G E O M E T R Y

SPACE AND SHAPE IN THE PRIMARY SCHOOL

Mathematics Science and Computing Department

Prepared by Tom Cooper
February, 1986.

ISBN: 0 86856 635 7

CONTENTS

OVERVIEW OF GEOMETRY

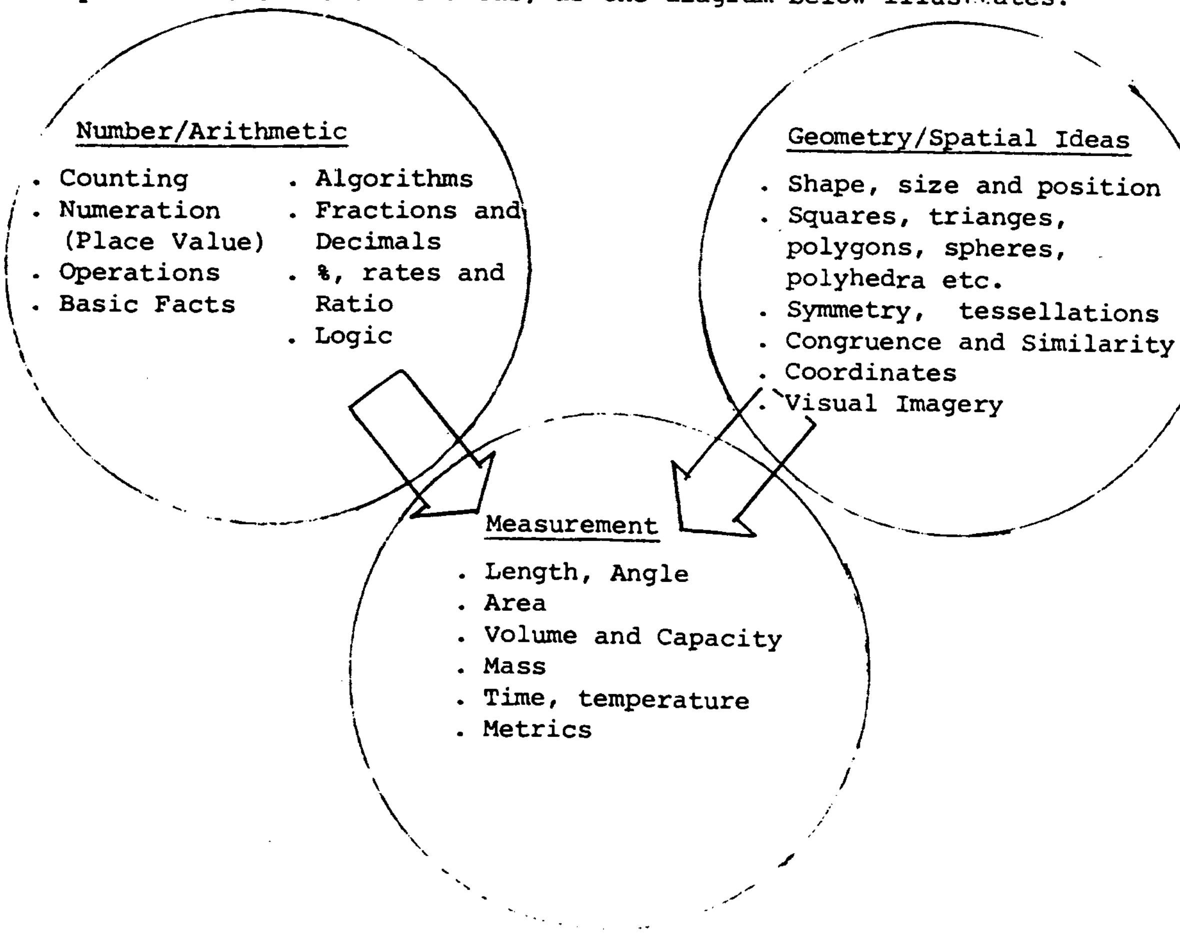
WHAT IS GEOMETRY?

Modern research is showing that human thinking has two aspects: verbal logical and visual spatial. Verbal logical thinking, associated in some literature with the left hemisphere of the brain, is the conscious processing that we are always aware of. It tends to operate sequentially and logically and to be language and symbol (e.g., number) oriented.

On the other hand visual spatial thinking, associated in some literature with the right hemisphere of the brain, can occur unconsciously without us being aware of it. It tends to operate wholisticly and intuitively, to be oriented towards the use of pictures and seems capable of processing more than one thing at a time - as such it can be associated with what some literature calls simultaneous processing.

On senses and the world around us have enabled both these forms of thinking, to evolve and develop. To understand and to modify our environment has required the use of logic and the development of language and number plus an understanding of the space that the environment exists in an-understanding of shape size and position that enables these things to be visualized (what we call geometry).

Mathematics, because it is a product of human thinking applied to problems in the world around, has, historically, and presently, two aspects at the basis of its structure: number and geometry. In fact it is useful to think of primary mathematics as three areas, as the diagram below illustrates.



Of course, modern research is also showing that both sides of the above model are highly integrated. Good mathematicians are able to drawn upon both sides and processes tend to cross over - witness the use of geometric materials and the role of visualising that is evident in modern methods of teaching number and operations.

But it is also evident that geometry has been neglected in schooling. It nowhere near gets that time and the emphasis of number and arithmetic. As such, schools make it difficult for children to have this balance between geometry and arithmetic that is the basis of mathematics and mathematical thinking. We have to do much more geometry in primary schools. This book is written in the hope that it will help this happen.

This book contains ideas for teaching geometry in primary mathematics classrooms. Geometry is unlike arithmetic. Most people agree that both subjects are important althouth this is often not translated into classroom practice. With arithmetic they not only agree that addition, subtraction, multiplication, and division should be taught; they even agree, to some extent, on when and how they should be taught. On the other hand, with geometry there are many different points of view as to how much, when, and why it should be taught.

It seems to us, that for many years, teachers and pupils have had rather a negative attitude towards geometry. This attitude probably stems from three main causes: teachers have a restricted concept of what geometry is, and what it is trying to achieve; pupils perceive the classroom geometry activities as being particularly dull and boring; and standard textbooks supply only a very narrow part of what should be covered in a rich geometric experience. We strongly believe that geometry can be presented in a stimulating and active manner, and in this book we present a wide range of activities to support our argument.

As we said before, it is very difficult to get two people to agree on a precise definition of geometry. However, the following quotations provide a basis for discussion and indicate clearly that the scope of geometry is far wider than the narrow horizons of Euclidean geometry that many of us remember from our high school days.

> *"Geometry is sometimes thought of an investigation or discovery of pattern and relationship in shape, size and place (position). These are observed in and derived from the immediate environment and the much wider world, both natural and man made."*

> *"Geometry is the exploration of space. A child, from the moment he is born, explores space. First he looks at it, then he reaches out into it, and then he moves in it. It takes a long time for him to develop the idea of perspective, of distance and depth, notions such as insides and outsides, back and front, before and after, and so on. In school, the development that has already taken place should naturally be encouraged and extended through the child's own experience."*

Stop and think how you use geometry in your life. When you read a map, you use measurements and make intuitive judgements about shortest distances. When you give directions, you refer to parallel streets, right turns, diagonal streets, and traffic circles. Arranging furniture draws on your intuition about shapes and relationships between them, and on your ability to visualise the effect on moving shapes. Moving a couch through a door can demand very precise measurements and then, perhaps, some real geometric problem solving. When you reinforce a sagging door or stabilise a windblown tree, you draw on knowledge or intuition about the stability of triangles and the instability of other shapes. There are, of course, many such everyday encounters with geometry.

Geometry also serves many people in their professional lives, for example, scientists, engineers, architects, fitters and turners, carpenters, elementary school teachers, etc. Think of the professional area of geometry involved in building a house. Drafting plans requires scaling and measurement, as well as problem solving with relationships between shapes. Each construction worker must read and understand the plans, and must solve a variety of technical problems that require at least an accurate intuition for geometry. We live, work and play in 3-D space. We have to know our way around it. We have to know our position or place in it. We have to be able to recognise things within it. We have to know shape and size. This is therefore the role of geometry in primary school, to prepare children to survive in and master the space in which they live.

Therefore, in this book, we will take a much wider view of geometry than traditionally found in schools. The knowledge of squares, triangles coordinates and 3-D solids is included. However, knowing how to read a road map, to follow a compass, to fit shapes together, to recognise land marks from any perspective is also included.

WHY TEACH GEOMETRY?

For this book we have taken a wide view of geometry - as the investigation of pattern and relationship in shape, size and position. It is the development of the skills and processes for children to be able to operate in, understand and manipulate their 3-D environment, from recognising and using shapes to following road maps.

It is our contention that much more geometry should be done at school particularly at the primary level. The arguments for this are three fold:

(1) Firstly, problem solving, particularly real world problems, should be a major focus in mathematics. Mathematics learning should prepare children for the problem solving that will confront them everyday. Many, if not all, real world problems involve shape, size or position (e.g., "how shall I redecorate my room?" or "where is the cafeteria?"). As such knowledge of geometry and problem solving experience in geometrical situations is essential.

(2) Secondly, geometry is crucial in many vocational areas, for example architecture, fitting and turning, commercial artist, carpentry, engineer, automotive repair etc. where the professional is required to image shapes and visualise how they fit together and relate to each other. One would want to add to this list airplane pilot, footballer, racing driver, explorer etc. In fact many of the basic living skills we would want all children to master depend on spatial ability, e.g., following a road map, building a pergola, determining which lounge suite is best for the room from 2-D pictures in a brochure, etc. How are these skills developed? There are some general methods and systematic rules in geometry that are essential. There are experiences that it is difficult for text books to supply. The best way seems to be to experience a wide variety of geometrical situations.

(3) Lastly, there is a particularly important reason for geometry. Geometry problems often involve the manipulation of shapes or the determining of position within a geometric framework. Success at such activities involves the manipulation of images to the mind or what is called <u>visual imagery</u>. As we have said before, research into how people think and do mathematics is giving some tantalizing evidence of the importance of this imagery. Study of mathematical geniuses has produced the surprising result that some of them (e.g., Einstein) thought "in pictures",

not in symbols. So has similar study of pupils in mathematics classes. For example, Kruteskii in his book The Psychology of Learning Mathematics, Wiley, has documented cases of pupils good at mathematics who use predominantly spatial ideas in their problem solving.

Studies of relationships between visual imagery and mathematical competence are giving some evidence of visual imagery's importance to mathematical ability. For example, Wheatley in his article in the Arithematic Teacher, November 1977, pp. 36-39 describes experiments where children are given mathematical problems to solve while connected to encephlographs. The results have shown that those children who are successful in doing the mathematical problems exhibited encephlographic activity in the brain that indicated both hemispheres of the brain (the intuitive wholistic and visual spatial side as well as the symbolic, sequential and logical side) were active in solving problem. Children who were not successful tended to only exhibit activity in the symbolic sequential and logical side of the brain. This right hemisphere activity was particularly noticeable during the early part of the problem solving where the children were looking for a way to attack the problem (and at the end when generalising).

This seems to give some indication that to be successful in mathematics, children's visual imagery and spatial ability must be developed. And this is done in geometry.

NOTE: This may be particularly important for girls because one reason put forward for the alleged male superiority in mathematics is low spatial ability, a shortcoming that an early focus on geometric experiences may overcome.

MAJOR CONCEPTS AND PROCESSES IN GEOMETRY

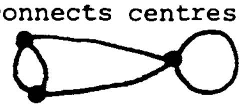
For this book, we have identified the following as the major focus of geometry in primary school mathematics. A detailed coverage of them is given in chapters 1 to 7 later in the book.

1. Position

Initially position can be defined in terms of everyday words such as "here, "there", "under", "over", "above", "below", etc. But sooner or later, numbers (and letters) can be introduced as a way of more formally determinining position (in terms of ordered pairs of numbers or letters, i.e., coordinates) - for example, "row C seat 4" or "2 blocks south and 1 block west", or "(2,3)".

Of particular interest in position, is direction. Discussion of direction can lead to defining straightness and curve - and so to straight lines, parallel lines and curved lines. Change of direction can be related to turn and amount of turn gives angle. A special area that needs to be followed is direction in terms of north, east, west and south i.e., compass bearings, and the use of this knowledge with knowledge of map reading to enable children to undertake orienteering and bush walking activities.

2. Networks, map reading and scale drawings

A very special skill that is needed regularly by adults and children in modern society is the ability to read maps - road maps, street directories and surface maps. A particular type of map that connects centres (or points) with lines (or arcs) is a network, e.g. 

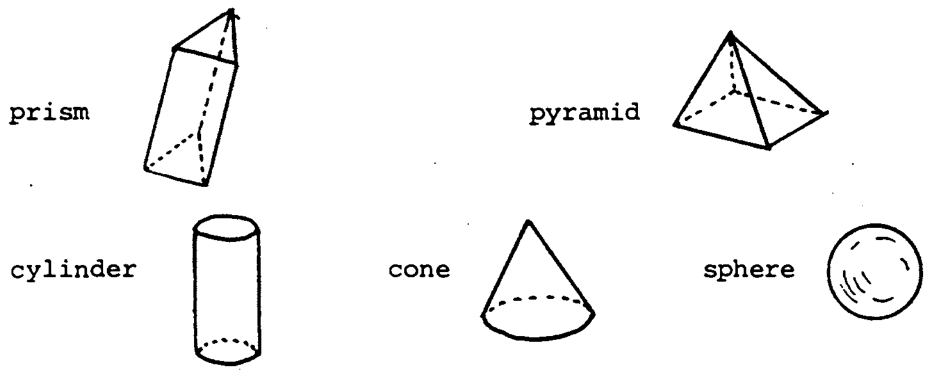
A network can represent towns and roads, centres and telegraph lines, or suburbs and bridges. A network divides space into regions. Travelling along, for example, roads between towns can be modelled by tracing over a network that represents the towns and roads. This enables many interesting travel problems to be studied by children at their desks.

A knowledge of coordinates, shapes, bearings and distance comes together with the ability to prepare and draw scale drawings (or maps) of homes, schools, parks, suburbs, etc. This does not only have to be a culminating activity - in fact getting children to draw their homes, themselves, etc can be an excellent starting point.

3. Three-dimensional shape (solids)

This is a most crucial area as we live, work and play in a three dimensional world. Initially we can use solids (bricks, lego, etc) to construct things. We can sort and classify solids in the environment by their properties - corners/no corners, rolls/does not roll, etc. We can study their surfaces and edges (leading into the study of line and two-dimensional shape).

The type of surface and edge a shape has, whether they are flat or curved or straight, determines their names. We particularly focus on five types of solid:

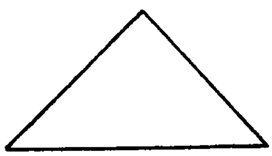


and solids which have flat surfaces and straight edges called polyhedra.

4. Two-dimensional shape

This is another crucial area because it is how we represent our three-dimensional world (e.g. in seeing, drawing, video, film, etc.)

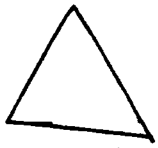
The concept of two-dimensional shape can be developed from three dimensional shape, from study of the surfaces of the 3-D solids. It can also be built up from subconcepts. For example the triangle below is composed of three straight lines (and three angles) which form a boundary.



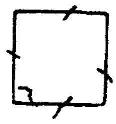
This triangle is closed and simple (lines do not cross) and bounds a region. Its lines are not parallel or curved. Formally, it is a simple closed boundary consistingof three straight lines.

Two-dimensional shapes are classified by their boundaries. When these are straight lines, they are called <u>polygons</u>. Polygons have special names based on the number of sides and whether these sides are parallel, equal in length or meet a special angles, e.g.

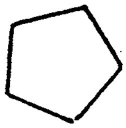
Triangle
(3 sides)



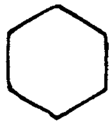
Square
(4 sides
all equal
meet at
right angles)



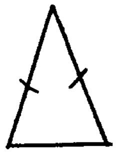
Pentagon
(5 sides)



Hexagon
(6 sides)

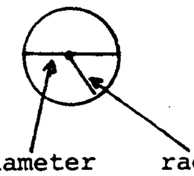


Isosceles
Triangle
(3 sides,
2 equal)



When sides are curved, special attention is given to the <u>circle,</u> e.g.

Circle



diameter radius

Semi Circle

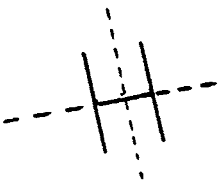
Shapes bound an area of a plane: called a <u>region</u>. The circular region is call a <u>disc</u>. A section of a the disc is a sector, e.g.,

Sector

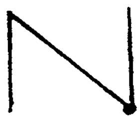
In polygons, lines that join non adjacent corners are called <u>diagonals</u>. Polygons can also be studied and classified in terms of these diagonals.

5. <u>Symmetry</u>

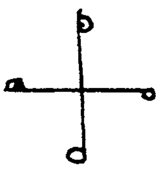
Shapes can be studied for both line and rotational symmetry and for the relationship between these <u>line symmetry</u> means that the shape can be in half along a line and <u>rotational symmetry</u> means that the shape looks the same after a part turn, e.g.



Line
Symmetry



No Line
Symmetry



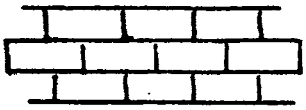
Rotational
Symmetry

No Rotational
Symmetry

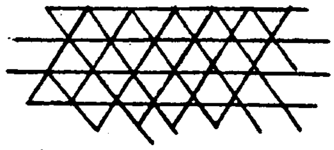
Shapes can be classified by symmetry. Symmetry can be related to reflection (flip).

6. Tessellations

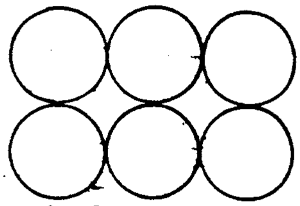
When a shape is such that many copies of it can be joined like tiles to cover space without overlapping or leaving gaps, the shape is said to tessellate, e.g.



Rectangles Tessellate



Triangles Tessellate



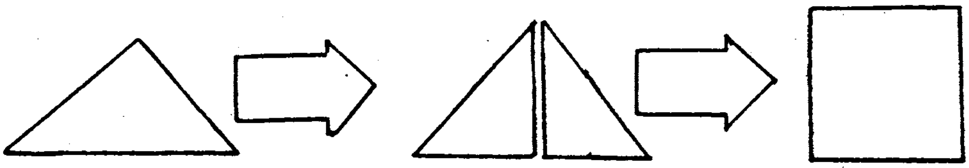
Circles do not tessellate

Tessellations can be studied for the angle properties of their shapes. Any tessellating shape can give rise to "graph" paper. Fabric designs result from a tessellating pattern as do Escher type drawings. Shape puzzles such as Hexiamonds and Pentominoes come from tessellating triangles and squares.

One can introduce tessellation through tiling. Shapes that tile without gaps on overlaps can be identified. Then we can look at tessellations of more than one type of shape. Three dimensional or solid tessellations (i.e., packages that pack well) can also be included in the study. Finally, tessellations can be used to identify shapes which form the basis of shape puzzles.

7. Dissections

When a shape is cut into smaller parts and these are reassembled to form a second shape, the act of reversing this process is considered to be solving a dissection, e.g.



We can begin dissections by focussing on the second phase of this process - joining sections to form a whole (e.g., jigsaw puzzles). These can be made by cutting a shape (e.g., a circle or a square) into pieces which children must reassemble. Tangrams and soma cubes are also a part of this section.

8. Similarity and perspective

Two shapes are similar if one is an enlargement of the other, that is, same shape but different size, e.g.

Similar shapes are like enlargements in photography or the action of light through a film.

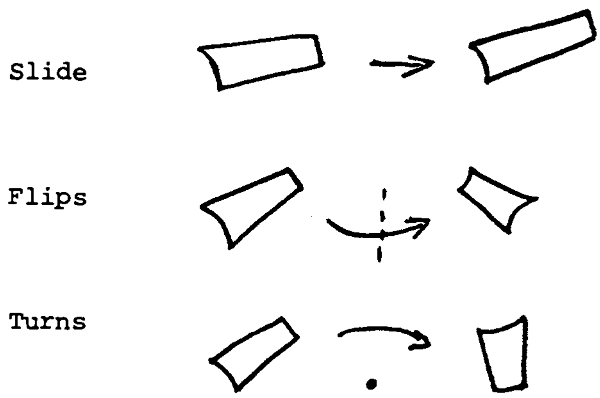
A more complicated projection, an extension of similarity, is perspective - the way we see the world. Perspective drawing and shadows provide experiences through which we can study this.

Children must experience similarity and perspective relationships and construct shapes that are projections (both similar and perspective) of each other. In this way they can discover the properties of such shapes.

9. Congruence and slides, flips, and turns

Two shapes we congruent if they are the same size and shape (they differ only by orientation), e.g.

Congruence can be used to characterize equality of length (i.e., equal length sides are congruent sides) and shape types. For example, isosceles triangles has two congruent sides. Congruent changes can be studied through slides, flips, and turns, e.g.



TEACHING GEOMETRY

By its very nature, geometry does not have the dominating sequential nature of arithmetic and much more teacher choice is available in determining appropriate teaching sequences. There are also many experiences in geometry not directly connected to the development of rules and general procedures but rather to the development of imagery and intuition and as such may not be recognised as important by teachers.

Geometry, as for the remainder of mathematics, has two aspects or functions: as a language and as a tool for problem solving. Too often in the past teachers have focussed on the language aspect - developing the names for various shapes (such as prisms, polygons, cylinders) rules for relationships such as similarity (e.g., equal angles) and procedures for constructions (e.g. bisecting an angle). Yet some of the more interesting activities (e.g., dissections, tessellations, constructions) are associated more with development of problem solving skill.

Geometry can be one of the most exciting and interesting sections of mathematics. It provides an opportunity for motivating children that should not be missed. Children enjoy doing it. It can be colourful and attractive. Pattern and shape can be created and admired. Success can be enjoyed by the majority.

Overall framework

Although there is contention in any framework for teaching geometry, the curriculum ideas on which this book is constructed and about which more will be said in Chapter eight are four fold.

(1) The focus of geometry should be from and to the everyday world of the child.

(2) There should be a balance between experiences which enable children to interpret their geometric world and process with visual images. Geometry should be within a problem setting.

(3) The children's activity should be multisensory (using actual physical materials and moving and transforming them) and structuring (recording results on paper in words and pictures). The "typical" geometry classroom would have groups, physical materials and pens and books ready to record. There should also be opportunities for children to display what they have made.

(4) Teaching activities should be based on three levels of development:

(i) the experiential level, at which children learn through their own interaction with their environment;

(ii) the informal level, in which certain shapes and concepts are singled out for investigation at an active, non-theoretical level; and

(iii) the formal level, where a systematic study of specific shapes is undertaken, with important properties identified and labelled. (Note that this does not include proof of results-rather, practical exercises which suggest that the properties seem perfectly reasonable.)

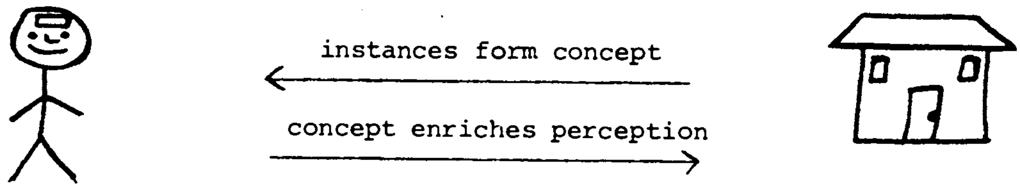
At the experiential level children should be allowed to learn through their perception not the teacher's words. Shape can be labelled and described but not broken into its component parts. Children should not be expected to be accurate in their statements. At the informal level, experiences can include analysing shapes and constructing shapes from their properties. The sub concept approach discussed later would be appropriate here. At the formal level properties such as congruence and similarity can be investigated. There should be no attempt at deductive proof and posing abstract systems.

NOTE: The three levels described above are similar to the first three Van Hiele levels.

Teaching approaches

There are four interesting, at times alternative and at other times complementary, approaches to organising the teaching of geometry that are worth discussing here. We look at these approaches again in Chapter Eight (and, to some extent, in Chapter Seven) later in this book.

(1) The environmental approach (or 3-D approach).
Here the starting point and organizing imperative for teaching is the environment, the everyday three-dimensional world around the children. Ideas are first developed from instances in the world and these ideas then improve the observation powers, as the following shows:



Concept of rectangle

Real world instance of a rectangle

In developing geometric concepts and processes, children are given practical experiences classifying, constructing and manipulating solids. Discussion of these experiences leads to identifying key distinguishing properties. For example corners, flat and curved surfaces, edges, ability to roll, etc.

2-D shapes are presented as they are encountered in the 3-D solids, for example, rectangles from table tops, circles from clocks, triangles from roofs of houses, etc. These 2-D shapes are then investigated for their properties.

Discussion of the positioning of 3-D solids leads to language development relating to position, e.g., here, there, etc. After measurement is developed, we can introduce coordinates.

After the practical experiences with the 3-D solids, there is more specialised investigation of these solids. As appropriate, certain broad unifying concepts are developed which allow more efficient and action oriented investigation of properties, e.g., symmetry, similarity and congruence can be encountered very early. Tessellations and dissections are used to develop visual imagery. Overall, the development follows these four stages:

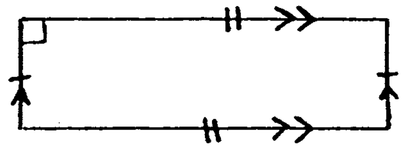
(i) intuitive experience with concepts and materials.

(ii) development of appropriate language, more detailed investigation of particular instances, identification of attributes and properties, construction of shapes through a variety of techniques and the combination of shapes to give larger ones.

(iii) development of a systematic scheme of properties, which shows shared properties and connections between concepts and shapes.

(iv) applications of the above understandings to problem situations and the development of visual imagery.

(2) The subconcept approach (sometimes called the topological approach). Here the starting point for teaching is to analyse a task into its prior abilities and an order they should be developed in children. Then the ideas are taught to build up the concept or processes from its subconcepts and subprocesses. For the rectangle below, the subconcepts are line (straight and parallel), turn and angle, right angle, closed, simple (lines do not cross over as in a figure 8), boundary.

Rectangle 

We start with the concepts of boundary, line and angle. Lines are joined at angles to give paths. The ends of these paths are joined to give a closed boundary. This closed boundary encloses a region. Simple closed paths are two dimensional shapes. In this way the various 2-D shapes are seen as products of their paths and (unlike the 3-D approach) it is not necessary to investigate these 2-D shapes for their properties to the same extent. For example, a figure formed from four equal straight lines (composed of parallel pairs) with four right angles is a square. These do not have to be found as the properties of a square.

The resulting 2-D shapes are then investigated and classified by unifying concepts such as symmetry. At the same time, they are used to assemble 3-D solids and used in shape puzzles.

Position concepts can develop out of the study of the relationship between shapes, (e.g., flips, slides, turns) and out of the actions involved in solving shape puzzles and constructing larger shapes from smaller. 3-D solids are constructed from their 2-D surfaces and then studied for their properties.

(3) The thematic approach.
In this approach, geometrical activities are organised around a central theme. For example a theme such as "squares" could be chosen. Then activities which include the following could be undertaken by children:

(1) The square shape and its properties;
(2) Concepts inherent in squares, such as parallel lines, right angles, congruent sides;
(3) Tessellating with squares and square graph paper activites;
(4) Pentomino puzzles (pentominoes are shapes formed from 5 squares);
(5) Tangram puzzles (a dissection puzzle based on a square);
(6) Three-dimensional shapes formed from squares (e.g., cubes, square pyramids);
(7) Soma cubes (a puzzle formed from cubes);
(8) Fabric design and escher type drawings based on squares tessellating;
(9) Patterns and artistic designs based on flipping, sliding and turning squares (e.g., modulo art);
(10) Coordinates (on square graph paper);
(11) Scale drawings and similarity constructions (enlargements) using graph paper; and
(12) Topics from areas other than geometry (e.g., areas, magic squares, square numbers).

The thematic approach is usually very exploratory or experimental. Instruction is organised so that children experience a variety of activities - learning by doing. In discussion, connections between different aspects of the theme can be highlighted making the thematic approach very schematic.

(4) The transformational approach.

In the 1950's, Piaget, studying children's spatial development, put forward the hypothesis that children develop through three stages in their geometric understanding and he identified each stage with a transformational geometry. Transformational geometries are a way of characterising the different types of geometries (one of which is the Euclidean geometry of straight and parallel lines and points we all know well) by the type of changes (or transformations) in shape that they correspond to. In other words a transformational approach looks at change in shape, size and position in our three dimensional world and designs instruction around looking at these changes by moving through the three geometries (topological, projective and euclidean) that Piaget identified as representing how children develop in spatial thinking.

This seems rather complicated but in practice it is a rich and exciting way of studying geometry and all of traditional geometry can be encompassed within it. As we live, we move, we move things and change their shape and we represent things (e.g., drawings, T.V.). These result in change and their are few types of change: size, shape, perspective or view, and location. When we consider school geometry as the study of space experiences, we look at three types

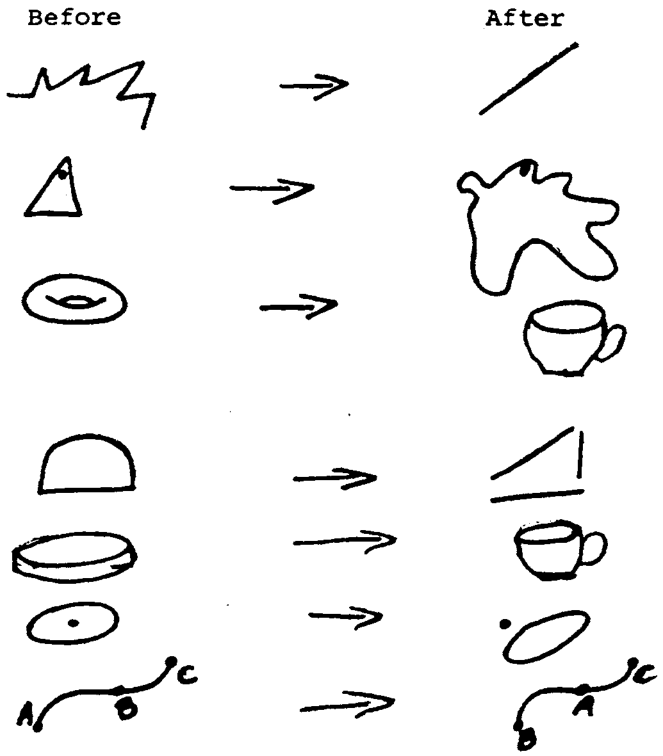
- change in shape (a geometry without straightness or length, involving change in position, size and shape - called topology)
- change in perception (a geometry with straightness but not length, involving change in position and size - called projective)
- change in location (a geometry with both straightness and length, involving change only in position or location - called euclidean).

Note: The Piagetian hypothesis that geometry experiences should move prescriptively topology to projective to euclidean has not been supported by research. Some topological notions, such as invariance and networks, appear late and some projective and euclidean notions appear early. The important thing for this book is that the transformation approach focuses attention on change, a powerful method of developing geometric ideas.

Topology deals with change through twisting, bending, stretching, moulding with no breaking, cutting, tearing, jointing or punching holes allowed. Topology is the study of those aspects or concepts that remain the same during these changes.

Topological transformations can involve changes in size or shape as well as location. The changes in shape can be extreme. Any transformation is topological as long as it does not involve tearing (breaking) or pushing separate points together. For example, a baker with a rolling pin effects a topological change on dough for making e.g., bread, rolls, etc.

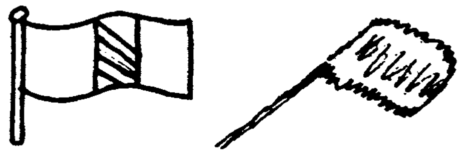
Topological change can be quite drastic. Yet shape is recognizably the same in some respects after such change. Things inside remain inside. Curves that are closed remain closed. The order along a curve remains the same. Things that are separated (or disconnected) remain separated. The number of regions enclosed remains the same.



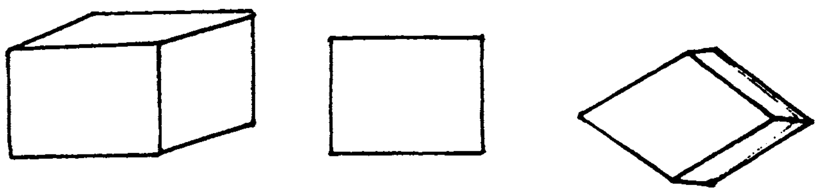
are topological change

are not topological changes.

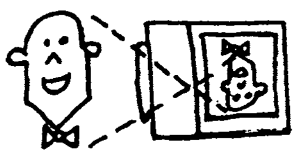
Projective transformations are a little harder to describe. Examples of projective transformations are those that result from visual phenomena - such changes as the change in shape which occurs from an object to its shadow:



or which result from viewing an object from different perspectives:



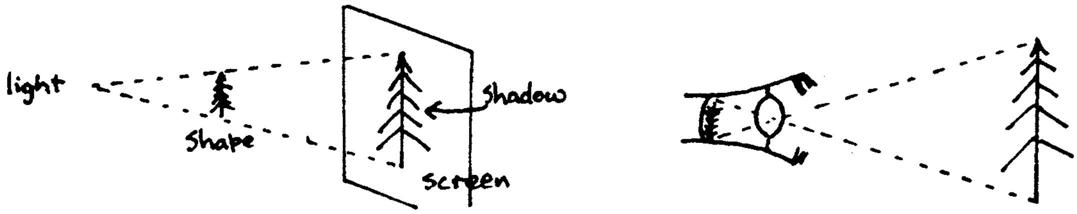
Actually, projecting slides and taking pictures involve projective transformations.



Notice that these projective transformations can involve change in size and some changes in shape, as well as changes in location.

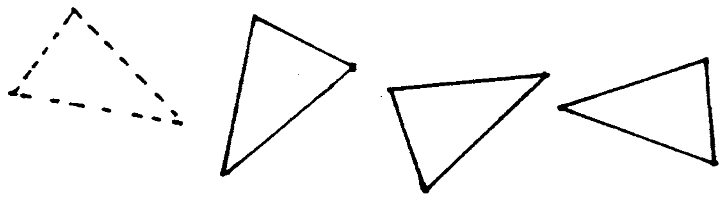
Projective geometry is best studies by considering shows cast in parallel light (sunlight) and divergent light (torchlight, candlelight). For such shadows, straight lines remain straight (and parallel lines remain parallel for sunlight), yet angle, length and size change (though the number of angles or corners remains the same).

This casting of shadows is a similar process to that of a camera photographing a scene (and to our eyes viewing the scene).



Perception and perspective (how we see and draw the world around us) are part of projective geometry. In other words such concepts as right/left and above/below (for example) are part of projective geometry.

An Euclidean or rigid transformation of an object changes its location but does not change its size or shape. For example, each of the solid-line triangles below is a rigid transformation of the dotted triangle. So you are performing a rigid transformation anytime that you relocate an object without changing its size or shape.



Euclidean geometry, therefore, is the study of changes that leaves shape and size unchanged. These consist of flips (reflections about a line, such as in a mira or mirror), slides (changes in positions without change in orientation - called translations), and turns (rotations about a point - the centre). An object changed only by flips, slides and turns remains the same in size and shape, although its orientation and position may change.

SEQUENCING GEOMETRY INSTRUCTION

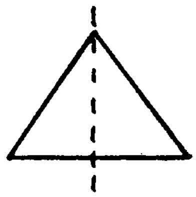
Each of the approaches above does supply some frameworks for sequencing geometry concepts and processes. But at this stage, it is prudent to be eclectic in their use - using any approach (and all approaches) depending on the needs of your class. Two points may provide some ways in which to do this.

(1) Spiralling the curriculum
Geometric concepts and process (even as seemingly straight forward as "rectangle") are not developed completely in anyone year - they must be returned to many times so that other geometric understandings, e.g., symmetry, can give them a deeper and more complex meaning. The various approaches can be used within this spiral. For example, the environmental approach could be used in the early years to first bring rectangle to children's notice. Children could observe and learn to recognise it from the instances in the world (such as doors, table tops, windows, walls, bricks) where it appears. Then later, once line and angle have emerged, the subconcept approach could be used to more deeply study rectangle and its properties. A thematic approach could then be used in a middle primary situation to integrate symmetry and tessellation with rectangle and to look at properties of diagonals. With a subconcept approach three dimensional solids that are constructed with rectangular faces, could be more formally studied, thus completing the circle back to the environment. These

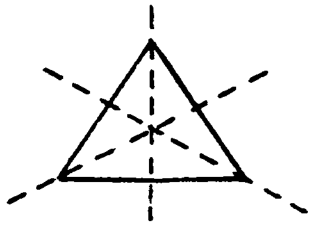
deeper understandings of rectangle could then be drawn into a transformational style look at how the rectangular shape is involved in change in the environment - to similarity, congruence and artistic patterns.

It should be noted that approaches are not the only thing revisited within this spiral. Concepts and processes themselves will also be returned tc. For example, an intuitive understanding (from the environment and simple shapes) can be developed for line symmetry in the early years. Later this line symmetry can be used to categorise shapes.

Example: Isosceles triangle - 1 line of symmetry



Equilateral triangle - 3 lines of symmetry



Later still, symmetry can be again visited to develop the rules and relationships associated with symmetric change (e.g., the number of lines of symmetry equals the number of rotations of symmetry if the number of lines is greater than two).

(2) Diagnosis
With choice available to the teacher, the basis for instruction lies in diagnosis. Each approach has a different effect on different children. The teacher should use the approach that suits his or her children and the objective he or she has, for the children, regarding the topic being covered.

This is also the case for the choice of topic. If teachers have in their minds a list of concepts and processes that children should acquire then they can choose the next geometric topic so as to keep a balance in what the children are learning. Teachers should also keep in mind that a child needs to understand a concept, be familiar with the processes that are associated with the concept and be able to apply these concepts when choosing what next to do. We can highlight this by looking at activity types.

<u>Activity Types</u>

Once a content objective in geometry has been decided on, exemplars (materials or pictures) have to be chosen to help introduce the ideas. In geometry, the following can be used to exemplify geometric concepts:

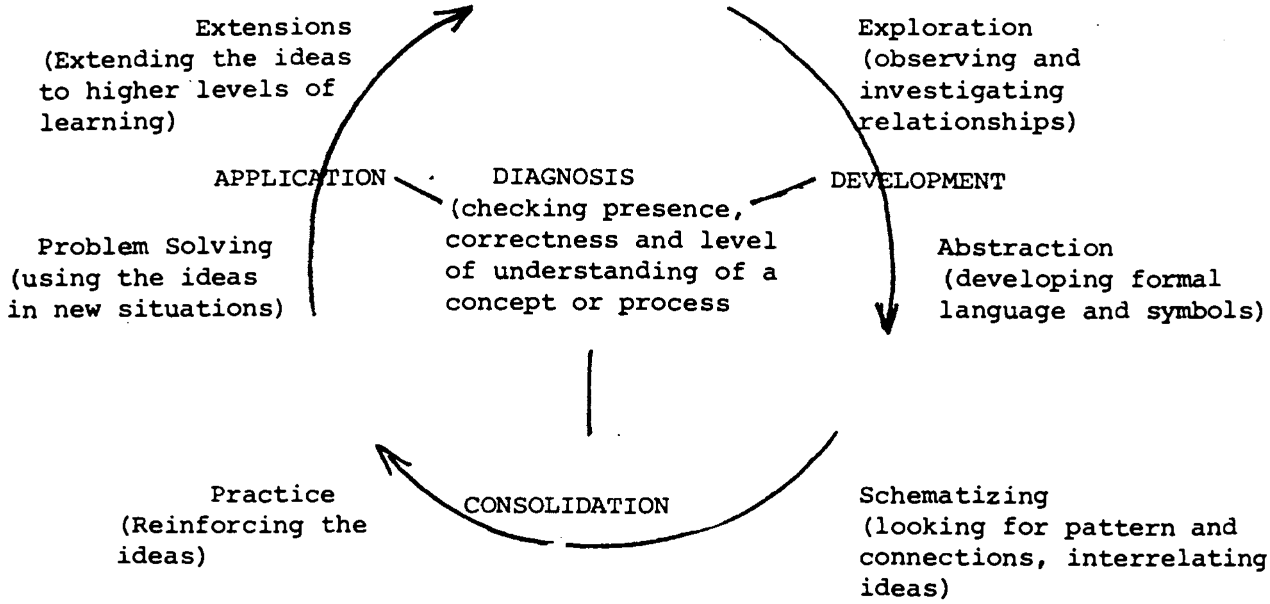
geoboards	(boards with arrays of nails on which shapes are made with rubber bands)
geostrips	(plastic strips with holes joined with paper fasteners)
miras	(plastic mirrors which reflect and transmit light)

cardboard, paper, scissors, glue, tape

straws, string, plasticine, toothpicks or skewer sticks

protractors, rulers, compasses, silva compasses, inclinometers, etc.

Once the appropriate materials have been chosen, behaviour that may indicate that the content has been learned must be identified and kept in mind during teaching and for any evaluation. Finally, a sequence of activities, as in the diagram below, should be moved through.



There must be a balance of activities that develop the ideas, consolidate the ideas and apply the ideas. Consolidation (and development) should ensure that the new ideas are integrated into the children's existing knowledge to form conceptual schemas - that the new ideas are not retained as isolated pigeonhold pieces of information but interconnected into the existing knowledge. Development should have exploration as its starting point. It should be active not imitative teaching, e.g.,

Teach Actively	Not Imitatively
1) Children explore ↓	1) Teacher describes (and "explains") the idea ↓
2) Children record and analyse results, looking for pattern ↓	2) Teacher demonstrates the idea or the procedures that use the idea ↓
3) Children "discover" ideas/procedures ↓	3) Children memorize the idea/procedures ↓
4) Children practice these ideas and the procedures that depend on them	4) Children work similar examples using the idea/procedures

Development activities will therefore consist of:

a) looking around and observing the environment and describing what is seen;

b) exploring shapes etc. (and experimenting) in order to gain insights into properties and uses; and

c) informally analysing shape, size and position in order to make inferences from which to refine and extend knowledge, an analysing which involves dissection, construction and recognizing patterns.

It is important that activities be given that go from model to language and language to model and that both examples and non examples of a concept be given. Furthermore it is better if children experience many embodiments (examples using different materials or showing different types) of a concept.

CHAPTER ONE: BEGINNINGS

In this chapter we look at the starting points of geometry. In unit one we discuss children's early experiences with space and shape: with movement through space, with identifying shapes (both 2-D and 3-D) and with intuitive understandings of symmetry and perspective. In unit two we discuss how the primitive concepts of proximity, separation, spatial order and enclosure and the early concepts of boundary and turn (and the associated concepts of line and direction) may be developed. In unit three we have a look at the transformational approach to these beginnings, seeing how change may be used with young children to develop a basic grounding in spatial ideas.

UNIT 1: SPACE AND SHAPE

Focus:
Shape is all around us - regular and irregular. As young children explore their world, they need to look at and feel shapes: the straight edge, the flat surface, the rounded sides, etc. They can experiment with how shapes behave: rounded shapes roll, flat surfaces stand still, etc. They can try to fit shapes together, or side by side, etc.

Space is all around us - even though it is easy to forget it. Children use shape: they play with balls, kites, balloons, bubbles, etc. They run around and move their arms. They are most aware of space when they do not have enough. Children squashed together for story time soon complain that there's "not enough room".

This unit focusses on how we can use childrens curiosity about the shapes around them and the space they move in to begin geometry.

Background:
The early work of geometry is intertwined with language, drama, art and play. By constructing things, moving through and in equipment, playing in playgrounds, dancing and painting, children learn the meanings of such words as near, far, under, above, inside, high, steep, etc.

Position:
Children experience too much and too little space and they can experiment with their bodies making themselves "as small as possible" or "able to fit through the window". With large equipment (swings, monkey bars, etc) and games of position ("I'm the king of the castle"), children can develop appropriate vocabularies for position. They can throw or roll balls. They can run and hop and jump. They can imitate (e.g., "simon says") or give and follow instructions (e.g., "blind man's buff", "pin the tail on the donkey").

Of particular importance in more formal work on position is the giving of instructions (e.g., "how do we get to your place"). A doll can be hidden and one child has to tell the others how to find it. The children have to learn how to use stable items in the environment as points of reference. Children can be asked to describe or draw their home.

Shape:
Children begin to abstract three dimensional shapes by seeing, handling and using them. We must be aware that children may not see shape as we intend. For example, a ball being given the role of "spaceship" in a game may not be being seen as a sphere.

The attention of the child should be focussed on the properties of the shapes he handles but it is not always appropriate to ask "what shape is this".

Two dimensional shape is also abstracted from commonality seen in day to day life. Children notice things that "are the same". Often it is not strictly true that the things are the same, but this should not be pedantically pointed out and the looking for similarities must be encouraged. Covering surfaces, printing, constructing solids from surfaces and cutting solids like boxes open are also good activities. Sorting shapes and discussing the sets formed will help develop vocabulary and help the children learn to identify the shapes. Making patterns with shapes will reinforce them. This can lead to tiling and tessellations.

Line Symmetry:
Ideas of symmetry and axes of symmetry might be developed through experience of folding, pattern making and play with mirrors. Patterns can be made by cutting folded paper and discussing "ink blot" type pictures.

Perspective:
Perceiving similarity and difference can also involve seeing things from different perspectives. When sitting around a table it is often difficult to match when something is upside down. A car or house is different when viewed from the side or back.

Get the children to look and draw things from strange positions (e.g., standing on chairs or when they are upside down). Try to get children to imagine - how would this look to a giant? or a worm?

<u>Materials:</u>
Paper, pen, scissors, tape, cardboard, paint or ink, blindfolds.

<u>Activities:</u>

1. A list of equipment that might be used for movement and position is:

   - climbing fences, monkey bars, the natural environment
   - boxes, chairs, tables, corners, blocks
   - balls, bats, skittles, balloons, aeroplanes, kites
   - dolls house, trains, toy cars, lego

   When the equipment is available children can:

   - set out and tidy up the equipment
   - organise it to enclose themselves or a doll
   - play in it or with it
   - play group, ring or dance games (e.g., "ring a' ring o' roses", "what's the time Mr. Wolf", etc)
   - dance to music through it and with it.

   The language that might be developed includes:

   - up, down, over, between, alone
   - straight, curve, high, through
   - further than, nearer, across, beside, behind
   - run, stop, jump up, sit down
   - backwards, side ways, twisted
   - quick, slow.

(1) Can you brainstorm up more equipment, child activities, language that would usefully be part of early geometry?

(2) Can you describe a rich outside environment for early geometry work - what would it have? Sand pit? Containers? Water? What would you do in it?

(3) Blindfold one of your group. Then guide her or him from a starting to an end point in the room (or outside) by verbal directions only. Be careful, have someone holding her/his hand - its not as easy as it seems.

2. (1) Children can learn about three dimensional shapes by seeing:

a) how they behave (e.g., building blocks);
b) properties (e.g., sorting, feely bag);
c) matching shapes;
d) fitting shapes together;
e) covering shapes;
f) using space inside shapes (e.g., sand tray); and
g) altering shapes (e.g., plasticine).

Brainstorm up a corner of a room that would be able to satisfy these learning experiences. What materials would you have to have? For example, water, bottles, buckets, clothes, animals, junk bag, wrapping paper, boxes, tracks, sand, clay, rubber material, playdough, etc.? How would you organise the corner?

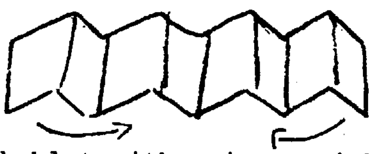
(2) Children can learn about two dimensional shapes in a similar manner to 3-D shapes with the same activities plus other special activities of matching shapes, covering surfaces, printing and taking rubbings and shadow puppets.

Can you brainstorm some ways we could get 2-D shape from 3-D shape (e.g., blowing paint over hands placed on paper)?

3. (1) Find 5 items inside the room and 5 items outside the room that exhibit symmetry. Check they are symmetrical by drawing them, cutting out the drawing and seeing if it folds in half.

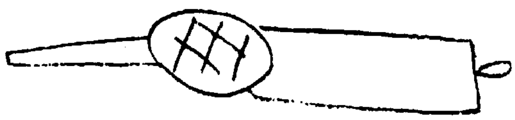
(2) Fold a piece of paper in half, then half again and half again. Then cut off corners, cut out holes in the centre, cut shapes out of sides. Open out the paper, discuss the result.

(3) Repeat (ii) above for a long sheet of paper. Concertina the paper before folding, e.g.



(4) Make an ink blot with paint or ink.

4. (1) Alan said this drawing was an elephant.



From what direction was the elephant drawn? What could the artist have been in?

(2) Can you do 3 drawings of common things from strange (?) perspectives?

(3) Rotating shapes, using mobiles, placing things upside down are all examples of activities designed to widen children's perspective. Can you think of anymore?

5. (1) Draw a floor plan of the house or unit in which you live from memory.

(2) Draw a plan of Carseldine Campus form memory.

(3) What are the difficulties for you in doing these plans? What extra difficulties will young children have?

Teaching Hints:
The basis of teaching geometry in early years is the imagination of the teacher and the curiosity of the children. The teacher has to set up a rich environment (and this must be done with resources and methodology) and to use the interests of the children - to build on their comments and questions.

Teachers have to use the play of the children. Satisfy the children's desire to play inside something. Use tea chests, crates, boxes, barrells, planks, tyres, rope, etc. Allow children to build with blocks. When construction occurs, children, count, compare and match. They find which surfaces fit best, they enclose area and make boundaries. Packing away gives experience of limited space.

When describing shape with children restrict description to the more common names but do try to relate surfaces to solids and to use names such as rectangle, circle, flat, straight, curved, etc.

Integrate geometry with measurement. The words big, bigger than, small, etc. must be questioned and replaced if possible, by more meaningful statements about length, breadth, width, etc.

Use children's drawings. This is particularly important. They show their grasp of size, shape, position, perspective and symmetry. Analyse them geometrically as well as artistically. Make children aware of shapes around them. Use junk, collage and model making. Develop words like curved, vertical, diagonal, round, square, etc. Encourage children to make patterns. Discuss symmetry.

Because our links are jointed we can bend them to fit in various shapes. Use the children's bodies. Sit on chairs, bend, hold arms, squash together, spread out, etc. Arrange objects, alter chairs and tables, use a dolls corner and cars and trains. Get children to stretch to high shelves and bend down to low. Make and suspend mobiles.

Use the natural world. Grow flowers and other plants. Look at tadpoles and frogs, fish, etc. Fit packages inside each other. Allow the children freedom to explore.

Discuss similarities and differences - this is the basis of learning. Do not just label. Discuss "what would happen if ..." (rolled it, stood it on this slope,tried to stand it upside down, etc). "What if" questions are crucial.

Copy pictures - paint and draw. Look at shapes from a variety of perspectives. Encourage children to look for shape in the street, at home, etc. Use variety. Discuss with children what they have done and what they see.

