Input	Middle	Output
8	24	4
10	30	10
20	60	40

4. Numbers are then put into the middle or output and students act out how to find the other sections of the table (using backtracking where necessary).

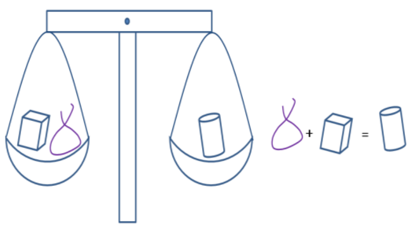


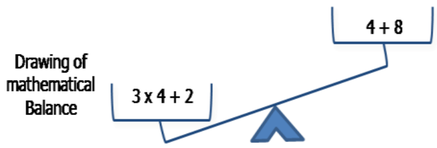
Input	Middle	Output
?	120	?
?	?	130

Modelling equations with masses/balances and length/lines

Mass/balance

1. The materials/pictures begin with real balances (which can only really cover adding and some subtracting) and move on to pictures of “mathematical balances” (which can cover all operations) added with symbolic equations.
2. Equals is shown by the balance being “in balance”, and not equals by the balance being “out of balance” (see examples below).

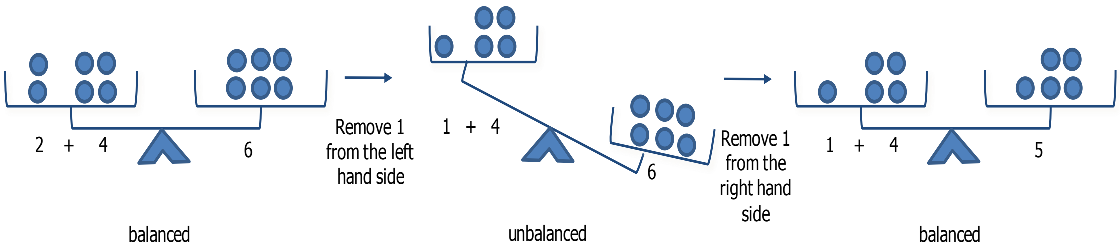




$3 \times 4 + 2 \neq 4 + 8$
$3 \times 4 + 2 > 4 + 8$

Equation: $3 \times 4 + 2 = 16 - 2$

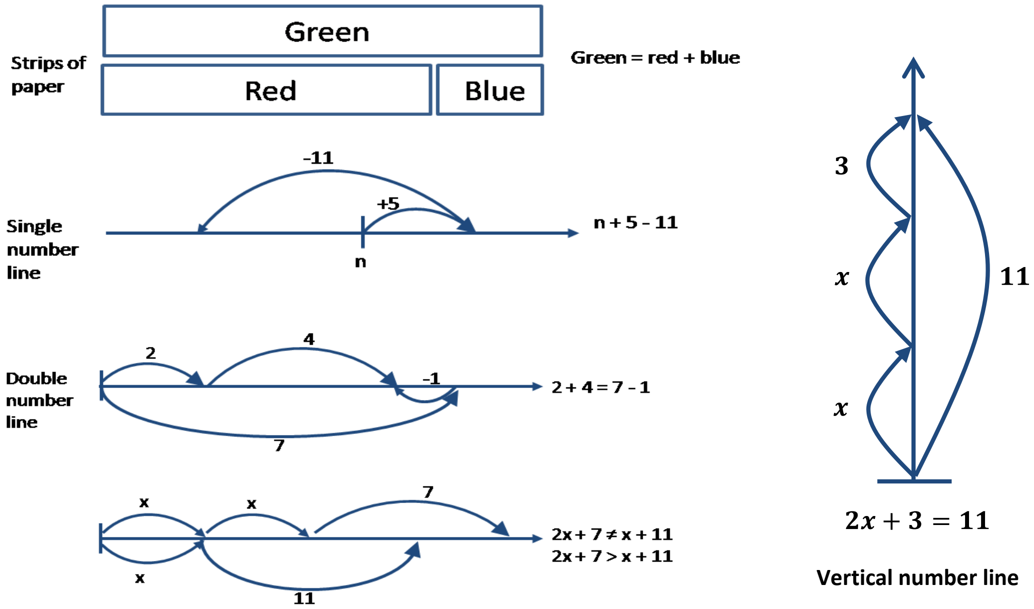
3. The balance is good for developing the balance rule and, in picture form, handling all operations.



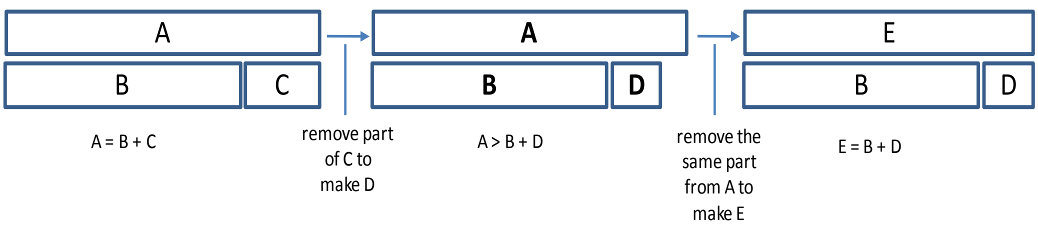
4. Virtual balances are also available online which provide a combination of the movement of the real balance with the range of operations available when using the picture balance.

Length/lines

1. The materials for this model are strips of paper and single and double number lines (see below for diagrams). All these length models can handle addition and subtraction and the line and double number line can handle simple multiplication and negatives as well. However, more complex operations are not possible. The number lines can be horizontal or vertical. The vertical line has the advantage of enabling an = sign to be placed under it and a left-hand and right-hand side to be identified (as in an equation).

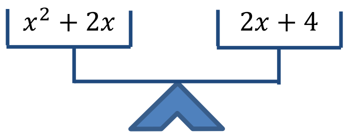


2. Length is also a useful material for teaching the balance principles, for example:

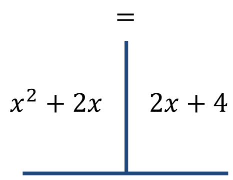


Advanced models

1. As we go up the years, the equations become more complicated. Physical balances and number lines can be used to show and solve simple equations, usually linear. But it is difficult to use them to show subtractions, divisions, complicated multiplications, and nonlinear examples such as quadratics. At this point, we need to move on to imaginary extensions of balances and lines, where anything is possible. One can subtract, divide, multiply, go into negatives, and so on. Anything mathematical is allowed.



Maths balance



Maths ruler

2. One simply thinks of the expressions being in balance or of the same length. So we can subtract $2x$ from both sides and then square root both sides. For example:

$$x^2 + 2x = 2x + 4$$
$$x^2 = 4$$
$$x = \sqrt{4}$$
$$x = +2 \text{ or} - 2$$

Subtract $2x$ from each side

Square root both sides

(remember negatives)

3. Once we have unknowns or variables, we can introduce models for unknowns. The following are useful:
   - a bag covered in question marks into which weights can be placed to act out e.g. ? + 3 = 11;
   - boxes of various shapes into which counters could be placed to act out e.g. ∆ + O + 4 = ∆ + 7; and
   - cups □ and counters O to act out equations with variables (cup acts as variable and counters as ones) e.g. □ + □ + OOO = □ + OOOOO.

Critical teaching points

Understanding is demonstrated by the following critical points:

- seeing same and different as leading to equals and not equals, without the weakness of believing RHS has only one thing in it – this means seeing equals as "the same value as" and not as "where we put the answer";
- understanding change, and seeing simple relationships as being change as well as relationship, and changes as being relationships;
- knowing the strength in using the beam balance and masses and number lines for what they can tell us in the early years (balance is a good metaphor for equals and a function machine is good in acting out change), but knowing that we have to become more abstract later; and
- understanding that ideas are taught through sequences of models that change over the years to be more abstract but also more directed and formal.

Test Item Types

This section presents instructions and the test item types for the subtests associated with the units. These will form the bases of the pre-test and post-test for this module.

Instructions

Selecting the items and administering the pre-post tests

This section provides an item bank of test item types, constructed around the units in the module. From this bank, items should be selected for the pre-test and post-test; these selected items need to suit the students and may need to be modified, particularly to make post-test items different to pre-test items. The purpose of the tests is to measure students' performance before and after the module is taught. The questions should be selected so that the level of difficulty progresses from easier items to more difficult items. In some modules this will follow the order of the units and subtests, and in other modules it will not, depending on the sequencing across the module and within units. The pre-test items need to allow for students' existing knowledge to be shown but without continual failure, and the post-test items need to cover all the sections in a manner that maximises students' effort to show what they can do.

In administering the pre-test, the students should be told that the test is not related to grades, but is to find out what they know before the topic is taught. They should be told that they are not expected to know the work as they have not been taught it. They should show what they know and, if they cannot do a question, they should skip it, or put "not known" beside questions. They will be taught the work in the next few weeks and will then be able to show what they know. Stress to students that **any pre-test is a series of questions to find out what they know** before the knowledge is taught. They should do their best but the important questions come at the end of the module. For the **post-test**, the students should be told that **this is their opportunity to show how they have improved**.

For all tests, **teachers should continually check to see how the students are going.** Items in later subtests, or more difficult items within a particular subtest, should not be attempted if previous similar items in earlier subtests show strong weaknesses. Students should be allowed to skip that part of the test, or the test should be finished. Students can be marked zero for these parts.

Information on the Module A2: Functions and equations item types

This section includes:

1. Pre-test instructions;
2. Diagnostic Mapping Points;
3. Observation Checklist and Teacher Recording Instrument;
4. Test item types; and
5. Additional resources.

Pre-test instructions

When preparing for assessment ensure the following:

- Students have a strong sense of identity; feel safe, secure and supported; develop their emerging autonomy, interdependence, resilience and sense of agency; and develop knowledgeable and confident identities.
- Students are confident and involved learners, and develop dispositions for learning such as curiosity, cooperation, confidence, creativity, commitment, enthusiasm, persistence, imagination and reflexivity.

When conducting assessment, take the following into consideration:

- Student interview for diagnostic assessment in the early learning stages is of paramount importance.
- Use materials and graphics familiar to students' context in and out of school.
- Use manipulatives rather than pictures wherever possible.
- Acknowledge the role of using stories in this early number learning, enabling students to tell stories and act out understandings to illustrate what they know.
- Playdough and sand trays are useful for early interview assessment situations.

Ways to prepare students for assessment processes include the following:

- In **individual teaching times**, challenge students' thinking. "Challenging my thinking helps me to learn by encouraging me to ask questions about what I do and learn. I learn and am encouraged to take risks, try new things and explore my ideas."
- In **group time**, model and scaffold question-and-answer skills by using sentence stems to clarify understandings and think about actions. Encourage students to think of answers to questions where there is no one correct answer, and to understand that there can be more than one correct answer (e.g. *How can we sort the objects?*).
- In **active learning centres**, use activities such as imaginative play, sand play, playdough, painting, ICTs and construction to think and talk about different ways of using materials, technologies or toys. Ask questions and take risks with new ideas.

Other considerations:

- Preferred/most productive assessment techniques for early understandings are observations, interviews, checklists, diary entries, and folios of student work.
- Diagnostic assessment items can be used as both pre-test and post-test instruments.

Remember:

Testing the knowledge can imply memory of stuff; asking the students **what they can do with knowledge** requires construction and demonstration of their understanding at this early understandings level.