UNIT 2: PRIMITIVE CONCEPTS

Focus:
At the basis of geometry are the primitive notions of proximity, separation, spatial order and enclosure. Identifying when we are near other things, the order in which these things (including ourselves) are, and when these things prevent our movement (enclose us), underlie much of later geometry.

This unit looks at these primitive concepts and at the early concepts of boundary and turn (and line and direction) that emerge from them.

Background:
Early spatial concepts emerge as a result of a child interacting with his/her environment. By constructing things, moving through and in equipment, playing in playgrounds, dancing, painting, etc. children learn the meaning of such words as near, far, under, over, above, inside, outside, high, steep, etc. Piaget has suggested that four particular spatial concepts (he calls them topological concepts) are of special importance:

a) Proximity - the nearness of one object to another. Very young children are only interested in things near them, that they can touch and manipulate. As they grow, children engage in activities that help them recognise that out-of-sight objects exist, and to clarify the distinction between near and far.

b) Separation - Children's drawings often show that they have an incomplete notion of an object consisting of separate parts. For example, young children often draw a body with arms and legs all emerging from a common point, with no body to separate them.

To assist the development of separation, we can use pictures showing a river separating two parts of a town, or a railroad track separating two buildings. Similarly, partitioning sets into subsets involves physically separating the elements of the set.

c) Spatial Order - (Sometimes called seriation). Children gradually develop the ability to order a variety of objects, beginning with obvious criteria for solid objects, e.g., length or size, and proceeding to more abstract criteria such as age or number. The ability to reverse an order comes rather more slowly.

d) Enclosure - Toys and pictures can be used to develop children's understanding of enclosure. A rural scene that uses toy plastic fences, horses, cows, barns, and so on, can be used to show animals and buildings enclosed by a fence.

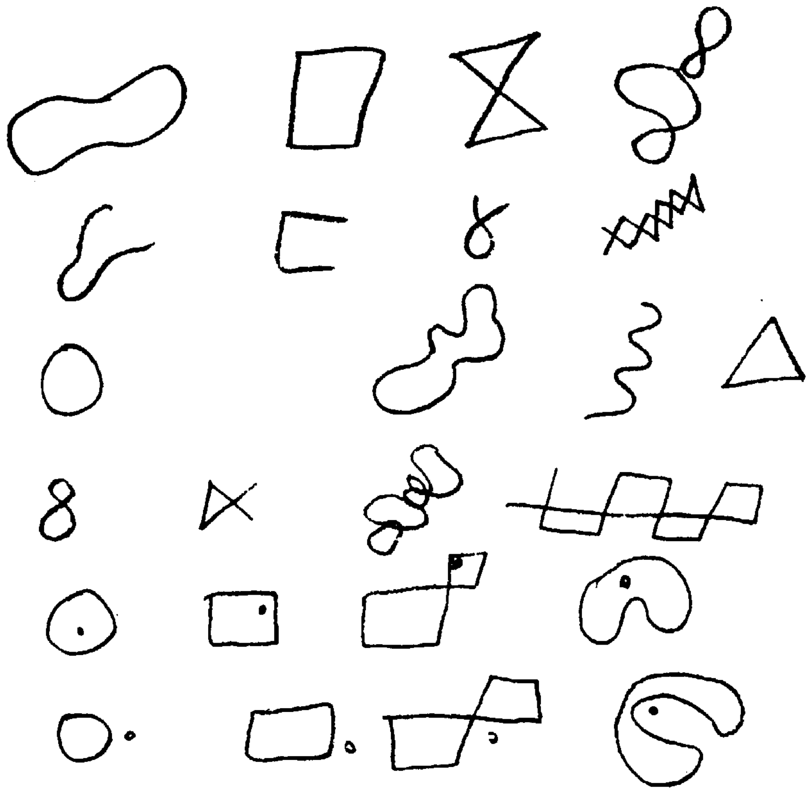
The special instance of a person being between two other people is also an example of enclosure. Obviously, objectives and people can be enclosed in a three-dimensional sense as well, as those with claustrophobic tendencies will confirm.

These basic concepts underlie all geometry. Of special interest are two early geometry concepts which emerge from them and which are the basis for shape: boundary and turn. From boundary comes the notions of surface, edge and line and so the concepts of three and two dimensional shapes. From turn comes direction and angle and so coordinates and a deeper understanding of shape.

Boundary:
One of the most fundamental concepts of geometry is that of boundary - something that separates one region from another. All three and two dimensional shapes are boundaries; for example, a triangle is a closed boundary composed of three straight lines.

A first contact with 'boundary' can come through consideration of things that enclose space. Good examples of this are, the classroom, a car, a box, a cupboard, the child's skin, a bucket, etc. The children can construct "something that encloses space" using paper, cardboard, glue, scissors, etc., or junk. Don't forget the two dimensional examples. Lines can be drawn on the floor or playground to divide it into regions. Games can be played where children enclose space (and other children) by holding hands.

In conjunction with boundary the concepts of closed/open, inside/outside and simple/non simple can be developed. Initially act these out with the children - draw large diagrams on the floor and walk them, have the children hold hands, open and close the door of the classroom, crawl in and out of boxes, etc. Later use drawings on paper:



closed
(can't get out of any section)

open
(can get out of one section)

simple
(does not cross)

non simple
(boundary does cross itself)

inside

outside

(inside and outside refer only to closed boundaries or the closed sections of boundaries)

Once boundary (both two and three dimensionally) has been developed, then the different types of boundary can be looked at, e.g., straight line, curve, surface, corner (vertice), edge, and the different ways of bounding something can be looked at.

Turn:
Many things in the everyday world turn: wheels, door handles, drills, records, etc. Children can turn themselves: quarter, half, three quarter and full turns. To walk without turning is to walk in a given direction, in a straight line. Standing without turning is facing a direction.

Direction can be equated to straightness - to a straight line that emerges from a point (called a ray). The amount something turns, i.e., the measure of the change in direction, gives <u>angle</u>.

<u>Materials</u>:
Paper, cardboard, glue, tape, scissors.

<u>Activities</u>:

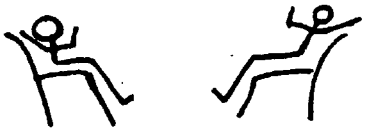
1. Childrens movement can be a valuable way to introduce geometrical concepts. Games such as "cat and mouse" (where a ring of children let through the 'mouse' and join hands to stop the 'cat') lead onto notions of closed and open boundaries. 'Mr. Here' type games (where a doll is hidden and children look for him using instructions, such as "He is near the door, above the table, ..."), teach the meaning of geometric words. Children can stand up <u>straight</u> or bend over at an <u>angle</u>. Children can hold their hands in front of them as below:

   and walk forward, without turning (a straight line). Curves can be drawn on the floor or playground and 'walked' in this way (so the difference between curved and straight can be seen). Polygons can be walked in this way too (so the difference between them and, say, circles can be experienced - polygons do all their 'turning' at points).

   When faced with the angle at the corner of a polygon, children can move one hand to the new direction (leaving the other in the old), turn their body then finally move the second hand to the new direction. This emphasises the 'turning'. Two children can walk forward without turning to experience parallel lines.

   One child can sit in a chair and strike a pose. Another child can sit facing her and act like a <u>mirror</u>.

   (Junk can be piled up to do this as well.)

   Symmetry can be shown by: 

   and non symmetry by: 

   With thinking, I am sure we can all find many ways to use body movement, dance, P.E., games, etc., to give the child experience of such concepts as -

   boundary, closed, open, inside, outside, order, straight line, curve, symmetry, angle (turning etc).

   Write down one activity for each of these concepts.

2. Bounding your body

(1) The ceilings, floor and walls of this room enclose space. Your skin encloses space (well almost!). Write down six more ways of enclosing space.

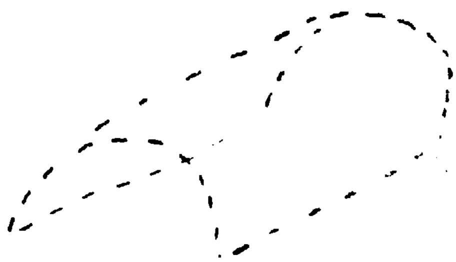
(2) Stand up. You are allowed to move in any direction but you must always be 2 metres from your first position. You are not allowed to jump or burrow into the ground. You are not allowed to bend over. You must keep your hands by your side. You are then enclosed in a space by:

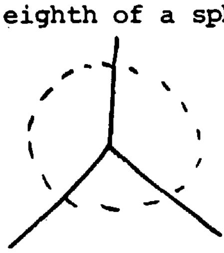
(i) the floor
(ii) an invisible ceiling of your height
(iii) an invisible wall 2 metres from your starting point (in all directions).

What is the shape of this region? Yes! It is a cylinder.

Can you 'imprison' yourself in a tunnel? In an eighth of a sphere?

e.g.





Make up your own set of rules to 'imprison' yourself in your own shape.

(3) Suggest ways to use the body and its movement to reinforce or develop basic geometric concepts, e.g., line, angle, open, closed, symmetric, etc.

List both the concept and the activity.

(4) Surfaces separate space: some are flat (planes), some are curved:

(i) Write down six plane surfaces and six curved surfaces that you can see.

(ii) Use the paper, scissors, glue and sticky-tape to enclose space with a shape of your own choice. Make it interesting and/or different.

(iii) Make a box to enclose space.

(iv) Enclose space using four plane surfaces only. (Note: Things that enclose space or separate one part of space from another are called boundaries of space.)

(5) Look again at your box.

(i) How many plane surfaces does it have? Plane surfaces such as these are called faces. Look at a face. Trace around its boundary with your fingers.

(ii) How many straight parts are there? These are called edges. The four edges make up the boundary of this face. Other surfaces have boundaries too.

(iii) List four other boundaries of surfaces you can see in this room.

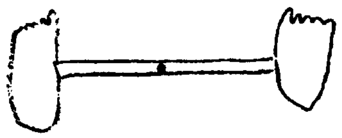
(6) Is there any other way to separate space or parts of surfaces (regions) other than boundaries? Can we have boundaries on a paper by using a piece of string?

3. (1) Make up a list of things that turn.

(2) The quarter turn can be introduced by activities with the child's own body or a clock face. Road signs highlight quarter turns. Road maps can be followed, turning <u>left</u> or <u>right.</u> Directions can be given for other children to follow. Children can describe where they are, e.g., "John is on my right ...".

Make up some situations and questions to highlight the half and full turn.

(3) Clockwise and anticlockwise.
This can be introduced by using cut outs of shapes such as



The children cut out a copy and turn it around the dot. They can draw the resulting shape after each quarter turn. Design a series of instructions and questions using this shape that would enable children to experience clockwise and anti-clockwise and develop meaning for the language.

Would it help if the model being used was a clockface?

<u>Teaching Hints</u>:

It is important that activities, questions and explanations be such that there is a focus on intuitive understanding or an understanding that the child has in his/her own words. We have used many geometric words in this (and other units). They are for you the teacher not necessarily the children. The important thing is that the children can express the concept in their own words - that you know they have an adequate understanding of what lies behind the concepts and processes.

With these primitive and early concepts children will be strongly influenced by their own limited experience. For example in developing proximity notions such as "near to", a child may believe that Grandfather is close (because he knows the journey) while the dentist is far away because (the journey is unfamiliar - though shorter). The class room or playground gives a good basis for discussion as does a model of the local suburb.

Spatial order can also be developed from such experiences. Children should know the order they pass land marks on the way to school. Lining up and putting materials away will also support this. More formal activities of copying sequences will reinforce that there is an order along a line, curve, or route. Modelling and playing in doll houses and model villages (with cars and trains) will also supply good experience of spatial order.

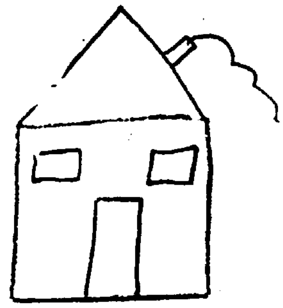
Lines maybe difficult for children, particularly straight lines. Once again lining up and tidying away will help, as will arranging furniture and discussion of how posts and houses are often in rows. So too will such activities as gardening and running races.

UNIT 3: CHANGE AND TRANSFORMATION.

Focus:
Piaget has hypothesised that childrens spatial development develops through the transformational geometrics (geometries of change): topology ⟶ projective ⟶ euclidean (see overview).

In this unit we look at how these three types of geometric activities can form the basis of the beginnings of geometry instruction.

Background:
Developmental research has indicated that, in the early years, children may not be ready for concepts such as straightness and length. Hence the traditional geometry of Euclid, which is based on straight lines and length, may be difficult for them. Hence strong arguments have arisen for involving other geometrics in the early years, particularly the transformational geometries identified by Piaget. Below Helen Mansfield of W.A.C.A.E. gives reasons why.



Have you ever watched small children draw a house? The windows are usually well above the door, but the door is almost always exactly in the centre between the windows. There on the roof is the inevitable chimney, not vertical, but perpendicular to the roof, with smoke curling down towards the ground. Children perceive the world in exactly the same way as we do, and yet their drawings show that their understanding of the physical properties of objects in space is not the same as ours. How does the child's conception of the properties of space develop?

Learning is an active process. The child learns about the space around him by moving around in it. He learns about the physical properties of objects in space by handling them. Some of the earliest spatial concepts developed by the young child are to do with position. He crawls into the cupboards, and under the couch, pulls himself onto a bed, plays peek-a-boo from behind his fingers and pushes his teddy through an open window. Through these activities he is developing mathematical concepts such as enclosure, separation and continuity. These properties of space are the concern of that branch of geometry which we call topology. No matter whether the window is wide open or just a little open, it still bounds a region through which the teddy can be pushed.

As the child comes to learn more about his spatial environment, he gradually acquires new skills which will enable him to co-ordinate his understanding of shapes into new schemes. Given a simple model of one or two houses, a church and a tree, the young child believes that everyone looking at the model will see exactly the same as he does. He is unable to imagine the same model from a point of view other than his own. When, at about the age of eight, he is able to reconcile other points of view, he has made a significant advance in his understanding of the space around him. An object is no longer in isolation, but is thought of in relation to other subjects in space, and space comes to be thought of as having a general system of organization. The concepts of projective geometry such as straightness and parallelism, are now within the grasp of the child.

All teachers are familiar with the problems that children encounter in conserving the properties of objects as they undergo various transformations. The acquisition by the child of the ability to conserve does not take place at the same time in all areas. For example, the child may conserve number at about 5 years old, but not conserve area until he is 7 or 8, or volume until he is 11. When he is about 7, he may be able to conserve length. He will recognise that if one of two sticks of equal length is pushed out of alignment, then both sticks are still of equal length. The ability to recognise that no matter how the parts of a shape are rearranged, or how a shape is moved around in space, its length or area or volume remain fixed is crucial to the child's understanding of Euclidean geometry. In Euclidean geometry, the measure of properties such as line segments, angles and areas is of central concern, but measurement skills do not develop in the child until his middle primary years. When he is able to bring conservation and measurement skills to his understanding of the world around his the child has made another very significant advance. He can recognise the organisation of the space around him, and he is now in a position to impose his own organisational structure on that space. He develops the ability to see space with reference to systems such as horizontal and vertical axes. The mathematical concepts of Euclidean geometry can now be understood.

At this stage of his development, the child is now ready to look beyond the properties of individual shapes, and to move from concrete situations to a more abstract representation of the properties of space. He is now ready to describe the properties and positions of shapes with reference to an abstract system such as co-ordinates. As well, he is now able to examine the properties of that abstract system in its own right. Similarly, he can move beyond a concrete understanding of transformations on shapes, and look at the abstract structure underlying those transformations. He can understand concepts such as reversibility, recognising that a transformation can be reversed by its inverse transformation (i.e., that a change can be undone by the opposite change).

One of the most fruitful ways in which we can learn the development of the child's conception of the space around him is to observe his representation of space in his drawings. Let us return to the child's drawing of a house. Very young children may make a solid scribble for a house. They are able to represent the ideas that a house occupies a particular space, separate from its surrounds, and that is about all. As the child develops into his preschool year, his house will show the child's increasing awareness of properties which are the concern of topology. Windows and doors will be drawn within the boundary. The door will be between the windows and so on. At this stage however, the house will be out of proportion. The windows will be above the door and the flower growing next to the house may be taller. The child has not yet grasped the central notion of projective geometry, though, for the car next to the house will have four wheels shown, even though it is drawn side on to the child, simply because the child knows that cars have four wheels. Later on, when the child is able to see space within a stable general framework of horizontal and vertical axes, the chimney will no longer be perpendicular to the roof, but will be perpendicular to the ground.

Some of the evidence we have suggests that children's understanding of the properties of space develops from topological concepts through projective to Euclidean concepts, but while the evidence is tantalizing, it is by no means conclusive. Even quite young preschool children, for example, will draw the windows of their house as squares of about *the same size, yet congruent angles and shapes are the concern of Euclidean geometry. Perhaps this confusion stems partly from the fact that in schools and in society as a whole we have tried to hurry children into an understanding of Euclidean concepts and have given scant regard to topological or projective ideas, leaving these to chance to develop.*

While it is not entirely clear how children develop spatial concepts, it is clear that a very much greater range of different geometrical concepts can and should be presented to young children.

The activities that follow are divided between the different topologies under the headings of:

(1) geometry without straightness or length;
(2) geometry with straightness but not length; and
(3) geometry with straightness and length.

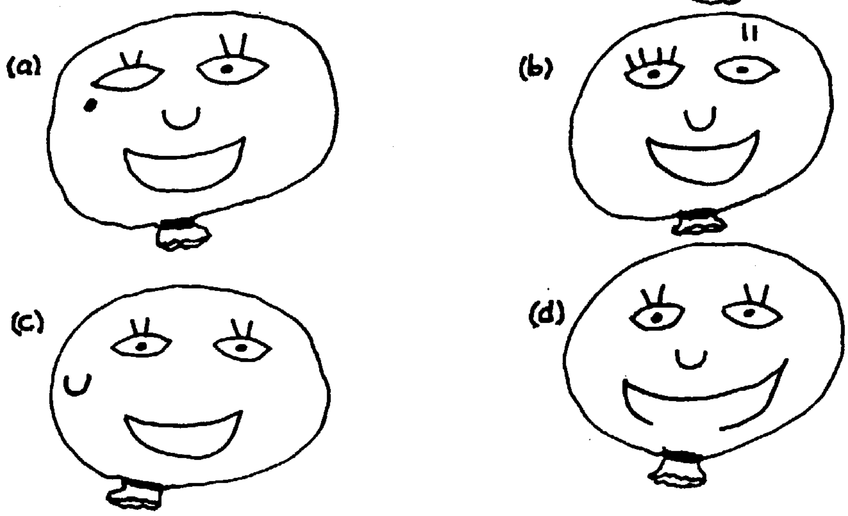
It is hoped that the suggestions will supply impetus to teachers to present additional geometry ideas to their classes.

Materials:
Balloons (two types - round and long and thin), felt pens, rubber bands, slide projector, torch or candle, screen (moveable), paper, cardboard, scissors, tracing paper.

Activities:
1. No straightness or length.
In the past, little attention has been paid to this geometry type even though children have some experience of it.

(1) Draw a face on the round balloon.
Blow the balloon up.
Bend and distort the face.
What faces can we make? Draw them.

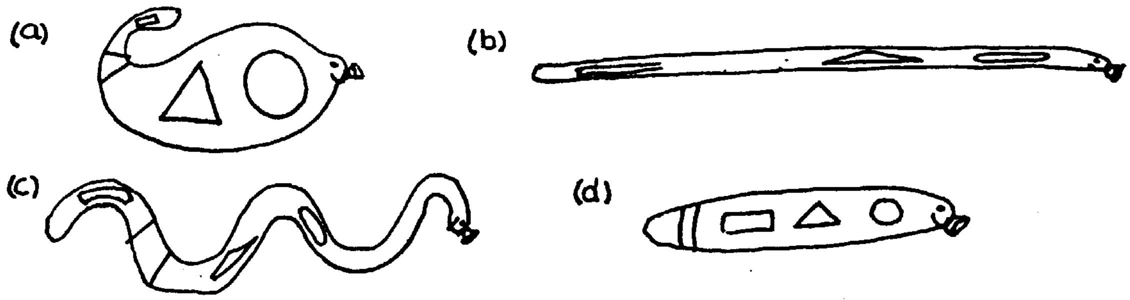
(2) Can we make the following faces?



(3) Can you make the faces bigger? Smaller?

(4) Draw a snake design on the long thin balloon.
Blow the balloon up.
Twist and distort the balloon.
Draw what you get.

(5) Can you make the following?



(6) What do the results in (2) and (4) above mean about our ability to recognise

(a) a friend that we have not seen for years?
(b) a car after it has been in a smash?
(c) scones after they have been cooked?

Why are we able to do recognise such things?

(7) Without actually doing it, could you list the letters (upper case, capitals) that you could make from

(a) a whole rubber band? 

(b) a cut rubber band? 

Get two rubber bands and check your ideas.

2. Straightness, no length.
This is the geometry of seeing and of shadows.

(1) Go out into the sunlight. Make your shadow move. Make it jump. Make it as long as possible. Make it as fat as possible. Trace around your partners shadow on a piece of butchers paper. Label and pin up in the maths room.

(2) Get into a group. Move around in the sunlight. Don't "bump" each others shadow. Can you do it? Now play "catch" with your shadows.

(3) Set up a slide projector (or torch or candle) to shine on a screen. Cover the screen with paper. Cut out a shape (from cardboard) like this - with a sharp point, or angle, and 2 holes.

(4) Cast shadows with your shape on the paper on the screen. Tilt the shape - tilt the screen. Make interesting shadows. Draw them on the paper covering the screen.

(5) Can you make the shadow bigger than the shape. Smaller?

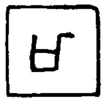
(6) Can you make a shadow where

(a) the sharp angle is a right angle?
(b) the sharp angle is an obtuse or blunt angle?
(c) there are more than 2 holes?
(d) there are less than 2 holes?
(e) there are a different number of sides than 4?

(7) As a result of these experiences what can we believe from our eyes. Where can we be fooled?

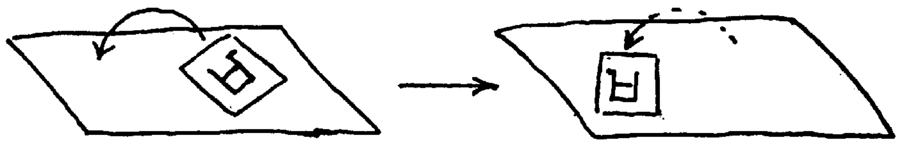
3. Straightness and length
This is the most familiar geometry to adults.

(1) Draw the shape below on a piece of tracing paper.



(2) Place this tracing paper over other paper.
Then slide, turn and flip (see below) the shape on top of the paper.

This is a flip - can be in any direction.



What shapes can you make. Draw them.

(3) Can you make a bigger shape? Smaller?

(4) Can you make the following?

(5) What does this say about

(a) how we should drive a car?
(b) how we should carry a desk through a door?

Teaching Hints:
The primary focus here is to extend ideas of geometry from shapes and names to what happens when things change. We have to remember that

(1) Children can learn to anticipate that certain things will happen in certain ways (and that some things will never happen);

(2) It is not necessary to use words to describe what happens precisely (imprecise words, pictures and motioning with hands is sufficient); and

(3) Learning by doing and discussion and settling differences by demonstration builds confidence and self reliance.

The idea is that we should assist children to

(1) Develop self reliance (children checking their own conclusions);

(2) acquire the ability to predict what will happen; and

(3) develop an expectation that mathematics makes sense.

Working with materials, discussing outcomes and manipulating pictures will achieve these ends, ends that it must be emphasised are long range.

CHAPTER TWO: THREE DIMENSIONAL SHAPE

In this chapter we look at how three dimensional or solid shape can be taught It is the first area (other than beginnings) that this book covers because of its primacy in the early years. Nevertheless, it should be read in conjunction with chapter three on two dimensional shape because a full understanding of solids can only be acquired in conjunction with an understanding of plane shape.

In unit 4 we will reacquaint ourselves with the various types of solids that make up the study of three dimensional shape. Unit 5 will look at how children can be instructed so as to enhance concepts of and skill with three dimensional shapes. Unit 6 covers a wide range of methods of constructions. Unit 7 looks at properties of three dimensional shapes and how knowledge of these might be developed in children.

UNIT 4: IDENTIFYING SOLID SHAPES

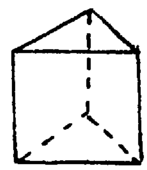
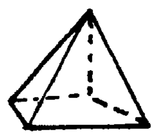
Focus:
Solid shapes on three dimensional shapes have three dimensions - length, breadth and height (or thickness). 3-D is just an abbreviation of three dimensional. The world around us is full of solid shapes - people, animals trees, buildings, toys, etc. In fact, almost anything in the world has three dimensions. Even a leaf or a piece of paper has three dimensions - although the third dimension (thickness) is very small.

This unit describes the various types of three dimensional or solid shapes covered in the primary years.

Background:
Formally a solid or 3-D shape is something that encloses a portion of space. It may or may not be filled (or "solid"). For example, a match box, whether, it is filled or not, is a solid shape.

True solid shapes must be completely closed. That is, they must have a "top", a "bottom" and "sides". (These words have been placed inside inverted commas because they are not the proper words to use). We shall now develop the correct terms.

Solid shapes can be divided into two types: polyhedra (which have flat "sides" and straight line edges, i.e., they have polygons for surfaces) and non polyhedra (which have surved survaces). Important examples of polyhedra are:

prisms  and pyramids 

and important examples of non polyhedra are:

cylinders  cones  and spheres 

The major terms with regard to these solid shapes are:

SURFACE: All solid shapes are enclosed by surfaces which are either *curved* or *flat*.

FACE: A *flat* surface of a solid shape.

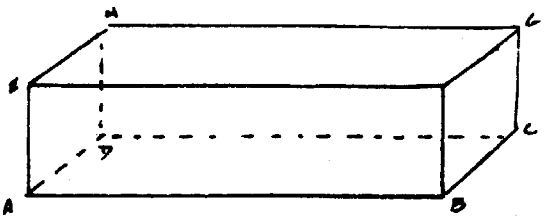
BASE: A *face* on which the solid shape rests. (A sphere has no base but all prisms and cylinders have two bases. Confusing, isn't it?)

EDGE: An edge is formed on a solid shape when one surface meets another. An edge may be *curved* or *straight*.

VERTEX: A special *point* on a solid shape when three or more *straight edges* meet. (The cone also has a vertex but this is a special case.)

Note that the plural of *vertex* is *vertices*.

In a *solid shape*, the *vertex* is the *point* usually formed by the intersection of at least *three edges* of the solid shape.



Point A is the vertex formed when $\overline{EA}$, $\overline{DA}$ and $\overline{BA}$ meet. Similarly, points B, C, D, E, F, G and H are all vertices. Therefore the term VERTEX (or VERTICES) is used with both plane and solid shapes.

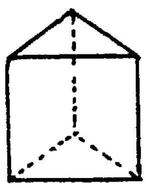
Prisms:

There are many types of prisms because the *base* of a prims is a *polygon* and there is an infinite number of poloygons. EACH PRISM IS NAMED ACCORDING TO THE SHAPE OF ITS BASE.

Each prism is made up of a certain nimber of *flat surfaces* called FACES. The two shaded faces of each of the prisms illustrated below are called BASES. All prisms must have TWO BASES which are PARALLEL and CONGRUENT (i.e., same shape and size). Note that the *bases* of all prisms are in the shape of a *polygon* and that a prism is named according to the SHAPE OF ITS BASE. All the other FACES of a PRISM must be RECTANGULAR or SQUARE in shape.

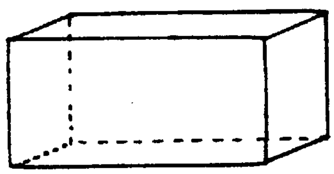
Examples:

(1) Triangular Prism



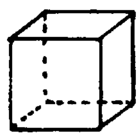
Has 5 FACES. (i.e., 2 bases and 3 other faces) The BASES (shaded) are TRIANGULAR. The other three FACES are RECTANGULAR. Has 9 EDGES and 6 VERTICES.

(2) Rectangular Prism



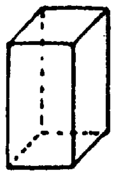
Has 6 FACES. (i.e., 2 bases and 4 other faces). The BASES (shaded) are RECTANGULAR. The other four FACES are RECTANGULAR. Has 12 EDGES. and 8 VERTICES.

(3) Cube

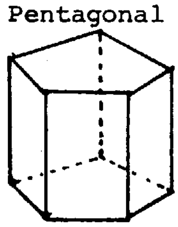


Has 6 FACES (all congruent). (i.e., 2 bases and 4 other faces) The BASES (s are SQUARE. The oth 4 FACES are SQUARE. Has 12 EDGES and 8 VERTICES.

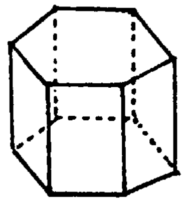
(4) Square Prism



(5) Pentagonal Prism



(6) Hexagonal Prism



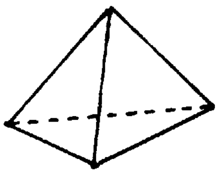
Note: All prisms have at least 5 faces, 2 bases (at each end of the shape), which are congruent poloygons and parallen, and all other faces rectangular.

Pyramids:

Pyramids differ from prisms in that they all have only one base and have all their other faces triangular. One end of them is a point or vertex. Each pyramid is named according to the shape of its base.

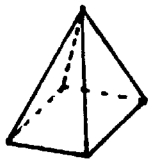
Examples: (all the bases are shaded)

(1) Triangular Pyramid



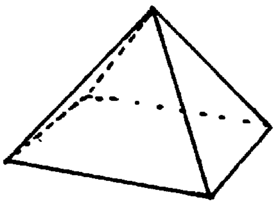
1 base (triangular) and
3 faces (triangular)

(2) Square Pyramid



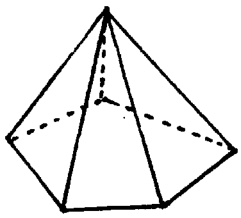
1 base (square) and
4 faces (triangular)

(3) Rectangular Pyramid



1 base (rectangular)
4 faces (triangular)

(4) Pentagonal Pyramid

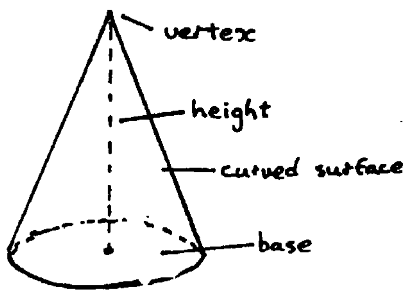


1 base (pentagonal)
5 faces (triangular)

Cones:

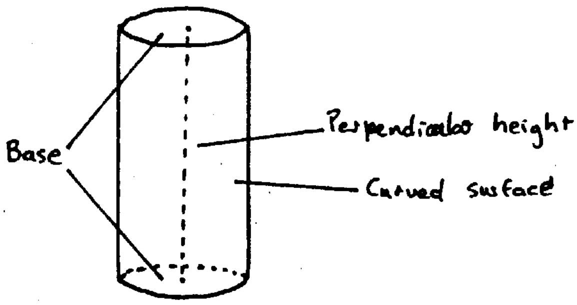
When the number of sides on the base of a pyramid increases without limit until the base finally becomes a "smooth" curve (a circle) without vertices, we have a cone.

A cone has a base which is circular, a pointed top which is called a vertex, a curved surface from circle to vertex and a height which is found by a perpendicular line from vertex to base, as in the diagram below.



Cylinders:

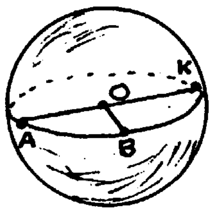
A cylinder is similar to a cone in having bases that are smooth curves. However, the cylinder is related to the prism in the same way the cone is related to the prism. When the sides of a prism increase without limit the base becomes circular and we have a cylinder. A cylinder has two congruent and parallel bases, which are circular, a curved surface joining both bases and a height which is found by drawing a perpendicular from top base to bottom base, as the diagram below shows:



Sphere:

The sphere is the most difficult solid to construct but probably the easiest to identify. Anything shapes like a round ball is a sphere, e.g, a tennis ball, golf ball, cricket ball, etc.

Look at the diagram of a sphere below.

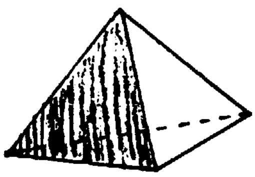


It has a completely curved surface. It has no polygonal base, no edges, no vertices. The CENTRE (O) of the sphere is the same distance from any point on its surface. The line segments, OA and OB, are called the RADII of the sphere, and the line segment, AK, is called a DIAMETER of the sphere.

The Platonic Solids:

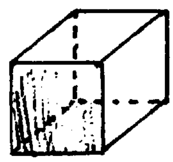
These are polyhedra that are composed of congruent surfaces. There are only five of them and they are:

(a) Tetrahedron (four)



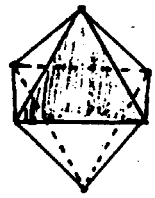
4 equilateral triangles for faces.

(b) Hexahedron (six)



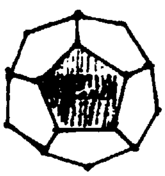
6 squares for faces.

(c) Octahedron (eight)



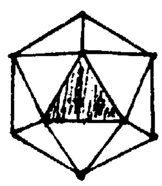
8 equilateral triangles for faces.

(d) Dodecahedron (twelve)



12 regular pentagons for faces.

(e) Icosahedron (twenty)



20 equilateral triangles for faces.

(In each figure above, one face has been shaded. The number of faces is the clue to their names.)

Materials:
Cardboard, scissors, glue, tape, copies of nets on the next 7 pages.

Activities:

1. Use the nets at the end of this unit to construct (in your group) one each of the prism, pyramid, cone and cylinder.

2. Observe and handle each shape and fill in the following table with YES or NO as appropriate.

SOLID SHAPE	ALL FLAT SURFACES	SOME CURVED SURFACES	HAS IT A SHARP POINT	WILL IT ROLL

3. Count the vertices, edges and surfaces of your shapes and place this information on the following table.

SOLID SHAPE	VERTICES	EDGES	SURFACES

4. Find examples in the classroom and outside in the real world of prisms, pyramids, cones and cylinders and list them in the table below.

PRISMS	PYRAMIDS	CONES	CYLINDERS

Which is the most prolific in modern society?

Teaching Hints:

It should be noted that this unit containing the formal definitions of solids has been only given first to acquaint you the teacher with these solids. It is not the intention that such information be given directly to children as the starting point for study of 3-D shapes. This should not be done.

The following units will explain how to teach 3-D shapes. As they will reiterate, formal language should follow intuitive experiences and all rules and general principles should be discovered.

The use of models is a potential pitfall. Many teachers use pictures (often line drawings of geometric figures) in a "show-and-tell" fashion unrelated to real-life situations. Research indicates that children need opportunities to manipulate models, such as regions and their posting holes, to construct models on geoboards and with paper and other materials, and to draw pictures of them, as well as to talk about them. Your geometry lessons should be more than "show-and-tell" activities by including provisions for these kinds of activities, regardless of the grade level at which they are presented.

A potential trouble spot for children new to the study of geometry is the confusion caused by unfamiliar words and terminology. Unless care is taken, they will have trouble understanding the words' meanings and uses. Keep these points in mind as you deal with geometry:

a) Include only essential terminology.

b) Keep the terminology as simple as possible. However, do be accurate in using it. It is all right to say "square corner" with primary children, but the terms "right angle" and "90° angle" should be used with older children.

c) Be sure parts of words are understood. For example, when children study quadrilaterals, they should learn that quadri- means "four", while lateral means "side" (both are from the Latin language); so, quadrilateral means "four-sided figure".

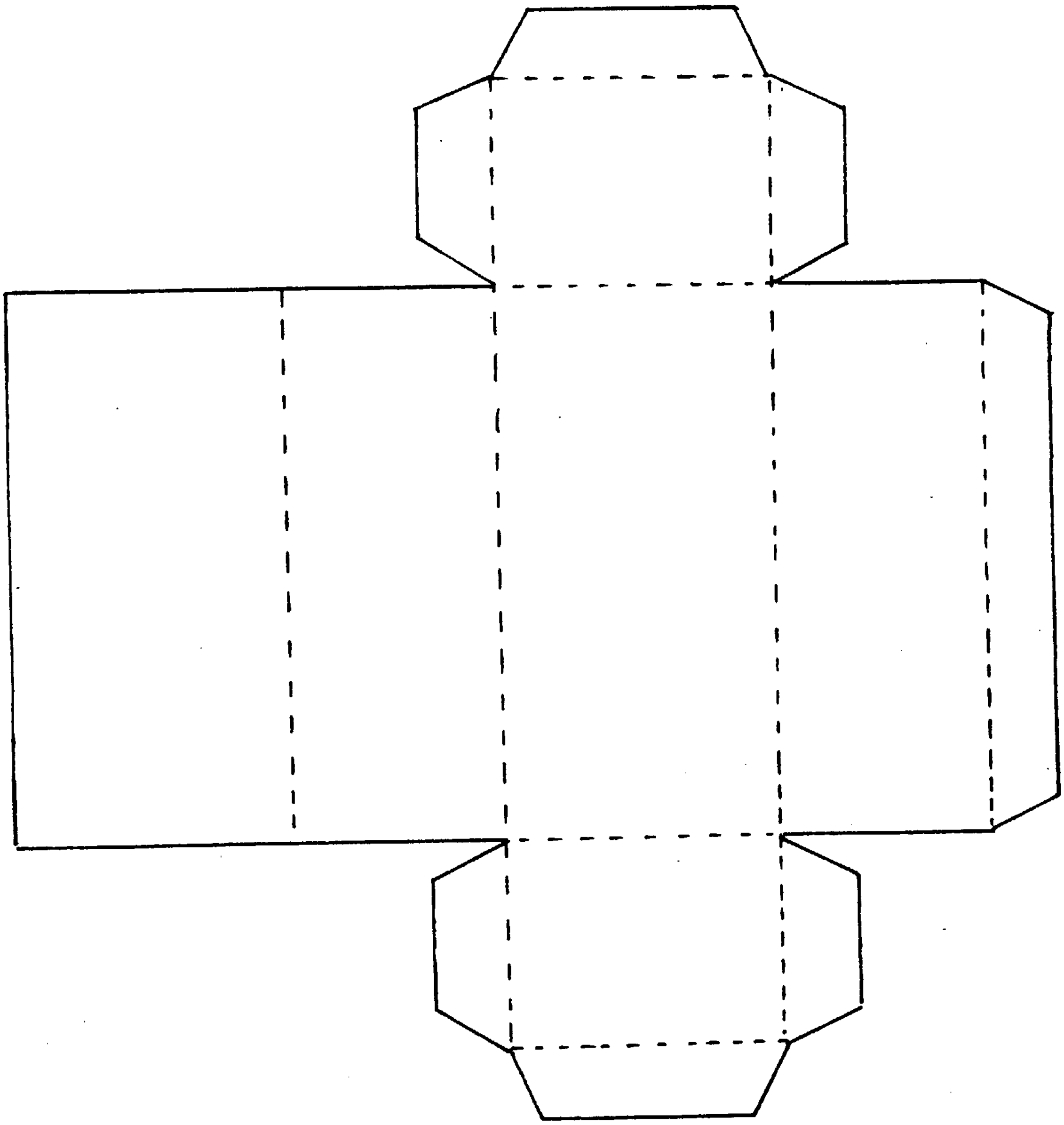
d) Once a word is introduced, use it frequently in discussions and written material. Direct children to write the name themselves (beside a drawing if possible).

e) Do not emphasise terminology at the expense of understanding of the concepts they represent.

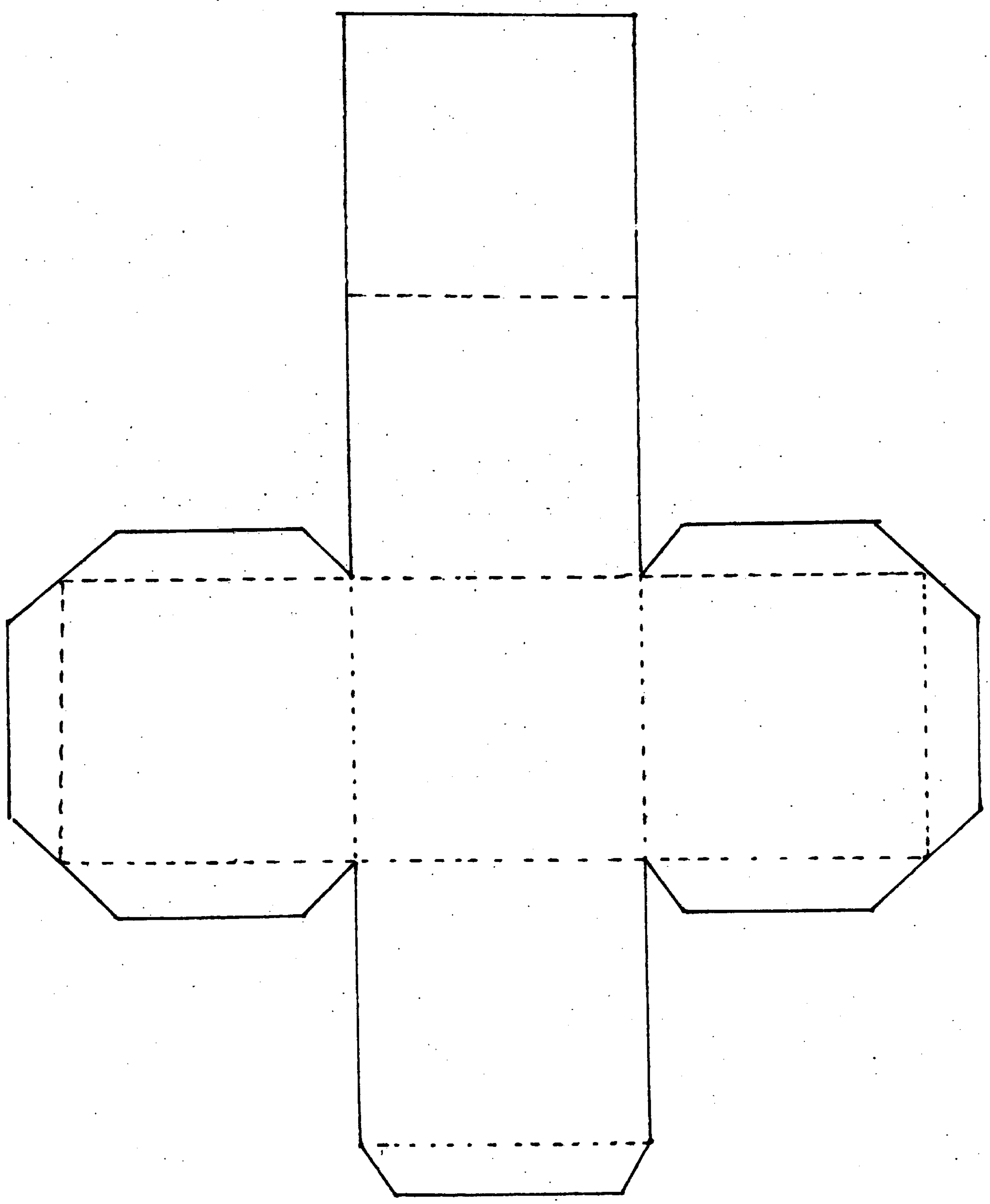
NETS

RECTANGULAR PRISM

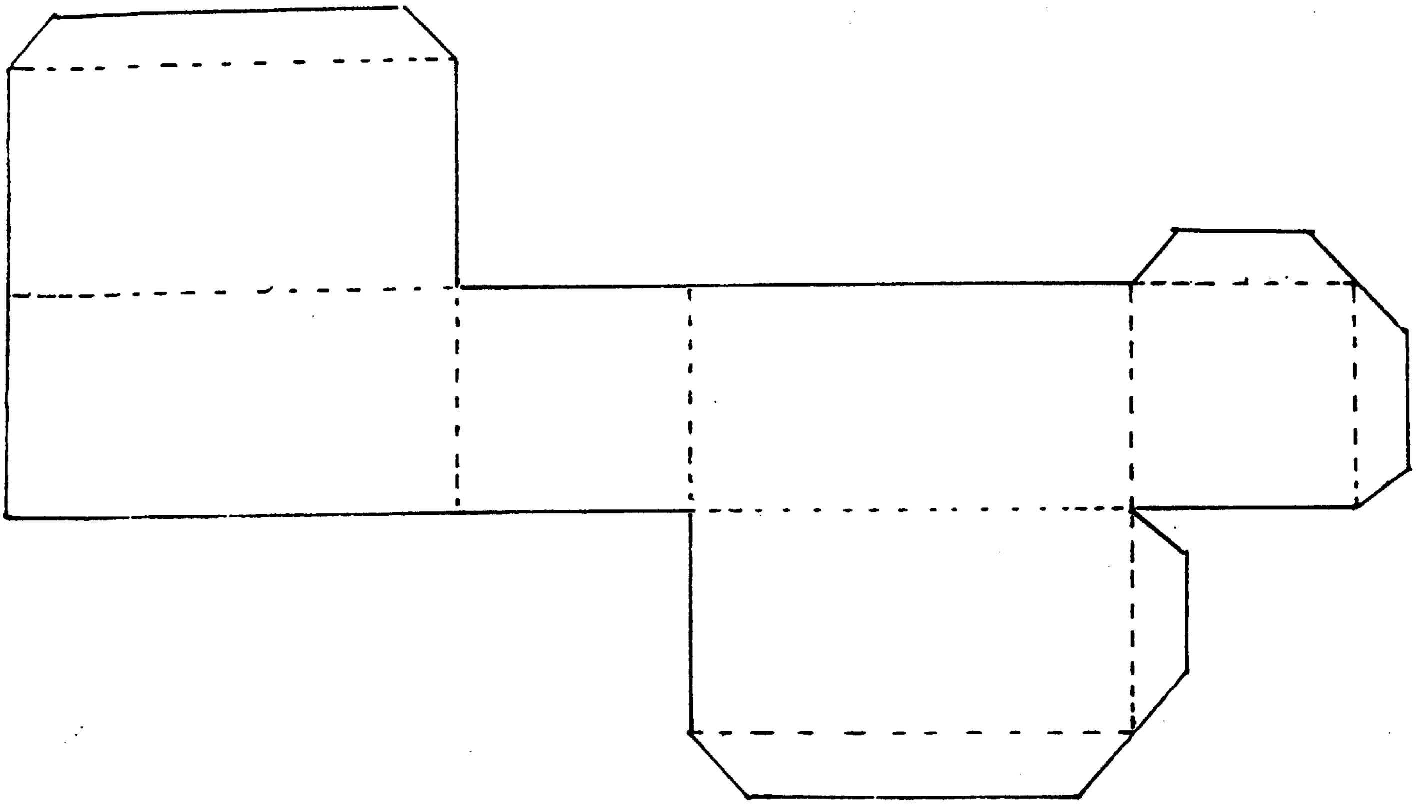
To assemble this net (and others), cut on the unbrocken lines and fold on the broken lines. Place glue on the shaded areas.