A2 Functions and Equations: Diagnostic Mapping Points

1. Early functions

Notion of change

- Set up a function machine. Have an input, an output and a rule. Choose an unnumbered change and put this in the rule box. Choose some objects to enact the change on.

  Can students discuss what is happening? Ask: *Can you to think of things to change? What changes can you think of?*

- Repeat activity with different examples (use lower case to capital, red follows blue, etc.). Give students a whiteboard/paper to record their changes.
- Have students make up their own action.

  Can the students work the reverse change?

Unnumbered functions and inverses

- Ask students to select an output card and to place it at the output side of the function machine. Ask: *What could the input card be?* Can the students "think backwards"? Ask: *What do we get if we undo the action?*

Guess my rule

- Think of a change rule and prepare input and output cards for this rule. Ask students to pick an input card and place it on the input side. Teacher puts an output card on the output side. Ask: *Can you guess the change rule?*

Multiple change

- Think up two changes that could occur one after the other. Have two function machines ready. Have students enact the change sequence. Do the students understand if they go forwards it is:

  first change → second change?

  But if they reverse it is:

  reverse of first change ← reverse of second change?
  (i.e. reverse of second change → reverse of first change)

  Can they record on their whiteboard the change using the arrow?

2. Early equations

Same and different

- Can students identify objects – what is the same and what is different? Can they describe the resulting groups?

Meaning of equals

- Use the human balance. Use assorted materials. Ask: *Are they the same? Are they different? How else can we say it when it balances?* (Do they use "equals"?)
- Have students label a balance with equals. Add something to one side of the balance and ask: *What has happened now? How else can we say that?* (Do they use "unequal"?) Label.
- Repeat activity with greater than and less than; add labels.

Unnumbered equations

- Use a balance scale. Ask students to find objects to balance. Can they "read" what is happening? Can they "join" two items together to balance what is on the other side? Have students make/draw the "equation", e.g. picture of block + ball = pencil case. *Note:* The mathematical language for "joining" is "add" or "plus".

Properties of equals using mass model

- *Reflexivity*. Equivalent objects or collections A = A. Use a balance beam. If two things are identical in all ways no matter what the situation, then they are equal.
- *Symmetry*. Two different objects of the same mass A = B. Use a balance beam. Rotate balance beam so B = A and ask: *What do you see?*

  Have students repeat with different objects. Ask: *What is on the LHS? What is on the RHS? Does the LHS = RHS? Does RHS = LHS? What does this mean for exercises or "sums"?*
- *Transitivity*. Have student find three items of the same mass. Using the balance: Can they show that the first item equals the second? Then that the second item equals the third? Ask: *Does the first equal the third?* Can they show this with the balance? Ask: *Is this always the case?*

  Understanding these three rules becomes the basis of equivalence.

3. Early numbered functions

Give students a real-world problem. Can students:

- Consider the problem as change and then draw a function machine?
- Act out change with the function machine?
- Fill in input–output table?
- Reverse the change?
- Develop inverse?
- Use arrowmath notation (as on right)?
- Do activity without machines – just with input–output table?

$6 \xrightarrow{+5} 11$

Change

$8 \xleftarrow{-5} 13$

Inverse change

Repeat with a two-step real-world problem.

4. Early numbered equations

- Replacing items throughout with numbers; replacing paper strips with Unifix.
- Relate real-world situations to equations.
- Use balance rule to explore what happens when a change is made.

A2 Functions and Equations: Observation Checklist			
Unit	**Concept**	**Knows**	**Can construct/do/tell/solve**
1. Early functions	Notion of change	Change and no change	Language of change; input/output Reverse the change
	Unnumbered functions and inverses	Inverse of change	Function machine Inverse
	Guess my rule	Input/output	Work out the rule
	Multi-change	There can be two or more changes	Simple arrowmath to track
2. Early equations	Same and different	Equals/unequals and equivalence	Language for same and different
	Meaning of equals	Words and symbols	Balance; greater than/less than
	Unnumbered equations	Structure and principle	Focus on relationships not answers; can reverse the activity
	Properties of equals	Three properties of equals A = A; A = B means B = A; A = B and B = C means A = C	Make with balances and/or paper strips
3. Early numbered functions	Unnumbered to numbered functions	How to show change +/− with materials	From counters to pictures to number cards and reverse this Map on input–output table
	One-step operation functions	Can work the change with numbers	Do change with real-world examples and reverse the change for four operations
	Multi-step operation functions	That there can be two changes	Map on a three-column input–middle–output table
	Arrowmath notation	Think of operations as change and apply arrowmath	Use arrowmath notation
4. Early numbered equations	Unnumbered to numbered equations	Getting two sides in balance	Go both directions; addition is joining and also addition is turnarounds
	Real-world situations to equations	Deconstruction; stories	Equation from real-world situation
	Balance rule	What happens when something is added or removed?	Different ways to balance the equation again

A2 Functions and Equations: Observation Checklist Teacher Recording Instrument				
Unit	**Concept**	**Knows**	**Can construct/do/tell/solve**	**Comments**
1. Early functions	Notion of change	Change and no change	Language of change; input/output Reverse the change	
	Unnumbered functions and inverses	Inverse of change	Function machine Inverse	
	Guess my rule	Input/output	Work out the rule	
	Multi-change	There can be two or more changes	Simple arrowmath to track	
2. Early equations	Same and different	Equals/unequals and equivalence	Language for same and different	
	Meaning of equals	Words and symbols	Balance; greater than/less than	
	Unnumbered equations	Structure and principle	Focus on relationships not answers; can reverse the activity	
	Properties of equals	Three properties of equals A = A; A = B means B = A; A = B and B = C means A = C	Make with balances and/or paper strips	
3. Early numbered functions	Unnumbered to numbered functions	How to show change +/− with materials	From counters to pictures to number cards and reverse this Map on input–output table	
	One step operation functions	Can work the change with numbers	Do change with real-world examples and reverse the change for four operations	
	Multistep operation functions	That there can be two changes	Map on a 3-column input–middle–output table	
	Arrowmath notation	Think of operations as change and apply arrowmath	Use arrowmath notation	

Unit	Concept	Knows	Can construct/do/tell/solve	Comments
4. Early numbered equations	Unnumbered to numbered equations	Getting two sides in balance	Go both directions; addition is joining and also turnarounds	
	Real-world situations to equations	Deconstruction; stories	Equation from real-world situation	
	Balance rule	What happens when something is added or removed?	Different ways to balance the equation again	

Subtest item types

Subtest 1 items (Unit 1: Early Functions)

1. Circle the changes that can be reversed

   egg → omelette

   break up jigsaw → piece together jigsaw

   water → ice

   baby → adult

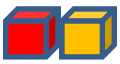
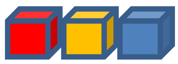
2. Fill in the empty boxes in the table. The first one has been done for you.

	Input	Change	Output
	 potato	cook it →	 chips
(a)	 eggs	cook it →	
(b)	 beanie	wear it →	
(c)	s	add “at” →	
(d)		add “at” →	flat

3. Guess my rule. Draw a line to join the input/output boxes that match.

In	Change	Out
		

In	Change	Out
		

In	Change	Out
		

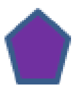
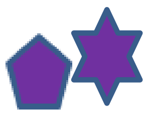
Cook it

Add a blue block

Wear it

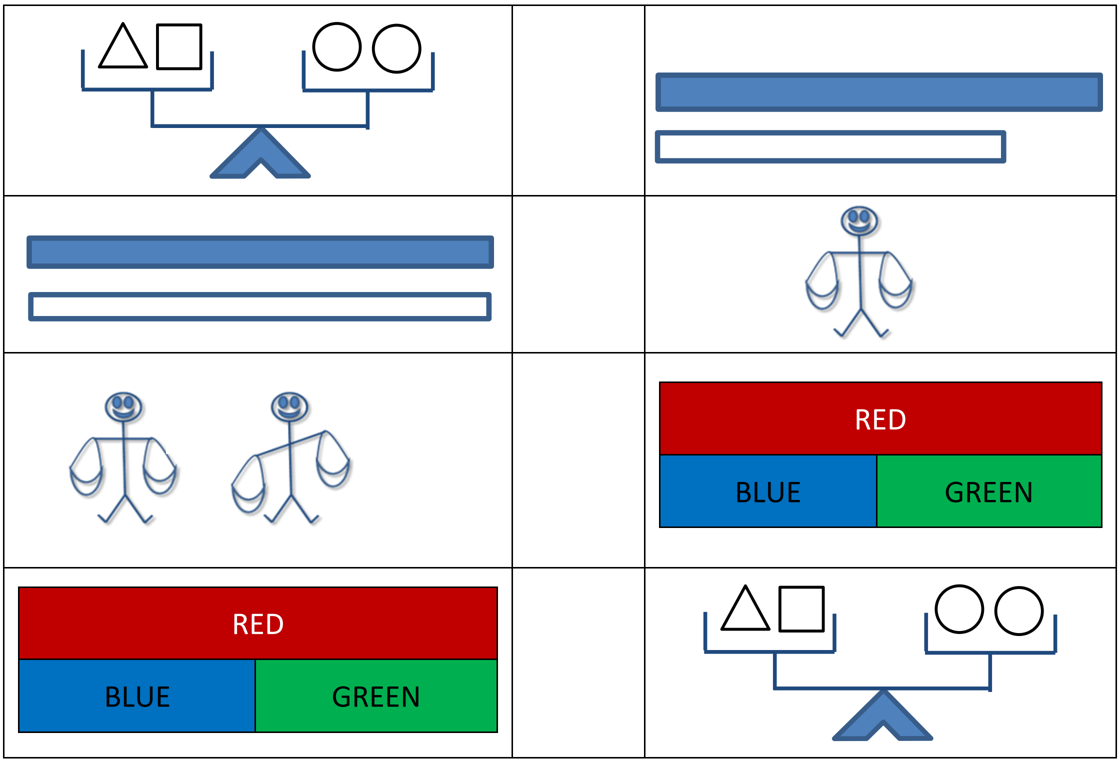
4. Write the two change rules in the function machines below.

In	First change	Out
		

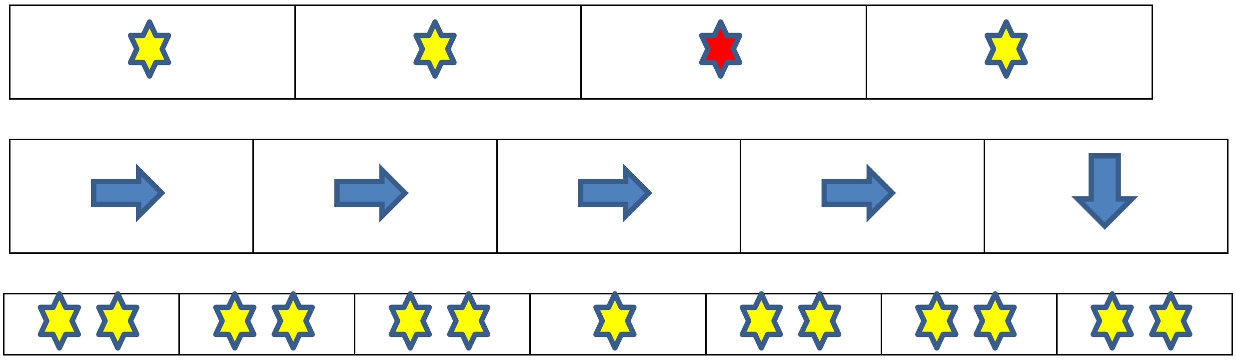
In	Second change	Out
		

Subtest 2 items (Unit 2: Early Equations)

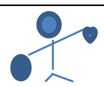
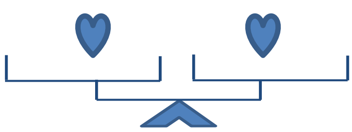
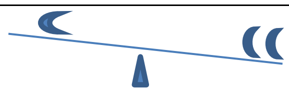
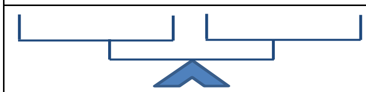
1. Draw a line between the boxes that are the same.