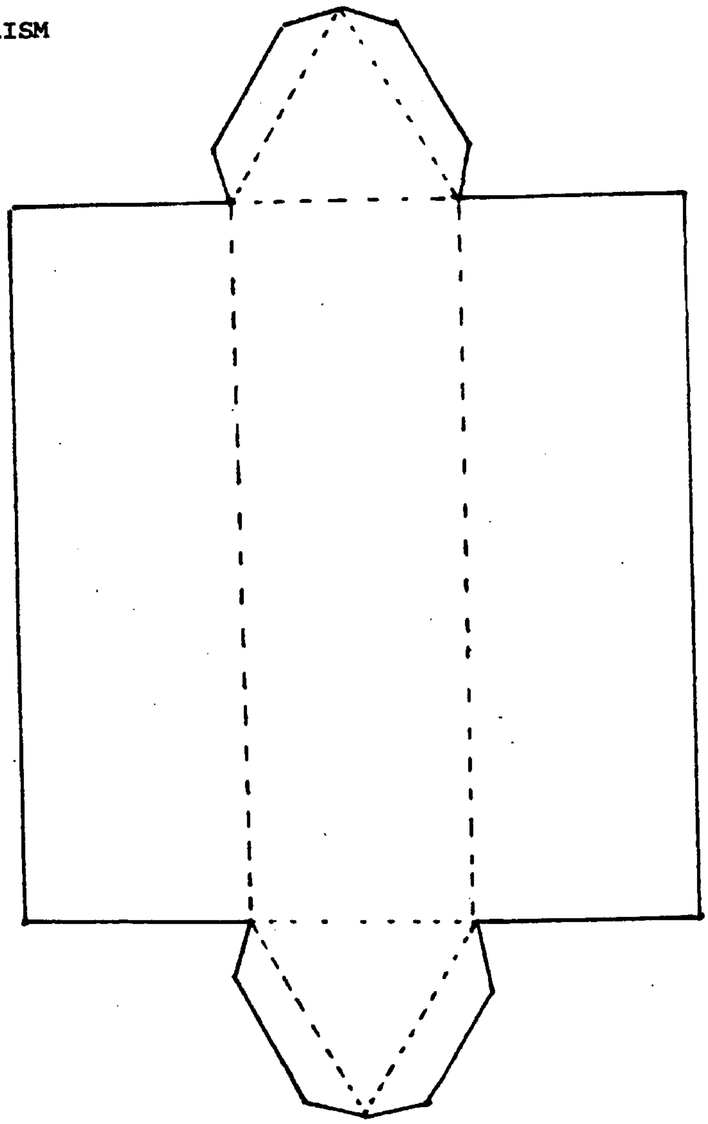
CUBE



SQUARE-BASED PRISM

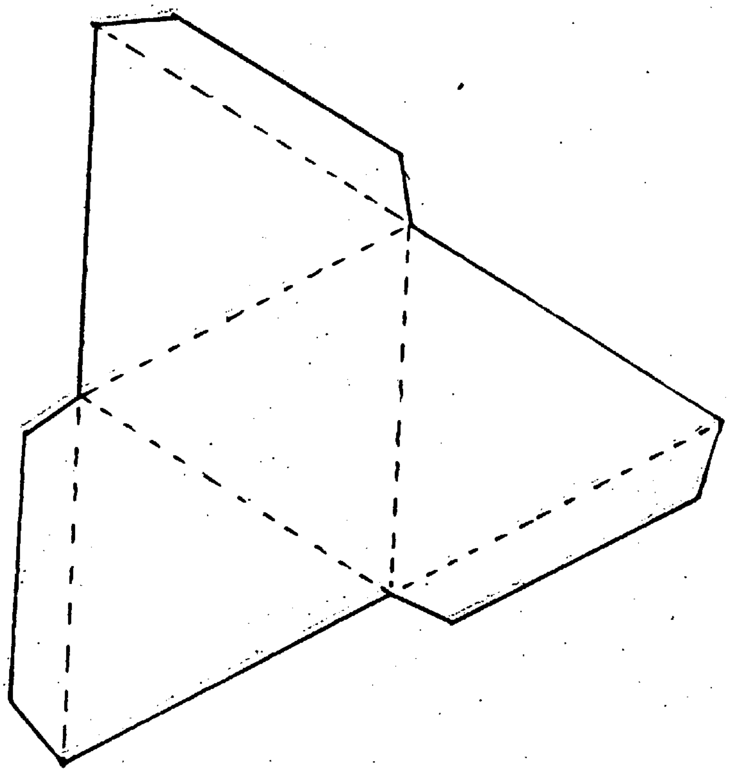


TRIANGULAR-BASED PRISM

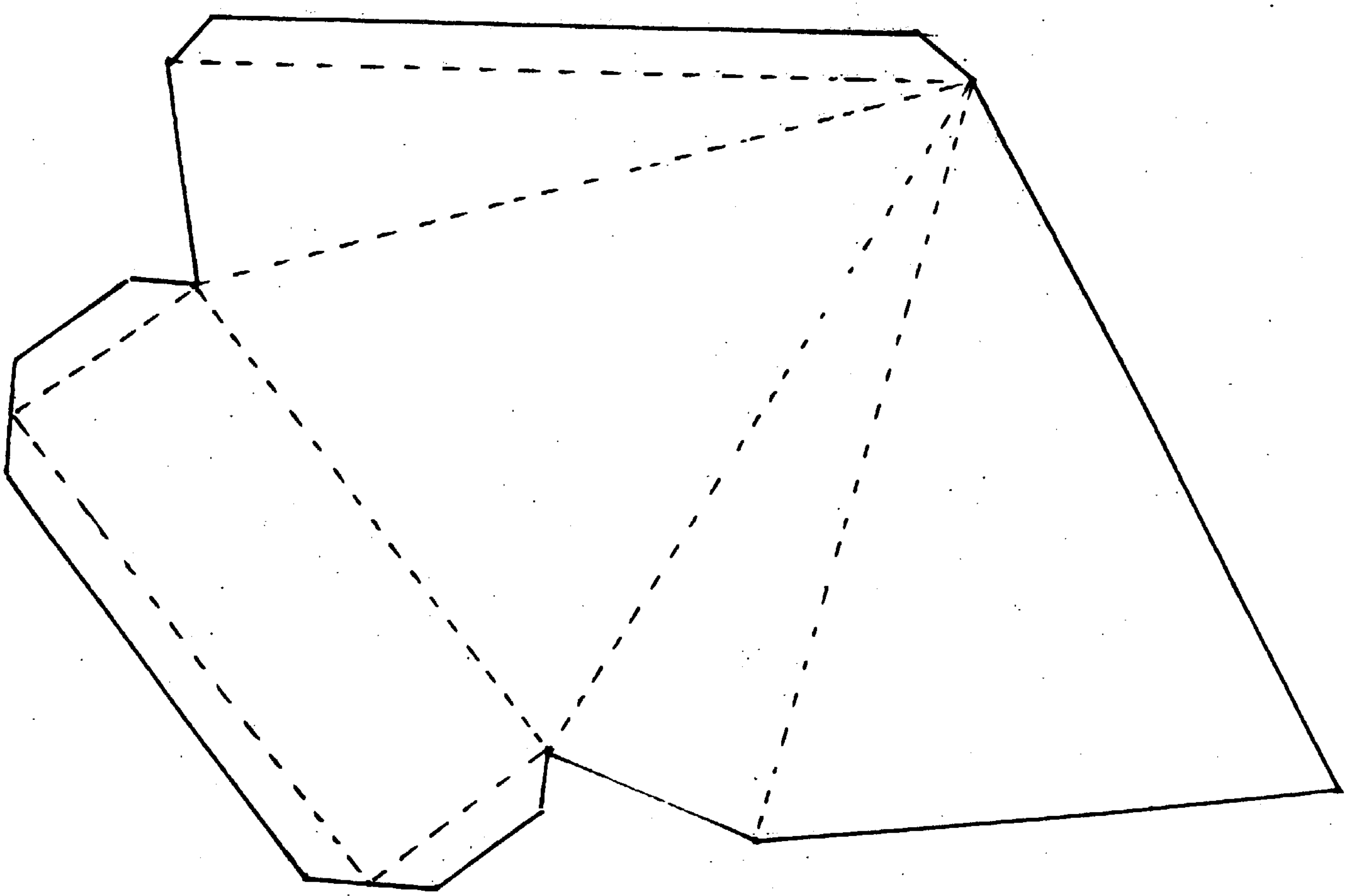


TRIANGULAR-BASED PYRAMID

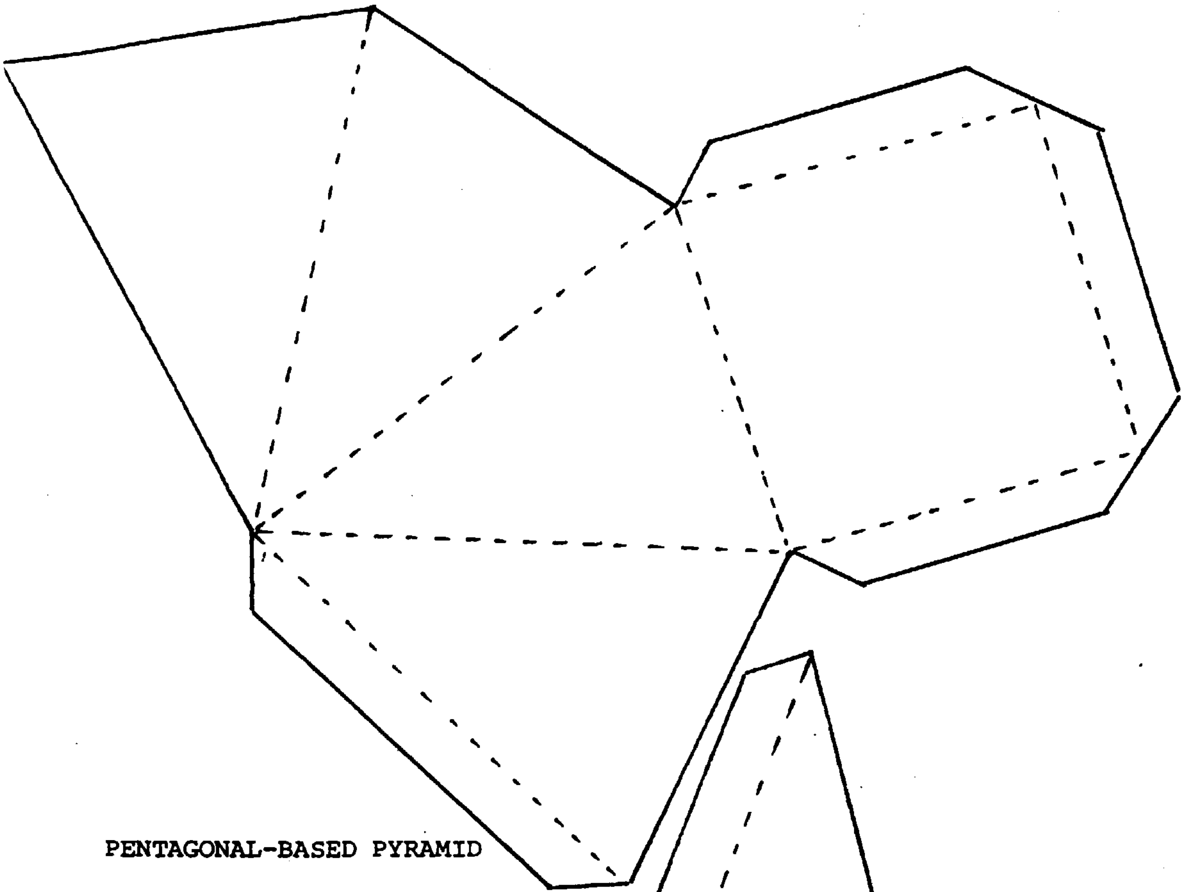
When this net is assembled you will find that it is a special triangular-based pyramid called a TETRAHEDRON



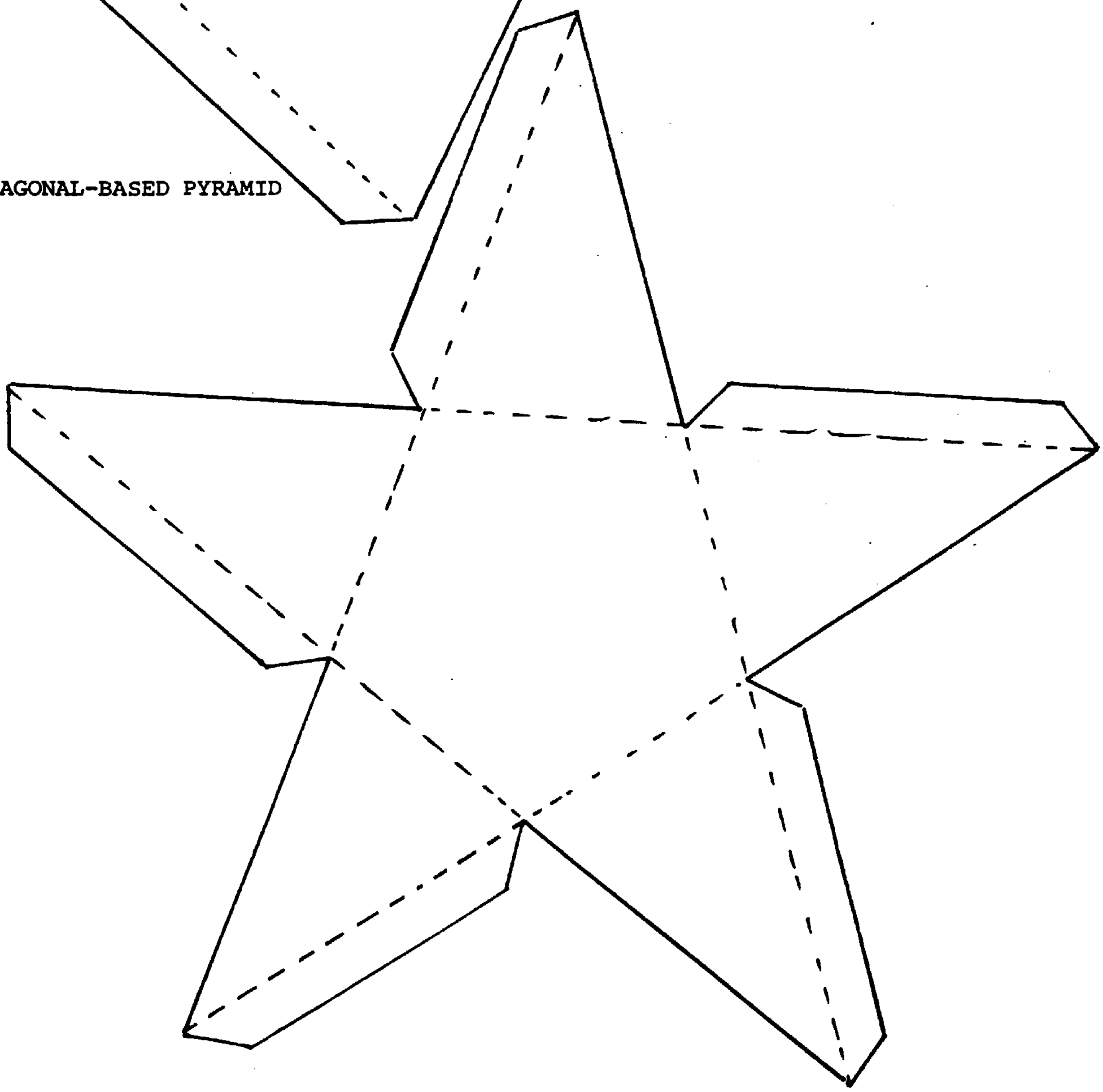
RECTANGULAR-BASED PYRAMID



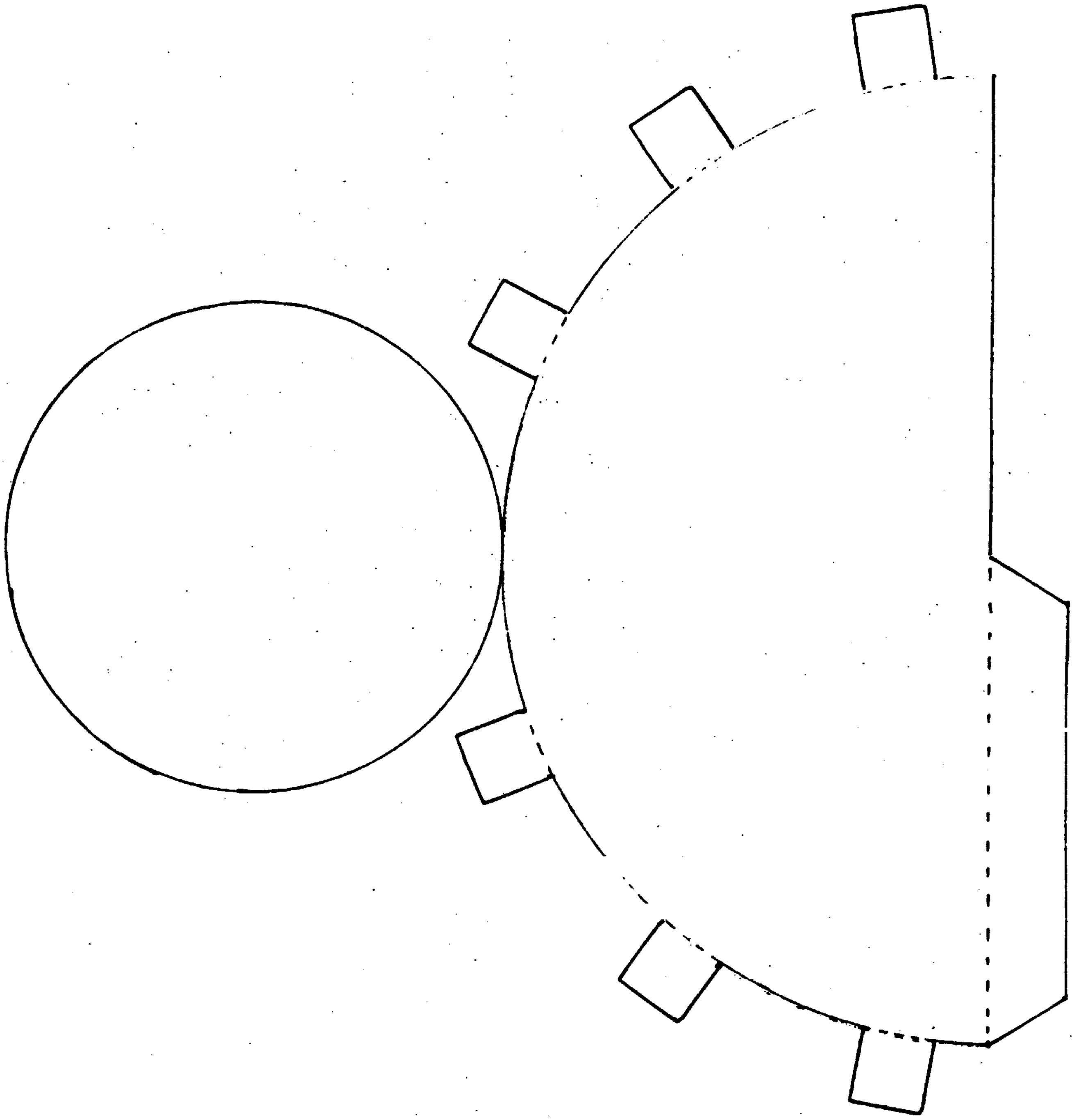
SQUARE-BASED PYRAMID



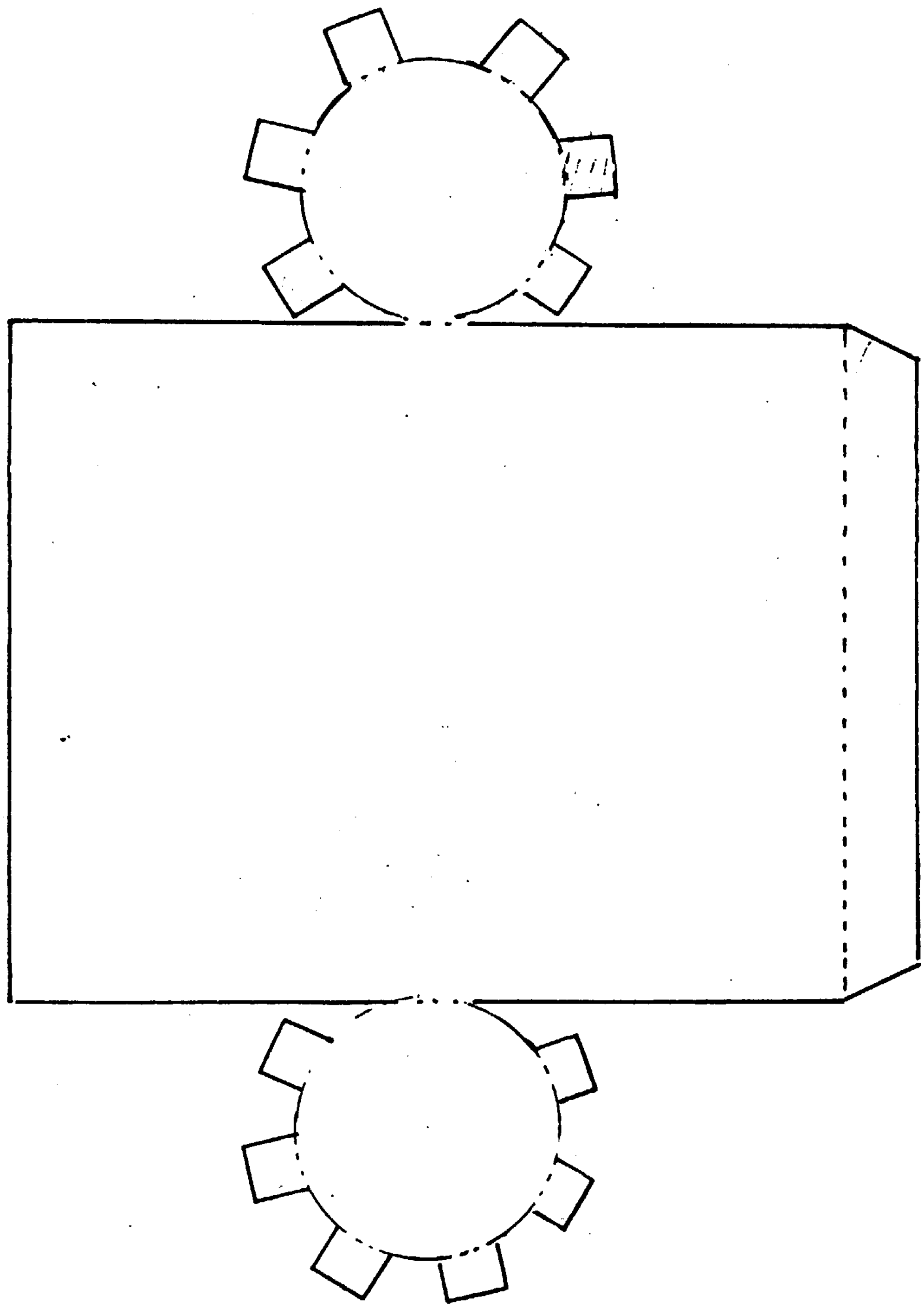
PENTAGONAL-BASED PYRAMID



CONE



CYLINDER



UNIT 5: TEACHING THREE DIMENSIONAL SHAPE

Focus:

Three dimensional shapes make up the world around us. In this unit we will look at teaching activities to develop understanding of three dimensional shape.

Background:

In this activity you are encouraged to look around you, to observe many different shapes, to look at a few of these more carefully, and to describe them. This is a most primitive geometric endeavour both from a historic point of view and from the standpoint of developing geometric concepts in children. You will look at your environment in order to discover its geometric potential. You will observe shapes and hopefully become sensitized to similarities and differences among them.

Materials:

Pen and paper, scissors, a collection of containers such as cereal boxes that are constructed by folding and fastening a plane shape.

Activities:

1. Go on a "shape scavenger hunt". That is, carefully investigate your daily environment (where you live, eat, work, play) for shapes of objects. Record as many shapes as you can. In particular, make a list of the five most common shapes that you find and of the five most unusual shapes.

2. Go on a "shape walk" with your class. As a class, investigate your classroom and portions of your school and town to observe shapes of objects. Compile a list of objects with common and uncommon shapes.

3. Choose three objects from your list that are distinctive. For each object list as many characteristics or properties as possible.

4. Fill in the answers to the following questions for one of your three objects.

   a) Name of object is ______________________________.

   b) Name(s) of the shape or shapes of the object is/are __________
   ____

   c) Is the object a surface or does it have a surface? __________

   d) Does the surface of the object have flat regions? __________
   If so, how many? ______________________

   e) Does the surface of the object have curved regions? __________
   If so, how many? ______________________

   f) Does the object have straight edges? ____________________
   If so, how many? ____________________________

g) Does the object have curved edges? ____________

If so, how many? ____________

h) Does the object have corners (vertices)? ____________

If so, how many? ____________

i) Is the object symmetrical? ____________

j) Does the object have congruent parts? ____________

k) Would the object roll on the floor? ____________

Why? ____________

l) Would the object slide on the floor? ____________

Why? ____________

m) The object is made of ____________

n) Write the name of an object that has the same shape as your object but which is made of different material ____________

____

o) Are the two objects the same colour? ____________

p) Are the two objects the same size? ____________

q) If you ask yourself questions b through l about the new object, would you give the same answers as for the first object? ____________

r) In what ways are the two objects geometrically different?

____

5. Answer the following questions concerning shapes in your environment?

a) A flying jet sometimes causes a vapor trail. What shape does the trail take? ____________

b) A child drops a pebble in a pond. What do you see? ____________

____

c) A child drops a pebble from a high building. What kind of path does the pebble follow? Where the pebble hits the street, what angle does the pebble's path make with the street? ____________

d) What shape are stop signs? ____________

e) Have you ever heard of a cubic planet? ____________

f) Why are things the shape that they are? (Pick an interesting object and analyse it.)

6. Containers or solid shapes* are probably among the earliest shapes actually constructed by humans. It seems likely that humans have always wanted to carry more than they could conveniently hold in their hands.

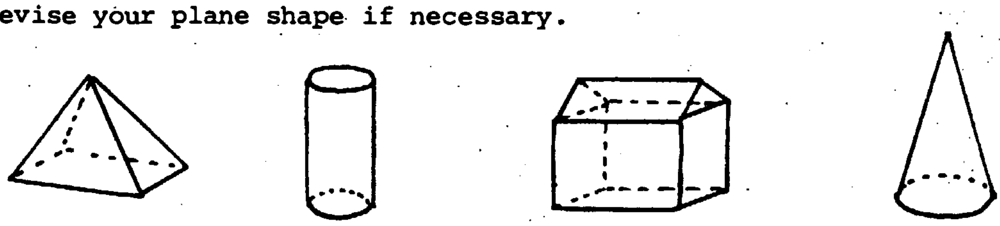
   (1) Go out and collect a solid shape supplied by your lecturer.

   (2) Look over the solid shape supplied to you and roughly sketch the plane shape from which you think its surface was constructed.

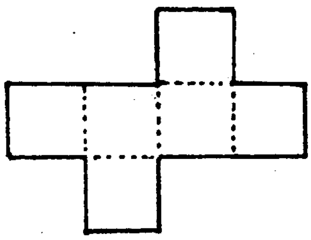
   (3) Dismantle (dissect) the shape to see if your sketch was correct.

   (4) Carefully draw a different plane shape that can be folded and fastened to form the same solid shape. Cut it out, fold it, and fasten it to check yourself.

7. Choose from the solid shapes pictures below one that you have not yet worked with in this activity. Draw a plane shape that you feel can be folded and fastened to construct its surface (dot in the fold lines). Cut, fold, and fasten your plane shape to check yourself. Revise your plane shape if necessary.



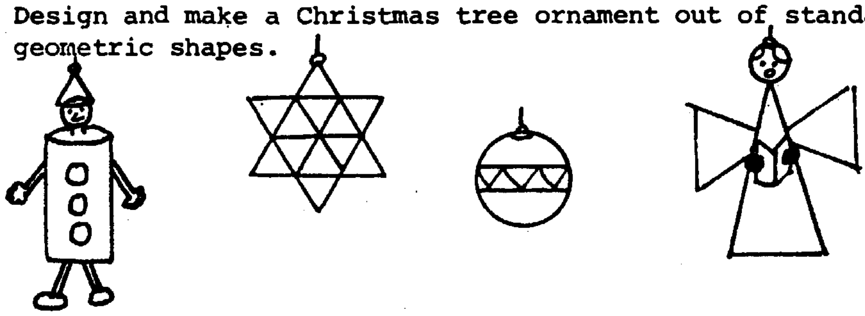
8. The shape below can be folded (along the dotted lines) to make a cube.



It is not the only shape that can. Construct several, using graph paper, different shapes that can be folded to make a cube. Draw each one that you find (be sure to draw in the dotted lines).

9. Children love to make things. Many of the things that they make are shapes that can have some potential for geometric analysis.

   (1) Design and make a Christmas tree ornament out of standard geometric shapes.



   (2) Describe briefly how you, as an elementary teacher, might plan a sequence of experiences near Christmas, Thanksgiving, or Easter that would combine geometry and art objectives.

10. In the above activities, you are seeing that many solid shapes can be analysed in terms of plane shapes. This fact provides one justification for the focus on plane geometry in much of the school curriculum. As we have said before there are many who feel that more emphasis should be put on solid shapes, especially with young children. It may also be true that the connection between solid and plane shapes should be brought out more clearly to geometry students at all levels.

    (1) Look at the "house" shape in 7 above.

    (2) List prerequisite knowledge of two dimensional shapes (and other concepts) that is required to fully formally define that shape.

<u>Teaching Hints</u>:

The above activities reflect (except for 10?) the environmental approach to teaching geometry. The subconcept approach is also applicable and activities for this will be found in unit 4.

It is important that care is taken with the examples children experience to understand three dimensional shapes. The teacher should ensure that children see

(1) many different examples of a solid type; and

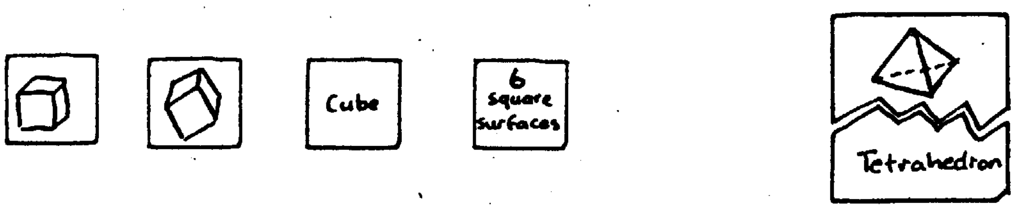
(2) examples that are not that type of solid as well as examples of the solid.

Furthermore activities should be structured so that both the following are done:

(1) the teacher says or writes the names of the solid - the child finds an example or model of it; and

(2) the teacher shows a model - the child says or writes its name.

Keeping a scrapbook of pictures, drawings, names and descriptions of various solids will help this as will reinforcement games of matching different drawings of solids with their descriptions and names (e.g., a gin rummy style card game or mix and match cards, as below).



Finally, it is essential that children have a <u>balance</u> of activities that describe (<u>interpret</u>) solids and that <u>construct</u> solids.

UNIT 6: CONSTRUCTING SOLIDS

Focus:

This unit explores methods of constructing three dimensional shapes. As such it also offers examples of building three dimensional shapes from their two dimensional surfaces.

Background:

In this activity we can lay the foundations for being able to analyse solids by their component surfaces. It also enables us to simulate one of the important vocational skills: building. Such a skill is now also of use to "handy men" and "handy women".

The basis of such building or construction is the surfaces or edges of the solid shape. Cardboard is used for surfaces and straws and toothpicks for edges.

Materials:

Cardboard, rubber bands, hole punch, tape, straws, string, plasticine or marshmellows, toothpicks.

Activities:

1. Read the following techniques for constructing solids.

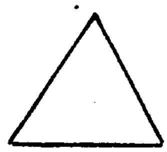
(a) *Nets*

This is the simplest way to construct 3-dimensional shapes. Patterns, complete with tabs to use for glueing and dotted lines to show where to fold, are transferred to cardboard. Then these are cut out, folded, glued or tapes to produce the final shape. A collection of nets is included in this book on pages 37 to 43 in unit 4.

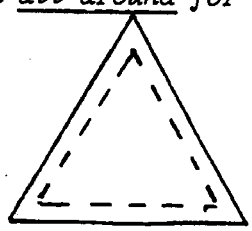
(b) *Rubber bands and cardboard*

This method is an excellent one for primary children. It differs from nets in that each face of the 3-D shape is made separately with tabs all around and then the faces are joined with rubber bands. This is how it is done for the tetrahedron (which has 4 equilateral triangle faces).

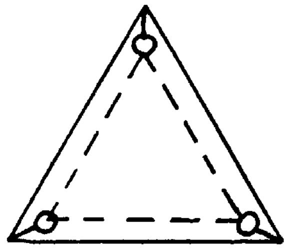
First we draw the equilateral triangle with which we wish to construct each face.



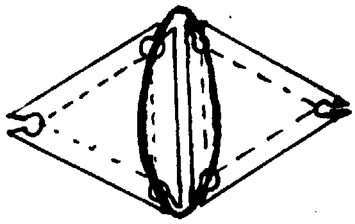
Then we add extra width all around for the tabs (about 1-2cm?)



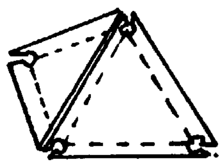
Four copies of this larger triangle are cut out (with inner triangle inscribed as above). Then a hole is punched at each corner of the inner triangle and from this a slit is made to the outer triangles corner.



The 'tabs' are folded out along this dotted line and triangles are joined with rubber bands as below.



The four traingles are joined at edges thus. The 'tabs' (held together by rubber bands) stick outwards.



In a similar manner

- *a cube can be made from 6 squares*
- *a dodecahedron from 12 pentagons*
- *a octahedron from 8 triangles (equilateral)*

and so on for most polyhedra (solids with polygon faces).

(Note: A polygon is any 2-D shape where edges are straight lines. A polyhedra is any 3-D solid where faces are polygons - cubes but not cylinders).

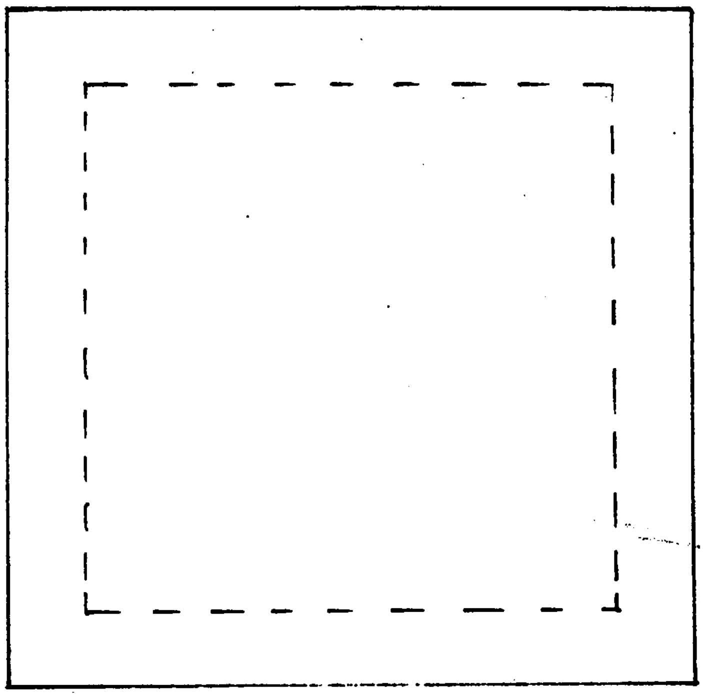
It is possible to make cylinders and cones like this if one "breaks" their "tabs" as below:

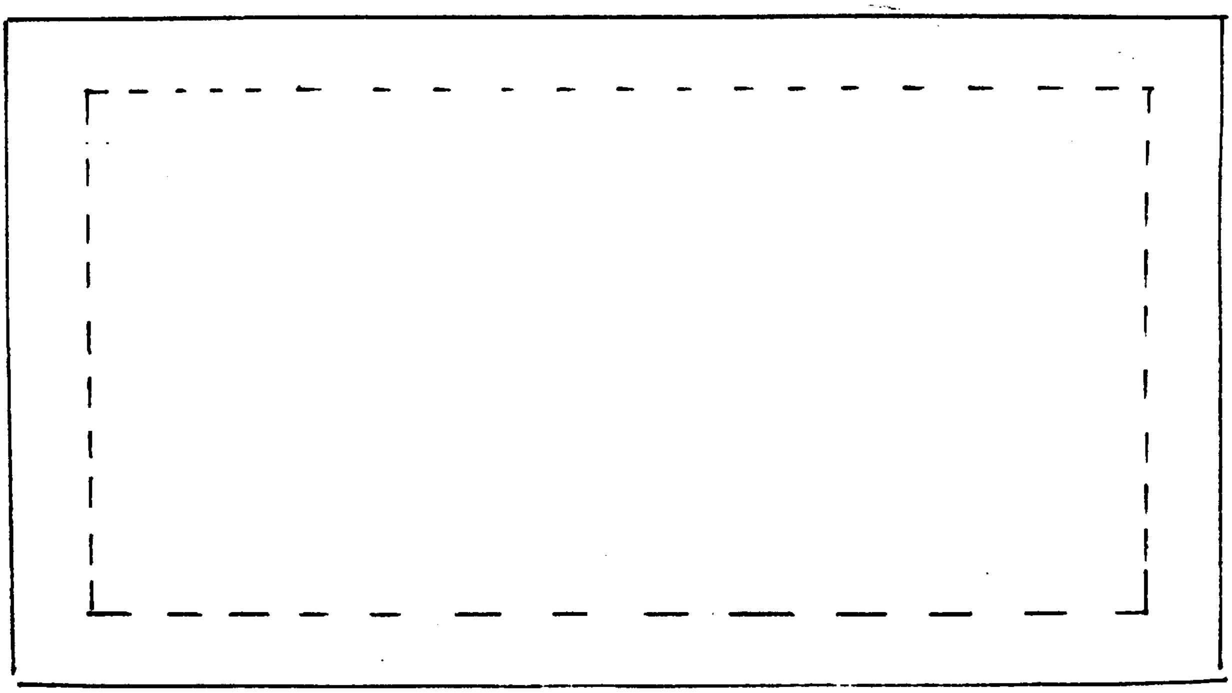
The tabs could be stapled, but rubberbands are better because they can be undone.

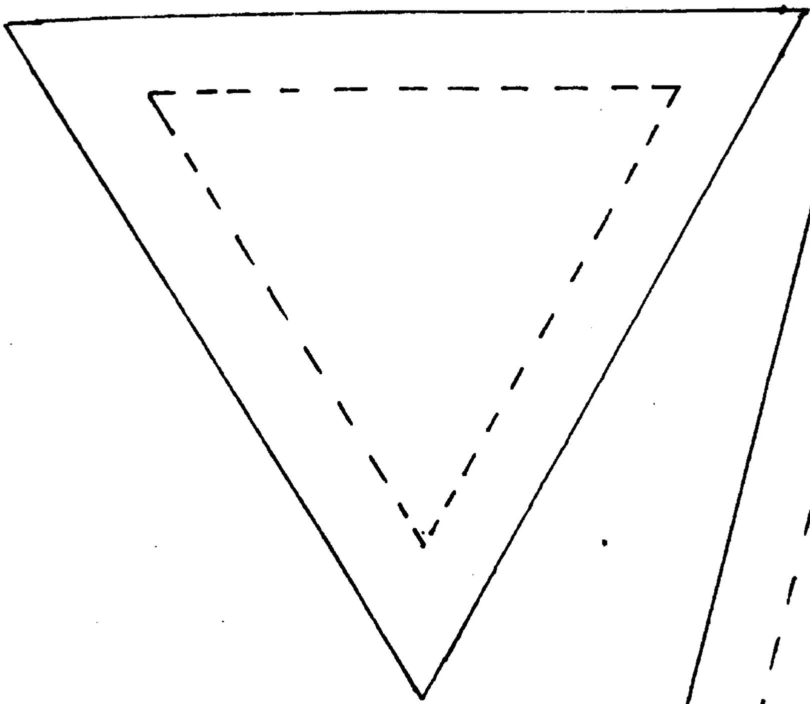
If triangles, pentagons, hexagons and squares are made with same length edges and rectangles with one pair of opposite ends at this same length (and so on), then the children can join nearly anything to any other shape and make their own 3-D constructions (making solids of their own choosing).

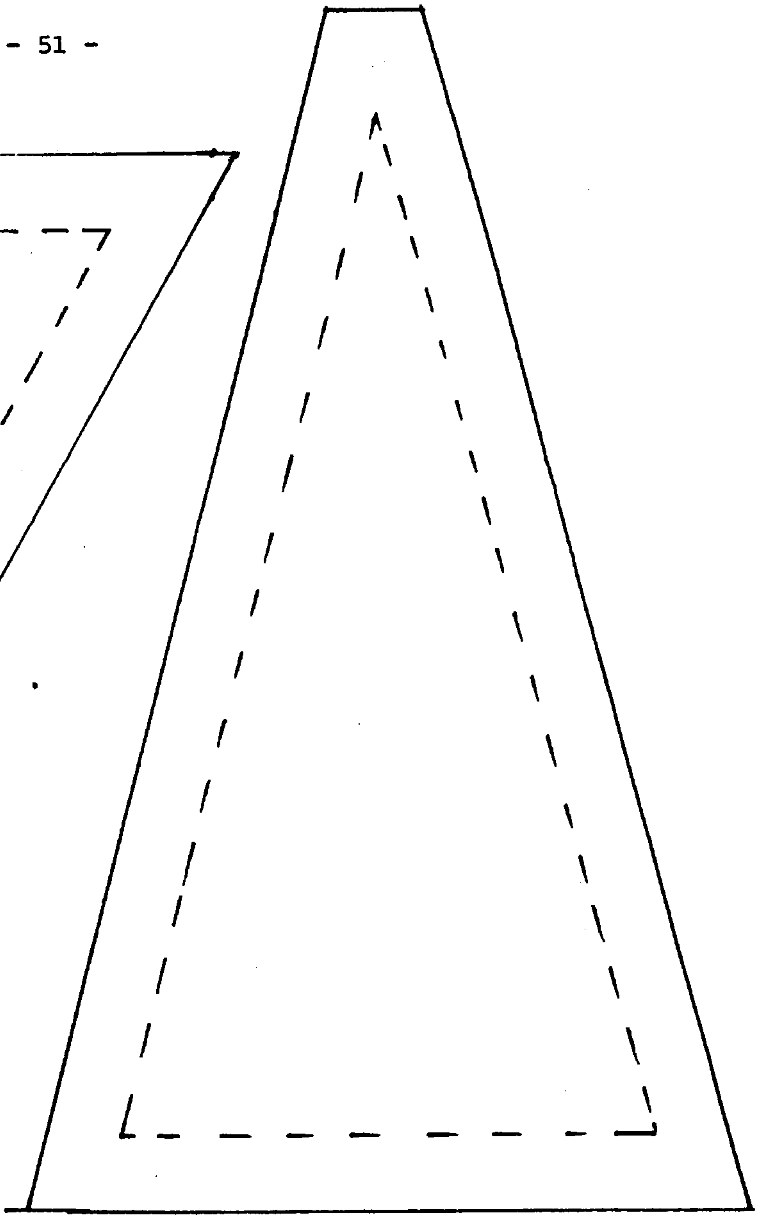
On the following pages are examples of shapes suitable for this construction technique.

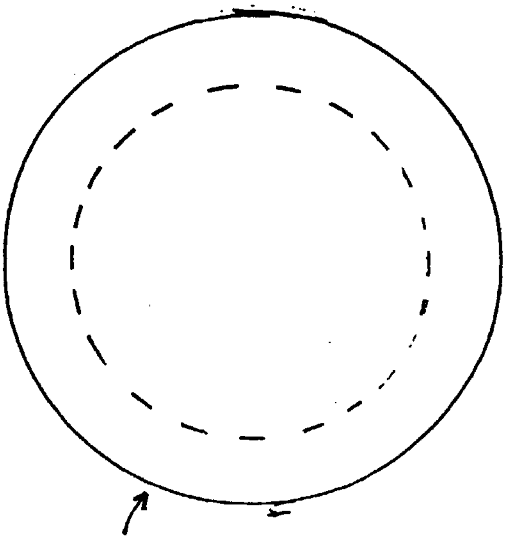
Examples of shapes for rubberbands and cardboard construction technique:





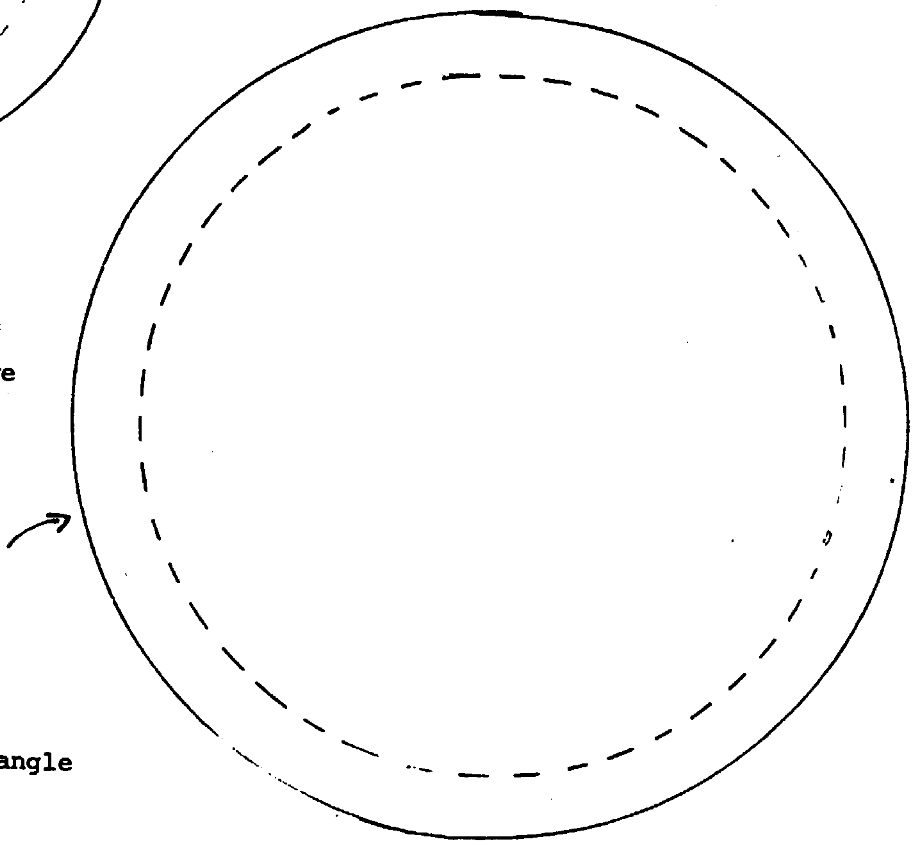






This circle will
fit the longside
of the rectangle
and the large triangle

Two sides of the square
and the small triangle
will be needed.



This circle
will fit two
of the long
sides of the
rectangle and
the large triangle

(c) *Straws and string*

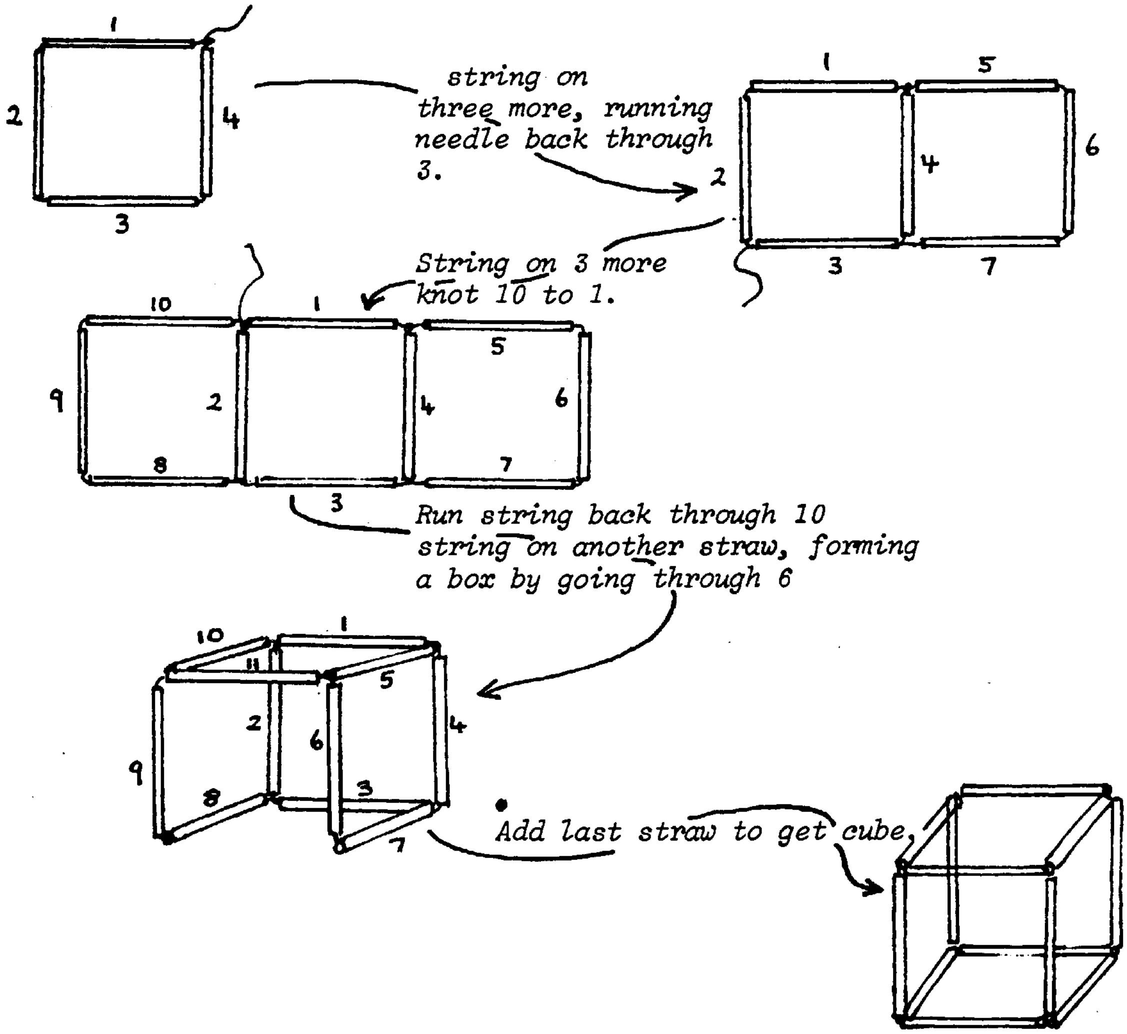
Another effective way of making solid shapes is to join straws with string (cotton, fishing line). The string is threaded through the straws. The resulting solid is "open" in that one can see inside - the straws simply "frame" the shape, supplying just the edges. A needle or piece of wire can be used to string the straws.

By cutting straws appropriately (to match edges of solids) we can join them to make most polyhedra. Some particular examples are given below:

Cube (hexadron)

12 straws - 6cm long	
24 straws - 12.3 cm long	*different colours for each length*
12 straws - 20cm long	*is required.*

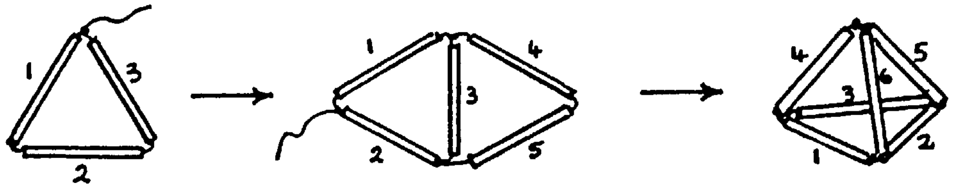
Take the twelve 6cm straws to make the cube as below: String on four straws



The cube will have a tendency to collapse (due to lack of rigidity). We can use the 24 straws of 12.3 cm long to add a square pyramid (using 4 straws) to each of the 6 faces of the cube. These 6 pyramids will end in 6 sharp points or corners. We can use the twelve 20cm straws to join the 6 points to make a second larger cube. This is called the dual.

Tetrahedron

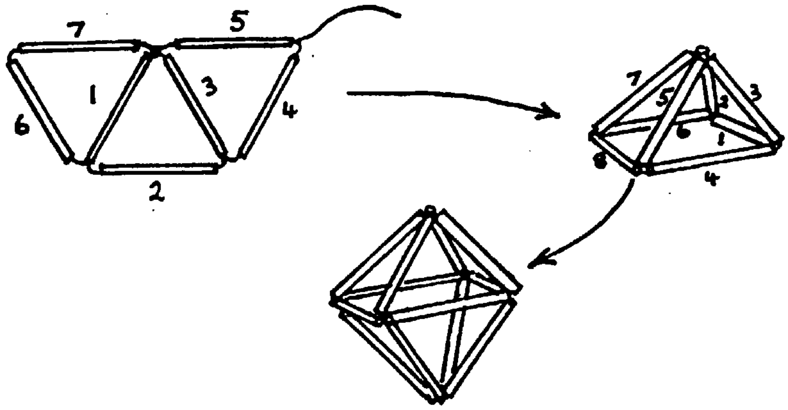
Six 5cm straws, twelve 11.5cm straws, six 20cm straws.
Using the 5cm straws



Octahedron

Twelve 7cm straws, twenty four 11.5cm straws, twleve 16.4cm straws.

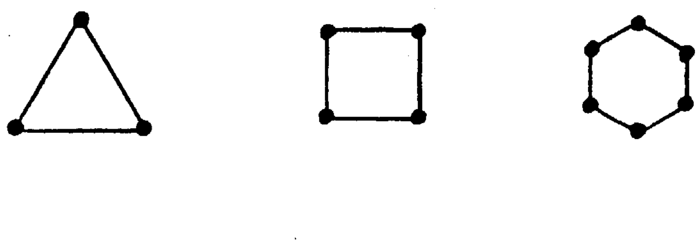
Follow the numbers in order



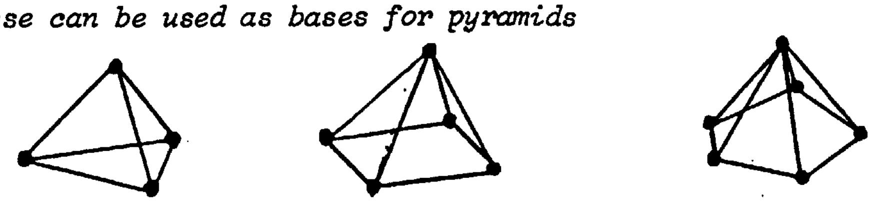
Toothpicks and marshmallows (or plasticine)

Another method of making 'open' solids (like those above with straws and string), is to use toothpicks and marshmallows (or balls of plasticine can substitute for the marshmallows). The idea is to use the marshmallows (or balls of planticine) as the corners and the toothpicks as edges of the solid shape. The toothpicks are stuck into the marshmallows (balls of plasticine).