2. Circle the box that is different in each row.



3. Circle either equal or unequal.

		equal or unequal
		equal or unequal
		equal or unequal
		equal or unequal

Subtest 3 items (Unit 3: Early Numbered Functions)

1. (a) Circle which change is true for this story problem.

Three ducks swam on a pond. Two more came along. How many on the pond now?

$3 \xrightarrow{\boxed{+2}} 5$ OR $3 \xrightarrow{\boxed{+1}} 5$

(b) Draw the arrowmath notation to show the change in this story.

6 birds on a fence and 2 flew away. How many on the fence now?

(c) Draw the arrowmath notation to reverse the change in the bird story.

2. Fill in the empty boxes.

(a) $\boxed{2} \xrightarrow{\boxed{+3}} \boxed{}$

(b) $\boxed{} \xrightarrow{\boxed{+4}} \boxed{7}$

3. Fill in the empty boxes:

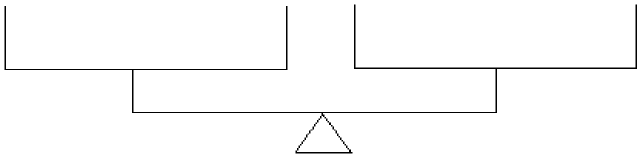
(a) $\boxed{3} \xrightarrow{\boxed{\text{double}}} \boxed{} \xrightarrow{\boxed{-2}} \boxed{}$

(b) $\boxed{} \xrightarrow{\boxed{\text{double}}} \boxed{} \xrightarrow{\boxed{+4}} \boxed{8}$

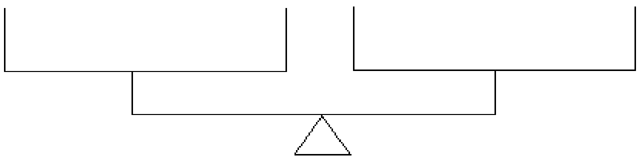
Subtest 4 items (Unit 4: Early Numbered Equations)

1. Use the stories below to fill the trays on the scales.

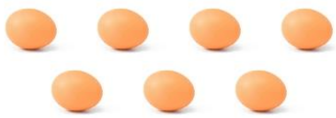
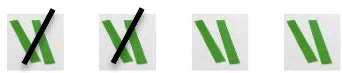
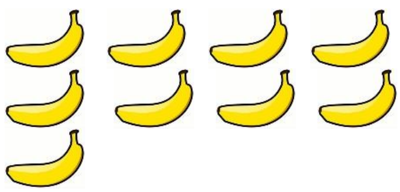
   (a) Jay had 3 apples in a bag and 1 apple in his hand. He had the same number of apples as Meg who had 2 apples in each hand.

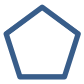


   (b) Elly caught 4 swordfish and 1 starfish. She had the same number of fish as Oliver who had 3 garfish and 2 starfish.

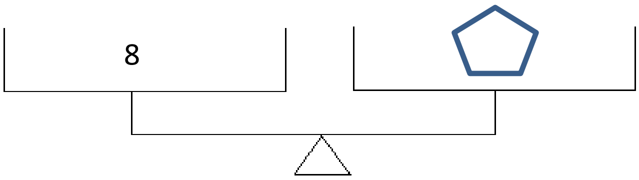


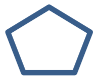
2. With your pencil join the story to the picture and the equation that match.

Story	Picture	Equation
3 Unifix cubes, 2 more added to group. How many altogether?		1 + 3 = 4
1 star appeared at sunset, then 3 more stars after dark. How many stars?		4 + 3 = 7
A hen laid 4 eggs, another hen laid 3 eggs. How many eggs altogether?		9 = 4 + 5
Tom cut 8 strips of paper to measure his desk. He only used 4 strips. How many strips left over?		3 + 2 = 5
I was given some bananas. Alex gave me 4 and Jan gave me 5. How many bananas do I have altogether?		8 − 4 = 4

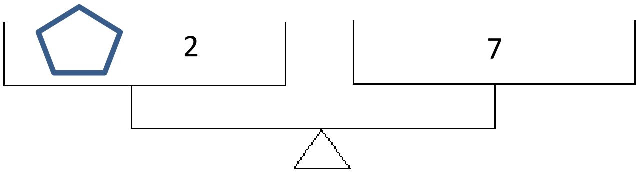
3. Work out what  equals by using the balance rule.

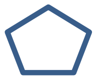
(a)



 =

(b)



 =

Additional resources

a	A
d	D
f	F
j	J



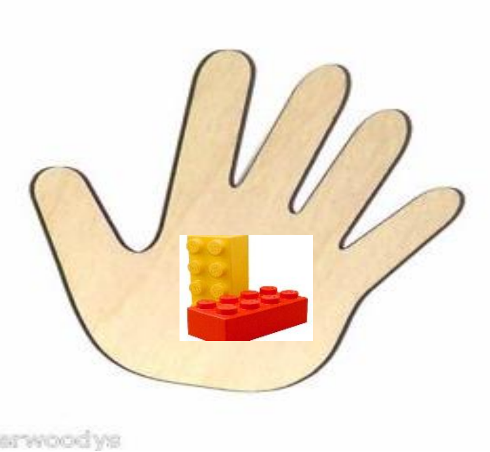





Unit 1: Early Functions

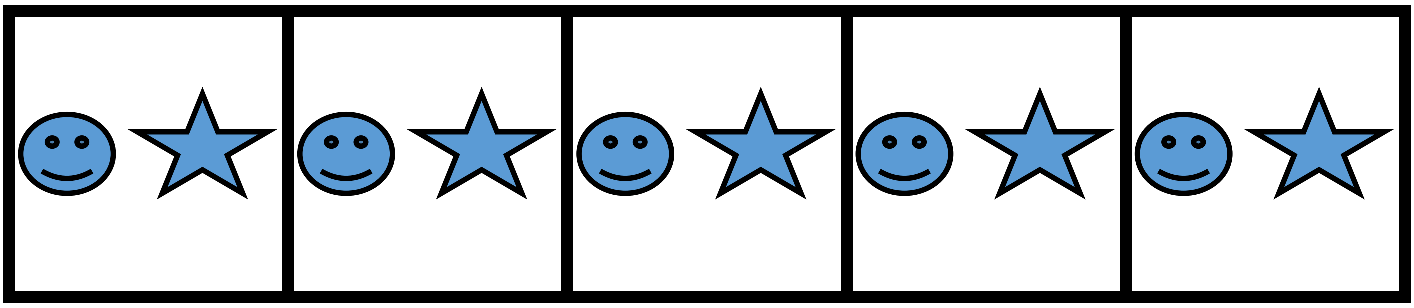
Notion of change

What could be in this egg?
What change happens?



Unnumbered functions and inverse

Input: handful of Lego blocks
Rule: pick up yellow and red block
Output: what do you have now?
Inverse: put the red and yellow back



Unnumbered functions and inverse

Cut up pattern cards
Input: 2 repeats of pattern
Rule: pattern repeat (face, star)
Output: 3 repeats of pattern

The kids seemed to understand this better than the Lego blocks above. I think the Lego became numbered.

Unit 2: Early Equations

Cards for the balance scales

=
More than
Greater than
Less than

Appendices

Appendix A: AIM Early Understandings Modules

Module content

1st module **Number N1: Counting** *Sorting/correspondence *Subitising *Rote *Rational *Symbol recognition *Models *Counting competencies	**2nd module** **Algebra A1: Patterning** *Repeating *Growing *Visuals/tables *Number patterns	**3rd module** **Algebra A2: Functions and Equations** *Functions* *Change *Function machine *Inverse/backtracking *Equations* *Equals *Balance
4th module **Number N2: Place Value** *Concepts* *Place value *Additive structure, odometer *Multiplicative structure *Equivalence *Processes* *Role of zero *Reading/writing *Counting sequences *Seriation *Renaming	**5th module** **Number N3: Quantity** *Concepts* *Number line *Rank *Processes* *Comparing/ordering *Rounding/estimating *Relationship to place value*	**6th module** **Operations O1: Thinking and Solving** *Early thinking skills *Planning *Strategies *Problem types *Metacognition
7th module **Operations O2: Meaning and Operating** *Addition and subtraction; multiplication and division *Word problems *Models	**8th module** **Operations O3: Calculating** *Computation/calculating *Recording *Estimating	**9th module** **Number N4: Fractions** *Concepts* *Fractions as part of a whole *Fractions as part of a group/set *Fractions as a number or quantity *Fraction as a continuous quantity/number line *Processes* *Representing *Reading and writing *Comparing and ordering *Renaming

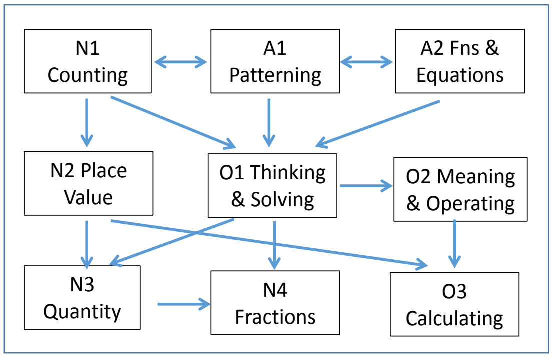
Module background, components and sequence

Background. In many schools, there are students who come to Prep/Foundation with intelligence and local knowledge but little cultural capital to be successful in schooling. In particular, they are missing basic knowledge to do with number that is normally acquired in the years before coming to school. This includes counting and numerals to 10 but also consists of such ideas as attribute recognition, sorting by attributes, making patterns and 1-1 correspondence between objects. Even more difficult, it includes behaviours such as paying attention, listening, completing tasks, not interfering with activity of other students, and so on.

Teachers can sometimes assume this knowledge and teach as if it is known and thus exacerbate this lack of cultural capital. Even when the lack is identified, building this knowledge can be time consuming in classrooms where students are at different levels. It can lead to situations where Prep/Foundation teachers say at the end of the year that some of their students are now just ready to start school and they wish they could have another year with them. These situations can lead to a gap between some students and the rest that is already at least one year by the beginning of Year 1. For many students, this gap becomes at least two years by Year 3 and is not closed and sometimes widens across the primary years unless schools can provide major intervention programs. It also leads to problems with truancy, behaviour and low expectations.