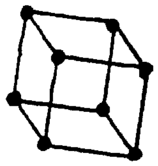
2-D shapes can be made

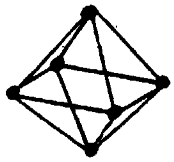


These can be used as bases for pyramids



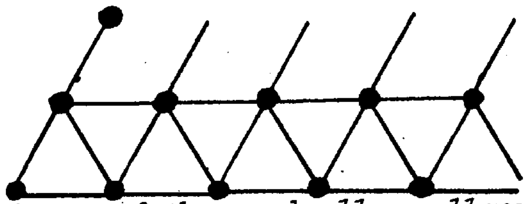
And other solids can be constructed.





By jointing toothpicks to marshmallows in triangles, it is possible to construct geodesic domes from patterns.

If the pattern below is bent around and empty ends of toothpicks placed in marshmallows, a dome results.



The softness of the marshmallows allows the stresses in the dome to be accommodated by the angle the toothpick enters the marshmallow.

(e) *Plasticine or modelling clay*

Solid shapes can be directly made by moulding modelling clay or plasticine.

2. Construct the following

(1) a cube using cardboard and rubber bands;

(2) a tetrohedron using straws and string;

(3) a triangular prism using toothpicks and marshmallows (or plasticine); and

(4) a cylinder using plasticine.

3. Solids with square and rectangular faces tend to collapse if made with straws and string (or with toothpicks and marshmallows).

(1) How could we make such shapes more rigid?

(2) What does this say about how houses should be constructed?

(3) Are there other construction techniques that are not suitable for certain solid shapes and vice versa? Why? List them below.

TECHNIQUE	SOLID SHAPES FOR WHICH TECHNIQUE IS UNSUITABLE
Nets	
Cardboard and rubberbands	
Straws and string	
Toothpicks and marshmallows	
Plasticine	

(4) Which construction technique do you prefer? Why? In particular what are the relative strengths and weaknesses of cardboard and rubberbands and straws and string?

4. Some constructions can be a lot of fun and involve problem solving.

(1) The pop-up dodecahedron.

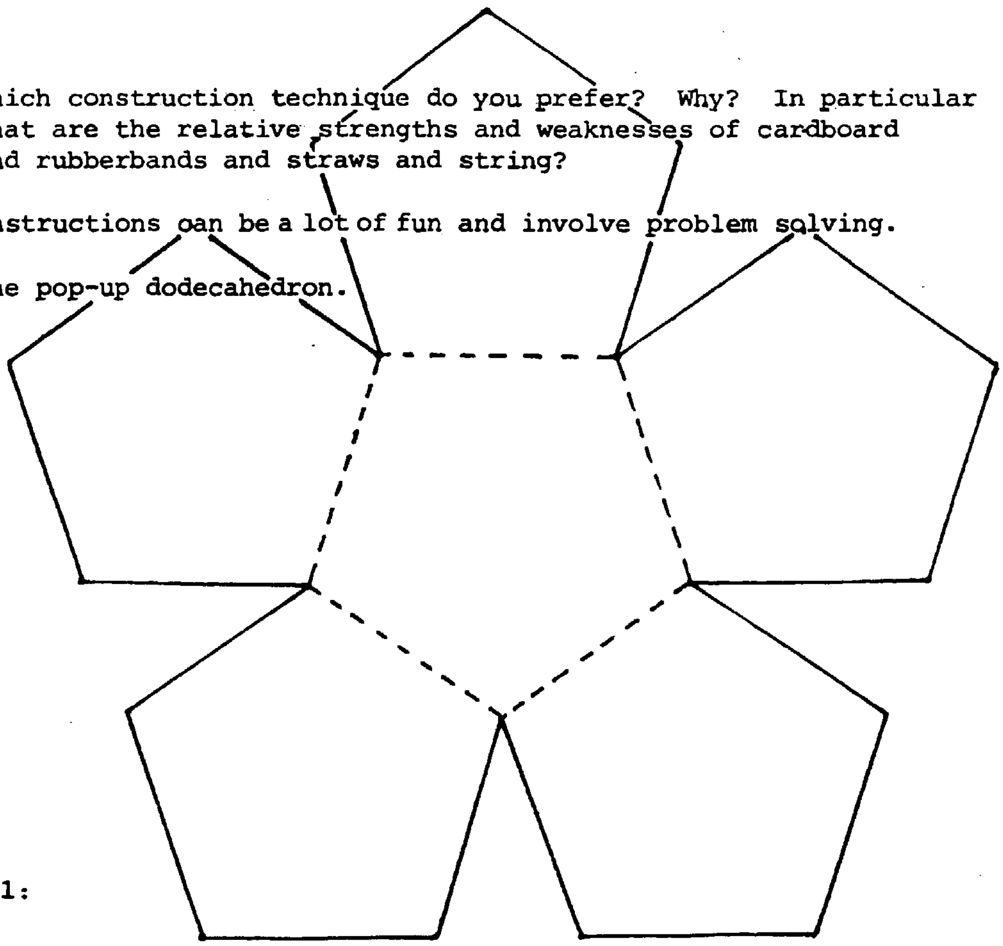


Figure 1:

Make 2 copies of figure 1 on tracing paper. Then put a piece of stiff paper or cardboard under the tracing paper, and poke a hole with the metal point of a compass at each of the points of the figure. Remove the stiff paper use a straightedge to draw all the line segments between the 20 holes. Then cut the figure out. Score each of the 5 sides of the pentogon in the centre of each copy, (along the dotted lines of figure 1) with your scissors.

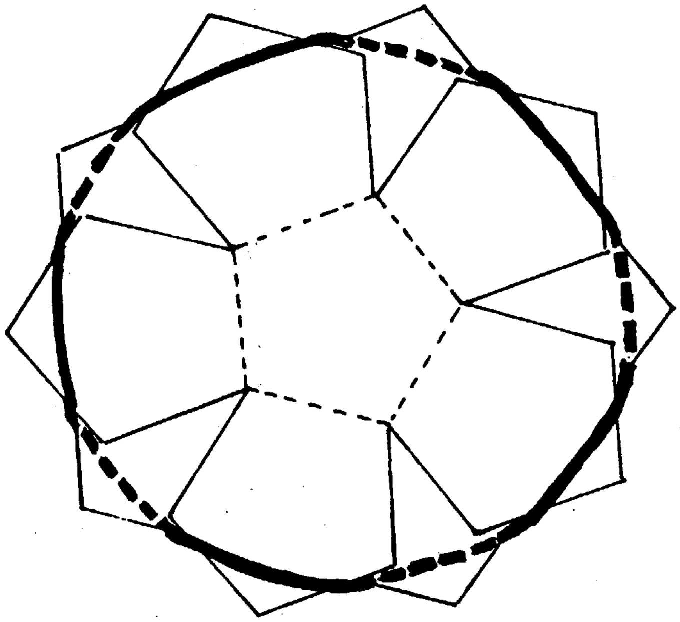


Figure 2:

Fold each of the 5 surrounding pentagons up to make a bowl shape figure. Now hold the 2 bowls facing each other and press them together so that they flatten out. Take a rubber bank and weave it alternately above and below the corners as shown in Figure 2 while holding the 2 pieces flat. If you carefully let go, the dodecohedron will pop into shape.

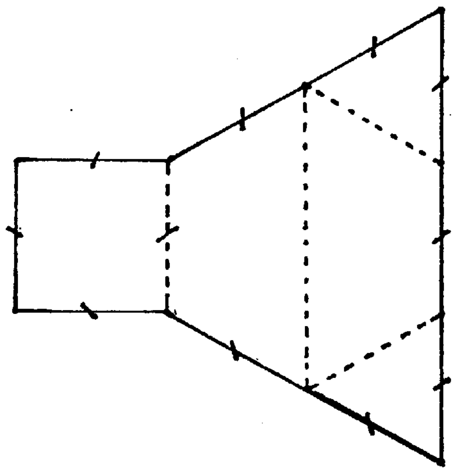
(Note: Figure 1 is best cut from heavy thick cardboard)

(2) Tetrahedron from the identical solids.

(a) Cut from cardboard two enlarged shapes, which are similar to the one below. (Note: the equality of the side lengths.)

(b) Form each net into a solid.

(c) Use the two solids to form a tetrahedron.



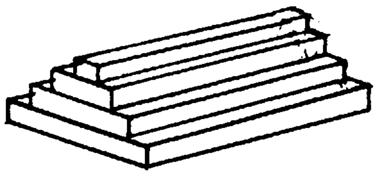
Teaching Hints:

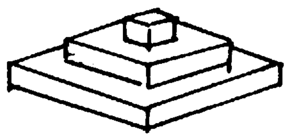
In these units on solids, we have focussed on activity to get children to look, observe, describe, experiment, analyse, dissect, construct and infer.

Construction techniques as in this unit should be used to achieve all these ends. A construction is a useful starting point to discuss what will happen in real life or for different solid shapes, i.e., to infer. A construction gives insight into what surfaces a shape will dissect to. Construction requires a focus on the faces, edges, vertices of the shape and their particular properties - a starting point for the formal analysis of a solid shape. Constructions, particularly when insufficient detail is given in instructions and the children have to work out what to do themselves, are great opportunities for experimentations, observing and looking.

Not all construction techniques are appropriate for a given situation. Open techniques, like straws and string, focus on edges and vertices showing the skeleton of the shape. Closed techniques like nets focus on surfaces. Open techniques require triangles for rigidity.

A particular technique very suitable for constructing solids by younger children is using blocks such as DUPLO and LEGO. Solids with rectangular and square faces are easy to construct. Solids with triangular faces are more difficult but can be reasonably simulated as follows:-





UNIT 7: PROPERTIES OF SOLID SHAPES

Focus:

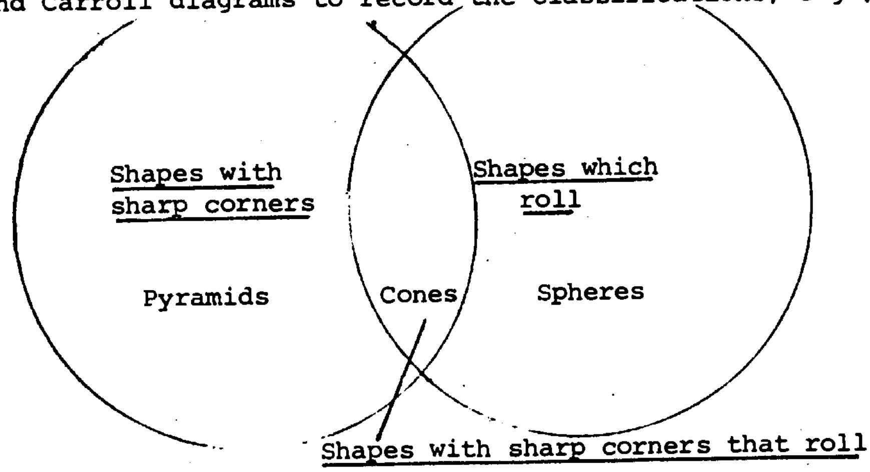
In this unit we investigate the properties of three dimensional shapes and some ideas for teaching such properties.

Background:

The properties that are worth investigating in solids have to do with the charactertistics of the surface of the solid (its vertices, edges and surfaces) the shape of cross sections, the way shapes pack together and the relative strength of different shapes.

Classifying solids:

Attributes upon which classification may be based include roll or not roll, rock or not rock, rough, smooth, pointed, vertices, faces, edges, regularity, volume, surface area, base area, shape of base, shape of faces, flat or curved surfaces, solid construction, open construction, hollow construction, mass, etc. Relationships such as the ratio of vertices to faces, edges to faces and edges to vertices can be explored. Use Venn and Carroll diagrams to record the classifications, e.g., a Venn diagram.



Euler's formula:

There is a relation between the number of surfaces, vertices and edges of a polyhedra called the Euler's formula. It is surfaces and vertices = Edges + 2.

Once this relationship is known, children can tell whether a polyhedra of a certain number of faces, edges and corners can exist.

Cross sections:

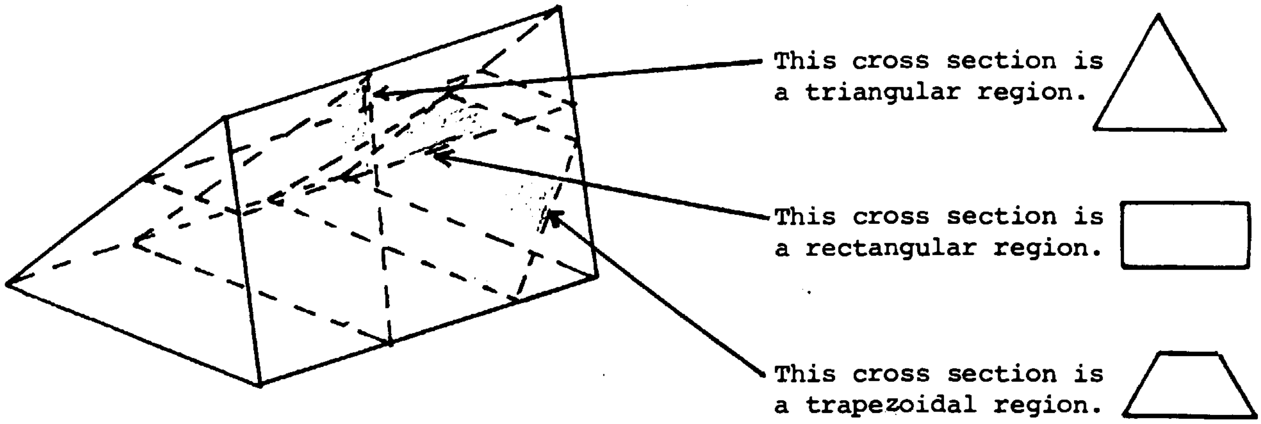
Cross sections are two dimensional shapes that are formed by cutting across a wolid shape. Children can

(1) cut solids made from potatoes, plasticine, etc and study their shape;

(2) predict the shape of cross sections; and

(3) predict which solids can have given cross sections.

As an example, this is some of the cross sections of a triangular prism:



Packing and Strength:

Which solid shapes tessellate - pack together without gaps or overlaps? This is a crucial characteristic of packaging - the boxes, tins, etc., into which goods are placed for sale. What solid shapes are strong? What shapes can hold material under pressure? Or can stand rough treatment? Again this is important for packaging.

Triangular, square and rectangular prisms and pyramids tessellate. Cones, cylinders and spheres do not. Yet spheres and cylinders are much stronger than prisms and pyramids because edges and vertices are weaknesses.

Studying supermarkets is an excellent way to tackle this area.

<u>Materials</u>:

1. (1) Select four different solid shapes from those provided by your Instructor and complete the following table.

MATERIAL (name)					
MOVEMENT (tick)	- Rolls				
	Spring				
	Bounces				
	Slides				
SHAPE (number)	- Vertices				
	Straight edges				
	curved edges				
	Flat surfaces				
	Curved surfaces				

MATERIAL Name)					
SURFACE TYPE (tick)	Squares				
	Triangles				
	Rectangles				
	Circles				
USES (tick)	Games				
	Containers				
	Building				

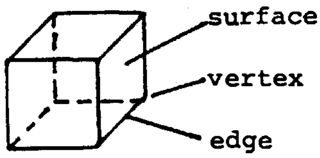
(2) Set up a Carroll diagram as below.

	All regular faces (equal length sides)	Some irregular faces (unequal length sides)
All flat surfaces or faces		
Some curved surfaces or faces		

Take a collection of solids, sort them according to the above criteria and write their names into the appropriate space of the diagram above.

(3) Game: One player describes a shape she or he has (a phrase at a time). The other players try to "guess" the shape. Players can start with equal piles of shapes each (hidden from other players). "Guessing" what the shape is, wins the shape for that player. The player with the most shapes at the end wins.

2. A polyhedron is a solid shape which consists only of flat (not curved) surfaces which intersect in straight edges which in turn intersect in vertices. For example, the cube below has six surfaces, each of which is a square; it has 12 edges which are equal straight lines, and it has eight vertices.



(1) Collect some polyhedra from your instructor.

(2) Fill in the following table for as many different polyhedra as possible. Look for any patterns in the numbers as you are doing so.

NAME OF POLYHEDRON	NUMBER OF SURFACES	NUMBER OF VERTICES	NUMBER OF SURFACES + NUMBER OF VERTICES	NUMBER OF EDGES

(3) Do you see any patterns in your table? Let s represent the number of surfaces, e the number of edges, and v the number of vertices. Write below the relationship, called Euler's relationship, that you see between s, e, and v.

(4) Draw or name solids with each of the following combinations of surfaces, edges, and vertices. Indicate if it is impossible to do so.

s	e	v	Name or Sketch a Shape
6	12	8	
8	10	6	
4	6	4	
3	6	5	

(5) Does the Euler relationship hold for solids that are not polyhedrons? Explain your answer.

(6) Can you find a relationship between the vertices and edges of polygons? A polygon is a plane shape that is closed and is made up of straight edges; e.g., triangles, rectangles, and pentagons are all polygons.

3. (1) Form solid shapes out of modelling clay and use a taut piece of wire to cut the shapes in order to observe various cross sections, or cut the solid shapes out of potatoes and then observe the cross sections which can be cut.

   (2) Draw a cube and then draw in

   (a) a triangular cross section

   (b) a rectangular (non square) cross section

   (c) a hexagonal cross section.

4. Manufacturers give thought to the construction of containers. Considerations such as the economic use of materials, ease of construction, appearance and the containers ability to be packed and stacked are considered.

   (1) Go to a supermarket (or to a collection of supermarket groceries).

   (2) Look at these packages. List which of them seem:

   (a) made of reasonably cheap materials;
   (b) easily constructed;
   (c) able to be packed and stacked easily;
   (d) to be able to tessellate (pack without gaps or overlaps); and
   (e) strong.

   Do you find any packages that satisfy all criteria?

5. (1) Why are manhole covers round and not rectangular?

   (2) Why is cheap housing square?

Teaching Hints:

Up to now we have mainly analysed solid shapes qualitatively and descriptively. In this unit, we include some quantitative analysis. As such these quantitative activities should be left to upper primary.

This section gives great opportunity for real problem solving. Children should be encouraged to go to supermarkets and construction sights to see three dimensional shape in action.

Rules such as Euler's formula should be discovered. Even upper primary children find it difficult to generalize without guidance. Hence offer the opportunity for rules to be discovered but include sufficient directions to guide the discovery. Seeing patterns is one of the most important mathematical processes. Every opportunity should be taken to experience it.

CHAPTER THREE: TWO DIMENSIONAL SHAPE

Our world is three dimensional. But we observe, perceive and represent the world two dimensionally. Our evolution and our experience has developed on ability to operate in our three dimensional world through two dimensional representations.

In this chapter we will study how to teach two dimensional shape. Unit 8 will look at lines and angles, the building blocks of two dimensional shapes. Unit 9 will survey the types of two dimensional shapes that should be covered in primary school. Unit 10 will look at how we should teach children about these shapes and unit 11 will cover how we should teach the properties (e.g., diagonals) of these two dimensional shapes.

This chapter should be seen in association with the two following chapters which will cover the concepts of symmetry and tessellation and discuss the role of shape puzzles and imagery. Chapter 7 (which looks at similarity, congruence and other transformations) will also have many activities on two dimensional shape.

UNIT 8: LINE AND ANGLE

Focus:

Line and angle are the building blocks of shape. They are also the basis to the way in which we represent our world and perceive what is going on. This unit will discuss sequences of activities to introduce these two concepts and the associated concepts of parallelness and perpendicularity. Discussion of bisection and congruent angles will also be included.

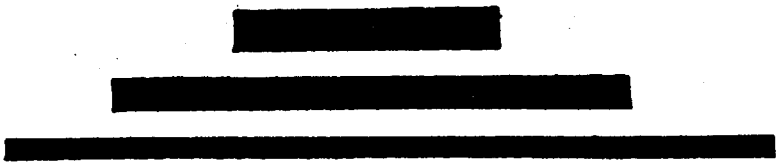
Background:

Points, lines and planes:

It is useful to begin by looking at line in relation to point and plane. We represent a point by a circular region but such a region never becomes a point, e.g.,

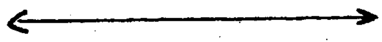
A mathematician thinks of a point as having no size at all. That is, we say a point has no dimensions. However, a small dot is a good "model" of the idea he has in mind. The point of a needle or pin, or the sharp top of a well-made pyramid or cone, would also be models of points. In notation, a point is named by using any capital printed letter, e.g., we call the point below A or just "A".

We represent a line by a long thin shaded rectangular region, which again never actually becomes a line, e.g.,



(Note: In Spatial Knowledge, we understand line to mean a straight line unless stated otherwise.)

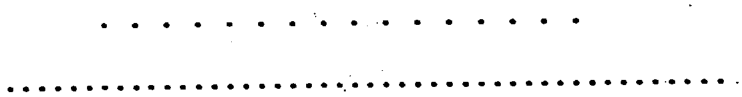
A line has length but no thickness. Therefore, we can say a line has one dimension. A line can be extended infinitely far in both directions. We could never find such a thing in practice but we use the diagram below as a model of a line.



The arrows at both ends show that the line can be extended without ending in both directions.

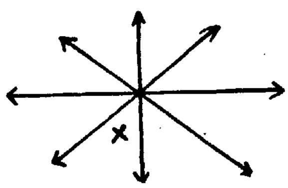
Other models of lines are the sharp edge of a ruler, or the edge of a box (if we imagine the edge, in these cases, extending without ending in both directions.)

The line can also be represented by a row of dots, but again, even if we keep adding tiny dots, we will never actually obtain a line, e.g.,

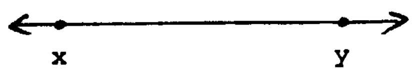


However, if we continued for long enough, the dots might be so close and extend for so far in both directions, that we would have quite a good model of a line. Of course, to "fill" the line, we would need an infinite set of dots.

In notation, we can not name a line with one point, e.g.,

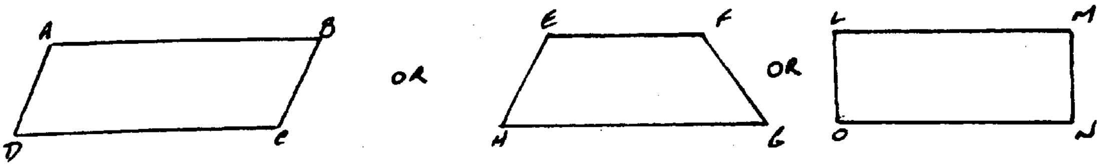


We have to use two points, e.g.,



To name the line, we use the two capital letters for the points and "$\longleftrightarrow$", e.g., we call the line above $\overleftrightarrow{XY}$.
It does not matter if we had called it $\overleftrightarrow{YX}$ but the convention is to use alphabetic order.

We may represent a plane by a quadrilateral whose covern points are named with capital printed letters, e.g.,



We name the plane by using the capital letters e.g., the left hand plane is ABCD.

A mathematician's idea of a PLANE is of a <u>flat</u> surface which is very thin and extending without ending in all directions.

A table top, a sheet of glass, a skating rink, are models of planes (if we imagine them extending without ending in all directions). Each page in this paper is a model of a plane. Each surface of a box, or a pyramid, is a model of a plane.

The mathematician does not state exactly what the words, <u>point</u>, <u>line</u>, and <u>plane</u> mean. Instead, he has a <u>picture in his mind</u> of what he means by point or line or plane - and he finds "models" of these things in the world around him which help him to use his ideas.

Examples:

(1) A town in Australia is a point on a map.

(2) Telephone wires stretched between poles are lines.

(3) The top of a table is a plane.

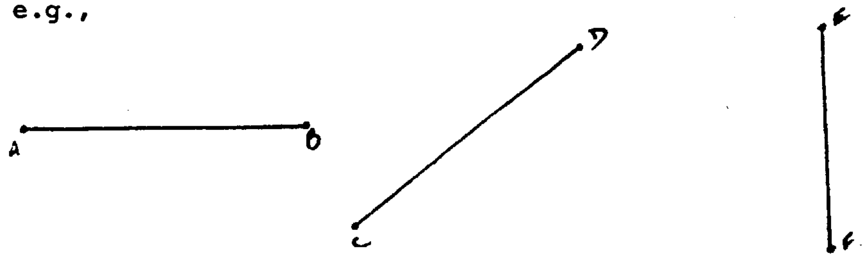
It should be noted that only three points, if they are not in a line, are necessary to define a plane.

Line Segments and Rays:

In real life, it is more likely that we shall need to draw, or name, only a part of a line.

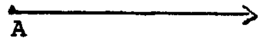
For example, when you rule a line in your Exercise Book, you are really only drawing a PART of a line. You start at one point and finish at another point.

This part of a line is called a <u>line segment</u> (the word "segment" means "a part of"), e.g.,



In notation, we present a line segment by the capital letters of the two points on the line and " ——", e.g., <u>the</u> line segment AB above is named by $\overline{AB}$.

A <u>ray</u> is a line that has a starting point but no ending point. In other words, it can extend in only <u>one</u> direction, without ending, i.e., a ray can be thought of as a "half-line", e.g.,



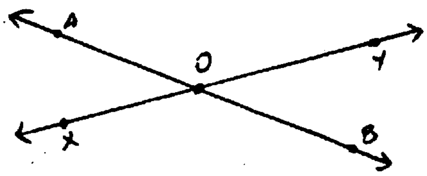
A good model of a ray is the light from a torch. Each ray of light has a starting point (i.e., the bulb), and extends in <u>one</u> direction in a straight line. We draw rays with two points, e.g.,

and name them using the two capital letters and " ——> ", e.g., the ray PQ above is named by $\overrightarrow{PQ}$

(Note: the beginning point of a ray is always named first.)

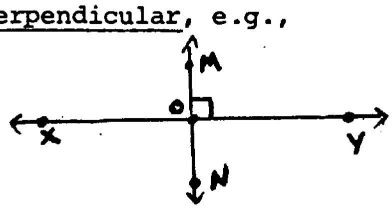
Types of Lines:

When two lines XY and AB, meet at point O, these two lines are called, intersecting, e.g.,



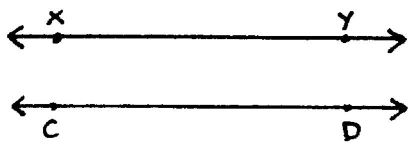
(We should note that any two lines which meet when they are extended are also called intersecting.)

When $\overleftrightarrow{XY}$ and $\overleftrightarrow{MN}$ intersect at point O at right angles, these two lines are called <u>perpendicular</u>, e.g.,



(Note the symbol " ∟ " for right angle).

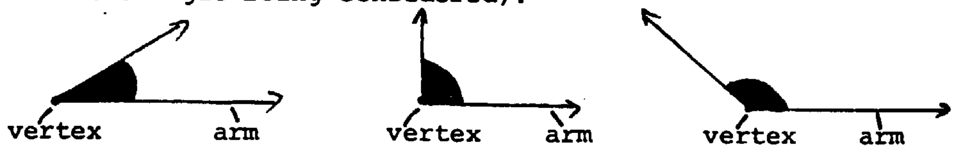
When $\overleftrightarrow{XY}$ and $\overleftrightarrow{CD}$ never meet, no matter how far they are extended in either direction, these two lines are called parallel, e.g.,



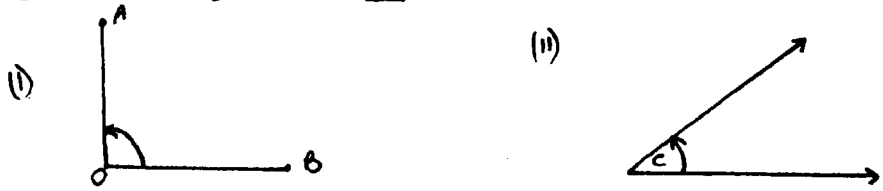
Parallel lines go in the same direction and are denoted by symbol " > " on each line. The notation for $\overleftrightarrow{XY}$ is parallel to $\overleftrightarrow{CD}$ is $\overleftrightarrow{XY} \parallel \overleftrightarrow{CD}$

Angles:

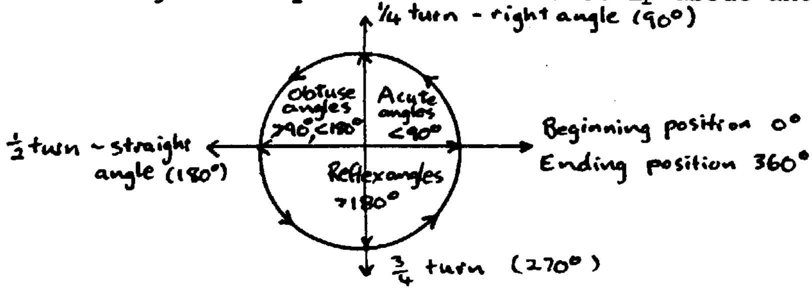
An angle is the opening between two rays that have a common end point, it is the amount of turn, the measure of the change of direction between the rays. The point is called the <u>vertex</u> and the two rays that form the angle are called the <u>arms</u> of the angle. For example, see the diagrams below (the shaded parts show the angle being considered):



We can name an angle by naming a point on each arm of the angle and the point at the vertex. The point at the vertex is always stated in the middle of the angle name. Therefore, in diagram (i) below, the angle would be named as $\angle AOB$ or $\angle BOA$. An angle can also be named by using a small printed letter as shown in diagram (ii). We would write this as $\angle c$. In both cases, the symbol for angles is $\angle$.



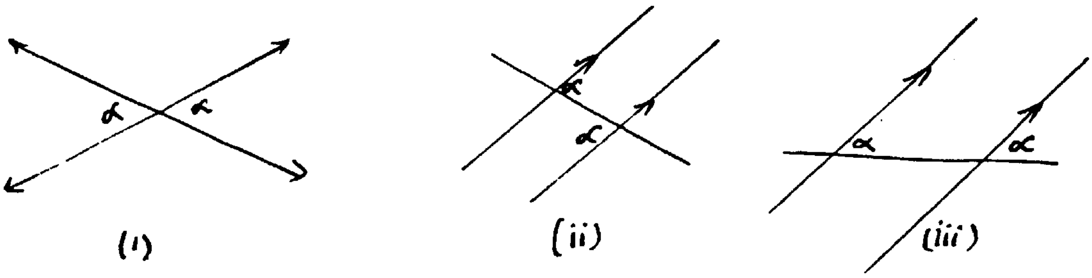
Here is a diagram showing the complete turn of one strip about another



As we can see, the unit of measure for angle is <u>degree</u>, which is 1/360th of a turn through a complete circle.

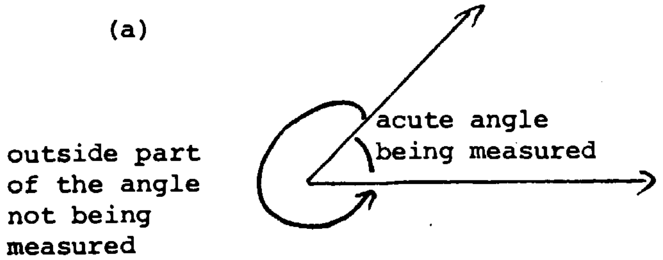
From the diagram above, you should notice that in a circle there is 4 right angles (i.e., 360° = 90° x 4); a circle is 2 straight angles (i.e., 360° = 180° x 2) and a straight angle is 2 right angles (i.e., 180° = 90° x 2).

The angles which have particular emphasis in primary schools are the right angles, the straight angle, the acute, obtuse and reflex angles, the vertically opposite angle (see diagram (i) below), the alternate angle (see diagram (ii) below) and the corresponding angle (see diagram (iii) below). Vertically opposite, alternating and corresponding angles are congruent (equal).

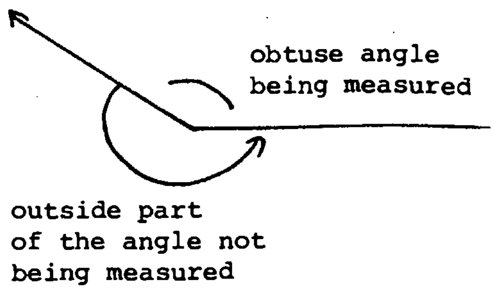


Angles are the amount of turn between the arms of the circle. But there is another angle outside that being measured between the arms, e.g.,

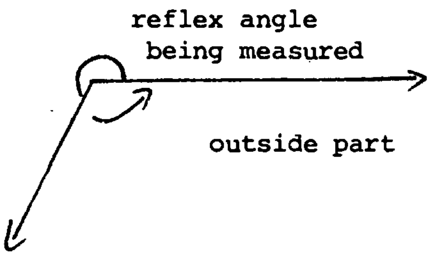
(a)



(b)



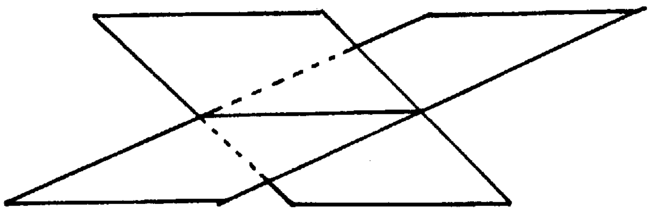
(c)



Hence there are two parts of the angles: the <u>interior</u> of the angle and <u>exterior</u> of the angle.

The intersection of planes:

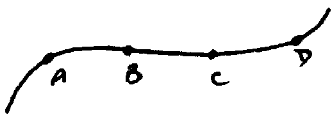
Two planes may intersect (just as two lines do). In the diagram below (which is only amodel), planes ABCD and plane EFGH intersect in $\overline{XY}$. (Note: planed intersect at a line while lines intersect at a point).



The plane of a ceiling intersects with the plane of a wall and the intersection is a line segment. The plane of a window intersects with the plane of a window-sill and the intersection is a line segment.

Curved Lines:

A two dimensional boundary that does not go on continually in the same direction, but turns constantly, is a curve. A curve can not be defined by a fixed number of points (unless it is regular, e.g., a circle or ellipse), it has to be drawn e.g.,



<u>Materials</u>:

Geoboards, rubberbands, paper, pen, cardboard, scissors, screen, sunlight, compass, protractor, ruler.

Activities:

1. (1) Place your hands out infront of you with palms together as below.

(2) Walk forward without turning.

(3) Walk forward turning constantly.

(4) Link up with a partner and walk forward side by side in the same direction without turning.

(5) Walk forward with your partner so your paths cross (don't bump into each other).

(6) Stand still looking over your hands and facing in that direction. Move your right hand clockwise in an angle. Leaving your left hand where it is, turn your body to face in the direction of your right hand. Now move your left hand to joint your right hand.

2. Geoboards.

Use a geoboard and rubber bands to do the following workcards.

Lines:

(1) Make the shortest line segment you can on your geoboard.

(2) Make the longest line segment you can on your geoboard.

(3) Make three line segments of different lengths.

(4) Make as many line segments as you can in one minute. How many times did your lines cross. What shapes did your lines make?

Angle: (Use string or rubberbands, restrict to a 5 x 5 geoboard)

(1) Make a "narrow" angle.

(2) Make a "wide" angle.

(3) Make an angle from a starting point at the edge of the geoboard.

(4) Make an angle from a starting point near the centre of the geoboard.

(5) Make an angle like the corner of a square.

(6) Make an angle with 2 nails between the rays.
0 nails between the rays.

(7) Make an angle with 6 nails outside the rays.

(8) Make an angle with 4 nails inside and 5 outside the rays.

Curves and Shapes: (Use wet string, restrict to a 5 x 5 geoboard)

(1) Make a simple open curve.

(2) Make the longest open curve you can with your string (all of it on the board.

(3) Construct a region that has only 1 nail inside it. (2, 3, 4)

(4) Construct a region that has no nails inside.

(5) Construct a region that has 1 nail outside (2, 3, 4)

(6) Construct a region that has 1 nail outside and 3 nails inside.

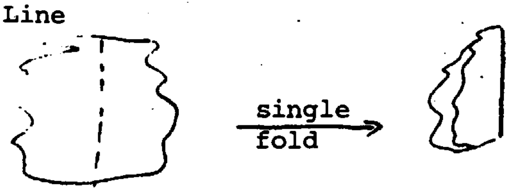
(7) Construct a region that has 3 nails on the boundary.

3. Paper folding.

Use scrap paper to complete the following paper folding activities.

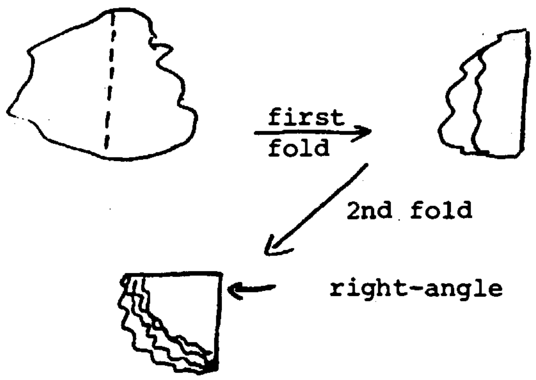
Representation with paper	Development

(1) Line



Testing for occurrence of (straight lines, flatness (of desks), straightness (of doors), etc.

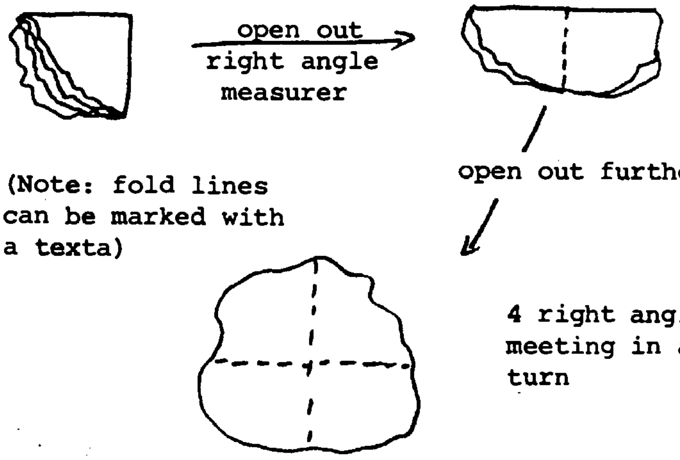
(2) Right angle (making a "right angle reckoner")



Testing for right-angles, corners (3 right-angles meeting) perpendicu (upright) walls and windows, etc.

Classifying (and forming angles - see below) equal to, smaller, large than a right-angle (acute, obtuse angles). Classifying triangles (one right-angle, all angles less than a right-angle - acute triangle, one angle larger than a right-angle - obtuse triangle).

2 right angles meeting in a straight line



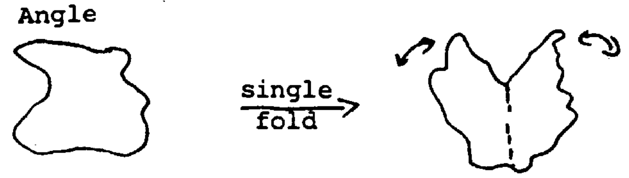
(Note: fold lines can be marked with a texta)

4 right angles meeting in a complete turn

Testing for where 2 right-angles meet in a straight line (door frame window, etc.).

Test for occurrence (window, etc.) Put paper on floor. Stand where fold lines meet. Turn from one fold line to the next, i.e., rotate body through right-angle or quarter turn (showing 4 such turns give a complete 360° turn). Introduce comp bearings of North, South, East and West.

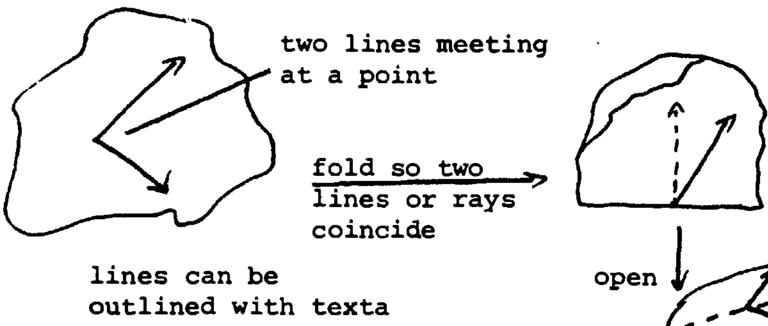
(3) Angle



Progress to angle as amount of "turn". Classify turns as more, less than right angle.

open and closing folded sheet (or book or hinged strips)

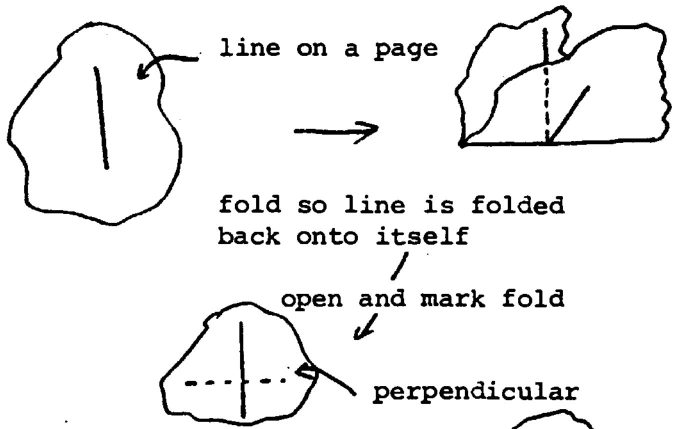
(4) Bisector of an angle



Quadrasect angles

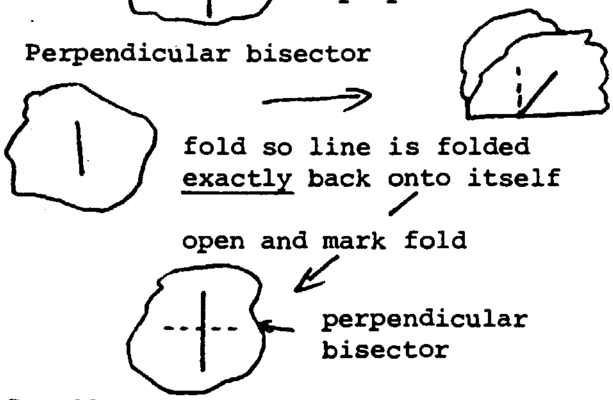
Use right angle and bisection to construct angle of 45°.

(5) Perpendicular

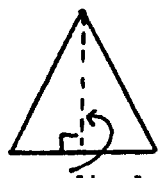


Construct perpendiculars from a point to a line (ensure fold goes through point). Construct squares, rectangles, et

(6) Perpendicular bisector

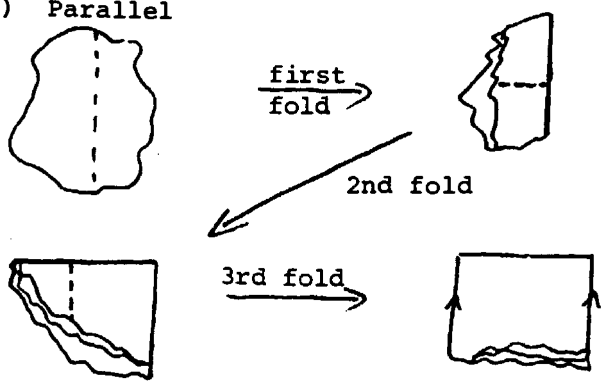


Construct isosceles triangles:



perpendicular bisector

(7) Parallel

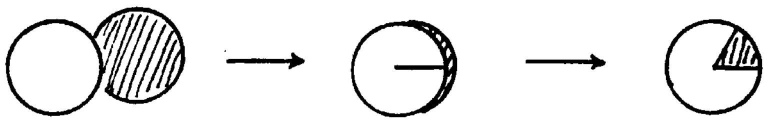


Testing for parallel lines (windows, doors, shelves) etc.

4. Which do you like best for line and angle activities: paperfolding or geoboards? Why?

5. Angle Wheel.

(1) Construct an angle wheel by cutting out two identical circles of different colours (shade one in). Put one circle on top of the other (the coloured one underneath) and cut a slit to the centre of both circles. With the slits in line and at 3 o'clock slip the lower right part of the top circle underneath the upper right part of the bottom circle as shown below.



(2) Turn the top circle anticlockwise and you will see an angle whose arms are the slits in the two circles.

(3) Use your angle wheel to make

(a) an acute angle
(b) a right angle
(c) an obtuse angle
(d) a straight angle
(e) a reflex angle.

Use your right angle reckoner. (It may be useful to mark, with lines, the 90°, 180°, 270° points on your top circle of your angle wheel).

(4) Which of the angles below are acute, obtuse or reflex?

6. Parallel lines can be advantageously studied by shadows in sunlight. Since the rays of the sun are aprallel, then parallel sides remain parallel regardless of the way we tilt shape and screen. (This does not happen in shadows by torchlight).

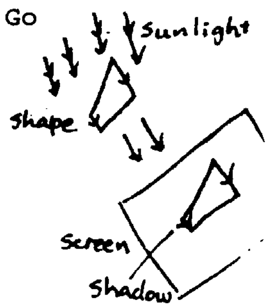
(1) Cut out a figure which has one pair of parallel sides. Go out into the sunlight.

(2) Cast shadows on the screen. Tilt screen and shape.

(3) Do the parallel sides remain parallel in the shadow?

(4) Go inside and cast shadows on the screen using a torch.

(5) Can you now get non parallel sides?



You can also study straight lines in this way. If your screen is flat a shadow of a straight stick is always straight.

7. (1) Identify as many occurrences of straight lines in your life as possible. For each occurrence:

   (a) Describe the occurrence;
   (b) Analyse why you think the line is straight.

   Do the same for parallel lines.

(2) Try to describe a straight line to a classmate without relating to physical objects.

(3) Briefly list those attributes of a straight line which distinguish it from a curved line. In what situations is each most useful? For example, which is easier to measure and why?

(4) Discuss a strategy that you would use to help young children learn what "straight" means.

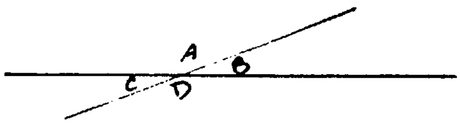
(5) What attributes of a straight line is Johny Cash evoking when he sings, "Because you're mine, I'll walk the line"?

(6) If you were walking on the equator, you would think that you were walking a straight line. Experiment with a globe to determine which curves on the globe share which properties of straight lines. For example, on a sphere what are the analogues to parallel lines?

8. (1) When two straight lines intersect in a point, several angles are formed. Are all of the angles always the same? Are all of the angles sometimes the same? Are some of the angles always the same? Are some of the angles sometimes the same? Experiment with some straight lines in order to arrive at answers to these questions. Illustrate your answers with examples.

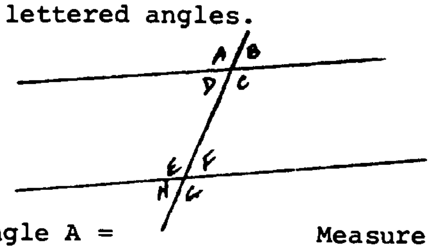
(2) Solve each of the following problems.

   a) Two straight lines intersect in such a way that the measure of angle A is 145°. Find the measures of angles B, C, and D. Use reasoning rather than a protractor to arrive at your answer.



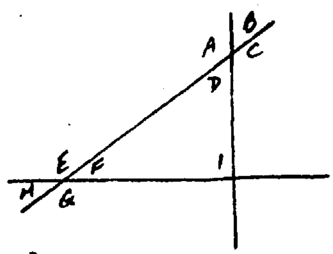
   Measure of angle A = 145° Measure of angle C =
   Measure of angle B = Measure of angle D =

   b) Two parallel lines are intersected by a transversal in such a way that the measure of angle F is 58°. Find the measures of the other lettered angles.



   Measure of angle A = Measure of angle E =
   Measure of angle B = Measure of angle F = 58°
   Measure of angle C = Measure of angle G =
   Measure of angle D = Measure of angle H =

c) Two perpendicular lines are intersected by a transversal in such a way that the measure of angle D is 36°. Find the measures of the other lettered angles.



Measure of angle A =
Measure of angle B =
Measure of angle C =
Measure of angle D = 36°
Measure of angle E =
Measure of angle F =
Measure of angle G =
Measure of angle H =
Measure of angle I =

(3) So far in this activity we have been analyzing relationships between infinitely extended straight lines. Most of the straight lines that occur in your life are finite and are often referred to as line segments. Describe and illustrate the possible relationships between two line segments in space. Check around to make sure that you haven't overlooked any.

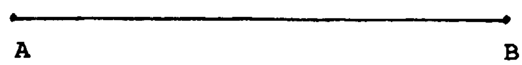
9. As a practical matter, one sometimes wants to reproduce accurately a certain geometric shape. A straight edge and compass can be convenient for doing this. Straight-edge-and-compass constructions can be fun for children, since they involve them in actually doing something and since they can produce an attractive and accurate result if some care is exercised.

Straight-edge-and-compass construction have the following as the rules of the "game".

(a) You can use the compass to make an arc of a circle.

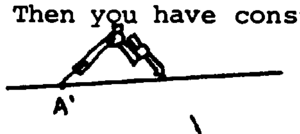
(b) You can use the straight edge to draw an "infinite" line or joint points. (No measuring or marking on the straight edge is allowed.)

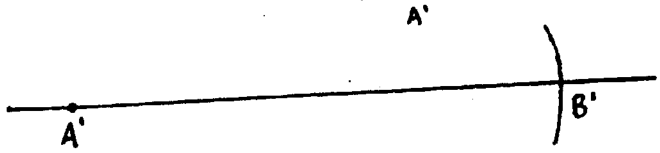
For example, suppose that you wanted to construct a line segment <u>congruent to</u> (same size and shape as) AB.



Step 1: Draw a line with your straight edge.

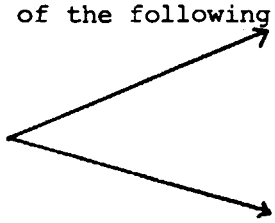
Step 2: Pick any point on this line and label it "A'"."Starting with point A',make an arc with radius equal to the length of the line segment AB. Call a point where the arc intersects the line B'. Then you have constructed A'B' congruent to AB.





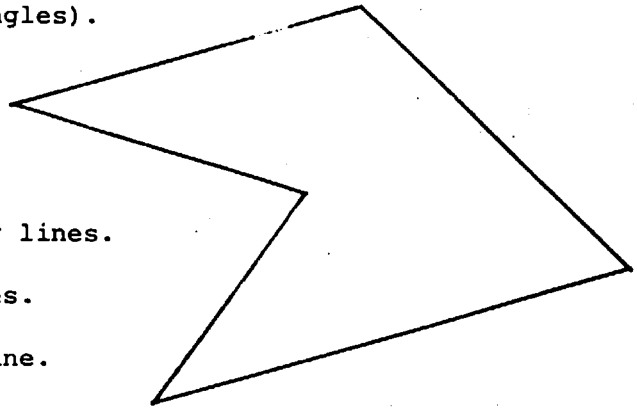
Use a straightedge and compass to do each of the following.

(1) Accurately reproduce this angle. Check your results by laying your constructed angle on top of this printed angle.



(2) Construct the bisector of the angle that you constructed in (1) (The bisector of an angle is a line that separates the angle into two congruent angles).

(3) Accurately reproduce this shape. (Check your construction).



(4) Construct two perpendicular lines.

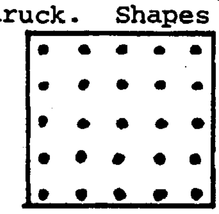
(5) Construct two parallel lines.

(6) Perpendicularly bisect a line.

Teaching Hints:

There are a variety of aids for teaching line and angle and many will do the job of others. Use this variety in your lessons. For example, paperfolding can do nearly everything that can be done with compass and straight edge. A particularly useful aid is the geoboard.

A geoboard is usually a square flat piece of wood into which equidistant rows and columns of nails have been struck. Shapes are outlined on this board with rubber bands.



Important: Ensure that no nails are loose before allowing them to be used. The rubber bands can dislodge a loose nail and turn it into a dangerous projectile.

The geoboard is one of the most widely useful aids for mathematics teaching. Some of the topics for which it can be used are constructing shapes, symmetry and tessellations.

It is desirable that each child in the class or the group be provided with the same type of board, so that comparisons and competitions are possible. For class work the nail and plywood types are suitable (perhaps an interested handy-man father could assemble one for each pupil). For small-group activities each child may use the plastic type with pegs and holes, inserting only those pegs required to give the desired number of pegs for the activities to be carried out.

Also required will be a number of strong, brightly coloured elastic bands for each child, for some activities each child will require a length of string, cord or strong yarn (about 69cm long).

Note: As with the diagrams in these notes, the geoboard can (?) be replaced with dot paper (paper with dots where the nails are) and rubberbands with coloured pencil lines.

Good activities with which to start children with geoboards are:

(a) Free Play - children create their own designs, shapes, patterns, pictures. Ask them to describe what they have done (name of shapes, number of boards used, etc.)

(b) Copying Patterns - teacher shows a pattern, children copy (or children do this with each other). At first allow children to see how pattern has been made, later only show finished produce.

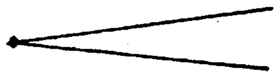
(c) Memory Patterns - Teacher shows or makes a pattern. Children study pattern (say for 1 minute). Pattern removed, children copy it from their memory.

The rubberbands can be replaced by <u>wet</u> string (makes it flexible and not "curl upwards"). The string can tie freely <u>between</u> nails. Most of the work with rubberbands can be repeated but with the following added:

(i) we can make curves or paths which are not based on straight lines and shapes with curved boundaries (i.e., not polygons)

(ii) we can make regions with <u>no</u> nails on their boundary

(iii) we can make rays by tying one end of the string to a nail and pulling tight. Hence we can make angles

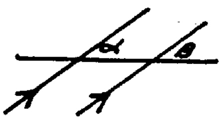


(e.g., sharp angles, acute and obtuse angles, right angles, "narrow" and "wide" angles, angles with 2, 3, ... nails between the rays and so on)

(iv) we can wind the string around a nail, pull it tight and wind it around another nail to show a straight line segment - then stretch out the ends of the string to show that this line segment is part of a longer line.

There are many <u>rules and techniques</u> with line and angle. Instruction should be such that children are guided to <u>discover</u> these. For example:

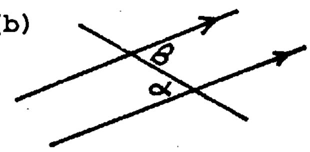
(a)



$\alpha = \beta$

(corresponding angles)

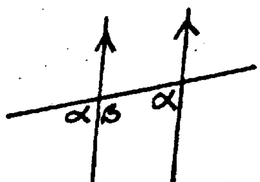
(b)



$\alpha = \beta$

(alternate angles)

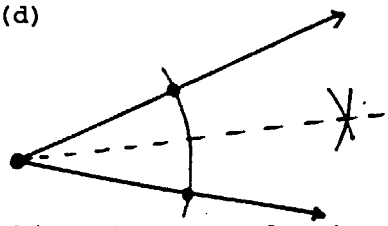
(c)



$\alpha + \beta = 180°$

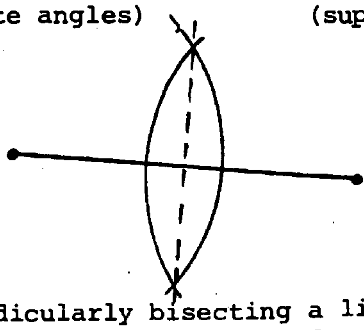
(supplementary angl

(d)



bisecting an angle with a compass and straightedge

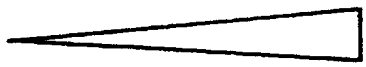
(e)



perpendicularly bisecting a line segment with compass and straightedge.

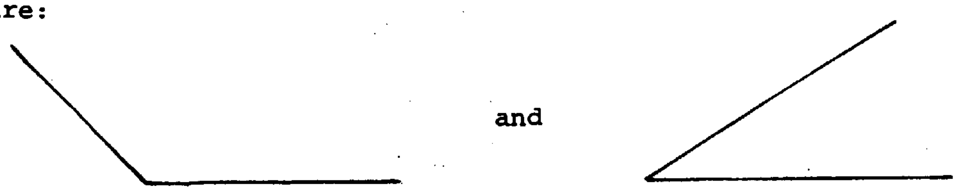
are all amenable to discovery teaching. Set up the situation and allow the children to explore the possibilities. The environment can be used. For example, trellises and shelving are good examples of parallel lines and their angle properties.

It should also be noted that angle can be considered as part of measurement. Therefore it is best if informal angle work procedes measuring with a protractor. Angles can be compared using paper cutouts. A "sharp" paper cutout, e.g.,



can be used to measure angle in terms of how many copies of it fit inside the angle.

Angles of various size can and should be constructed. Compass and ruler, circle wheels, geoboards and paperfolding can all be used. Of particular interest are:



and ... angles.

OBTUSE ... ACUTE

Paper folding can be used to construct a right angle measurer to determine (by comparison) angles which are larger (obtuse) or smaller (acute) than this right angle.

Finally units (degrees) can be introduced - as 1/360th of a complete turn.

UNIT 9: IDENTIFYING TWO DIMENSIONAL SHAPES

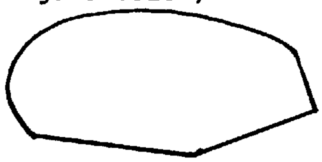
Focus:

In this unit we explore the range of two dimensional shapes that can be studied by primary children: polygons, such as triangles, squares, rectangles, parallelograms, hexagons, etc., and non polygons such as circles. Discussion on how to teach such shapes will be left to the next unit.

Background:

The study of simple two dimensional shapes is the basis of primary geometry. Below we describe the major shapes that are usually in the primary syllabus.

But before this, a general definition. A two dimension shape lies in a plane and is a closed simple boundary consisting of straight and curved lined segments. For example, the figure below (we should actually say the boundary of the figure below)



is a shape because

(1) it is a boundary in a plane;

(2) it is closed, not open like  ;

(3) it is simple, not crossing like a figure eight; and

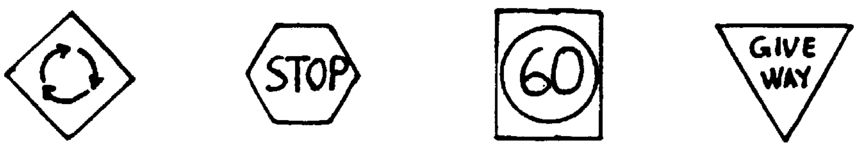
(4) it is composed of curved and straight lines.

Note: The point at which two straight lines or a curved and straight line meet or intersect is called a vertex. (similarly to three dimensional shape).

Polygons:

Any closed simple plan figure consisting of three or more vertices and straight line segments (sides) joining them is called a polygon. The name comes from "poly", a greek word for many, and "gonos", a greek word meaning angles.

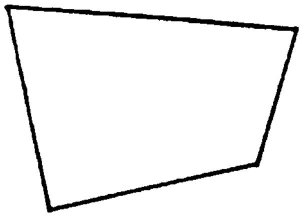
There are many examples of polygons in everyday life, e.g., road signs.



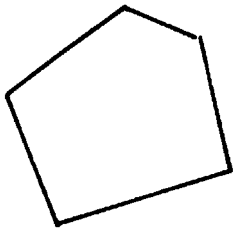
Polygons can be convex or non-convex. A convex polygon does not "Jut in", e.g.,

convex non-convex

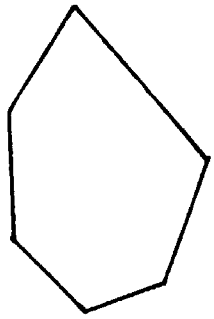
Polygons can be different sizes, and can have sides of unequal length and angles of unequal size. Polygons are named from their number of sides, e.g.,



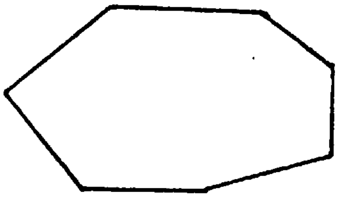
Quadrilateral
4 sides



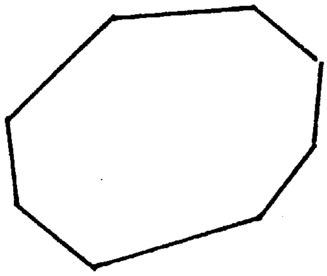
Pengagon
5 sides



Hexagon
6 sides

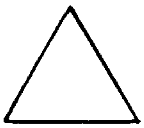


Heptagon (or Septagon)
7 sides

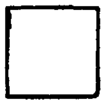


Octagon
8 sides

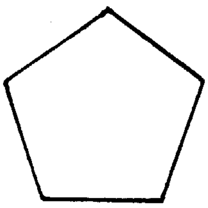
Polygons can have sides of equal length, and angles of equal size, and these are called REGULAR POLYGONS, as in the examples below:



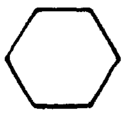
Regular Triangle
(equilateral
triangle)



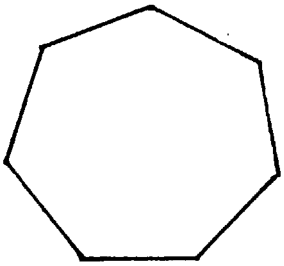
Regular Quadrilateral
(square)



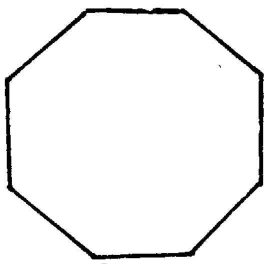
Regular Pentagon



Regular Hexagon

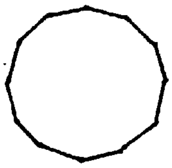


Regular Heptagon (or Septagon)



Regular Octagon

As you may have noticed, as the number of line segments is increased, the plane shape becomes more "circular"? You will see this more clearly when you look at the regular polygon, called a dodecagon, below (it is the same as a 50 cent piece);

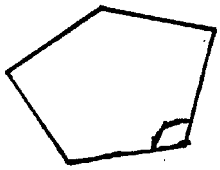


To summarize, the following diagram shows how polygons can be sorted into special groups.

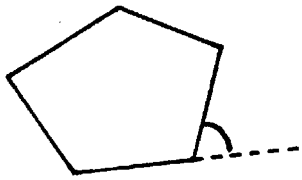
POLYGONS

Triangles (3 sides)	Quadrilaterals (4 sides)	Pentagons (5 sides)	Hexagons (6 sides)	Heptagons (7 sides)	Octagons (8 sides)	Nonagons (9 sides)	etc

For a polygon, we say that an interior angle of the polygon is inside the polygon while an exterior angle of the polygon is the amount of turn if a person was walking the polygon (i.e., the angle between one side continued and the next side).e.g,



interior angle

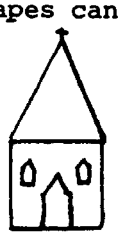


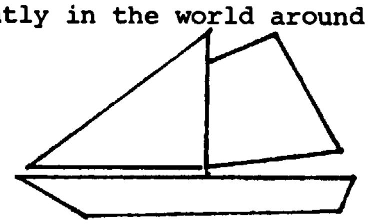
exterior angle

Triangle:

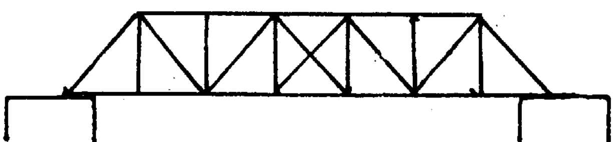
A triangle is a polygon of three straight lines. It is a plan figure (i.e., it has only two dimensions - length and breadth). It can be drawn on one surface.

Triangular shapes can be seen quite frequently in the world around you, e.g.,

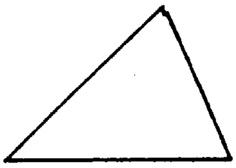




An interesting fact about triangles it that once a triangle is formed, its shape cannot be changed without changing the length of its sides. Triangles, then, are said to be rigid. Triangular shapes are often used to make buildings and bridges more rigid, e.g.,

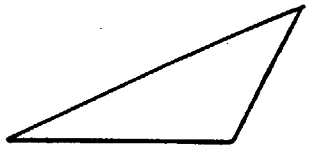


All triangles have three sides and three angles. Traingles are named according to the size of their largest interior angle, e.g.



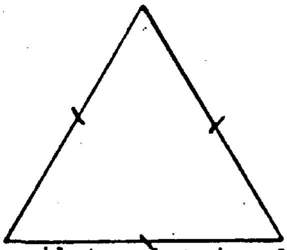
acute-angled triangle

right-angled triangle

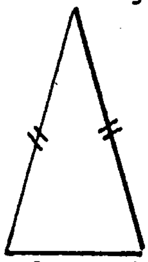


obtuse-angled triangle

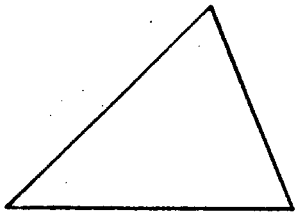
Right angled triangles obey Pythagoras' Thearem that the sum of the length of the hypotenuse squared (the hypotenuse is the side opposite the right angle) is equal to the sum of the squares of the lengths of the other two sides. Triangles can also be named according to the lengths of their sides, e.g.,



equilateral triangle
(all sides equal all angles 60°)

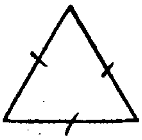


isosceles triangle
(two sides equal, two angles equal)

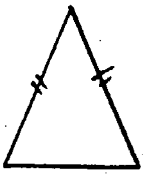


scalene triangle
(no sides or angles equal)

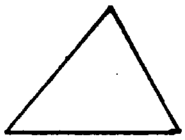
If you need to describe a triangle even more specifically than by the size of its largest angle or by the length of its sides, you would combine both ways, e.g.,



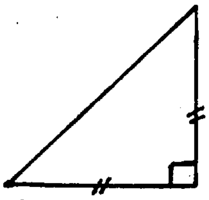
acute-angled equilateral triangle



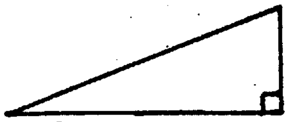
acute-angled isosceles triangle



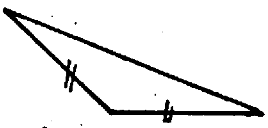
acute-angled scalene triangle



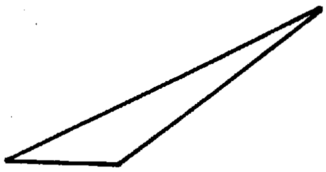
right-angled isosceles triangle



right-angled scalene triangle

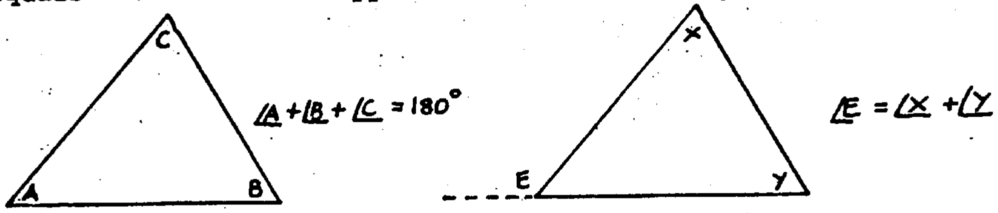


obtuse-angled isosceles triangle



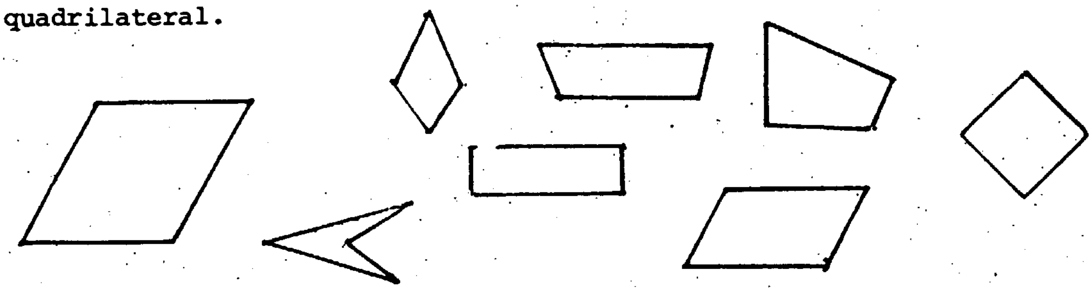
obtuse-angled scalene triangle

The interior angle sum of a triangle is 180° and any exterior angle of a triangle equals the sum of the opposite two interior angles, e.g.,



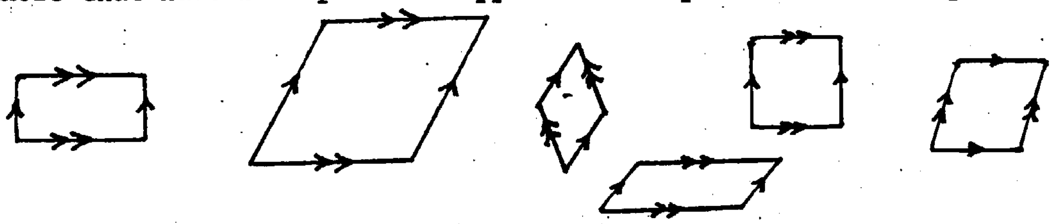
Quadrilaterals:

A quadrilateral is a polygon consisting of four straight lines. It comes from "quadri", meaning four, and "lateral", meaning side. Everything below is a quadrilateral.

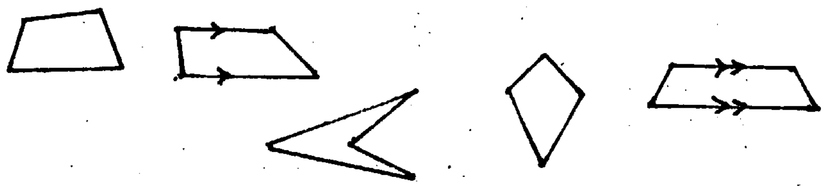


Quadrilaterals can be divided into

(1) those that have both pairs of opposite side parallel (called parallelograms)



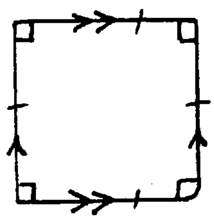
(2) those that do not, e.g.



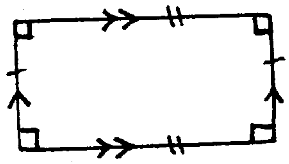
(Note: parallel lines are shown by arrow heads)

The important quadrilaterals are:

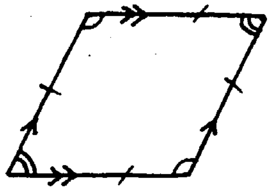
(1) the square - both pairs of opposite sides are parallel
- all sides are equal
- all angles equal 90°
- is a rectangle, a rhombus, a parallelogram and a trapezium



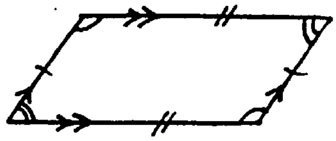
(2) the rectangle - both pairs of opposite sides equal and parallel
- all angles equal 90°
- is a parallelogram and a trapezium



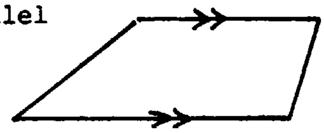
(3) the rhombus (this is sometimes called a diamond)
- both pairs of opposite sides are parallel
- all sides are equal
- both pairs of opposite angles are equal
- is a parallelogram and a trapezium



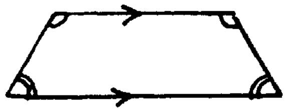
(4) the parallelogram
- both pairs of opposite sides are equal and parallel
- both pairs of opposite sides are equal
- is a trapezium



(5) the trapezium - a pair of opposite sides parallel



(the isosceles trapezium has the non parallel sides equal and angles adjacent to the parallel sides equal).



(Note: the use of the symbols , and)

The quadrilaterals have particular properties with regard to <u>diagonals</u> but these will be studied in unit 11.

As can be seen in the above, there is a relationship between plane shapes, quadrilaterals, trapeziums, parallelograms and special parallelograms as can be seen in the following:

Plane shapes

↓

Polygons
Closed simple shapes that have 3 or more vertices and line segments joining them

↓

Quadrilaterals
Polygons that have 4 vertices and 4 sides

↓

Trapeziums
Quadrilaterals that have a pair of sides parallel

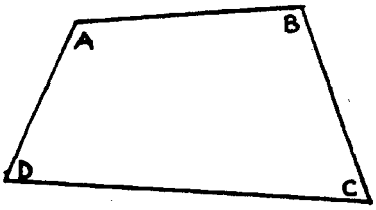
↓

Parallelograms
Trapeziums that have both pairs of opposite sides parallel

↓

Special parallelograms
Rectangles, Squares, Rhombus

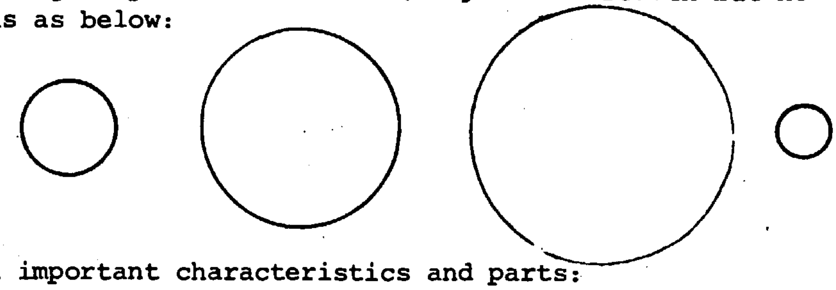
The sum of the interior angles of a quadrilateral is 360°, i.e.,



$\angle A + \angle B + \angle C + \angle D = 360°$

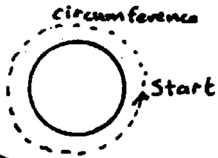
Circles:

A circle is a boundary which is equidistant from a point (its centre). It is a plane figure having only two dimensions (length and breadth but no thickness). It looks as below:



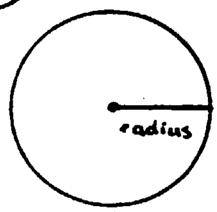
Circles have certain important characteristics and parts:

(1) Circumference - the distance around the circle (this is called the perimeter in any other shape).

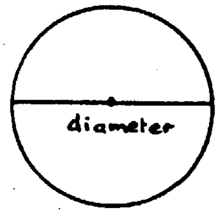


(2) Centre - the point that is equidistant from all points on the circle

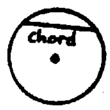
(3) Radius - the distance from the centre to every point on the circle
- is represented by a line as shown



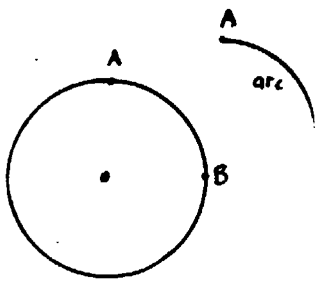
(4) Diameter - the distance across the circle through the centre
- is represented by a chord through the centre as shown
- there is an infinite number of diameters in a circle
- the diameters also represent the lines of symmetry of the circle



(5) Chord - a line segment drawn from one point of the circle to another



(6) Arc - a section of a circle, having two end points



(7) Semicircle - half the outline of the circle

Regions and discs:

Shapes are outlines or boundaries. We can recognise shapes, like circles and quares, by there outlines. However, shapes also enclose a certain amount of surface. This surface is called a region. Regions are named from the shape that bounds them. For example:

(1) a rectangle, as on the right, has a boundary of four line segments - we can measure the distance around it, a distance which we call perimeter; but

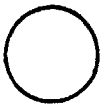
(2) the shaded interior of the same rectangle is called a rectangular region - we can measure this interior, a measurement which we call area.

Similarly a triangular region and a square region are shown below by (i) and (ii) respectively

(i)

(ii)

The circular region below has a special name - it is called a disc:



a circle is the boundary only

a disc is the boundary plus the region.

Discs have certain important parts:

(1) semi disc - half the boundary and half the region (including the centre)

(2) quadrant - one quarter of the boundary and one quarter of the region (including the centre)

(3) sector - part of the outline and part of the region bounded by an arc and two radii

(4) segment - part of the outline and part of the region bounded by a chord and an arc.

Materials:

Geostrips, compass, ruler or straight edge, pen, paper.

Activities:

1. Use geostrips to do the following:

(1) Construct a 3-sided, a 4-sided, a 5-sided and a 6-sided shape. What are their names.

(2) Construct a parallelogram, a rhombus, a trapezium, an isosceles triangle, an equilateral triangle, a scalene triangle.

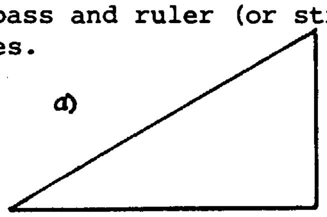
(3) Make a square. Attach 2 rubber bands to act as diagonals (they must be stretched). Check that the angles of a square are right angles. Check that the diagonals are the same length and bisect each other at right angles. What happens to the diagonals if we push the square out of shape?

(4) Do the same as (3) above for a rectangle.

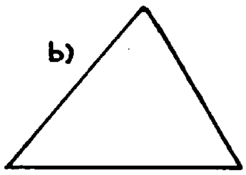
(5) Make a triangle exactly the same as another triangle. Make a triangle that looks exactly the same as these 2 triangles but is bigger.

2. (1) Use compass and ruler (or straight edge) to copy these two triangles.

a)



b)



(2) Use compass and ruler to construct triangles (if possible) of sides of lengths

a)	6cm	2cm	3cm
b)	3cm	4cm	5cm
c)	3cm	4cm	7cm
d)	8cm	4cm	6cm

(3) Can you think of a rule to predict from lengths of sides, whether a triangle is possible? Test your rule on these two triangles:

a)	5cm	7cm	8cm
b)	5cm	7cm	13cm

(4) Using ruler, compass and protractor can you construct a triangle for the following cases?

(a) Two sides 3cm and 4cm and angle 50° between them.
(b) Two sides 5cm and 3cm and angle 70° not between them.
(c) Two angles 45° and 75° and a side 5cm between them.
(d) Two angles 30° and 80° and a side 4cm not between them.
(e) Three angles 40°, 75° and 65°.
(f) Three sides 4cm, 6cm and 8cm.

(5) Which of the above cases defines a unique triangle?

(6) Can you use a ruler and a compass to construct circles for the following cases?

(a) A point on the paper, the circle passes through the point.
(b) A point on the paper, the centre of the circle is this point.
(c) Two points on the paper, the circle passes through both points.
(d) Three points (not in a line) on the paper, the circle passes through all three points.

(7) Which of the above cases uniquely defines a circle? Which defines only two possible circles?

4. Find examples of two dimensional shapes in the environment. Write them into the following table.

SHAPE	EXAMPLE
Square	
Rectangle	
Circle	
Parallelogram	
Trapezium	

5. Shapes are not only related to geometry. Shapes can be used for aesthetics, for traditional reasons, for ease of construction, because of utility of a specific attribution, for mystical or religious reasons and because of the laws of physics.

(1) For what reasons are circles involved in the following:

(a) people forming a ring around an object they all want to see;
(b) puddles of water are usually round;
(c) round dinner plates;
(d) round car wheels
(e) hand made bowls are usually round;
(f) basketballs have circular cross sections;
(g) pumpkins have circular cross sections;
(h) the master charge card has intersecting circles on it.

(2) What is the reasons for such cliches as:

(a) big wheels?
(b) social circles?
(c) the eternal triangle?

Teaching Hints:

As we did for three dimensional shapes in chapter 2, we have looked at the various two dimensional shapes before looking at how you teach them so that you will have more complete information on content to bring to the next unit. Two dimensional shapes should not be imitatively taught to children through "show and tell" demonstration. Children should be actively involved in experiencing them.

In fact the style of teaching most appropriate for two dimensional shape is reflected in the way we have in the activity for this unit, developed the rule for constructing triangles with certain lengths of sides. Instruction should be a 4 step process.

(1) The children should experience constructing triangles from directions.

(2) The children should discover a rule from that experience.

(3) The children should test their rule to see if it is consistent in other cases.

(4) The children should apply the rule in new situations.

The teaching of geometry is also facilitated through children constructing their own examples using appropriate equipment or material. One particularly good structured material for two dimensional shape is geostrips. Geostrips are like meccano consisting of strips and fastners. The strips may be of plastic or metal, as are meccano and other building strips, with special fasteners or nuts and bolts for joining, for example.

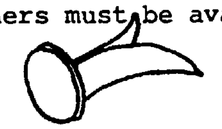
Just as useful are cardboard strips. These should be made of very firm strong cardboard, about 2cm wide and provided in a variety of lengths. The following are desirable.

(a) A number of strips all of the same length, and punched 1cm from each end. A suitable size would be 16cm long.

(b) A larger number of strips of many different lengths, several of each length, punched at each end as above, and also at equal intervals (2cm) along the length. It would be desirable, to facilitate selection by the children, to paint in the same colour all strips of the same length, and to use different colours for each different length.

(c) Some unpunched strips which the child may punch to suit his needs (also a paper punch with which to do so).

Plenty of paper fasteners must be available for hingeing the strips together.



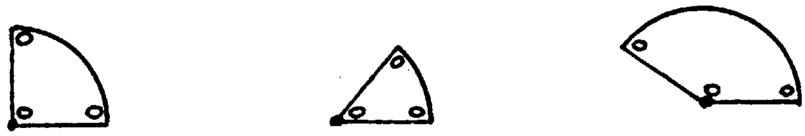
The children should work in pairs or singly in small group situations so that they can have access to the full range of strips.

As a beginning, children use the strips creatively to make pictures and shapes of their own choice. Then the strips can be used to construct shapes as directed by the teacher.

Shapes are the strength of geostrips. Children can make any polygon. Further activities that can be attempted are:

- making patterns of squares, triangles, etc.
- trying to make a circle
- making equal sided (regular) polygons
- making similar shapes.

Preceding this work on shapes, geostrips can be used to show line and path. Two strips can also be joined to make a hinged angle. Sector shaped pieces of cardboard, such as those below, can assist.

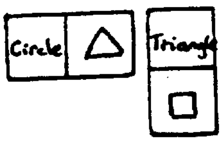


Strips can be connected to these to make angles which hold firm.

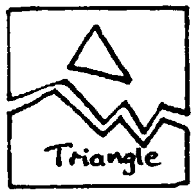
As a language, two dimensional shape must be consolidated with children. Recognising and naming shapes should become automatic.

Drill games such as:

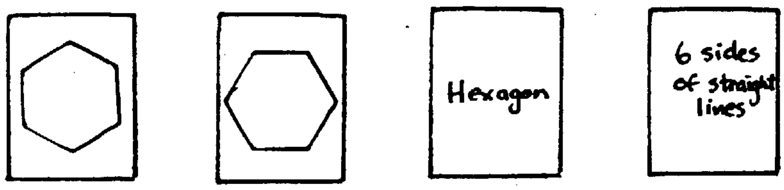
dominoes



mix and match cards



card decks for gin rummy, concentration and snap,



bingo, and flash cards should be used to reinforce and maintain recognition.

UNIT 10: TEACHING TWO DIMENSIONAL SHAPE

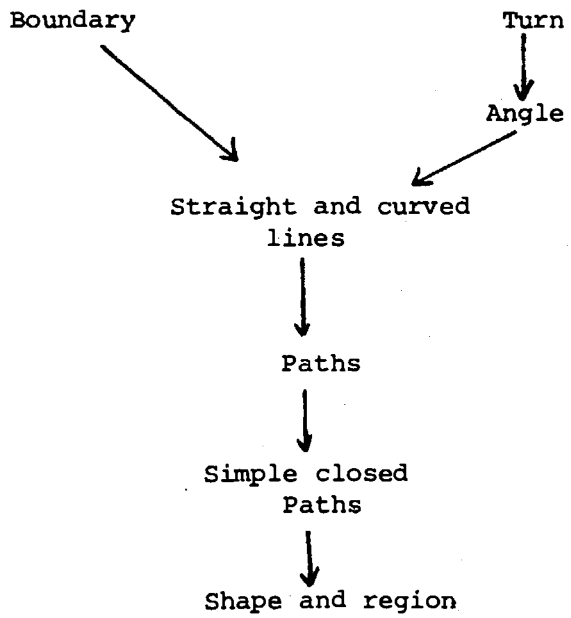
Focus:

In this unit we focus on methods to introduce and consolidate two dimensional shape. The main emphasis in the unit is on the subconcept approach. The environmental approach has been covered to some extent in chapter two and the transformational approach will be covered in chapter seven.

Background:

Children should be given many opportunities to construct two dimensional shapes, (by geoboards, geostrips, paperfolding, and compass and straight-edge), to analyse such shapes and to manipulate such shapes. The introduction of the two dimensional shape type should contain many opportunities for children to experience many examples (and non examples) of the type.

As we have said before, a two dimensional shape can be considered as a simple closed boundary composed of straight or curved lines with the inside being called a region. Therefore, in the subconcept approach, such a shape can be developed from these other concepts as the diagram below shows.



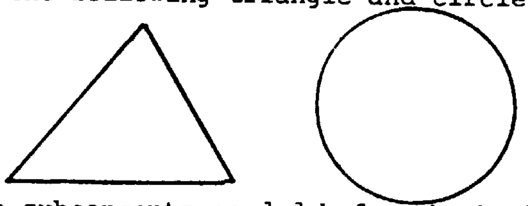
Alternatively the environmental approach uses the world around us to find and classify examples of shapes. It seems appropriate to use this environmental approach as the starting point for instruction but to fall back to the subconcept approach for more formal analysis. The beginning activity of unit 9 covers much of what can be done using the environment and we also have the activities in chapter two, hence we begin here with the subconcept approach.

Materials:

Geoboards, rubberbands, paper, rectangular paper, circle paper.

Activities:

1. (1) Look at the following triangle and circle.



List the subconcepts needed before both shapes can be fully understood by children.

(2) Look back at the geoboard activities in unit 8. Then use a geoboard and rubber bands to work the following activities.

PATHS AND REGIONS		
Use rubber bands	*(1)*	*Choose one nail. Choose another as far away as possible. Make a path of line segment. from one nail to the other. How many times did your lines cross? How many corners did you make?*
	(2)	*Repeat (1) above but return to the first nail. Try again without any crossings (to make a region). How many corners?*
	(3)	*Make a region with 5 nails on the outside (restrict yourself to 25 nails).*
	(4)	*If the geoboard has 25 nails and there are 2 nails on the inside of a region and 3 on the outside, where are the other 20? Make such a region on your board.*

SHAPES		
Use rubber bands	*(1)*	*Can you make a shape with 1 side? 2 sides? 3 sides? What is the 3 side shape called?*
	(2)	*How many triangles can you make with 2 rubber bands? With 3 rubber bands? Who has made the most triangles?*
	(3)	*Can you make a star from several triangles?*
	(4)	*Make a rhombus. Make a parallelogram. Make a trapezium.*
	(5)	*Make a square inside a square. Make a triangle inside a rectangle. Make a square inside a triangle inside a square. Make a parallelogram overlapping a trapezium.*

CURVES AND SHAPES:

Use string (wet)	*(1) Make a simple open curve.*
	(2) Make the longest open curve you can with your string (all of it on the board).
Restrict to a 5 x 5 geoboard	*(3) Construct a region that has only 1 nail inside it. (2, 3, 4).*
	(4) Construct a region that has no nails inside.
	(5) Construct a region that has 1 nail outside. (2, 3, 4)
	(6) Construct a region that has 1 nail outside and 3 nails inside.
	(7) Construct a region that has 3 nails on the boundary.

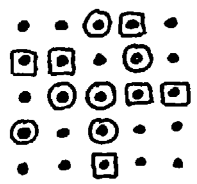
GAMES AND PUZZLES

(1) The Square Game (for two players)

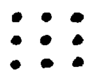
You will need a 5 x 5 nail geoboard and some small rings of two different colours for putting over the pins. Small pieces cut from drinking straws of two different colours are quite suitable.

The first player places a ring of his colour over any pin and the second player then places a ring of his colour over any other empty pin. Take it in turns until one player wins by making a square with rings of his colour at all four corners.

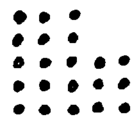
(2) In the square game on right, each of the players has had 6 turns. It is now red's turn. In this game, no matter what red does now, black is assured of winning. How ?



(3) Consider this 3 x 3 square of nails. Construct five differently shapes and different sized isosceles triangles using this square.



(4) Make this shape on a geoboard. Divide it into four parts, each part being the same size and shape.



2. If you join these activities to those on the geoboard in unit 8, we have a progression of activity.

line ⟶ angle ⟶ paths and regions ⟶ shape

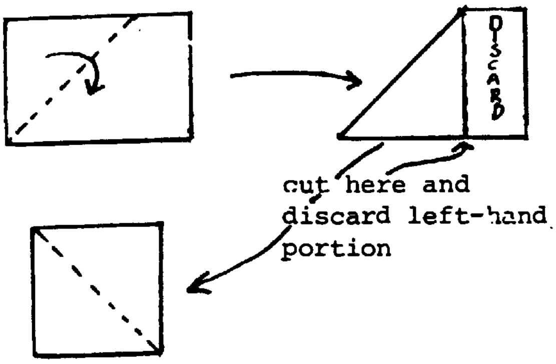
It is possible to ask children to construct, e.g., 5 different shapes composed of three sides. Everything that is constructed will be a triangle. Similarly it is possible to direct children to construct, e.g., 5 different shapes of 4 sides with one opposite pair of sides parallel. These examples can then be discussed and named with "trapezium".

What are the advantages of this approach over "show and tell"?

3. Use paperfolding to complete the following activities:

Representation with paper	Development

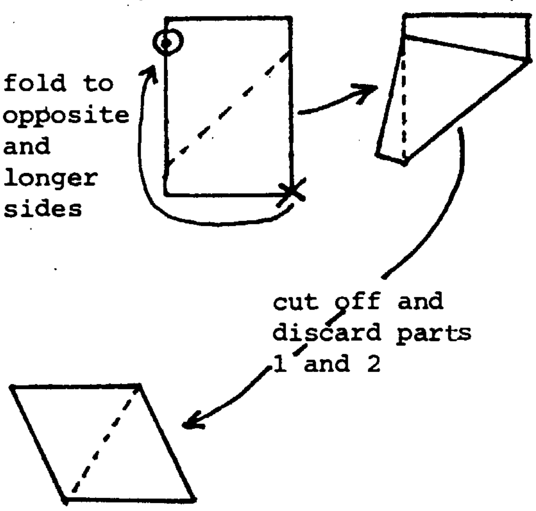
(a) Square (fold from a rectangle)



By folding, show opposite and adjacent angles and sides are equal.

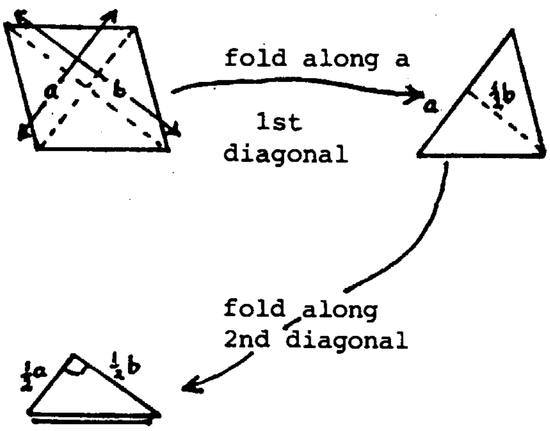
Fold along diagonals. Mark the diagonals in with a texta colour. By folding, show the diagonals bisect each other. Use right-angle measurer to show all angles are right angles, including where diagonals meet.

(b) Rhombus (folded from a rectantle)



By folding, show opposite angles and opposite and adjacent sides are equal. By folding show adjacent angles are not equal.

Fold diagonals. Mark in the diagonals with a texta colour. By folding show the diagonals perpendicularly bisect each other.



Area of resulting triangle

$= \frac{1}{2} \cdot \frac{1}{2}a \cdot \frac{1}{2}b$

$= \frac{a \cdot b}{8}$

Area of rhombus

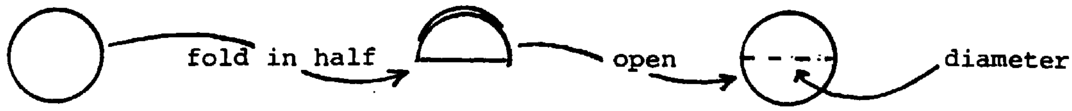
= 4 times area of triangle

$= \frac{a \cdot b}{2}$

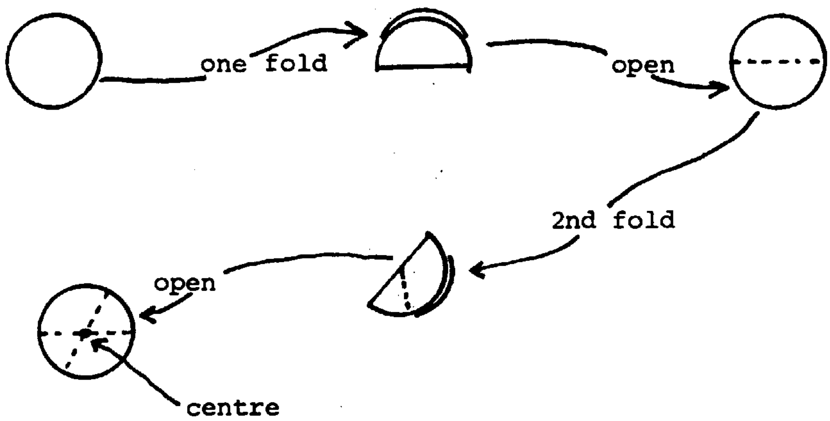
½ product of diagonals

Representation with paper | Development

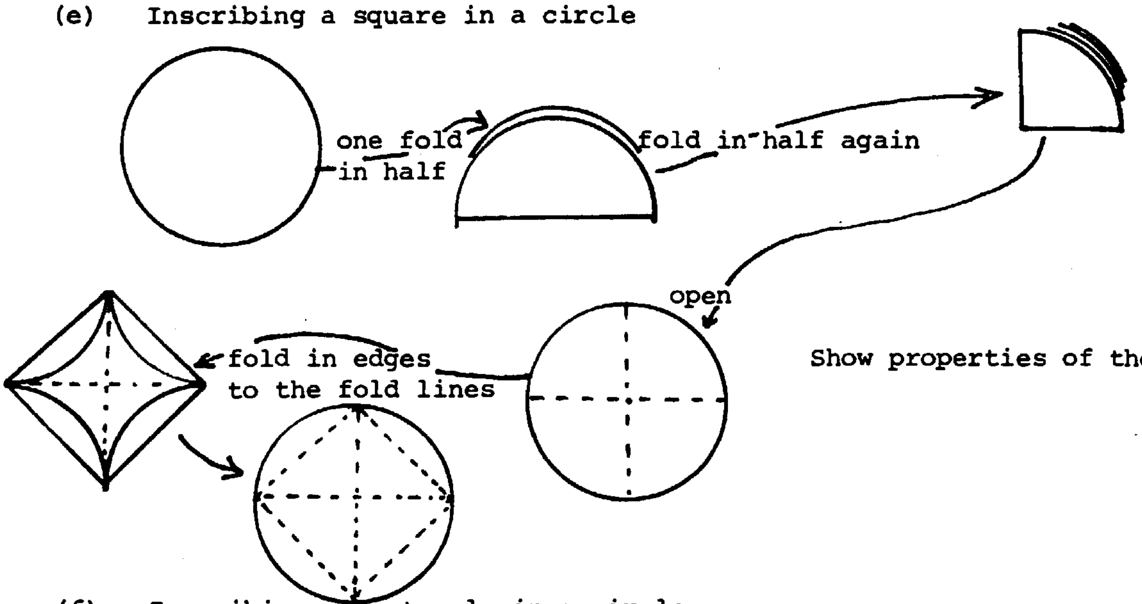
(c) Diameter of a circle



(d) Centre and radius of a circle

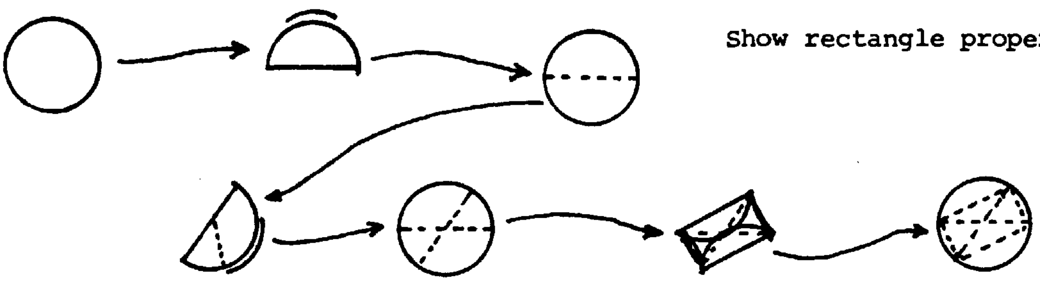


(e) Inscribing a square in a circle



Show properties of the square.

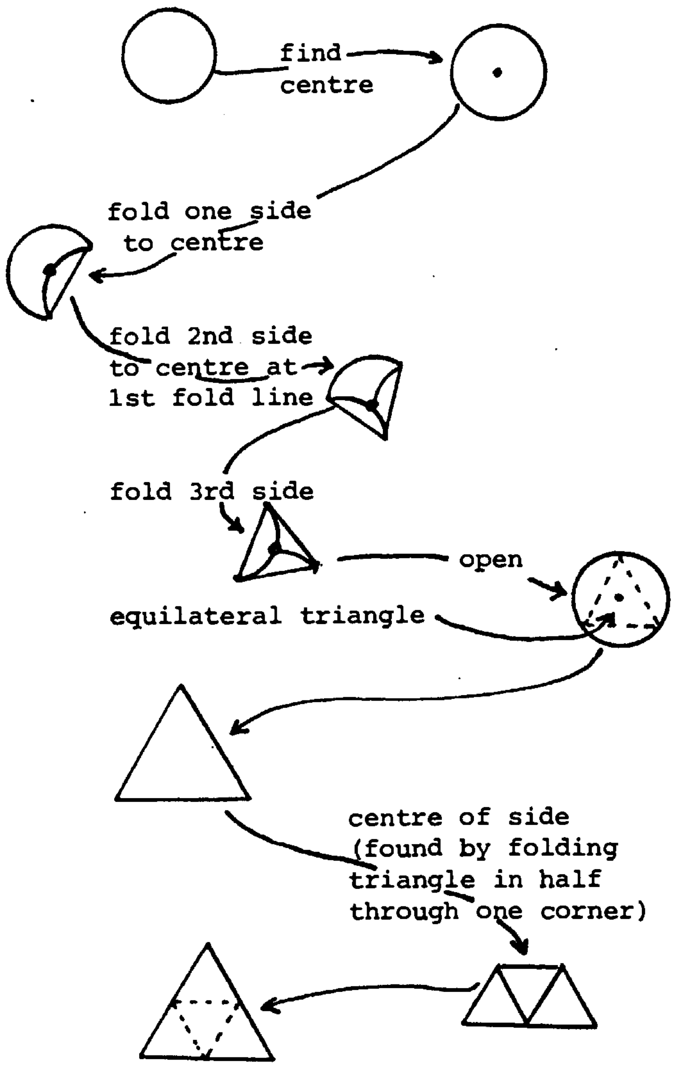
(f) Inscribing a rectangle in a circle



Show rectangle properties.

Representation with Paper	Development

(g) Inscribing a triangle in a circle

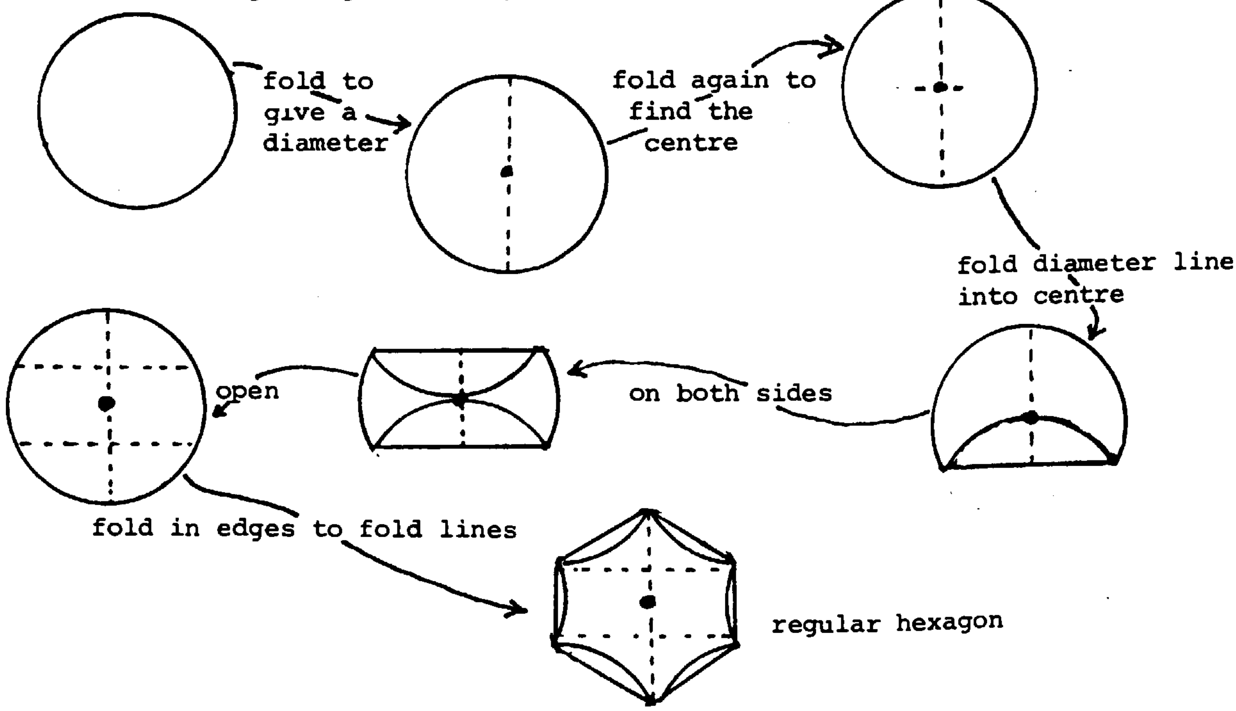


Show properties of an equilateral triangle by cutting off excess and folding (all sides and angles equal). Show 3 lines of symmetry.