Components. The AIM EU project was developed to provide Years F–2 teachers with a program that can accelerate early understandings and enable children with low cultural capital to be ready for Year 3 at the end of Year 2. It is based on nine modules which are built around three components. The mathematics ideas are designed to be in sequence but also to be connected and related to a common development. The modules are based on the AIM Years 7–9 program where modules are designed to teach six years of mathematics (start of Year 4 to end of Year 9) in three years (start of Year 7 to end of Year 9). The three components are: (a) Basics – A1 *Patterning* and A2 *Functions and Equations*; (b) Number – N1 *Counting* (also a basic), N2 *Place Value*, N3 *Quantity* (number line), and N4 *Fractions*; and (c) Operations – O1 *Thinking and Solving*, O2 *Meaning and Operating*, and O3 *Calculating*. These nine modules cover early Number and Algebra understandings from before school (pre-foundational) to Year 2.

Sequence. Each module is a sequence of ideas from F–2. For some ideas, this means that the module covers activities in Prep/Foundation, Year 1 and Year 2. Other modules are more constrained and may only have activities for one or two year levels. For example, Counting would predominantly be the Prep/Foundation year and Fractions would be Year 2. Thus, the modules overlap across the three years F to 2. For example, Place Value shares ideas with Counting and with Quantity for two-digit numbers in Year1 and three-digit numbers in Year 2. It is therefore difficult, and inexact, to sequence the modules. However, it is worth attempting a sequence because, although inexact, the attempt provides insight into the modules and their teaching. One such attempt is on the right. It shows the following:



1. The foundation ideas are within *Counting*, *Patterning* and *Functions and Equations* – these deal with the manipulation of material for the basis of mathematics, seeing patterns, the start of number, and the idea of inverse (undoing) and the meaning of equals (same and different).

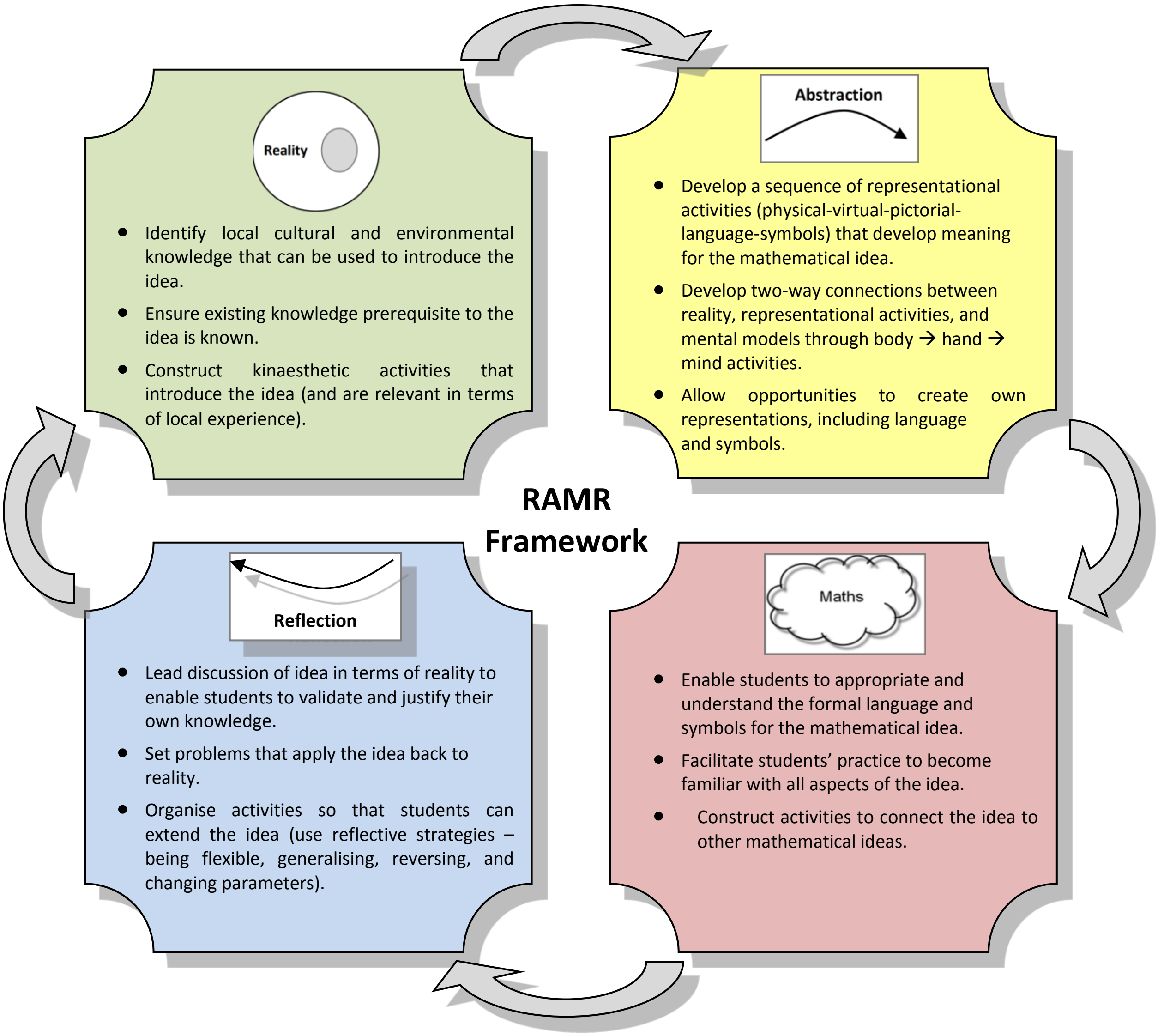
2. The central components of the sequence are *Thinking and Solving* along with *Place Value* and M*eaning and Operating* – these lead into the less important *Calculating* and prepare for *Quantity*, *Fractions* and later general problem-solving and algebra.

3. The *Quantity*, *Fractions* and *Calculating* modules are the end product of the sequence and rely on the earlier ideas, except that *Quantity* restructures the idea of number from discrete to continuous to prepare for measures.