Puzzle to fold equilateral triangle into 4 smaller equilateral triangles.

Construct a tetrahedron from the folded triangle. Construct a hexagon by folding the corners of the triangle into the centre.

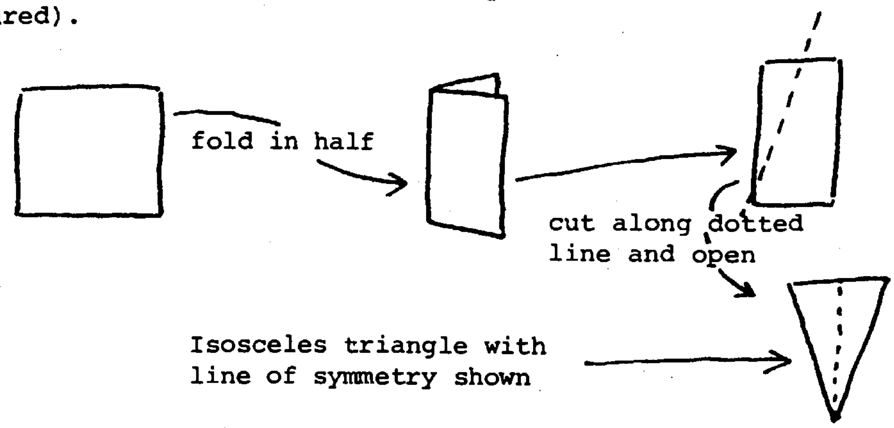
(h) Inscribing a regular hexagon



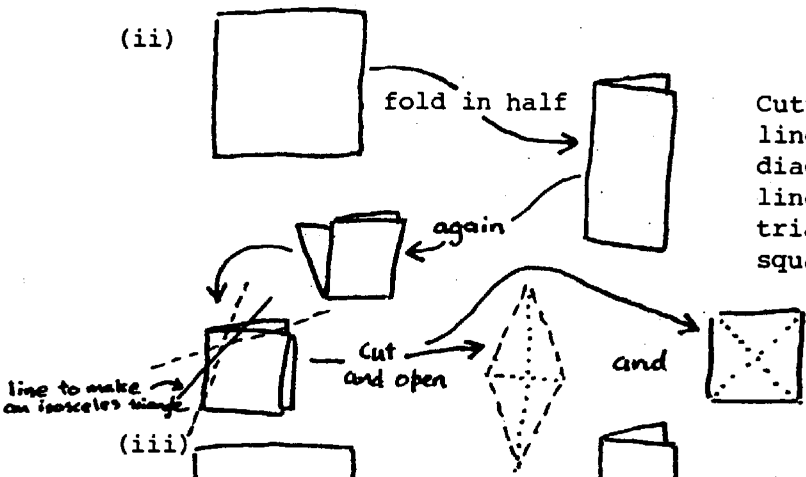
(i) Discovering properties of regular polygons by cutting and folding

You need squares of paper (preferably coloured).

(i)

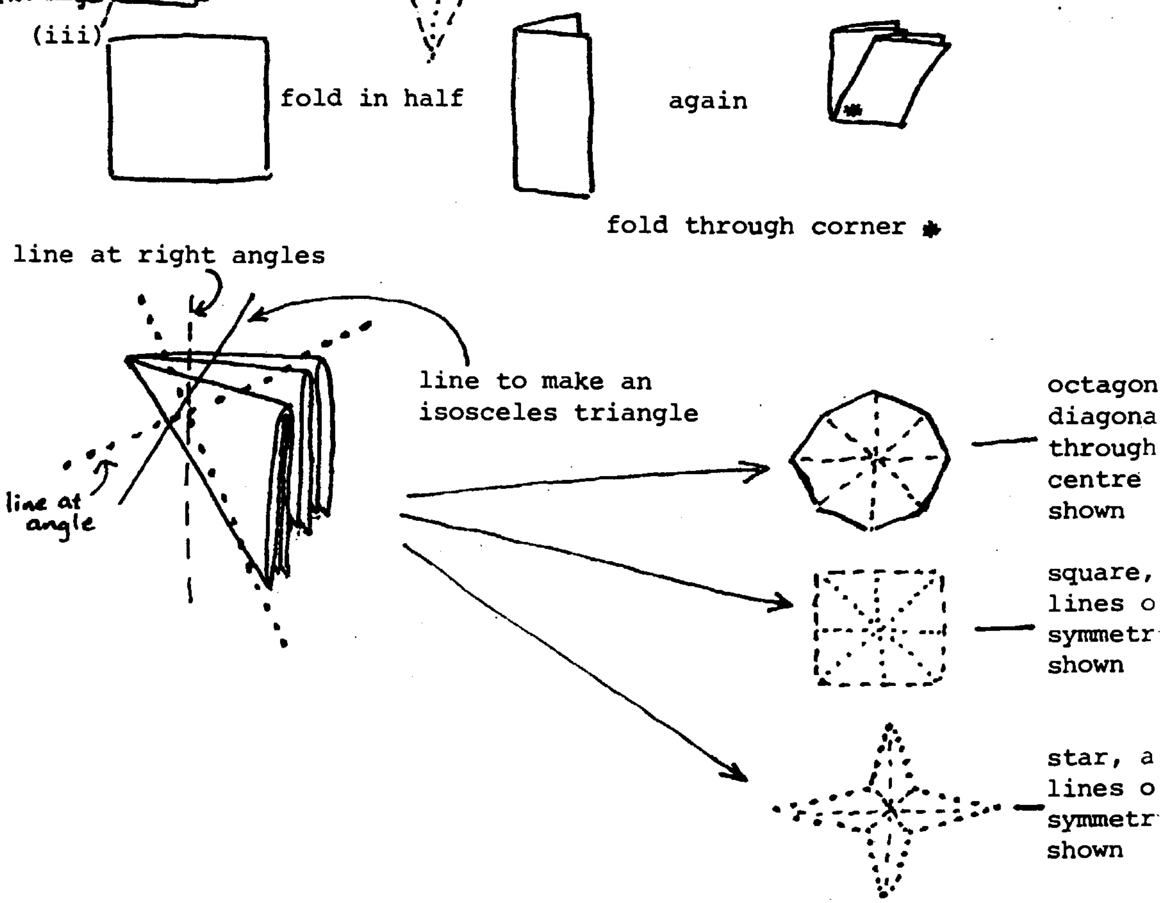


(ii)



Cutting along any angles (dotte line) gives rhombus with diagonals shown. Cutting along line at 45° or to give isoscele triangle (firm line) makes a square with diagonals shown.

(iii)



Teaching Hints:

As is obvious from the activities in this unit, we advocate children constructing shapes and testing them for properties. The two materials we have exhibited are geoboards and paper folding.

Geoboards are excellent for lines, Children can

- join two nails with a rubber band to make a straight line segment.
- make the shortest line, the longest line, make as many lines as possible without any touching, make as many different length lines as possible, make lines until teacher says stop (how many did you make? How many crossed another line?).

Children can join lines to make paths between 2 chosen nails (does not have to be shortest path as in lines above). Questions about their path can be asked (How many turns? How many rubber bands? Is it the longest path?)

The children can be restricted in their paths to a certain number of rubber bands, or turns, or nails, or number or crossings, (paths crossing itself). Children can focus on the different paths they can make or the different places they can end (given a common starting point). Children can make closed paths (from a nail back to that nail). We can use these paths to introduce:

- curve (not a straight line)
- open (did not rejoin to where started)
- closed (did rejoin to start)
- simple (path did not cross itself)
- region (area enclosed by a simple closed path)
- boundary (where the rubber band goes)
- inside (inside the region)
- outside (outside the region.

The children can now make regions (or simple closed paths). They can be asked to make them with even/odd sides, sides all the same or all different lengths, a certain number of nails inside, a certain number of nails outside or a certain number of nails on the boundary. Some of these can be quite challenging. Suppose your class has a 5 x 5 or 25 nail geoboards. Children find it difficult to make regions with, say, 2 nails on the inside and 3 on the outside (and 20 on boundary) - particularly if you specify which nails are which. Discussion should be allowed and good results displayed.

Obviously, we can use paths to introduce two dimensional shape. We look at shapes with:

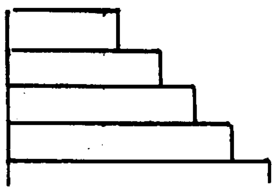
- one or two sides (can't be done)
- three sides (triangles)
- four sides (quadrilaterals, rectangles, parallelograms, squares, rhombi, trapezium)
- etc.

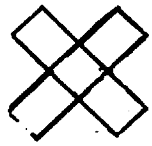
Once these shapes are named we can move onto

- how many we can make following certain rules (e.g., how many triangles can we make with 2 rubber bands)
- the different types of each shape (e.g., scalene, isosceles, equilateral triangle on a geoboard)
- dividing shapes into parts (e.g., a pentagon into triangles)
- more complex shapes (e.g., stars made from several triangles)
- and to area of shape by counting squares.

3 x 3 or 5 x 5 geoboards are easy to make and can do a surprising amount

- 76 triangles (8 isosceles, 28 right angled and isosceles, 16 right angled, 24 scalene) can be made on a 3 x 3 board
- 12 parallelograms can be made on a 3 x 3 board
- 28 trapeziums are possible on a 3 x 3 board
- non square rectangles are interesting on a 5 x 5 board (particularly if look at patterns such as:





Paperfolding is capable of doing a surprising amount of geometry also (see also unit 8 and further work in chapter 4). It is also a very cheap material - scrap paper can be used.

The subconcept approach to teaching geometry is quite appropriate for two dimensional shape. The prior knowledge is well learnt before shapes are attempted. The children construct the shapes before their names are given. In their constructions the children have already performed the right angles, parallel sides and equal sides that define a shape. For example, to get a parallelogram, children will be asked to make a shape of 4 sides, with opposite sides equal and parallel. Hence this is what the word "parallelogram" will be attached to. It can now be unnecessary to spend time teaching such properties. If open ended instruction is given, (e.g., "make 5 different shapes") children will construct many different shapes before a name is given making abstraction of commonality more complete.

The subconcept approach when used in an open ended problem solving way such as described above requires material which is capable of exploration with things being quickly and easily broken down and rebuilt. The geoboard and geostrips are excellent for this.

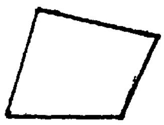
UNIT 11: PROPERTIES OF TWO DIMENSIONAL SHAPE

Focus:

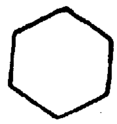
Many of the properties of shape that enable us to classify and name them (such as equal angles, parallelness, congruent sides, etc) have already beeh covered in this book. But there are many other properties and rules that are appliable to two dimension shape. In this unit we will look at some of these: particularly those to do with sums of lengths and of sides, angle sums, diagonals and rigidity. Other properties to do with symmetry, tessellation, similarity and congruence will be left to later chapters.

Background:

The number of sides is the major characteristic upon which names of shapes are determined, e.g.,



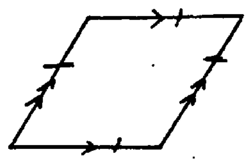
quadrilateral
(4 sides)



hexagon
(6 sides)

But names are also given for congruence of sides (and angles) and parallelness of sides, e.g.,

isosceles triangle
(2 sides and angles equal)

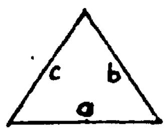


rhombus
(all sides equal, opposite sides parallel)

There are properties and rules for two dimensional shapes, that have to do with measures of their angles and sides and that are not needed for naming, that children should investigate. In this unit we will consider how some of these, which we have listed below, should be taught.

(1) All triangles have side lengths such that the length of any one side is always less than the sum of the lengths of the other two sides.

i.e.,

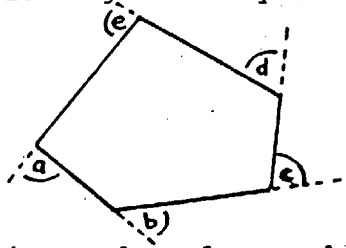


$c < a + b$ always

(Note: activities to discover this property will be found in unit 9.)

(2) The sum of the exterior angles of any convex pologyon is 360°

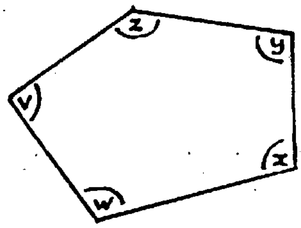
i.e.



$\angle a + \angle b + \angle c + \angle d + \angle e$

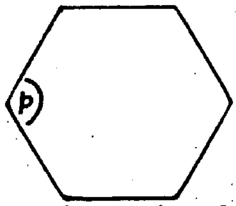
360°

(3) The sum of the interior angles of any polygon of n sides is 360° less than the preduct of the number of sides and 180 (or (n-2) 180°) i.e.,



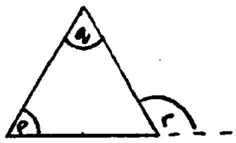
$\angle v + \angle w + \angle x + \angle y + \angle z = 5 \times 180 - 360$

(4) The interior angle for any regular polygon of n sides is 360 divided by the number of sides less than 180° $(180 - \frac{360}{n})$, i.e.,



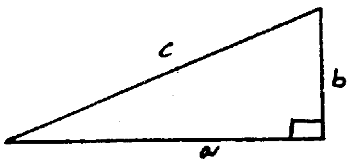
$\angle p = 180 - \frac{360}{6}$

(5) Any exterior angle of a triangle is equal to the sum of the opposite two interior angles, i.e.,



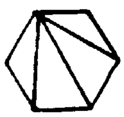
$\angle r = \angle p + \angle q$

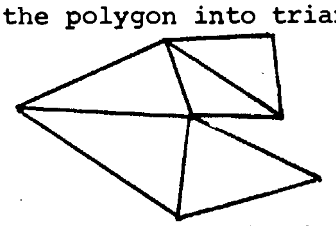
(6) Pythagoras theorem: for a right angled triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides (the hypotenuseis the side opposite to the right angle), i.e.,



$c^2 = a^2 + b^2$

For any polygon, the line between two non adjacent vertices is called a <u>diagonal</u>. Diagonals will divide the polygon into triangles, i.e.,





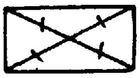
This <u>triangulation</u> will make the shape rigid and is the structure that brings strength to buildings, bridges and other constructions. For diagonals, we will study, in this unit, the following properties.

(1) Any poloygon of n sides requires n-3 diagonals to make it rigid.

(2) Any polygon of n sides can be triangulated into n-2 triangles.

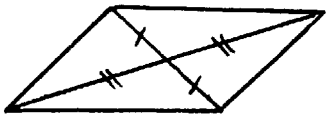
(3) Any polygon of n sides has a maximum of $\frac{(n-1)\ (n-2)}{2}$ - 1 different diagonals which can be drawn.

(4) Two equal diagonals bisecting each other means the quadrilateral is a rectangle and all rectangles have such diagonals, e.g.,

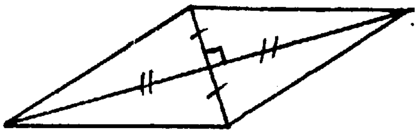


(5) Two equal diagonals perpendicularly bisecting each other means a square and all squares have such diagonals, e.g.,

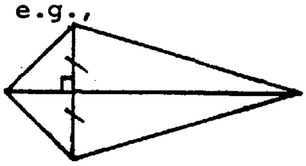
(6) Two unequal diagonals bisecting each other means a parallelogram and all parallelograms have such diagonals, e.g.,



(7) Two unequal diagonals perpendicualarly bisecting each other (intersecting at right angles) means a rhombus, and all rhombi have such diagonals, e.g.,



(8) Two unequal (or equal) disgonals intersecting at right angles such that one diagonal bisects the other, means a kite and all kites have such diagonals, e.g.,

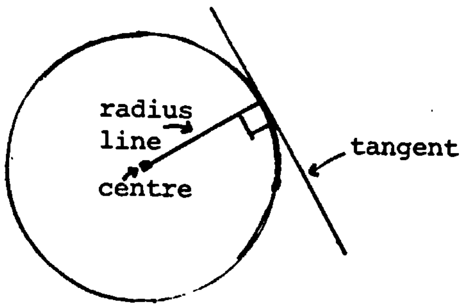


Note: This defines the special quadrilateral called a <u>kite</u>.

Can you work out a rule for the diagonals that is necessary and sufficient to make the quadrilateral a trapezium?

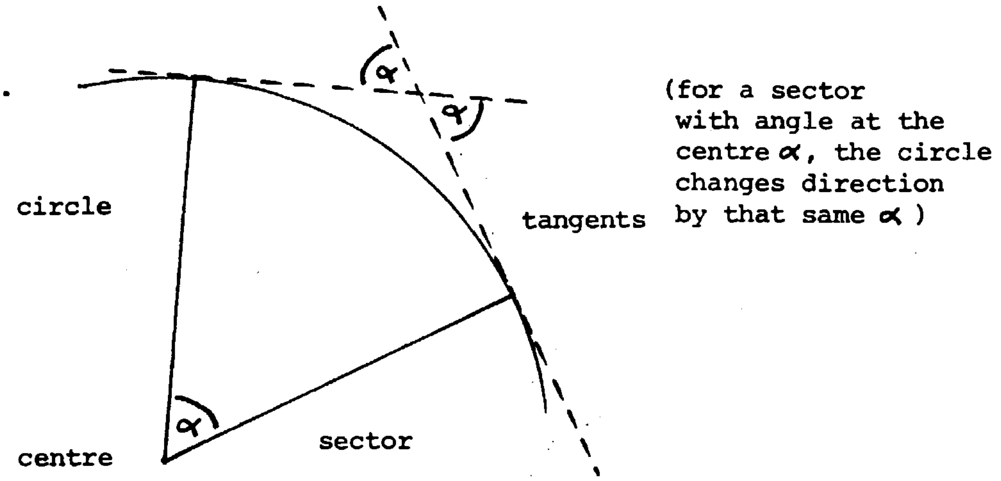
If a straight line is drawn perpendicular a radius, to meet it at the circle, that line is a tangent to the circle (see below). Such a tangent is in the same direction at this meeting point as the circle.

circle

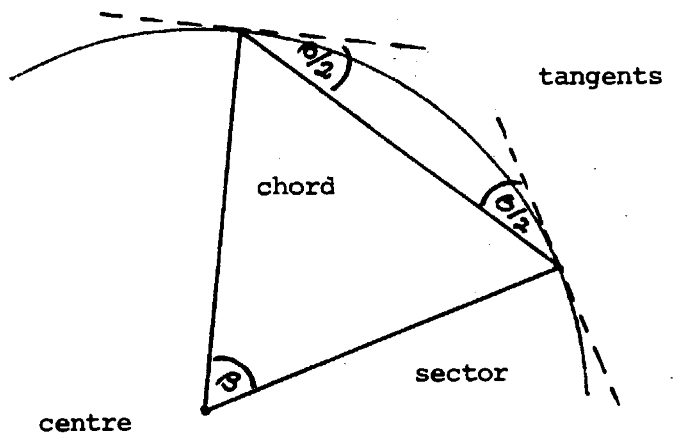


Tangents enable us to define two particular properties with regard to the direction a circle is moving in relation to sector angle as below.

1.



2.



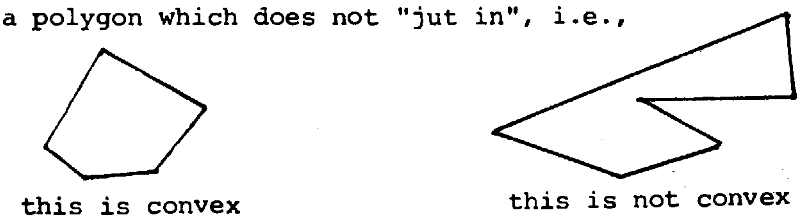
These properties are particularly useful for constructing shapes out of arcs and circles using LOGO on a micro-computer.

Materials:

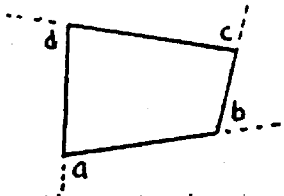
Geostrips, compass, ruler and straight edge, protractor, paper, scissors.

Activities:

1. (1) Draw a large convex polygon on the floor or playground. Remember this is a polygon which does not "jut in", i.e.,

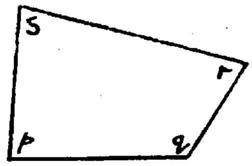


(2) Walk around this polygon. What angle do you turn? Hence what is the exterior angle sum of this or any polygon? For example

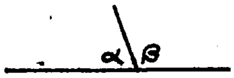


$\angle a + \angle b + \angle c + \angle d = ?$

(3) Now you have the exterior angle sum for your polygon, can you work out the interior angle sum? For example:



$\angle p + \angle q + \angle r + \angle s = ?$

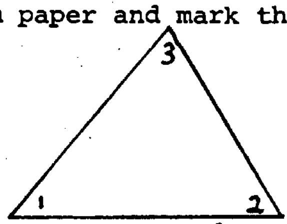
Rmember:  $\alpha + \beta = 180°$

(4) Can you use your rule in (2) above to calculate what each interior angle is for the following regular polygons:

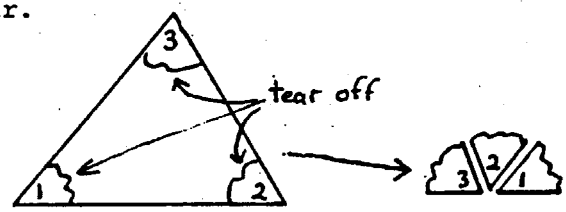
(a) an equilateral triangle?
(b) a square?
(c) a regular pentagon?
(d) a regular hexagon?
(e) a regular decagon?

2. Paper folding.

(1) Cut out a triangle from paper and mark the corners 1, 2, and 3 as below.



(2) Cut off or tear off these corners and assemble as below - paste them together.

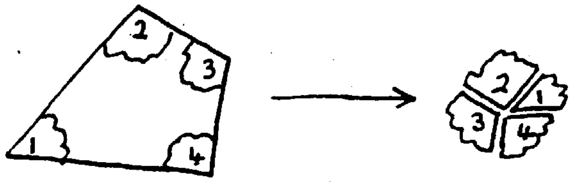


(3) What is the sum of these interior angles?

(4) Repeat this process for the following and calculate their interior angle sum:

(i) a square
(ii) a parallelogram
(iii) an irregular quadrilateral
(iv) a pentagon

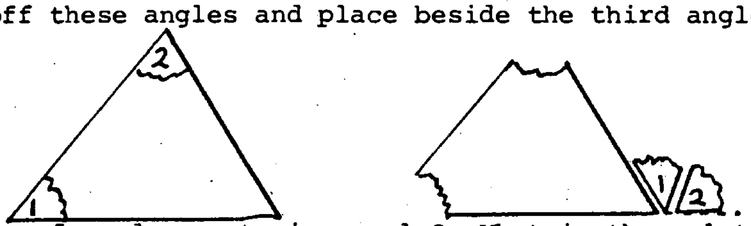
For example:



(5) What is the general rule for the interior angle sum of a polygon of n sides? Test your rule on a hexagon.

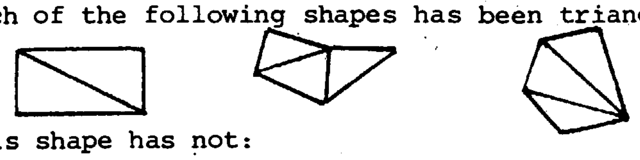
(6) Can you use this result to work out the interior angles of a regular polygon of n sides? Test your rule on a hectagon.

(7) Cut out a triangle from paper and mark two angles. Cut off or tear off these angles and place beside the third angle as shown:



Have you formed an exterior angle? What is the relation between this exterior angle and the opposite interior angles?

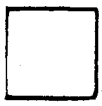
3. (1) Each of the following shapes has been triangulated:



This shape has not:

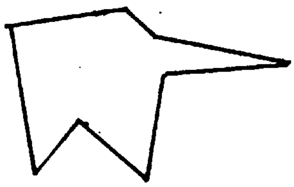
Triangulate these shapes

(a)

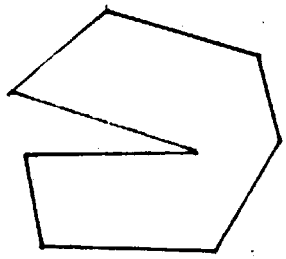


(two ways)

(b)

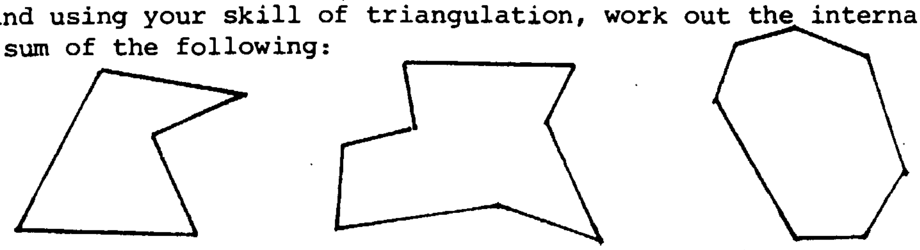


(c)

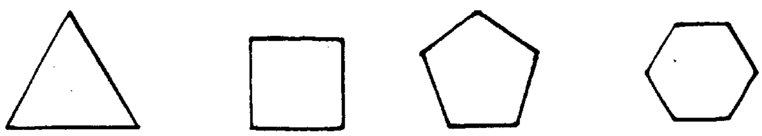


(2) Give an example of a shape that can not be triangulated.

(3) Remembering that the sum of internal angles of a triangle is 180° and using your skill of triangulation, work out the internal angle sum of the following:



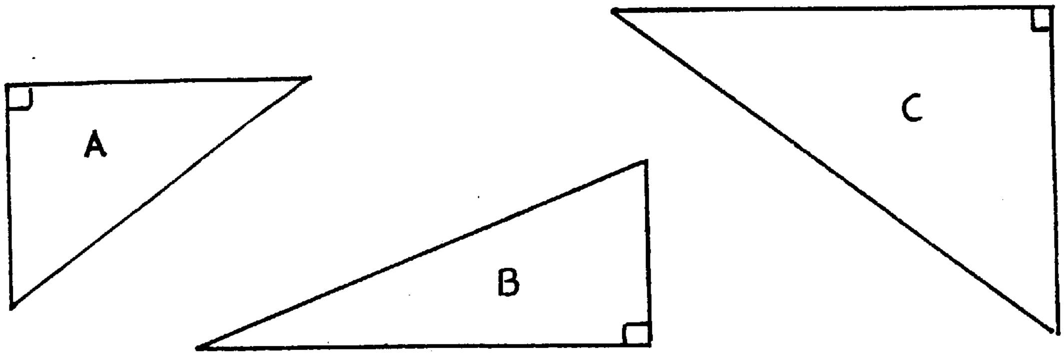
(4) In a similar way work out the internal angle sum of the following regular pentagons:



(5) Can you use the pattern in (4) above to work out the interior angle sum of a ten sided polygon (the decagon).

(6) Can we "quadrilate" shapes as we triangulate them?

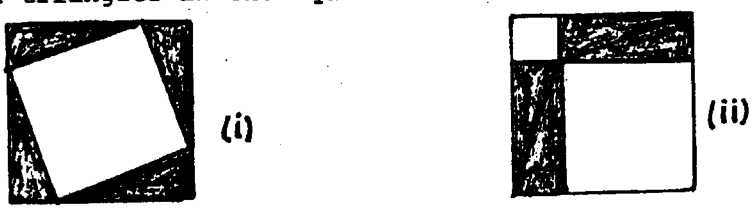
4. (1) For each of the right angled triangles below, measure lengths a, b and c and complete the table (use a calculator)



TRIANGLE	a	a^2	b	b^2	$a^2 + b^2$	c	c^2
A							
B							
C							

(2) Draw a square of sides 17cm. Cut out 4 right angled triangles of 5cm, 12cm, and 13cm. Shade these triangles.

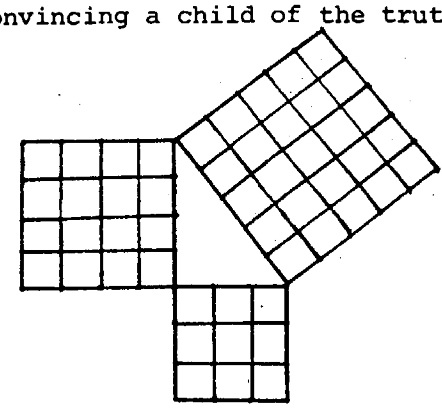
(3) Place the 4 triangles in the square in the following two ways:



(4) Look at the left over areas. What is the left over area in (i)? What are the two left over areas in (ii)? Is the area in (i) and the two areas in (ii) the same?

(5) Combining the result in (4) with that in (1), can you think up a rule that relates c (the length of the side opposite the right angle - the hypotenuse) to a and b (the lengths of the two other sides)?

(6) What are the advantages and limitations of the illustration below in convincing a child of the truth of the pythagonean Theorem?



5. Geostrips.

Use geostrips to do the following activities:

RIGIDITY

(1) Construct a door e.g.,

How can we stabilize the door - make it rigid?

(2) Construct a triangle, a rectangle, a pentagon and a hexagon. Which are rigid?

(3) Add the minimum number diagonals until the shapes in (2) above are rigid. Don't let the diagonals cross. How many triangles did this divide the polygon into.

(4) Complete a table such as below:

NAME OF POLYGON	*NO. OF SIDES*	*DIAGONALS TO MAKE IT RIGID*	*NO. OR TRIANGLES MADE*	*SUM OF INTERIOR ANGLES*
Triangle and so on	*3*	*0*	*1*	*180°*

(5) Can you see a pattern in this table that will enable you to find the interior angle sum of a 100 sided polygon?

(6) What real life instances, where triangles are used to give stability or rigidity, can you think of?

DIAGONALS

(1) Construct a triangle, a quadrilateral, a pentagon, a hexagon and a pentagon.

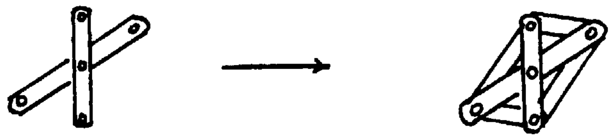
(2) Add in all possible diagonals to these polygons.

(3) *Complete a table such as below.*

NAME OF POLYGON	NO. OF SIDES	NO. OF VERTICES (A)	DIAGONALS FROM EACH VERTEX (B)	AxB	TOTAL NO. OF POSSIBLE DIAGONALS
Triangle	3	3	0	0	0
Quadri-lateral and so on	4	4	1	4	2

(4) *Can you see a pattern in this table that will enable you to find the total number of possible diagonals of a 100 sided polygon?*

(5) *Take any two equal strips and join them at their centre. Considering these as diagonals, join their ends as below:*



What shape does this give?

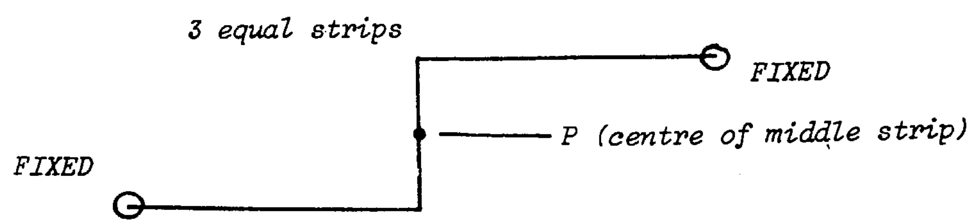
Repeat this for two unequal strips joined at the centre and then two equal strips joined centre of one to not centre and not end of the other. What shapes result in this?

Experiment with different diagonals to answer the following:

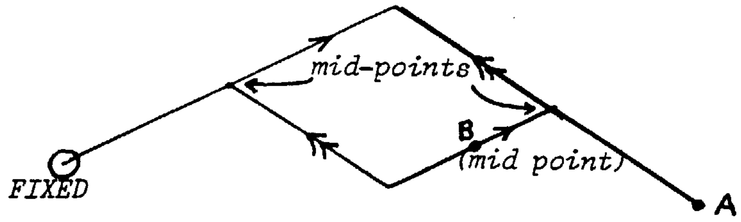
(a) *Can the diagonals of a rectangle intersect at right angles without forming a square?*
(b) *Can the diagonals of a parallelogram be equal without forming a rectangle?*
(c) *When do the diagonals of a parallelogram intersect at right angles?*
(d) *Can a kite have equal diagonals intersecting at right angles?*
(e) *Can a quadrilateral have equal diagonals intersecting at right angles and yet not be a kite, a parallelogram, a trapezium or any other special type?*

LINKAGES:

(1) *Construct a James Watt's Linkage. Trace out the locus of point P.*



(2) *Construct a pantograph.*

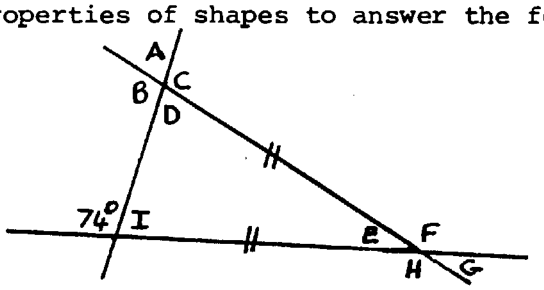


Copy a picture, placed under B, at A to enlarge it.
Copy a picture, placed under A, at B to reduce it.

6. Think of real world instances as you can where triangles have been used to stabilize shapes. Do this particularly for right angled triangles.

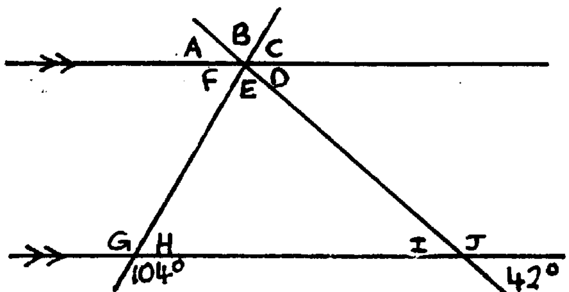
7. Use your knowledge properties of shapes to answer the following questions:

(1)



What are the measures of the other angles A through to I?

(2)



What are the measures of the other angles A through to J?

(3) Which of the following angles could be angles of a triangle?

1°, 62°, 117°	30°, 30°, 30°
45°, 45°, 45°	60°, 60°, 60°

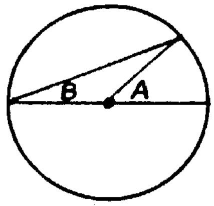
(4) Complete the following so that they could be the measures of the angles of a triangle.

40°, 40°, ___	180°, ____, _____
90°, 90°, ____	

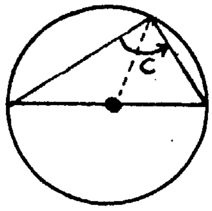
(5) One side of a triangle is 6cm long, another is 8cm long, and the angle between them is 90°. What is the length of the third side? Draw the triangle and estimate what the other angles are?

(6) If a triangle has angles of 30° and 40° and if the side between them is 5cm long, what must the other angle be? Draw the triangle and estimate the lengths of the other sides.

(7) Use what you know about angles and triangles to convince a classmate that B is half the measure of A.



(8) Using (7) above, can you convince a classmate that angle C is always 90°?



(9) Make up a question in the spirit of the above and give it to a classmate to do.

Teaching Hints:

The important strategies for teaching shape are variety in presentation and experience; and system in diagnosis to check that all concepts are being equally covered. There is a large variety of activities in this chapter to achieve this. They are based on two approaches.

Firstly, shape can be seen in the immediate environment. Rectangles can be seen in doors and windows, triangles and trapeziums in roofs, circles in clocks and wheels, squares in boxes and hexagons in tiles. Such shapes can be classified by the number of their sides, the lengths of their sides, their angles and any parallelness.

Secondly, using paperfolding, geoboards and geostrips, we can build up the notion of a shape and region from boundary, line, angle and path.

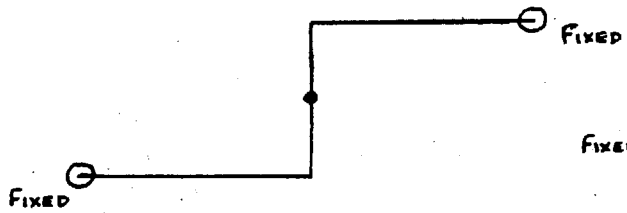
In the first of these approaches, open ended sorting and classifying tasks can gainfully be employed. In the second a much more structured approach, ensuring that each concept is developed, consolidated and applied before the next in sequence is moved onto, has to be used.

We have focussed on angle and length measurement to a large extent in this chapter. We have used them to classify shape. We have developed rules and properties from using them. There are however, other ways to analyse and classify shape which are more intuitive but have been left to later chapters, notably symmetry and tessellation. The design of a complete teaching sequence to study two dimensional shape will have to wait for these. In fact, later chapters will contain many activities that should precede many of those that are in this chapter.

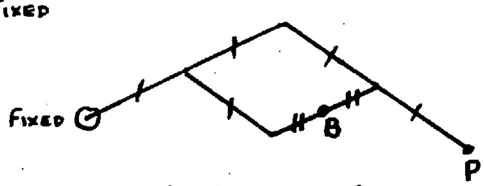
To return to the present, the use of geostrips in the unit for the study of diagonals is significant. There were two ways in which this was done. Firstly, the shape was made with the strips and then the diagonals were added in. Secondly, the diagonals were constructed and the shape built around them. Both these types of activity should be done with children.

The geostrips were useful in studying rigidity and the role of triangles. This study should move onto looking at real life instances (roofs, trusses, doors, etc). It is a very profitable activity to have a class visit a half finished house and then try to build a scale model of one themselves.

Linkages are also an interesting thing to study with the geostrips. The two linkages covered in this unit were:



James Watt's linkage



The Pantograph

The point P in James Watt's linkage transcribes a circle (or covers a disc). The pantograph will enlarge if B follows the original drawing and P has the pen and will reduce if P follows the original drawing and B has the pen.

<u>The programming language LOGO,</u> when used on a micro computer, is an effective way to explore, consolidate and apply shape concepts. Because instructions are given in terms of direction (angle) and distance, the properties of shapes that most relate to length of side and interior and exterior angle are most useful when using the language. The construction of polygons, circles and composite shapes strongly reinforces knowledge about angle sum and relationships between interior and exterior angles and between tangents and sector angle.

CHAPTER FOUR: SYMMETRY AND TESSELATIONS.

Much of the material covered in chapters 2 and 3 (three dimensional and two dimensional shape) is more suitable for upper primary than for middle and lower primary. For completeness, we defined and described the common solids and shapes that should be studied in the primary years at a formal level. The basis of this formal discussion was number, length, angle and parallelness.

There are characteristics of solids and shapes which are more intuitive and therefore can be part of children's activity earlier than the more formal aspects. Two such characteristics which have a large role in primary geometry are symmetry and tessellation.

In this chapter we focus on how to teach symmetry and tessellation: the range of activities available, the relationships which should be developed and the sequencing appropriate for children. In unit 12 we introduce line and rational symmetry and we look at its application in unit 13. In unit 14 we introduce the concept of tessellation and discuss where it can lead and we look at the application of tessellation to art in unit 15. In unit 16 we look at tessellating solid shapes, particularly looking at the role of packaging in modern society.

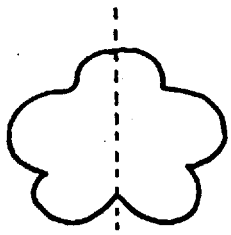
This chapter should be seen in association with chapters 3 and 5, the chapters which surround it, and chapter 7.

UNIT 12: LINE AND ROTATIONAL SYMMETRY

Focus:
This unit focusses on one of the most basic shape concepts: symmetry. The unit describes the two types of symmetry (line and rotational) which can be studied and discusses how they may be introduced to and developed in children.

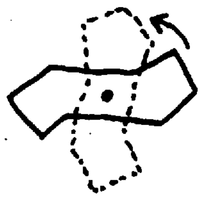
Background:
The two types of symmetry are

line symmetry



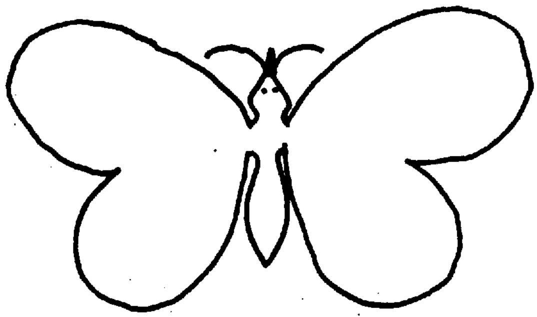
(the parts match if we fold along the dotted line)

rotational symmetry

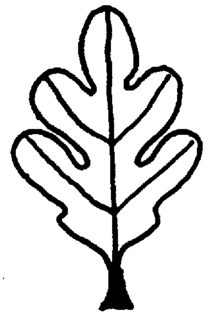


(a tracing of the design matches the design the design after being turned a part of a full turn)

For line symmetry the two sides of the shape are "balanced:, i.e. they are congruent (same size and shape). The line about which they are congruent is the line of symmetry (or axis of symmetry). For example



The butterfly has two wings balanced about its body. Its body is the line of symmetry. The butterfly is line symmetrical.

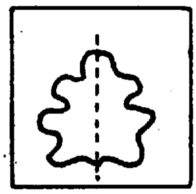


The leaf is cut in two halfs by the centre rib. The two halfs are congruent. The leaf therefore has line symmetry. The rib is the line of symmetry.

(1) Line Symmetry

Line symmetry can be introduced by paperfolding, mira, (or ordinary mirrors). Line symmetrical figures can be tested for their line symmetry by paperfolding:

e.g.



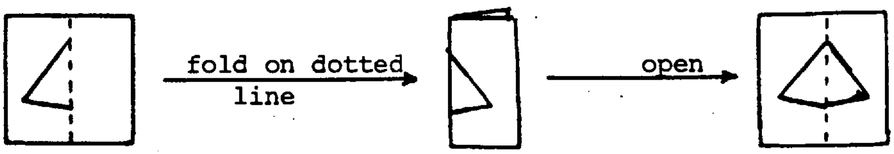
Fold on dotted line to show both sides are the same
(or fold a tracing of the shape in this way)

Objects can be found in nature or drawn on paper and tested for symmetry. Ink blots can be made as follows:

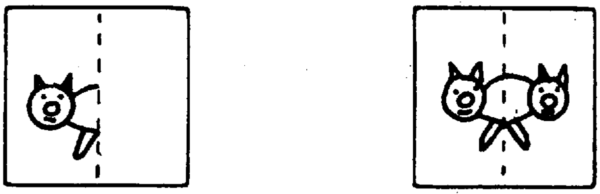
to begin line symmetry.

Line symmetric figures can be constructed by paperfolding:

e.g.

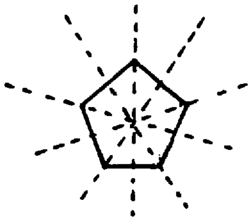


draw what you can see from the other side.

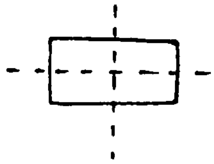


Obviously a diagram can have more than one line of symmetry,

e.g.



5 lines of symmetry



2 lines of symmetry

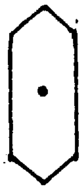
(2) Rotational Symmetry.

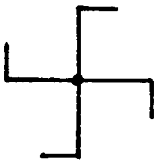
Tracing a design (onto tracing paper or thin paper) and then turning this tracing on top of the design (to see whether tracing and original match after part of a whole turn) is the best way to study rotational symmetry. But it should be noted that a shape such as:

is considered to have no rotational symmetry; yet a shape such as:

which matches after a half turn has <u>two</u> rotational symmetries (we count the full turn as well). Similarly

2 rotational symmetries

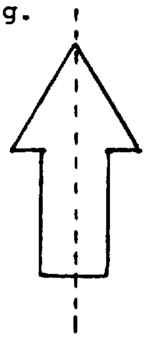




4 rotational symmetries

It is not possible to have 1 rotational symmetry - only 0, 2, 3, etc., are allowed. It should also be noted that the rotation to achieve the symmetry is about a point. Changing the point or <u>centre of rotation</u> changes the rotational symmetry.

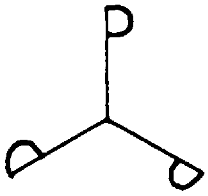
(3) Relationship between line and rotational symmetry.
It is possible for a figure to have line symmetry without any rotational symmetry, e.g.



1 line of symmetry

no rotations of symmetry.

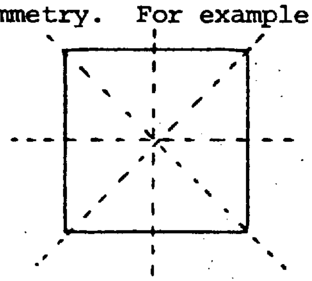
It is possible for a figure to have rotational symmetry without any lines of symmetry, e.g.



3 rotations of symmetry

no lines of symmetry

But if there are 2 or more lines of symmetry, then the number of rotations of symmetry equals the number of lines of symmetry. And the angle between consecutive lines of symmetry is half the angle between consecutive positions of rotations of symmetry. For example

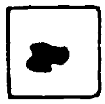


4 rotations of symmetry
4 lines of symmetry
Angle between consecutive lines - 45^{o}
Angle between consecutive rotations - 90^{o} ($360 \div 4$).

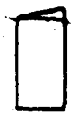
Materials:
Tracing paper, pen, mira, texta, ink

Activities

1. (1) Take a piece of paper and put a blob of ink in the centre



(2) Fold the paper in half



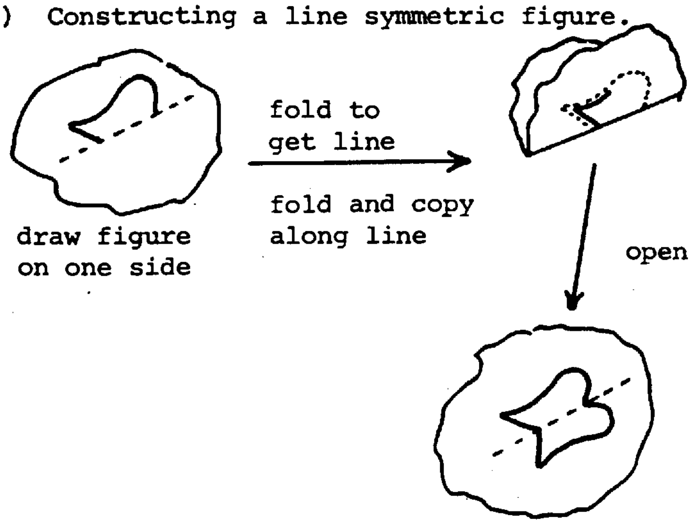
(3) Open out. What shape do you have?

(4) Draw the shape. Does it balance on either side of the fold line?

2. Use tracing paper to undertake the following paper folding activities.

REPRESENTATION WITH PAPER	DEVELOPMENT

(1) Constructing a line symmetric figure.

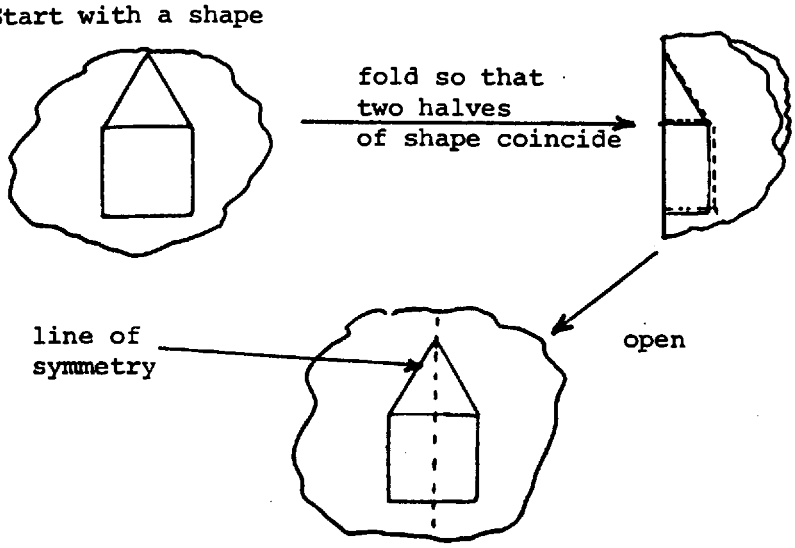


By completing "half pictures", determine what common items (e.g. human face, dogwalking) are sensible if symmetrical.

(see activity 5)

(2) Testing for line symmetry

Start with a shape

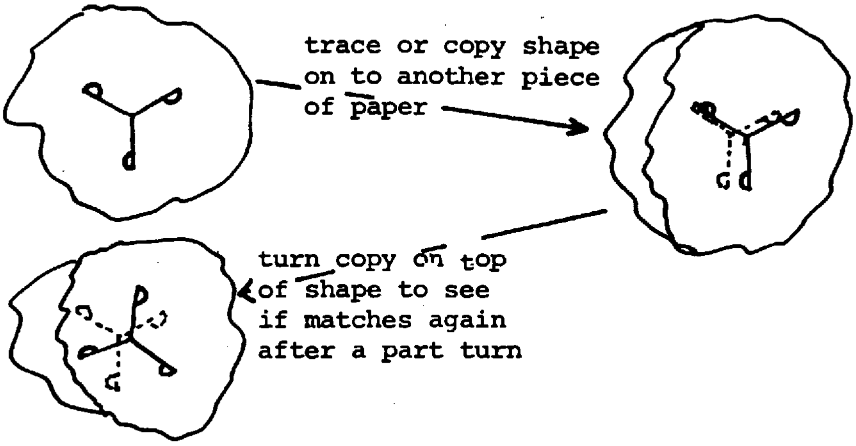


Determine how many lines of symmetry a shape has.

Classify shapes (e.g. isosceles, equilateral, scalene triangles) by their number of lines of symmetry.

(3) Testing for rotational symmetry

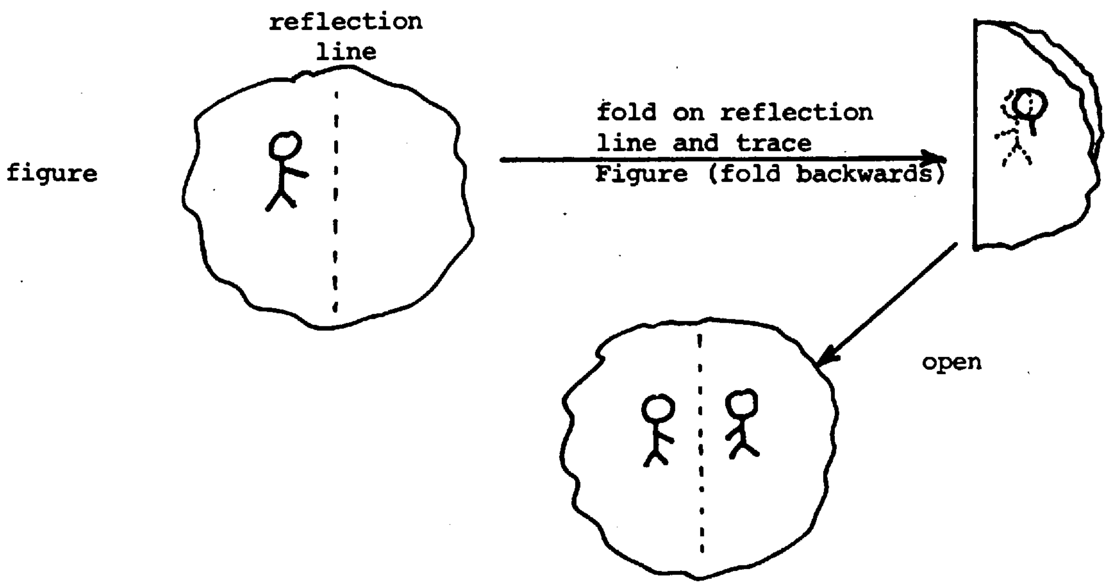
Start with a shape.

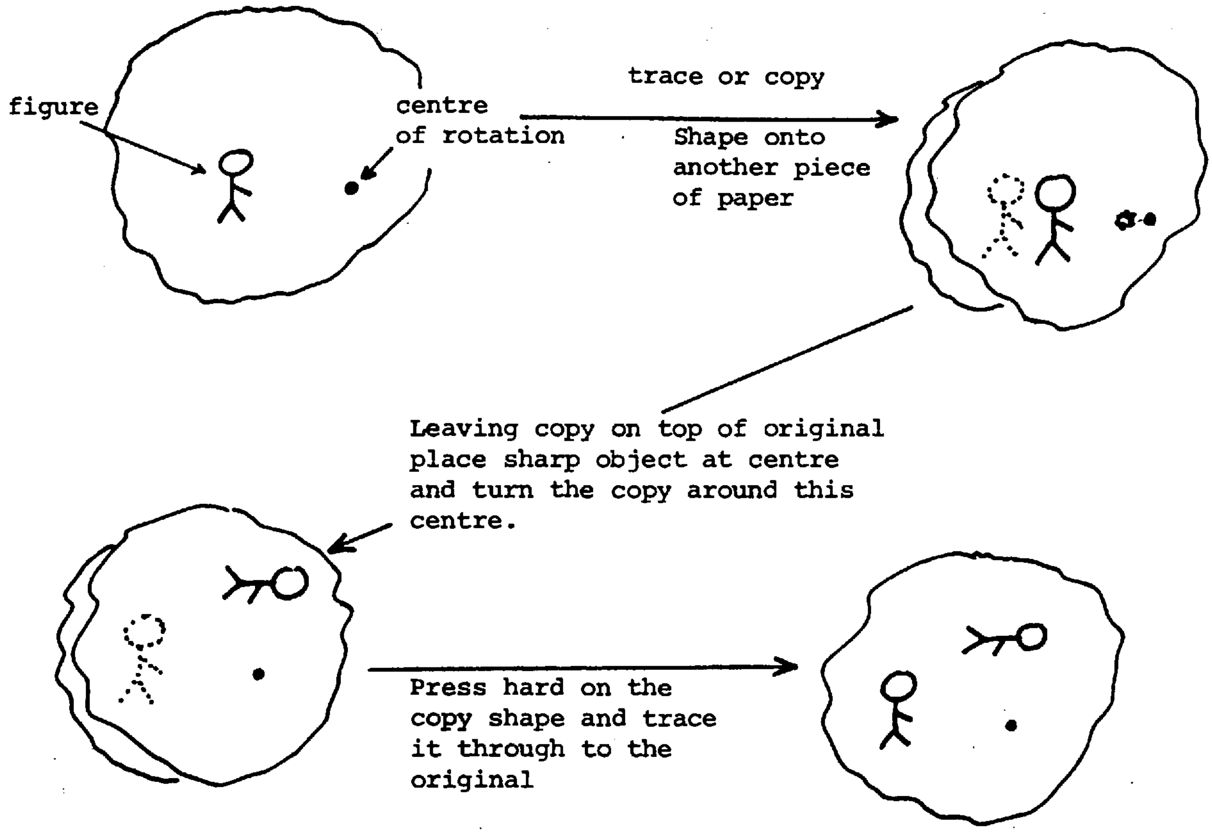


Determine how many rotations of symmetry a shape has.

Classify shapes (e.g. squares, rectangles) by the number of rotations of symmetry.

(4) Extension to reflection and rotation

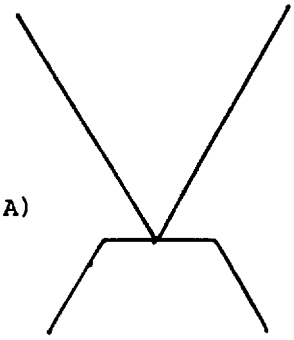




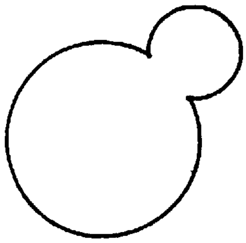
3. (1) Complete the following workcards using tracing paper. (Copy each design onto the paper.

CARD 1.

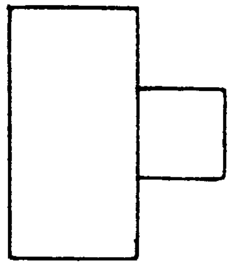
A)



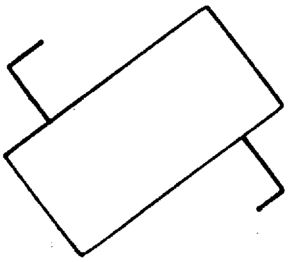
B)



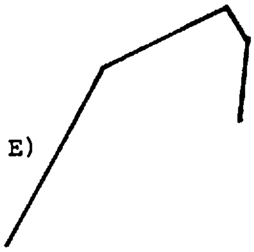
C)



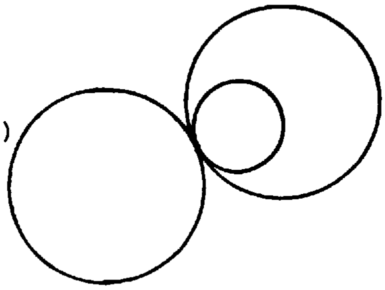
D)



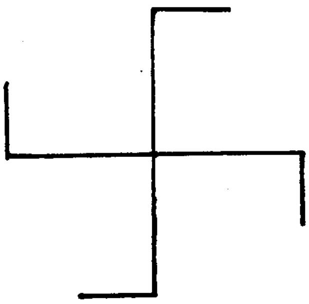
E)



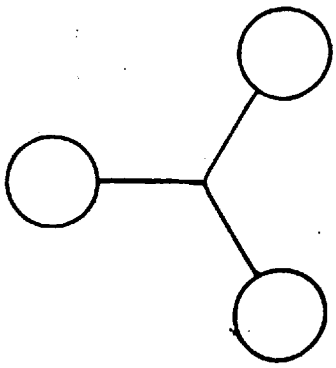
F)



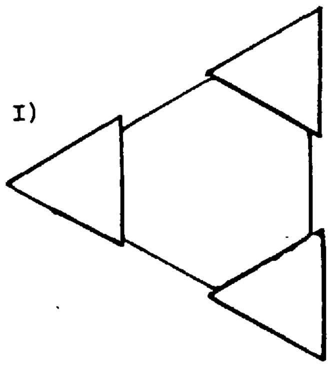
G)



H)

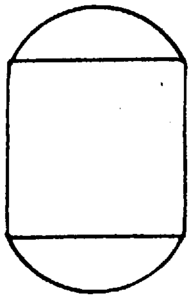


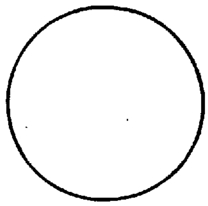
I)

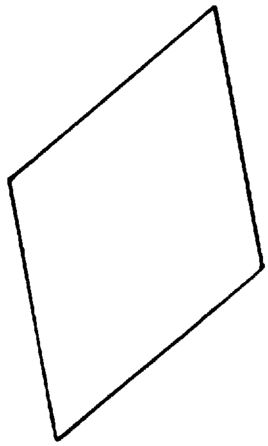


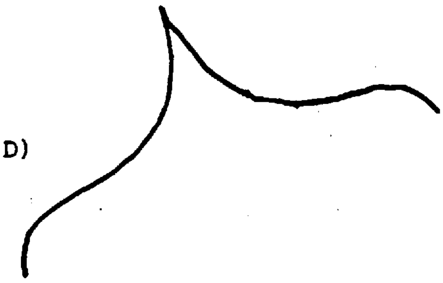
* *Which designs match themselves by folding along a line?*
* *Which match by turning a tracing part of a full turn?*
* *Do any not match either way?*

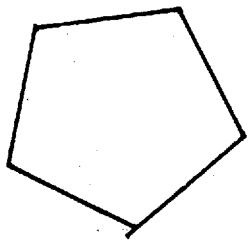
CARD 2.

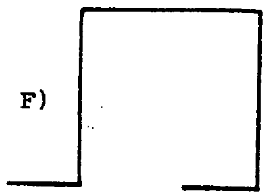
A) 

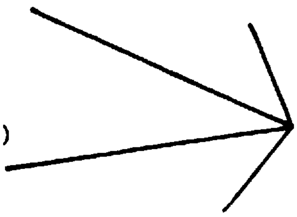
B) 

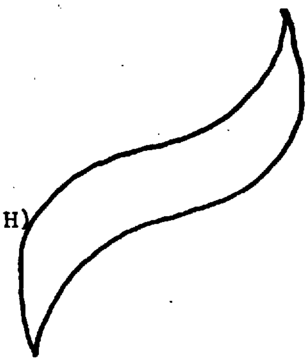
C) 

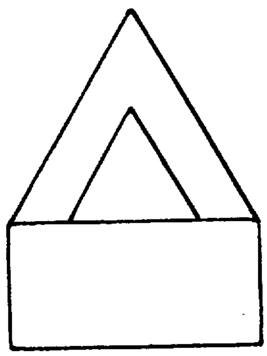
D) 

E) 

F) 

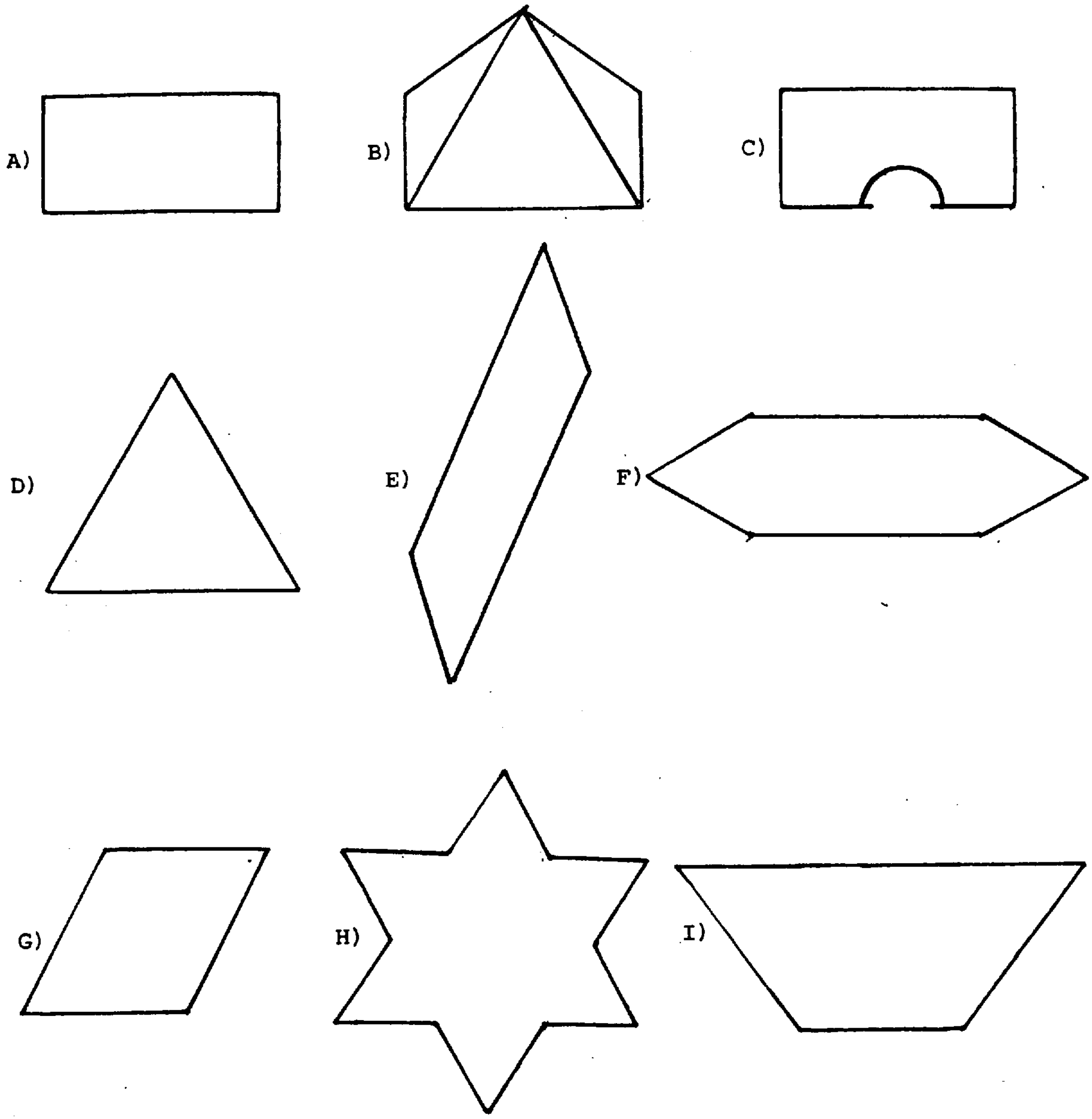
G) 

H) 

I) 

* *Which designs have line symmetry?*
* *Which designs have rotational symmetry?*
* *How many lines of symmetry do they have?*
* *How many rotations of symmetry do they have?*

CARD 3



**Which shapes can be turned part-way around and look just as they did before?*

** Do some do both?*

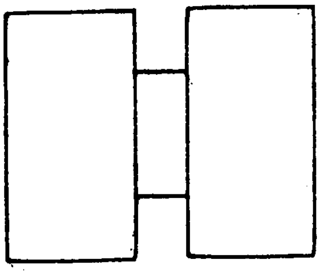
** What is different?*

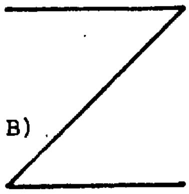
** Which can be folded so their parts match?*

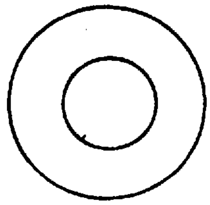
** What is alike about shapes A and C?*

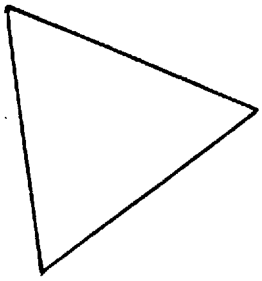
** What do you notice about shapes G and I?*

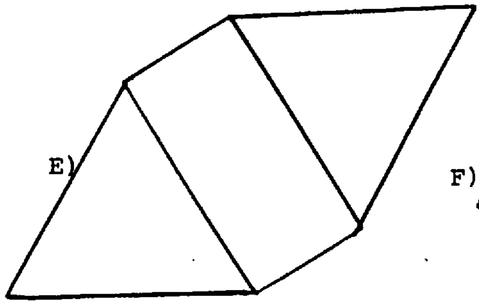
CARD 4

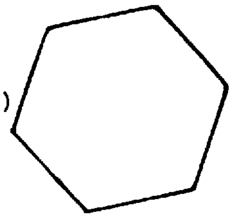
A) 

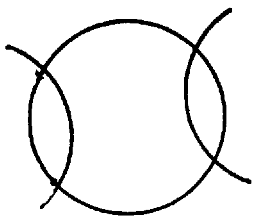
B) 

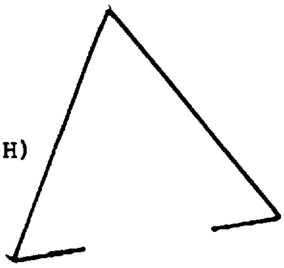
C) 

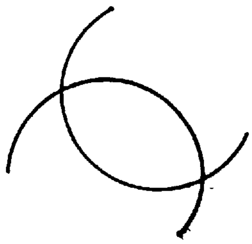
D) 

E) 

F) 

G) 

H) 

I) 

**For each design, find all lines of symmetry. Now, before you test it, do you think it has turning symmetry?*

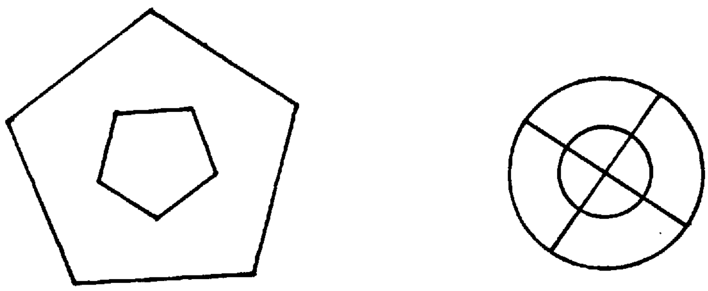
Test it.

Were you right?

**What do you notice about the number of lines of symmetry a design has and the number of part-turns it makes?*

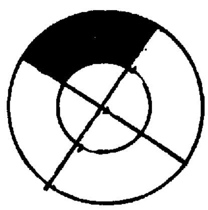
(2) Read the instructions. Notice how the language changes as we move card 1 through to card 4. Place each of the cards in the year level you feel most appropriate for the language and content in the card.

4 (1) Copy the following two figures onto tracing paper. Fold and rotate this tracing. How many rotational symmetries and lines of symmetry do they each have?



(i) (ii)

(2) Look at (ii). Shade it as below:



(iii)

(3) How many lines of symmetry and rotational symmetries does it now have?

(4) Shade (ii) so that it has 2 lines of symmetry and 2 rotational symmetries

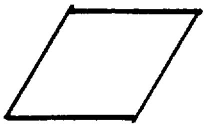
(5) Shade (ii) so that it has 1 line of symmetry and no rotational symmetry.

(6) A line symmetric figure can be constructed by folding paper. How can a figure with rotational symmetry be constructed?

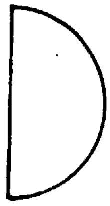
(7) Construct a figure with 2, 3 and 4 rotational symmetries?

(8) Complete the table below for the following figures.

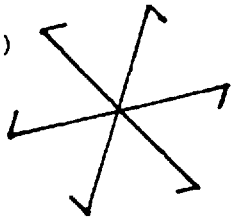
(a)



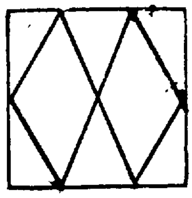
(b)



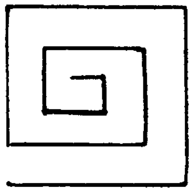
(c)



(d)



(e)



(f)

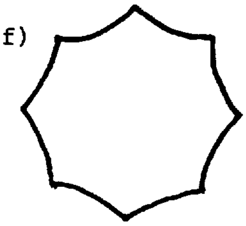


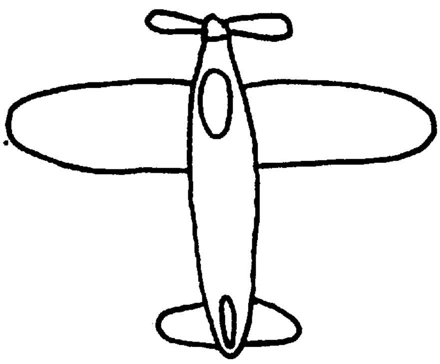
Figure	No of lines of symmetry	No of rotational symmetry

(9) From the table above can you see a pattern that relates the number of rotational symmetries to the number of lines of symmetry? (Look at the cases where the number of lines of symmetry is less than 2 and where it is greater than or equal to 2).

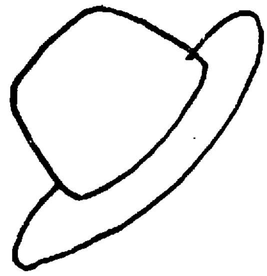
(10) Looking at (a) and (c) above, can you see a relationship between the angles between the lines of symmetry and the angles between consecutive rotations of symmetry?

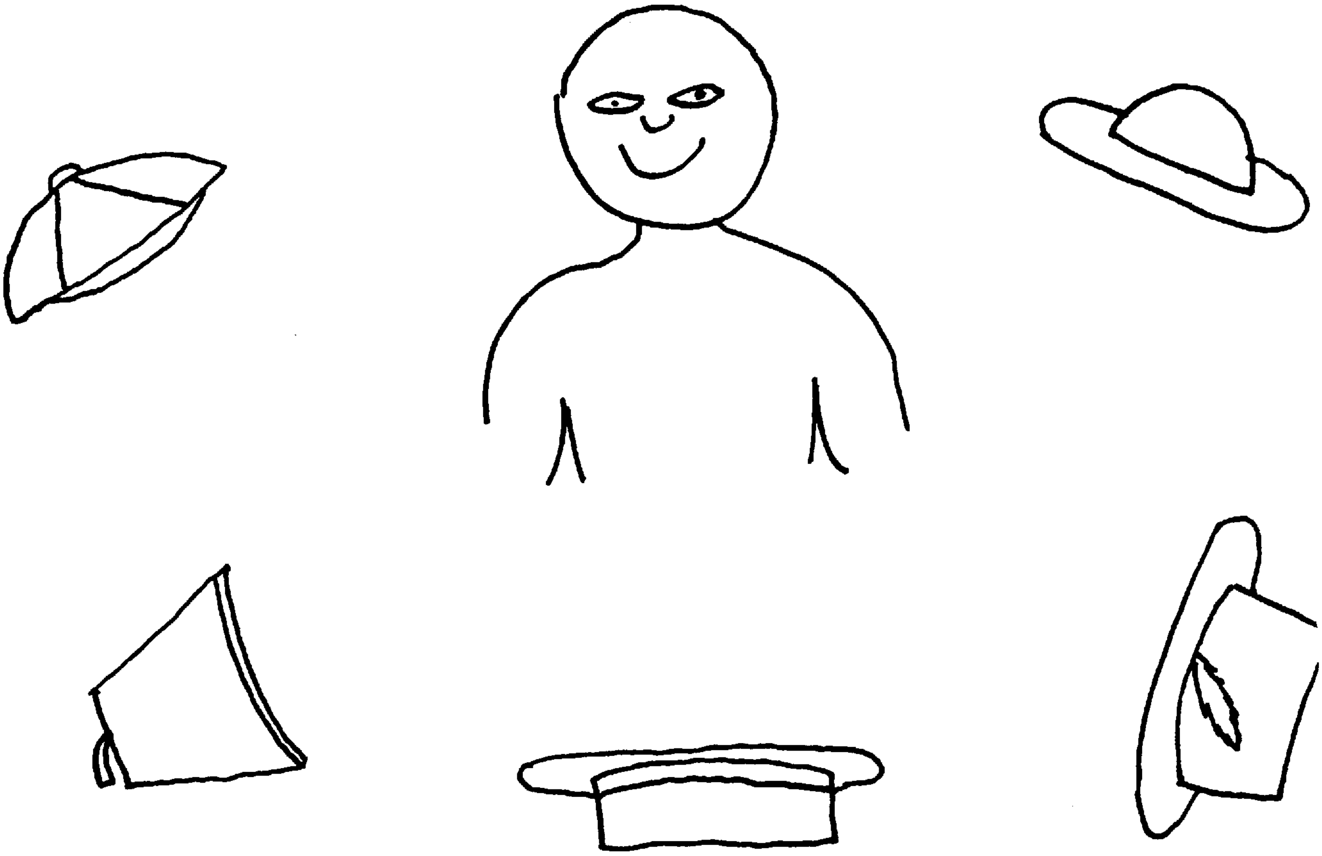
5. Use a mira to complete the following workcards.

Card 1. Park your aeroplane nose to nose with the one below. Take off backwards, circle the airfield and then land. Try other tricks.

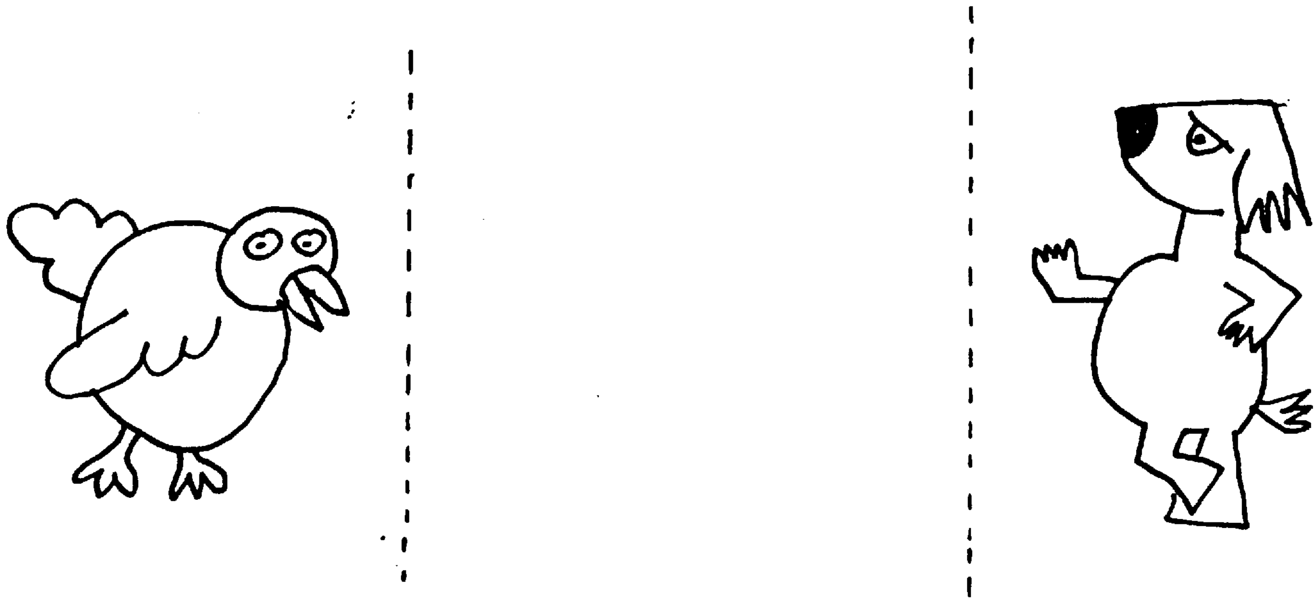


<u>Card 2</u> *Pick a hat. Which one looks best? Try all the wigs on the boy by moving your mira mirror around.*

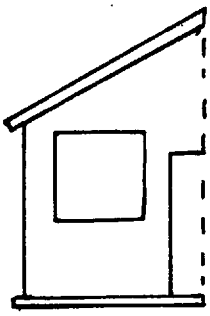


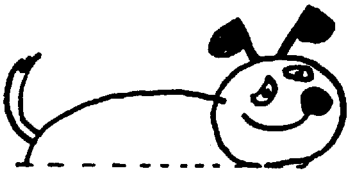


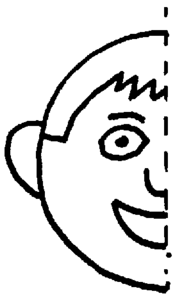
Card 3. *Reflect these shapes about the line shown. Use your mira. (Put a piece of paper on the other side of line. Put your mira on the line. On your paper draw what you see).*

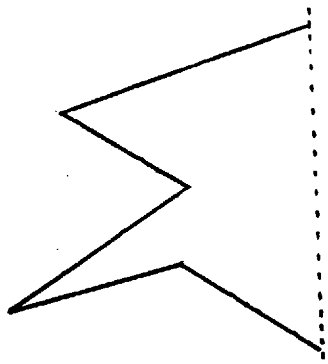


Card 4. *Place your mira along the lines. Draw the other half of the figures. Does the result make sense? Name the figures that do.*

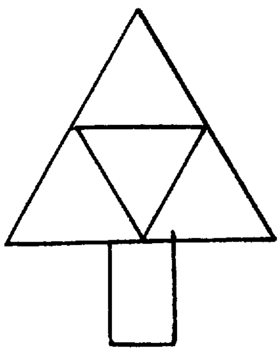


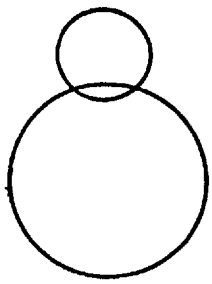


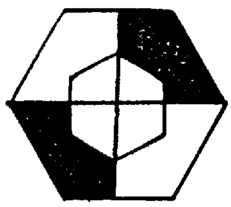




Card 5. A. *Test the figures below for line symmetry. Use your mira. Draw in all the lines of symmetry you find.*

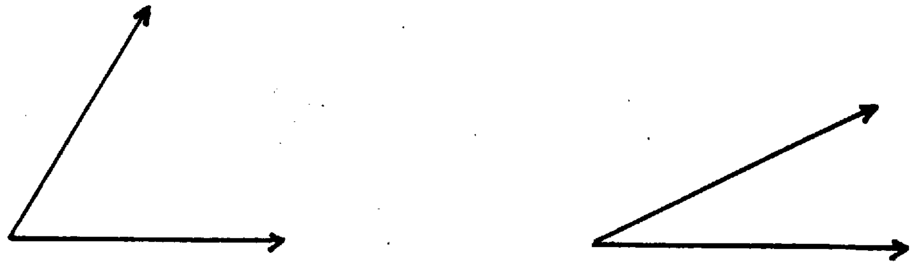






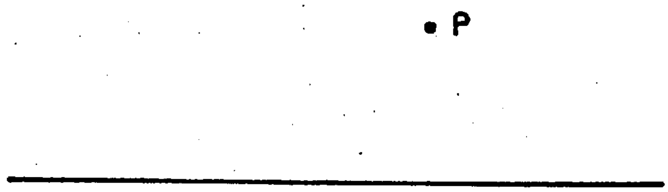
B. *Do the same for the capital letters of the alphabet.*

Card 6. A. *Reflect one ray onto the other and find the bisector of these angles.*



B. *Reflect the line exactly back onto itself to find the perpendicular bisector.*

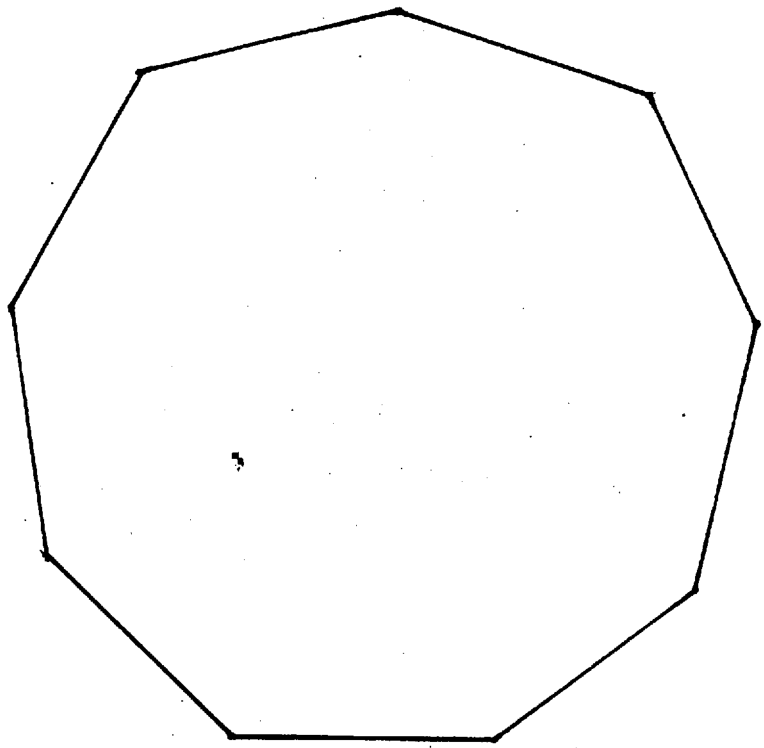
C. *Reflect part of the line back onto the other part, through point P, to find the perpendicular from P to the line.*



6. Look around the room. Look around the yard outside. List five items that have line symmetry. List five items that have rotational symmetry.

7. Games can reinforce symmetry concepts. Play the game POLYGON below.

POLYGON *2 - 4 Players*



* *PLACE A MARKER ON EACH VERTEX (CORNER) OF THE ABOVE REGULAR NONAGON*

* *PLAYERS TAKE TURNS*

* *A PLAYER MAY PICK UP ONE OR TWO MARKERS (THE TWO MARKERS HAVE TO BE NEXT TO EACH OTHER)*

* *THE PLAYER WHP PICKS UP THE LAST MARKER(S) WINS*

VARIATIONS: *(1) Play with regular - centagon*
heptagon
octagon
hexagon

(2) Player whp picks up last marker loses.

Questions: (1) Is it an advantage to go second?

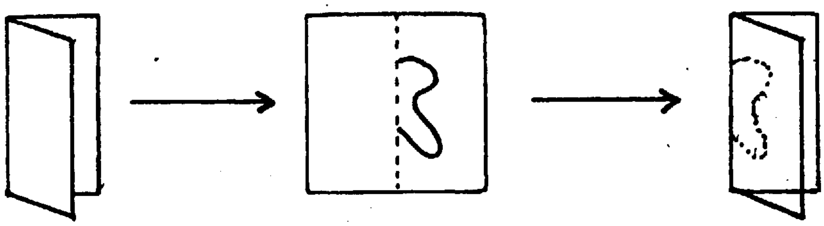
(2) What is the winning strategy?

8. Write the capital letters of the alphabet. Which have line symmetry? Which have rotational symmetry? Can you write a word which is line symmetric? Can you write a word which is rotationally symmetric?

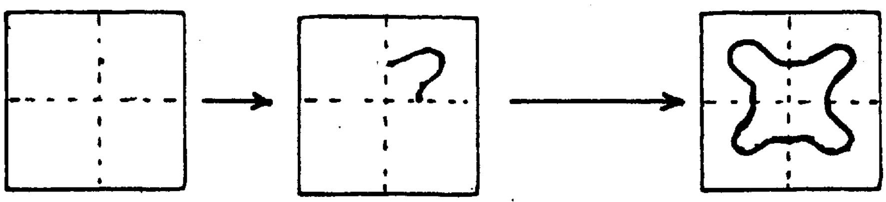
Teaching Hints:

Symmetry is something that has to be experienced. It is not sufficient to demonstrate and show. And it is important to give children symmetry activities that cover the following three activity types.

(1) Constructing symmetrical figures - drawing figures that are line or rotationally symmetric. Paper folding can be used to draw line symmetric figures e.g.

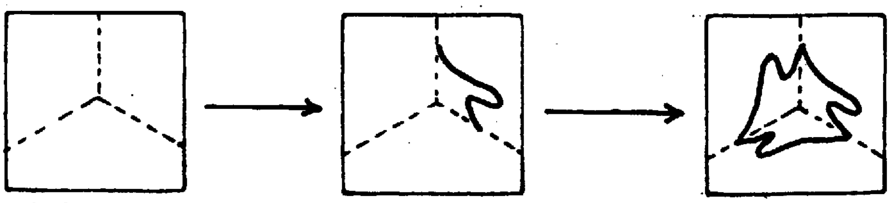


Fold — Draw on one side — Fold and copy



Fold twice — Draw twice — Fold and copy three times

For rotationally symmetric figures, divide the page into the number of sectors that you want for rotational symmetries and rotate through each sector. For example, for three rotational symmetries,

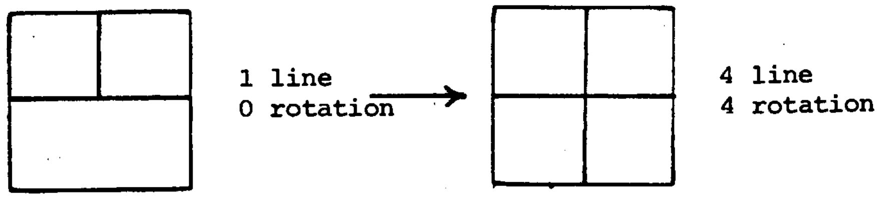


Divide page into 120° sectors — Draw in one sector — Rotate a copy of this into the other 2 sectors.

(2) Identifying symmetric figures - using tracing paper and paper folding and turning to identify the number of line or rotational symmetries a figure has.

(3) Transforming symmetric figures - taking a figure which has a certain identified number of line and rotational symmetries and making changes to the shape that change the symmetries.

For example:



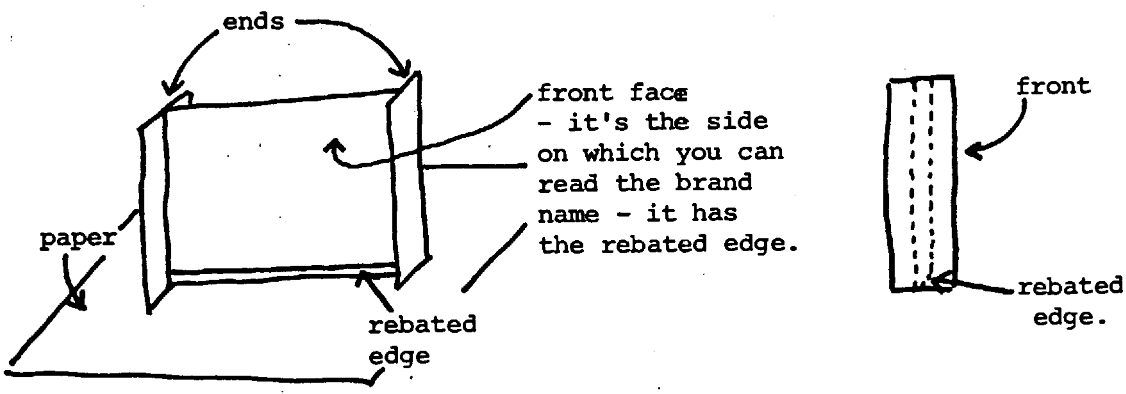
It is likewise important that children connect symmetry to the world around them:

- to natural things (leaves, pine trees, etc.
- to man made things (houses, cars, etc.)

It is interesting to look at how symmetry makes things attractive and then, beyond this, how a small non symmetric touch (e.g. the front of a Volvo) adds to a shape (see next unit).

A particular aid that is very useful for line symmetry, and which can emulate much of what paper folding and ruler and compass can do, is the <u>mira</u>.

A mira is a red plastic mirror shaped as below:-



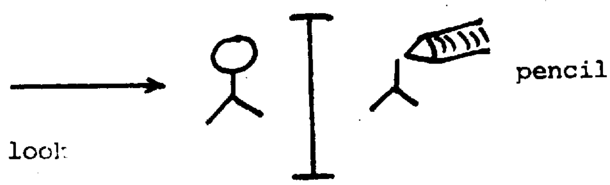
Front view

The mira is used on paper as shown. Its strength is that the red plastic allows you to look through the mira (to what is behind the mira) and to see the reflection of what is in front of the mira <u>at the same time</u>.

This means a mira can be used to <u>superimpose</u>, reflected and transmitted pictures (and thus to find symmetry) and to draw reflections.

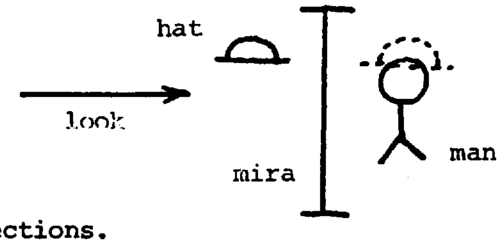
(1) Familiarization with the mira.

(a) Drawing shapes (actually reflections of shapes)
- place mira beside drawing, look through the mira over the top of the drawing, copy image with pencil behind the mira.



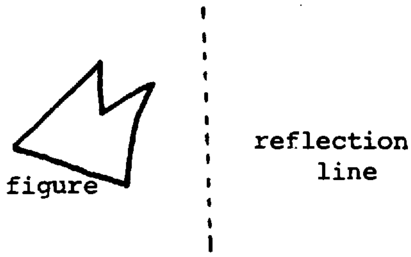
(b) Completing shapes

- for example, put the hat on the man by placing mira between hat and man and looking through the mira, over top of the hat.

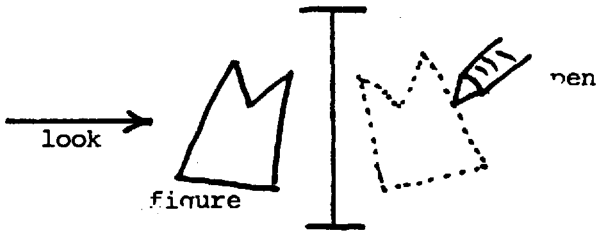


(2) Reflections.

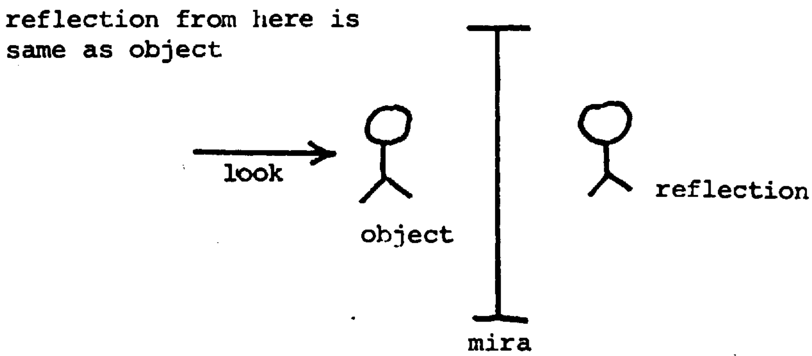
(a) Reflect figures about a reflection line by placing mira on that line



and looking over the top of the figure (through the mira) and copying the image behind the mira.



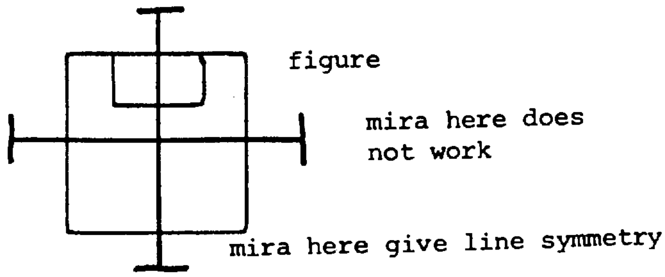
(b) Find the line of reflection between object and reflection (or find if a reflection has occurred or find if two objects are the same or congruent) by placing mira so object and reflection superimpose.



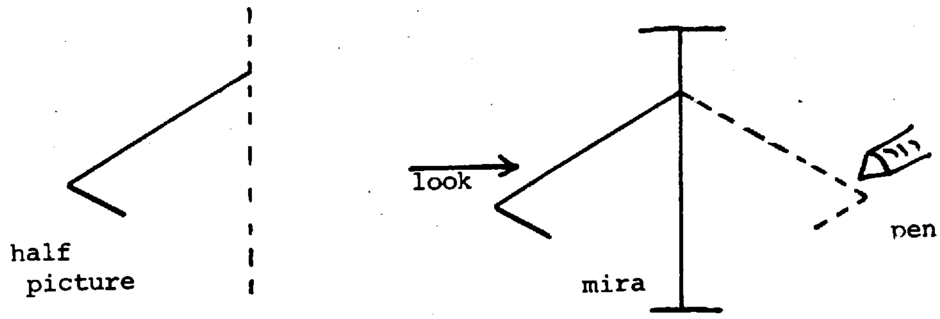
Drawing with pencil along rebated edge will give the reflection line.

(3) Line Symmetry

(a) Determine if figures are line symmetrical by placing mira so one half of figures matches other half. Drawing along rebated edge will give the line symmetry.

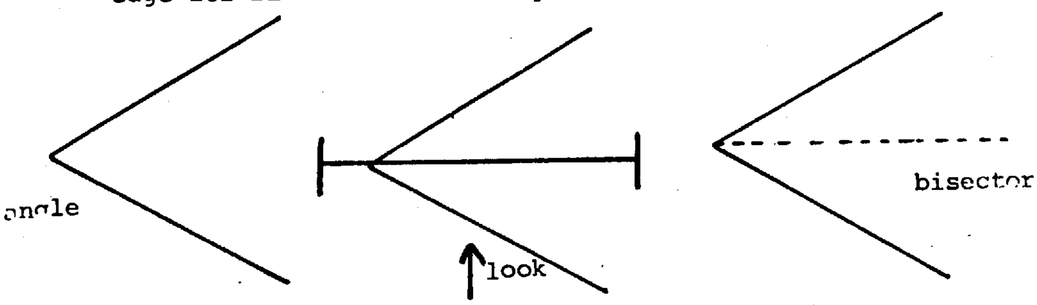


(b) Use mira to complete half pictures to a line symmetrical form

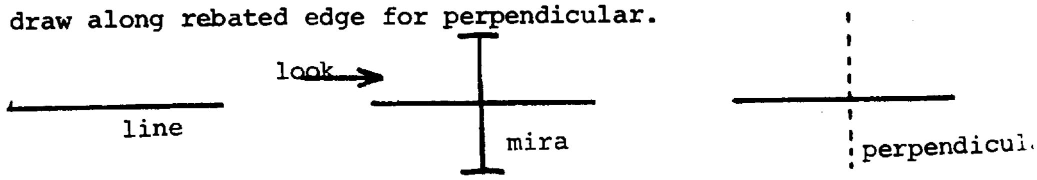


(4) Perpendiculars and Bisectors

(a) Place mira so one ray matches the other. Draw along rebated edge for bisector of the angle.



(b) Place mira so that a straight line is reflected onto itself and draw along rebated edge for perpendicular.



If line is matched exactly onto itself, this gives a perpendicular bisector. If mira is made to pass through a given point this gives the perpendicular from that point to the line.

(5) Congruence

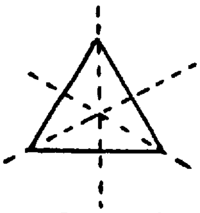
If looking through the mira shows one figure to be the same as a second, these two figures (or lines) are congruent.

UNIT 13: APPLYING SYMMETRY

Focus:
When symmetry is understood, it is an excellent method to use to classify and define two dimensional shape. It is also the basis of much of what our society calls beautiful. In this unit we investigate some of the applications of line and rotational symmetry.

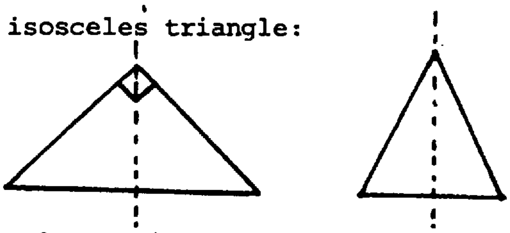
Background:
Triangles

An equilateral triangle.



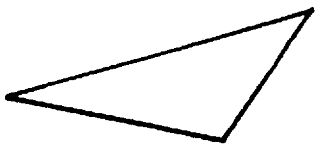
all sides equal
3 lines of symmetry (the medians of the triangle)
3 rotational symmetries.

An isosceles triangle:



Two sides equal
1 line of symmetry
no rotational symmetry

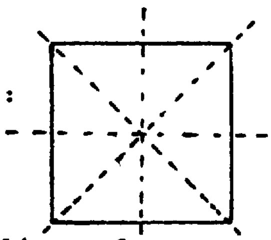
A scalene triangle:



no sides equal
no lines of symmetry
no rotational symmetry

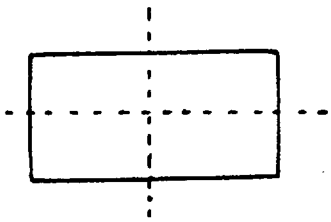
Quadrilaterals

Square:



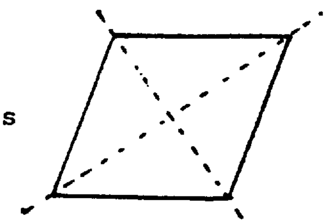
4 lines of symmetry
4 rotational symmetries

rectangle



2 lines of symmetry (not diagonals)
2 rotational symmetries

rhombus

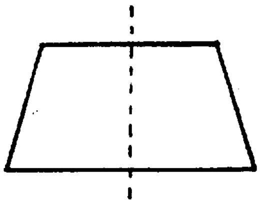


2 lines of symmetry (diagonals)
2 rotational symmetries

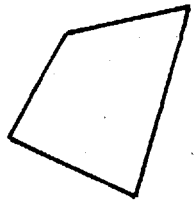
parallelogram

no lines of symmetry
2 rotational symmetries.

isosceles trapezium



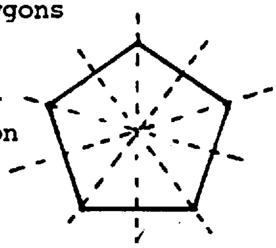
irregular quadrilateral:



one line of symmetry
no rotational symmetries

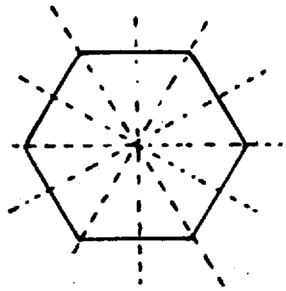
Other Polygons

Regular Pentagon



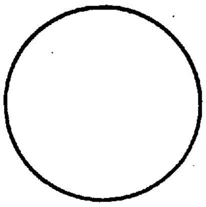
5 lines of symmetry
5 rotational symmetries

regular hexagon



6 lines of symmetry
6 rotational symmetries

Circle



infinite number of lines of symmetry
(all diameters)

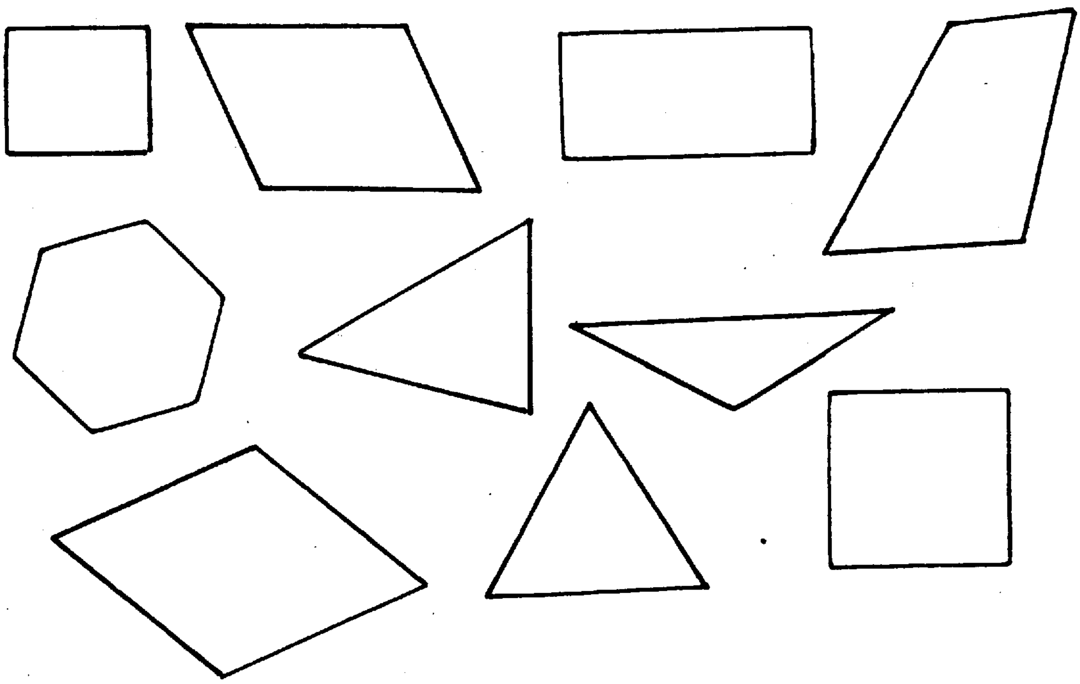
infinite number of rotational symmetries

Materials:

Tracing paper, pen

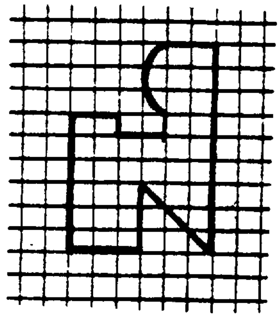
Activities:

1. (1) Check the following for line and rotational summetry.
   (2) On the basis of this check,classify each shape (read the background to this unit for the definitions).