Appendix B: RAMR Cycle

AIM advocates using the four components in the figure below, reality–abstraction–mathematics–reflection (RAMR), as a cycle for planning and teaching mathematics. RAMR proposes: (a) working from reality and local culture (prior experience and everyday kinaesthetic activities); (b) abstracting mathematics ideas from everyday instances to mathematical forms through an active pedagogy (kinaesthetic, physical, virtual, pictorial, language and symbolic representations, i.e. body → hand → mind); (c) consolidating the new ideas as mathematics through practice and connections; and (d) reflecting these ideas back to reality through a focus on applications, problem solving, flexibility, reversing and generalising. The innovative aspect of RAMR is that Reality to Abstraction to Mathematics develops the mathematics idea while Mathematics to Reflection to Reality reconnects it to the world and extends it.

Planning the teaching of mathematics is based around the four components of the RAMR cycle. They are applied to the mathematical idea to be taught. By breaking instruction down into the four parts, the cycle can lead to a structured instructional sequence for teaching the idea. The figure below shows how this can be done.



The YuMi Deadly Maths RAMR Framework

Appendix C: Teaching Frameworks

Teaching scope and sequence for functions and equations

TOPICS	SUB-TOPICS	DESCRIPTIONS AND CONCEPTS/STRATEGIES/WAYS
Functions and equations	Early functions	Change Function machine Input-output tables Unnumbered Relating stories to change
	Early equations	Same-different/equals-unequals Unnumbered contexts
	Early numbered functions	Unnumbered → numbered Relating stories to change Arrowmath notation One step and multi-step function activity
	Early numbered equations	Unnumbered to numbered contexts Mass and length models Relating real-world stories to equations Balance rule

Proposed year-level framework

YEAR LEVEL	ALGEBRA – FUNCTIONS AND EQUATIONS	
	Semester 1	**Semester 2**
Pre-Prep	*Functions: Meanings and notation* – Explore idea of change (e.g. look at how walking two steps changes 3 steps to 5 steps, a drawing to a painted drawing, cooking, and so on). *Equations: Meaning* – Introduce same and different; sort and classify.	*Functions: Backtracking* – Explore getting input from output for simple example ("What did I start with?"). *Equations* – Use balance and length to explore when things are same and different. *Balance rule*.
Prep	*Functions: Meanings and notation* – Explore function machine for unnumbered situations. Look at two machines for two-step changes. *Equations: Meanings* – Relate same and different to equals and unequals; use unnumbered contexts (e.g. weights and length) to explore equal and order relations. Introduce idea of equation/inequation to represent balanced and unbalanced situations.	*Functions: Backtracking* – Look at Output → Input as well as Input → Output. Relate to real-world situations. Discuss backtracking in two-step unnumbered situations. Recognise there are two steps back as well as forward. *Equations: Balance rule* – Discuss what happens to equals when weights are added or removed from sides of the balance, or lengths are added and cut off. Relate to MAST activity.
1	*Functions: Meanings and notation* – Explore function machine for unnumbered and simple numbered situations (addition and subtraction). Relate to real world; record Input and Output; introduce arrowmath notation. *Equations: Meanings* – Use balance and line models in unnumbered situations to relate real-world situations to equations and vice versa. Express equations with more than one object on either side. Move to numbered situations and express equations; relate to real-world situations; show 5=2+3 and 6-1=2+3 are correct equations.	*Functions: Backtracking* – Continue Output → Input activities for unnumbered and numbered situations. Discuss relation of forward to backward change for addition and subtraction changes. Introduce backward arrowmath. *Equations: Balance rule* – Look at adding and removing from either side. Express balance rule for unnumbered situations. Relate to MAST activity.
2	*Functions: Meanings and notation* – Relate real world to machine, Input-Output and arrowmath notation. Consolidate function machine work with addition and subtraction real-world situations. Undertake simple arithmetical excursions – discuss what they say about the operations. Undertake arrowmath activities with calculators. *Equations: Meanings* – Begin to record balance and length numbered and unnumbered situations as equations. Introduce role of addition in recordings. Explore symmetry and transitivity for complex equation situations. Relate to real world.	*Functions: Backtracking* – Relate backtracking to inverse of operations (addition/subtraction). Discuss and relate to real world. Continue with arrowmath. *Equations: Balance rule* – Introduce balance rule for numbered situations.
3	*Functions: Meanings and notation* – Introduce multiplication and division on function machines. Continue arithmetical excursions. Check students can interpret world in terms of change. Discuss two-step function machines. *Backtracking* – Look at how backtracking will help find an unknown; relate to inverse of operations (multiplication/division). Look at backtracking when two steps. *Equations: Meanings* – Move to pictorial situations. Ensure students can analyse real-world activity in terms of equations. *Balance rule* – Encourage students to generalise balance rule for adding and subtracting. Use with physical and pictorial models.	*Functions: Solutions* – Introduce unknown by considering what happens when Input is unknown – how can we think about the Output (focus on quasi-experimental and language descriptions). Explore unknown stories (e.g. I have a number ...? and Jack and Fred both gave me the same amount of money, I spent ...?). Use number lines to follow and solve stories. Look at backtracking from an unknown output. *Equations: Unknowns* – Introduce notion of variable and relate to equations (let students choose their notation for unknown). Explore balance stories with unknowns. Focus on quasi-experimental and language descriptions. Discuss using balance rule with unknowns.
Focus	Kinaesthetic: Body → Hand → Mind. Building early algebraic thinking: ideas, logic, techniques, habits of mind. Practising language.	Kinaesthetic: Body → Hand → Mind. Building early algebraic thinking: ideas, logic, techniques, habits of mind. Practising language and representation.

Accelerated Inclusive Mathematics Early Understandings Project

YuMiDeadly

Growing community through education

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