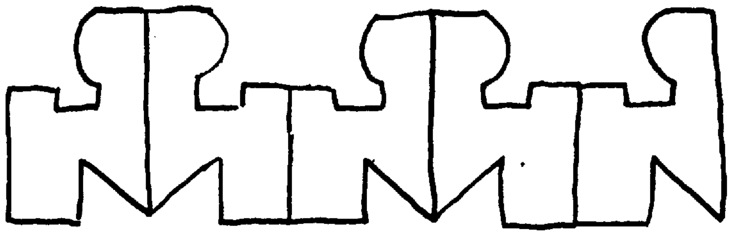
2. Frieze Patterns

A motif like that on the right can be extended into a frieze pattern by a series of symmetric changes.

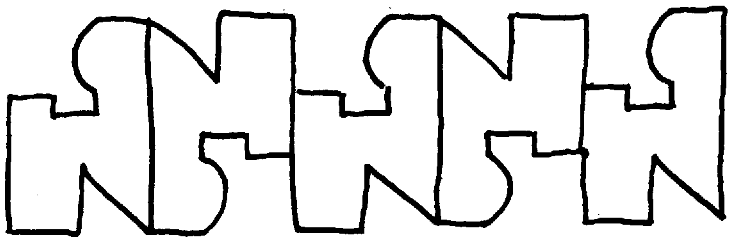


For example

(a) line symmetric changes



(b) rotational symmetric changes (half turns)



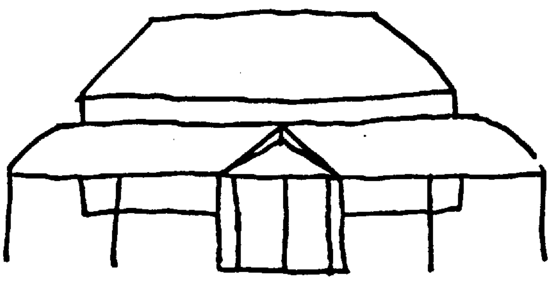
(1) Obtain some graph paper and draw your own motif. Look ahead. Try to draw a motif that will turn into a good pattern.

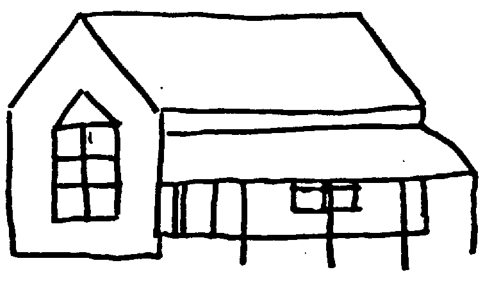
(2) Turn your motif into a frieze pattern by line symmetric and then rotationally symmetric changes.

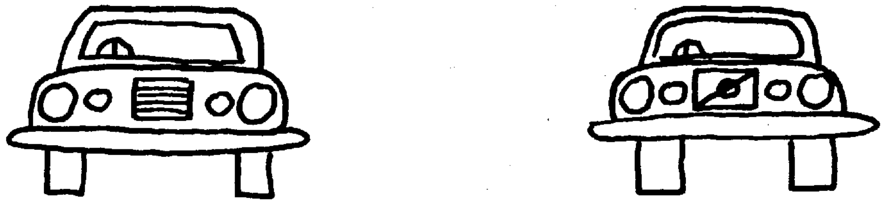
Teaching Hints:

This classification of shapes by symmetry is simpler for children than classification by equal sides, equal angles, parallel sides etc. Hence symmetry should precede the use of equal length, equal angles, and parallel lines in the classification of shapes.

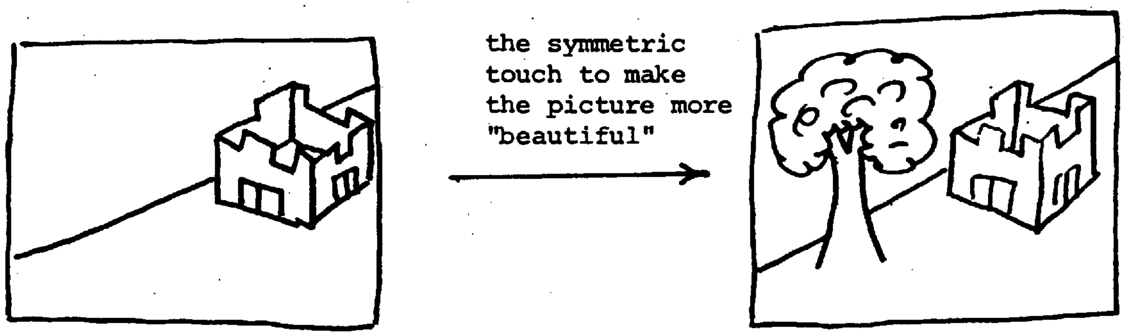
The focus on art is well worth following up. Many things that society finds beautiful have tremendous symmetry. The human face and body is one example. Buildings are another. Fussy design which would otherwise appear cluttered can be improved by symmetry - witness many of the older houses in Queensland. Beauty is also enhanced by a simple (and often major) non symmetric change. For example







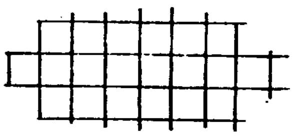
Children can be directed to study this and to use it in drawings - to develop their own work based on symmetry. It should be noted that the "symmetry' in many pictures is not strictly mathematical, for example



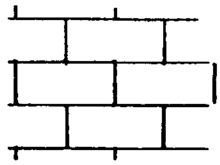
UNIT 14: TESSELLATIONS

Focus:
In a society that puts shapes together to build and cover and that packs shapes together to carry them around, shapes that fit and pack well are important. In this unit we focus on tessellations - shapes which fit together without gaps or overlaps.

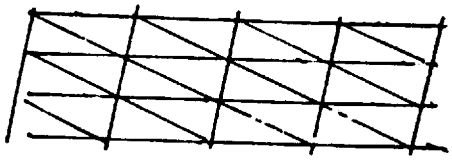
Background:



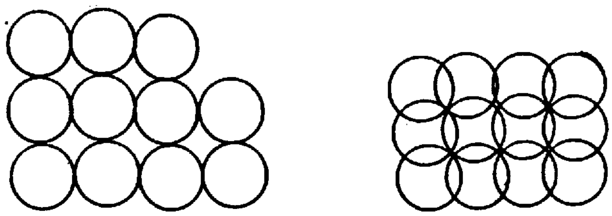
squares tessellate



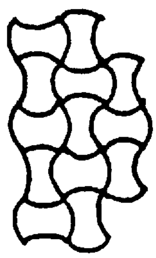
rectangles tessellate



triangles tessellate



circles do not tessellate

but this shape  does 

(1) Beginnings

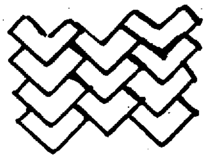
Give children tiles to see if they fit together (e.g. any triangle, squares, rectanges, rhombi, parallelograms, kites, trapeziums, any quadrilateral, regular hexagons tessellate; (circles, semicircles, pentagons don't)

Children can also

(a) make shapes by fitting tiles together; and
(b) cover pictures (with different sections using different coloured tiles).

(2) Determining which figures tessellate

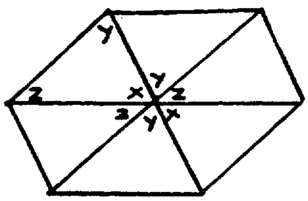
Start with a figure (e.g. ), make 20 copies and arrange to see it tessellates. The pattern produced must show that the tessellation will go on for ever.

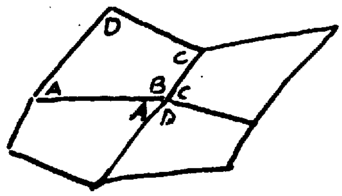
e.g. that  tessellates can be seen from 

but not from 

(3) Properties from tessellations

The tessellating pattern for triangle and quadrilateral.





gives rise to rules:

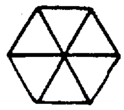
(a) sum of angles of a triangle $(x + y + z)$ $=180^{\circ}$
(b) sum of angles of a quadrilateral $(A + B + C + D)$ $=360^{\circ}$

Children can also be led to see that any shape that tessellates has an angle sum which is a factor of (or equal to) 360° or have corner angles which are a factor of 360°.

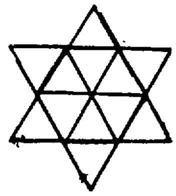
(4) Putting tessellating figures together to form shapes

The children fit the figures together to form required shapes (which can be labelled and coloured in. They can also:
(i) count the number of figures which make up the shape
(ii) identify the shapes by name
(iii) identify smaller shapes (and the number of different smaller shapes) within the shapes.

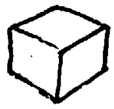
e.g. triangles
form a hexagon



or a star



diamonds form
a cube

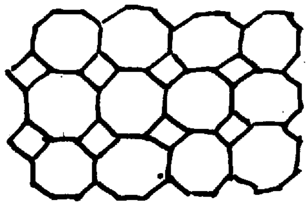


or a

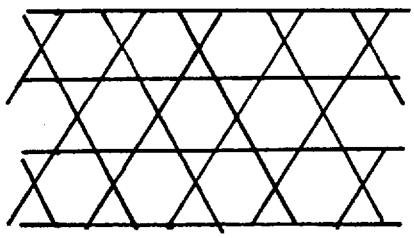
(5) Tessellations using 2 or more figures

If a figure does not tessellate (or even if it does) maybe it will if it is combined with another figure e.g.

octagon and squares



hexagon and triangle



(6) Area
Area is a measure of coverage so before the standard unit of square is introduced, children should measure area of a shape in terms of many, say, triangles, rhombi, hexagons, etc. fit inside the shape. Discussion can centre on how this relates to type and size of "unit", the need for a common or standard unit, and so on.

(7) Tessellation grids
Any shape that tessellates can make grid or graph paper, e.g. square graph paper, triangular grid paper, hexagon graph paper etc. This paper can be printed in sheets and used for:

(1) copying patterns in previous tessellations work
(2) inventing patterns similar to those made with individual figures
(3) outlining (in texta) shapes made from multiples of a basic tessellating figure, e.g. in triangular grid paper one could outline a large triangle, a large parallelogram, cube or other 3-D looking shape, and so on.
(4) finding or identifying particular shapes
(5) outlining shapes consisting of one tessellating figure, 2 figures, 3 figures etc. joined along edges
(6) outlining <u>all</u> possible shapes formed from a given number of the figures (say 5) joined along edges - called "islands"
(7) seeing if shapes made above tessellate themselves
(8) area.

(8) Art and Shape Puzzles
Tessellations are the basis of fabric design and artistic prints of the type drawn by M.C. Escher. More on this in Unit 15.

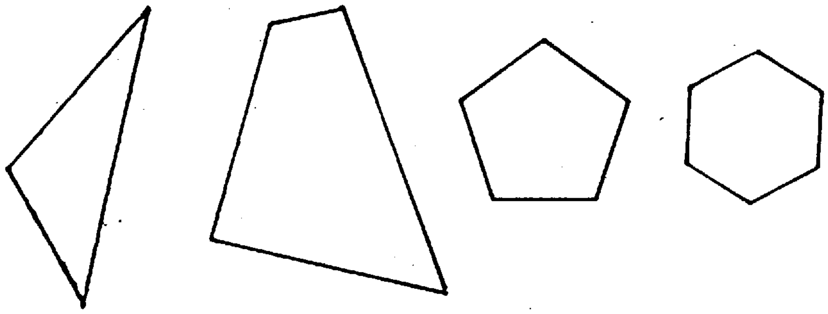
A islands of a given number of tessellating shapes can form the basis of jigsaw type puzzles. Pentominoes and hexiamonds are the most commonly known. More on this in the next chapter.

Materials:

Paper, cardboard, scissors, glue, coloured pencils, textas
Graph or grid paper made up of squares, rectangles, different types of triangle, rhombi, parallelograms and hexagons.
Templates (plastic) of various standard shapes.
A variety of tiles.

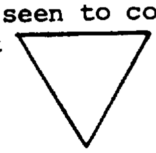
Activities:

1. (1) Would the following shapes tessellate?
Investigate by making multiple copies of each shape (suitably enlarged - note that the third and fourth shape are regular, but the exact proportions of the other two are not important).

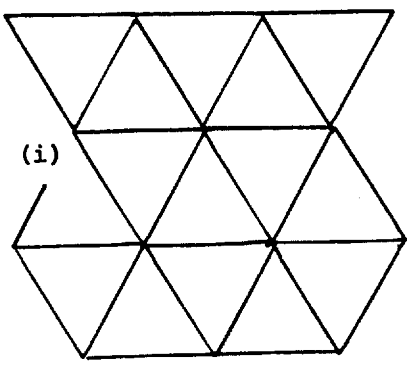


(2) Make 12 copies (paper will do, cardboard is better) of the following shapes. Determine which tessellate (fit together without gaps or overlap).

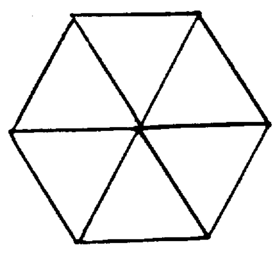
(a) square
(b) Rectangle (twice as long as it is wide)
(c) Rhombus
(d) Parallelogram
(e) Trapezium
(f) Irregular quadrilateral
(g) Equilateral triangle
(h) Isosceles triangle
(i) Right angle triangle
(j) Scalene triangle
(k) Regular pentagon
(l) Regular hexagon
(m) regular octagon
(n) Circle

Note: To ensure that you can convince anyone that your shapes will tessellate, the 12 have to be placed together in a pattern that can be seen to continue on for ever. For example, (i) convinces us that  tessellates, but (ii) does not.

(i)



(ii)



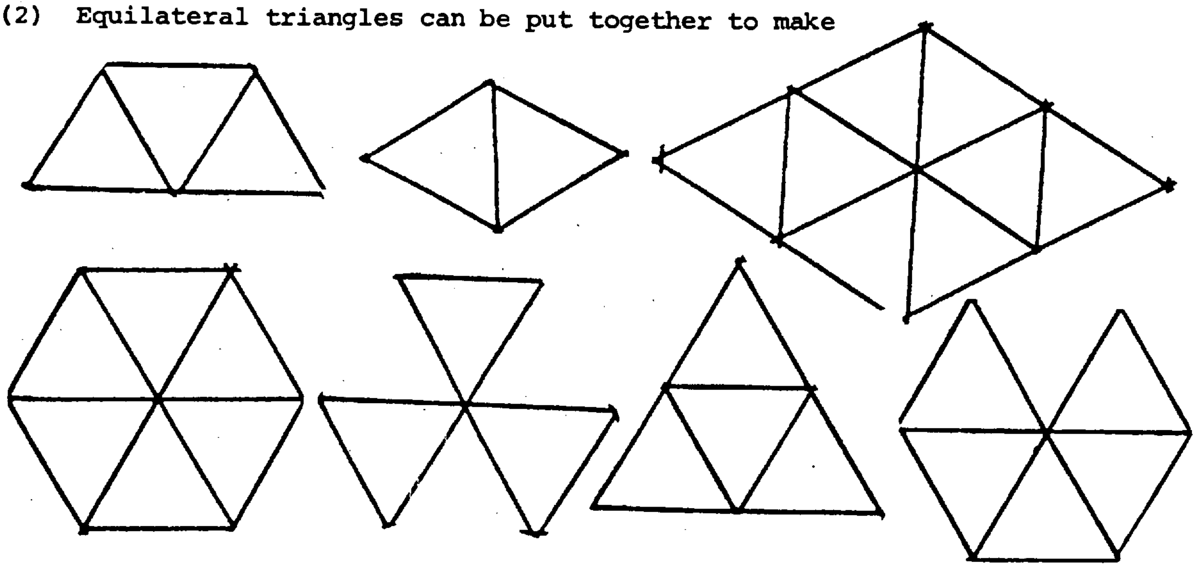
(3) Can you join rectangles to make 4 different tessellating patterns?

2. Obtain a template of different shapes. Draw a pattern (try to make it tessellate) using one of the shapes. Keep the edges together. Colour in the shapes to make the pattern more interesting.

3. (1) What are the best words to use with little children to start them off in tessellations? What is the best material?
   (2) Obtain a collection of tiles. What is the best way to fit them together to cover the surface? Make a pattern. Make a tessellation.

4. (1) Obtain cardboard and cut out copies of a shape or obtain a collection of like shaped tiles. Or obtain a template. Either by putting together or drawing, construct, copy and label more complex shapes that can be formed from the beginning shape.

   (2) Equilateral triangles can be put together to make



What other shapes can you make with equilateral triangles.

   (3) Construct seven shapes that can be formed from

   (a) right angled triangles (b) rhombi
   (c) rectangles.

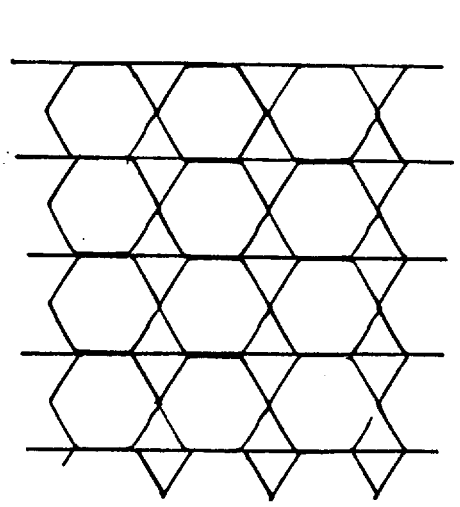
5. (1) Make a tessellating pattern from an irregular
   (a) triangle (scalene)
   (b) quadrilateral

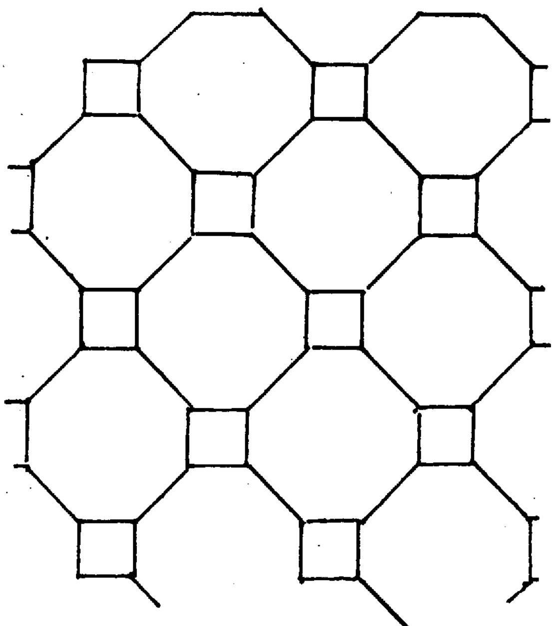
   (2) Mark the corners of a first triangle x, y and z and then place these numbers on the corresponding corners on the other triangles. Do the same for the quadrilateral, but mark the corresponding corners A, B, C and D.

   (3) Looking at where the triangles and quadrilaterals meet at the corners, what is $x+y+z$ (the interior angle sum of a triangle) and $A+B+C+D$ (the interior angle sum of a quadrilateral)?

   (4) Does the answer to (3) above give us a rule for which shapes tessellate? Use this rule to explain why hexagons tessellate but pentagons and octagons do not.

6. (1) Octagons do not tessellate but octagons and squares do. Similarly hexagons and triangles also tessellate.



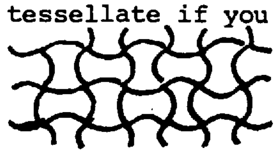


(2) Can you get a two shape tessellating pattern from
  (a) regular pentagons and triangles
  (b) hexagons and diamonds
  (c) regular hexangons and equilateral triangles (different to above)
  (d) regular pentagons and rhombi

(3) Experiment with a template joining two shapes along their edges. Which ones tessellate? Which ones form an interesting pattern?

Try a square and an equilateral triangle

7. A circle will tessellate if you cut some of it out,

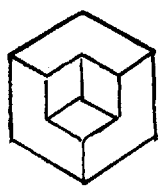
e.g. 

Can you change a non tessellating shape (e.g. pentagon) so that it tessellates? What about an already tessellating shape?

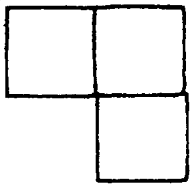
8. Use the square, triangle and hexagon graph paper at the end of this book to do the following

(1) What shapes do you see. Draw around them in texta colour.

(2) Can you draw (on the triangular grid paper)
  (a) hexagon
  (b) arrowhead
  (c) parallelogram
  (d) cube
  (e) a drawing of this solid
  (f) a 3-D drawing of building and streets.



9. (1) If we take say three squares and place them together, e.g.



we have an island of three squares.
Using shapes, graph paper or templates, construct and describe other islands.

(2) Construct all the islands (that are non congruent) for four triangles, four pentagons.

(3) Which of your islands above tessellates itself?
(choose one. Cut out copies of it. Try and fit them together)

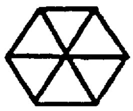
10. Play the game CURVO.

3 players.

Materials: *A set of picture cards (see examples given)*
Twenty of each of following shapes - hexagon, parallelogram, trapezium, rhombus and triangle (see below)
Die.

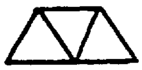
Rules: *Each player takes a picture card and the shapes, twenty of each, are placed in the centre. The object of the game is to roll the die, take the appropriate shape and place it on the picture.*

For a throw of 6 take the shape:



For a throw of 4 take the shape:

For a throw of 3 take the sahpe:



For a throw of 2 take the shape:

For a throw of 1 take the shape:

For a throw of 5 the player is entitled to remove any shapes to the value of the throw from the picture of the player on his right and place them on his own picture. If the player on the right has no appropriate shapes they are taken from the next player on the right and so on. The first player to complete his picture with shapes being the winner.

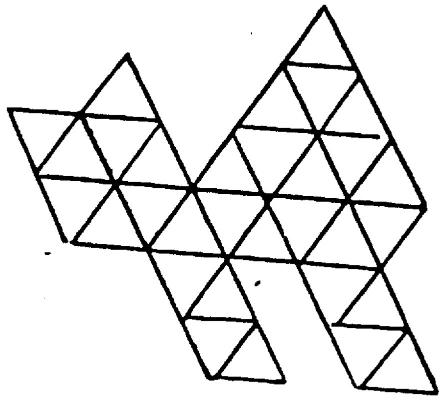
All shapes must be placed so that they don't overlap the edges of the picture or other shapes already placed. Once a shape is placed it may not be moved.

A player forfeits his turn if he cannot fit a shape to his picture or if there are no appropriate shapes left.

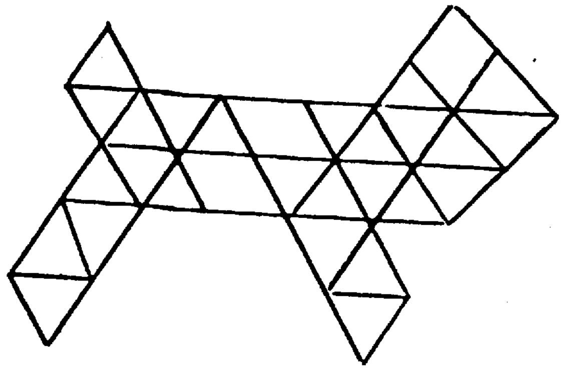
The game ends when one player completes his picture.

Scoring: 30 points bonus for complete picture.
1 point for each triangle covered at the end of the fame.

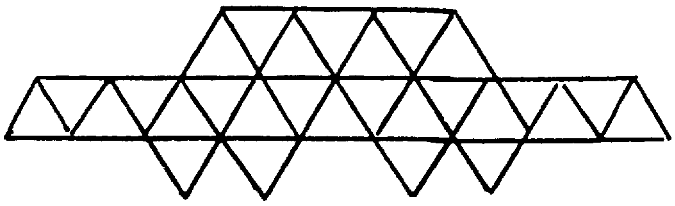
Examples of Picture Cards.



CAT



DOG



CAR

Children could develop and extend picture card set through free play activities.

Teaching Hints:

Begin tessellating activities with children via "tiling". Tessellation is a long word and as a definition difficult to get across in words. But nearly everyone is used to tiles and knows they have to be fitted together to cover space. The experience with the tiles lays the foundation for the understanding of tessellation. It is easy to act out what a tessellation is.

Templates, pens and coloured pencils is an excellent way to develop tessellation. Students can construct (and at the same time prepare a record) various tessellating patterns. The square and equilateral triangle is highly recommended for this technique. Then the shapes can be coloured in to enhance the pattern.

It is crucial that we, initially, are not pedantic about shapes being joined at edges. Children should be free to develop their own patterns. Then the special case where the pattern has no gap or overlap can be introduced as tessellation.

The shapes being tessellated should be strong, unbending and accurate. For this reason professionally prepared tiles, plastic or wooden shapes and templates are more successful with beginning tessellaters than their own class made copies.

It is important to end by stressing that so much geometric activity can develop from the tessellation concept. We start with simple 2-D shapes. Can we take this shape (or solid) and fit it with itself so that it covers 2-D space (or fills 3-D space) with no overlaps or gaps? If this can't be done for a shape, can we still form a tessellation(or tiling pattern) if we allow a second shape to be fitted in? The tiles in our kitchens and bathrooms, the bricks on our walls and pathways, the patterns on our curtains, shirts and dresses are all examples of such tessellations. M.C. Escher has produced artistic patterns from such tessellating beginnings.

Then once we have the tessellating shape we can produce tessellating grid paper with that shape repeated across it (e.g. square graph paper triangular grid paper, rectangular grid paper, hexagonal grid paper and so on). Along with this we have aids which focus on the intersection on the lines that make up these tessellating figures (e.g. geoboards, dot paper).

Now other avenues of exploration open. We can investigate area (and volume with solid tessellations) - both the areas of different shapes and shapes of a given area. We can investigate patterns in the tessellating grid paper. And the ideas that have been thought up for investigations using dot paper or geoboards is seemingly endless. Some good examples are in P.O'Daffer and S. Clemens Laboratory Investigations in Geometry, (Addison-Wesley).

Then we find another problem solving avenue opens. We can cut out all the shapes that are formed for a certain number of the tessellating shapes joined together (e.g. 5 squares gives 12 pentominoes, 6 triangles gives 12 hexiamonds). These sets of shapes can be explored to see if they tessellate. They can be joined together to form larger shapes (like jigsaw puzzles). There are whole books devoted to things that can be done for example with pentominoes. There is a whole range of possibilities (and this is widened if we bring in the solid tessellations and colour). Metamorposis by L. Mottershend describes in one chapter polyominoes, polyhexes, polyiamonds, McMahons 3 colour triangles, Polyaboloes, Polyominoids. There are others I know of (hexagonal animals, McMahon 3 colour squares). Martin Gardner has many publications in this area.

The major teaching approaches to use are:

(1) exploration and investigation - try things out, attempt to create new tessellations, copy and record, look for any patterns; and

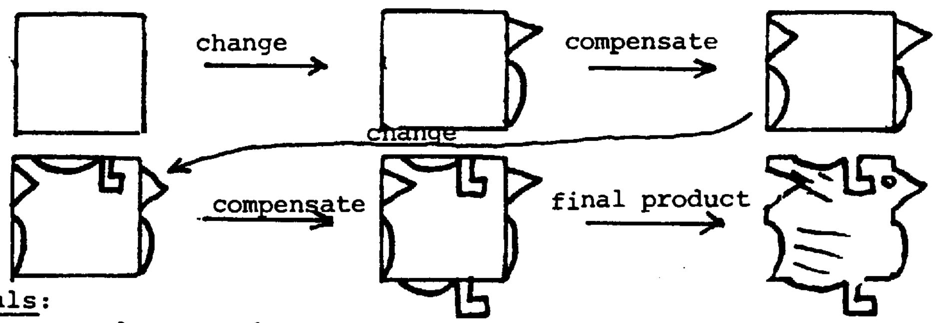
(2) identification in the world around us - look around for everyday examples of tessellations (e.g. bricks in walls, paving bricks beside shops, tiles at home).

UNIT 15: TESSELLATION AND ART

Focus:
This unit focusses on the application of tessellating shapes to fabric design and art.

Background:
All fabric design can be based on tessellations. A figure that tessellates has a design drawn inside it and then this design is repeated inside all the figures that make up the tessellation. The lines make up the tessellating figure are erased and the pattern left. This is called "hidden repeats" in the designer's language.

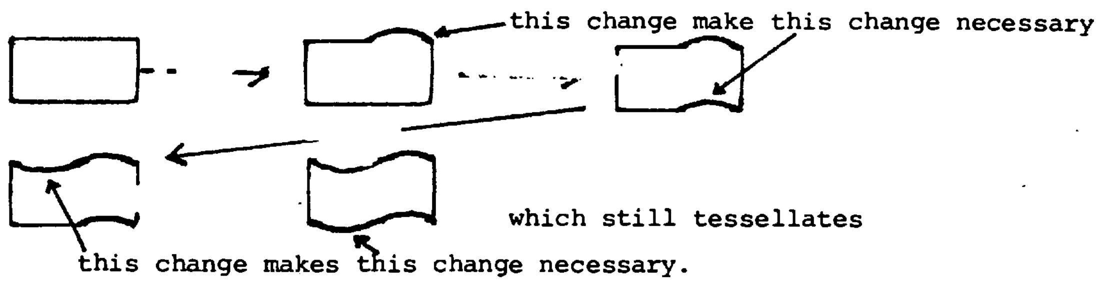
Escher prints showing tessellating figures (of men on horseback or frogs, etc) all start from a tessellating figure as the bird does below. Children can be challenged to develop their own figures.



Materials:
Pen, paper, ruler, graph paper.

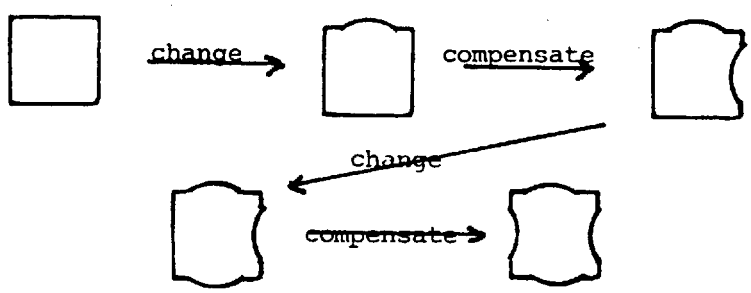
Activities:

1. A tessellating shape can produce an artistic pattern as long as what is done to one side is done "oppositely" to the opposite side, i.e. each change has a corresponding compensating change. For example:



   Starting with a hexagon, make a new tessellating shape using compensating changes that are opposite each other as we have for the example above.

2. Of course if the shape is notationally symmetric then the compensating changes can be done with notations. For example:

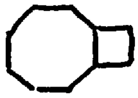


The final shape tessellates if each figure is rotated 90° before the join.

Starting with an equilateral triangle, do a similar change to the above to make a new tessellating figure.

3. Now you have the idea, create your own brick paver or tile. Make it as creative as you can. You may wish to involve 2 shape tessellating figures.

e.g.



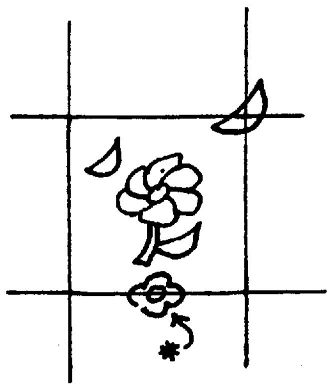
Escher was capable of making changes to tessellating squares, rectangles hexagons and triangles that produces tessellations of men on horseback, winged horses, frogs, birds, etc. Try to make a very creative Escher type drawing of your own. (You may find it useful to read Creating Escher-Type Drawings by E.R. Ranucci and J.C. Teetes - Creative Publications).

4. Tessellating shapes (or hidden repeats) are the basis of fabric design (carpets, curtains, linoleum, dresses, shirts).

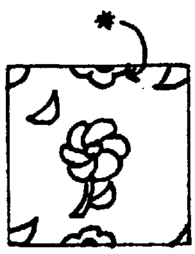
To construct them:

(a) Choose a shape that tessellates

(b) Draw "invented" design in one cell of tessellation.



(c)



Redraw design into shape (include all parts that protruded into other cells in corresponding positions, e.g.*)

(d) Repeat design as a tessellation to give overall result (could burn design into foam square with hot paper clip and then use as a stamp).

You can use triangles, squares, rectangles, hexagons as the basis of their design.

Carpets, wall paper, dress fabric, curtains can all be studies to find their basic tessellating shape.

(1) Construct your own design in a square, as above. Repeat it for a pattern.

(2) Look at a piece of fabric. Try to identify the hidden repeat in the design.

(3) Construct your own design but start from:

(a) a triangle tessellation;

(b) a hexagon tessellation.

Try to make the designs as creative as you can. What if you rotated or reflected the design as you repeated it?

<u>Teaching Hints</u>:
The opportunity to attempt a long term, highly creative pattern should be given to children when studying this area. It is one way to cater for individual talents of children.

Younger children may need a lot of "holding by the hand" to be able to create anything. The teacher can create the pattern and help a class or group to make it.

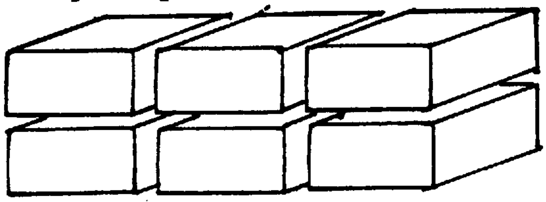
UNIT 16: SOLID TESSELLATIONS

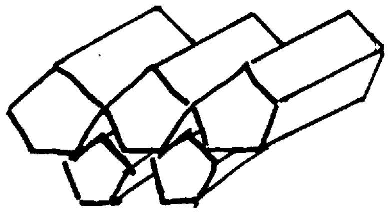
Focus:

There are some solid objects we can pile or pack together and they will fill up a truck or a room. Other objects, e.g. apples or basketballs, are fairly useless for this. This unit focusses on three dimensional tessellations, solid shapes which pack together without gaps or overlaps, and their role in modern society.

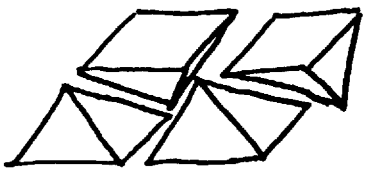
Background:

For a three dimensional shape to tessellate it has to have tessellating faces and flat surfaces. The square (the cube) and rectangular prism both tessellate as does the traingular and hexagonal prism. But the pentagonal and octagonal prisms do not. For example:





Similarly square, rectangular, triangular and hexagonal prisms tessellate while other polygon based pyramids do not. For example:

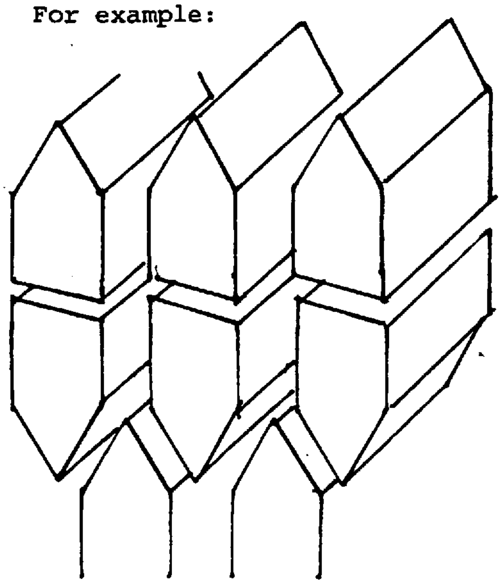


next layer upside down.

bottom layer of pyramids.

It is possible that a second (or third solid shape) could be mixed with a first to produce a tessellation. For example octagonal and square prisms will tessellate.

Solid shapes which are the combination of tessellating solids will also tessellate. For example:



Materials:

Cardboard, scissors, ruler, glue.

Example of solids (cubes, bricks, erasors, tetsahedron, etc.)

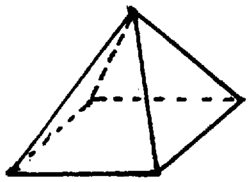
Activities:

1. By stacking cardboard copies or actual examples of the solid, determine whether the following tessellate:

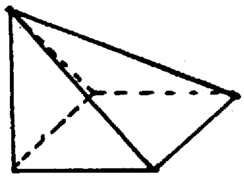
   (a) cube
   (b) rectangular prism
   (c) triangular prism
   (d) square pyramid
   (e) tetrahedron
   (f) sphere
   (g) cone
   (h) cylinder

2. A pyramid with its vertex directly over the centre of its base (e.g. example (i) below) will tessellate. Is the same true for a pyramid where the vertex is not over the centre of its base (e.g. example (ii) below)?

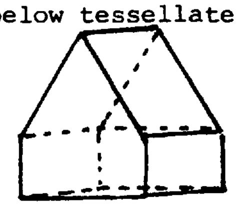
(i)

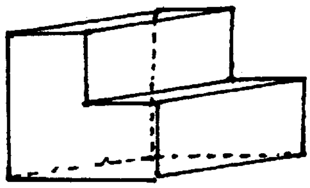


(ii)



3. The solids below tessellate:





   Draw or construct three other solids which are composites of pyramids and prisms that should also tessellate.

4. Consider the articles you find on supermarket shelves (visit a supermarket).

   (1) What is the most common shape? Why? Does it relate to tessellation?

   (2) Why are not all packages tessellating solids?

   (3) How do they arrange the packing and cartage of the non tessellating packages?

   (4) Are any compromises made in packages to gain some tessellating characteristics?

   (5) Read the teaching hints following. Now look again at your supermarket shelves and the questions above.

Teaching Hints:

Tessellations involving solids requires experimentation. It is so hard to draw or visualise what is going on. But examples are hard to make (and particularly hard to make well). Teachers (and schools) need to collect examples and store them. For example:

. erasors (rectangular prisms)

. blocks (cubes)

. triangular, rectangular, square and circular dowling cut into sections. (triangular prisms, rectangular prisms, square prisms and cylinders)

. L shapes dowling cut into sections

. tetrapack containers (tetrahedrons)

. small wood and plastic houses (cut off the chimneys of Monopoly houses)

. cans, bottle tops (cylinders).

. various types of small packets (rectangular prisms)

. balls (spheres).

Packaging in our society is a compromise between strength, cheapness, appearance and packing (tessellation). The best shape to pack (or tessellate) is a rectangular prism (the common box). It is also cheap to make because it can be folded from a net. But it is weak. Corners and edges are points of weakness. The strongest surface is a curve like in a sphere or a cylinder with semispherical ends. Also the sphere is the package where the least material (surface area) encloses the most contents inside (volume). The cube is the best in this regard when we consider prisms. Against this the tall square prism or cylinder appears to have the greatest volume of contents for an actual fixed volume. For example compare a litre size drink bottle with a 10cm cube (which is also a litre). The bottle appears to be much larger.

Sometimes the package has to be strong, (e.g. pressure pack cans, gas bottles). Therefore spheres or cylinders with sperical ends should be used. But these pack so poorly. Hence compromises are made. Have a look at a pressure pack can or a gas cylinder to see what they are.

The cylinder is, of course, the best compromise. It will stack to some extent lying down or standing. It is easy to make and relatively strong.

Materials that do not pack well are either placed inside another package (e.g. light globes) or packed, with something to stop them rolling around, into containers that do tessellate (e.g. cartons).

CHAPTER FIVE: SHAPE PUZZLES AND VISUAL IMAGERY

Visual imagery is one of the major end points of instruction in geometry. It is the major problem solving tool that goemetry instruction develops.

Visual imagery is the ability of the child to manipulate images in the mind to solve problems. Consider the following problem.

Jenny is sitting beside Graham and David at the round table. Anne is two places from Robert and David is on his other side. Sally is sitting next to Anne. Graham is closer to Anne than David. Describe who is sitting next to whom.

Try to solve it without visualising the table and the people sitting around it.

The ability to think visually is critical in mathematics achievement. Many educators argue that it is one of the major characteristics of problem solving performance. They recommend that geometry instruction to develop it as a better approach with low achievers than continued drill with their errors. Modern methods of instruction in numeration and operations utilize visual imagery in the way materials are used to develop the concepts and processes.

One of the good ways to develop visual imagery is through shape puzzles of a "jigsaw" type. In this chapter we look at such puzzles. Unit 17 looks at dissections and unit 18 looks at the special dissections e.g. tangrams and the soma cube. Unit 19 looks at the tessellation puzzles of pentominoes, hexiamonds and McMahon colour shapes. Unit 20 completes the chapter by looking at visual imagery and how it may be more directly developed.

UNIT 17: DISSECTIONS

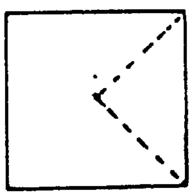
Focus:
We can take a completed picture (or solid shape) and cut it into pieces. The pieces can be given to another person and it is a problem for him/her to reconstruct the shape or solid (or to use the pieces to construct other shapes or solids). To make it even more difficult the pieces can be joined to make another shape and the child can be given the task of finding where to cut this second shape so to find the pieces that can form the first shape. Such is the foundation of the "jigsaw type" puzzles that are dissections. Much ingenuity has been used to find the most "diabolical" and fruitful ways to subdivide shapes so as to make the constructions difficult and to allow for a variety of constructions. This unit focuses on these dissections and how skill with the visual thinking that is involved can be developed with children.

Background:

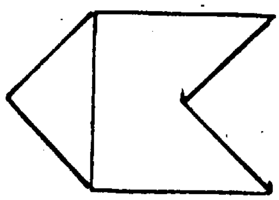
A simple dissection can be formed in the following way.

ren

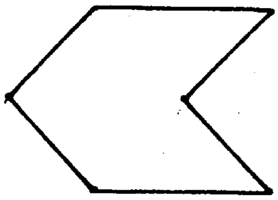
(a) Consider the square shown in the diagram below and the dotted lines as cuts for dissecting it.



(b) If this square is drawn on paper and the cuts shown made, the two resultant pieces can be re-arranged thus.

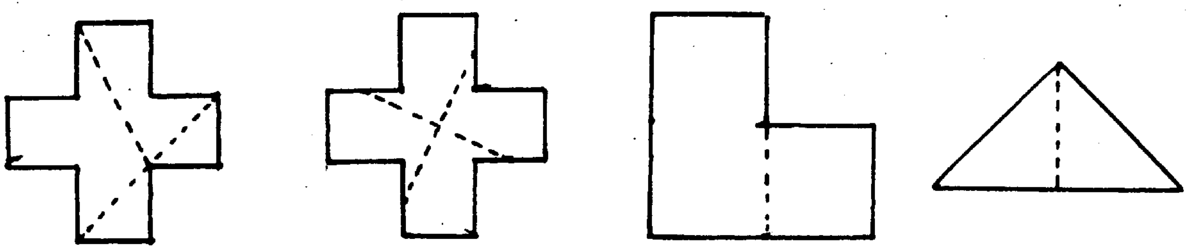


(c) If the boundary only of this figure is reproduced as below then



a dissection puzzle is apparent. The puzzle is to insert one straight cut to dissect the chevron into 2 pieces which can be re-arranged to form a square.

The above example is a dissection formed from a square. Here are some others:



Dissections can be formed from any shape: square, rectangle, circle, triangle, hexagon, etc. They can also be formed from 3-D solids, e.g. sphere and cube puzzles. There are two types of dissection:

type A: shape cut into pieces - children reform pieces as in a jigsaw puzzle; and

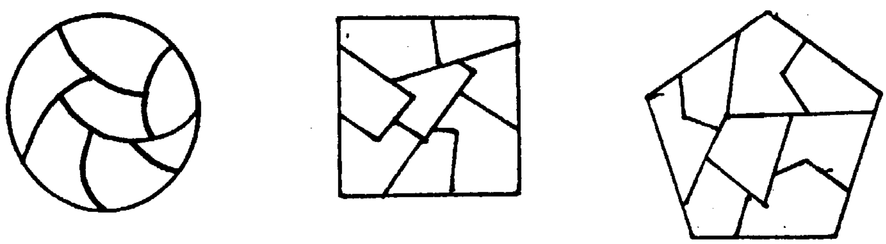
type B: shape cut into pieces, reformed into second shape - children determine where to cut second shape to form pieces to make the first shape.

<u>Materials</u>

Pen, paper, scissors, glue, tape.

<u>Activities</u>

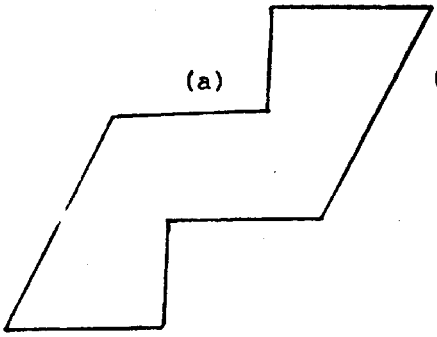
1. (1) Make large copies of the following:



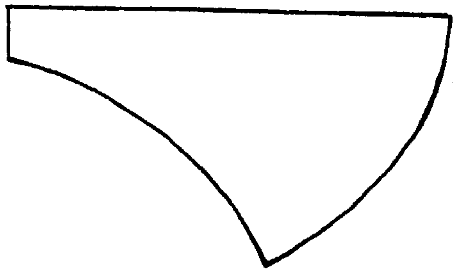
(2) Cut along the lines to form three jigsaw puzzles. Try to fit the pieces together to form the original shapes.

(3) Copy the following figures on to paper. Make one straight cut in each figure. Then fit the pieces together to form a square.

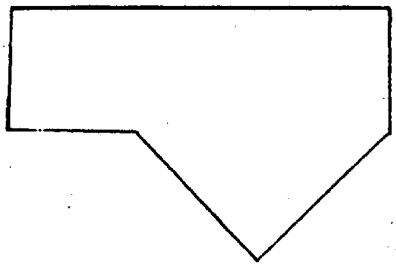
(a)



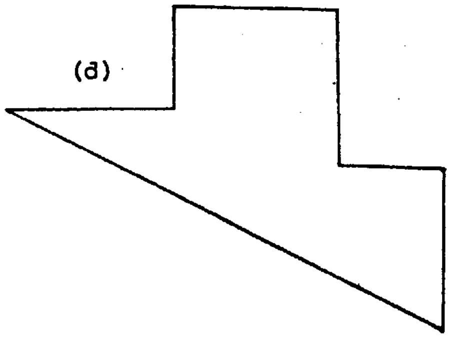
(b)



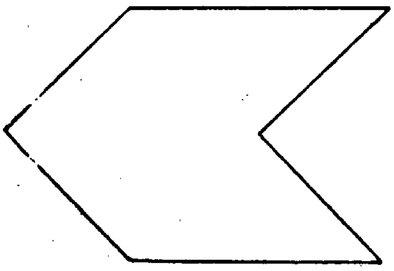
(c)



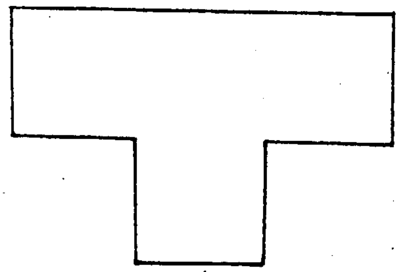
(d)



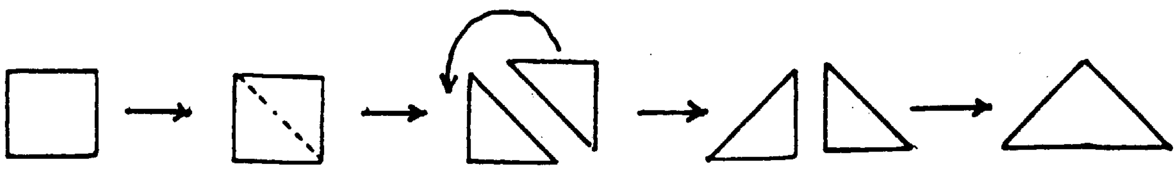
(e)



(f)

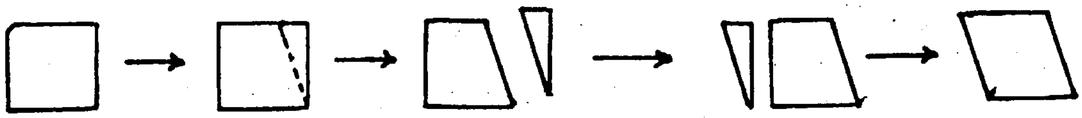


2. A dissection can be made as follows. We start with a square.



The result can then be given to a child to cut and form back into a square.

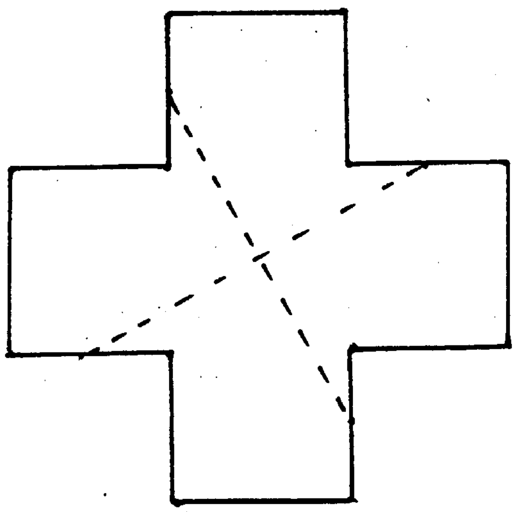
Here is another one.



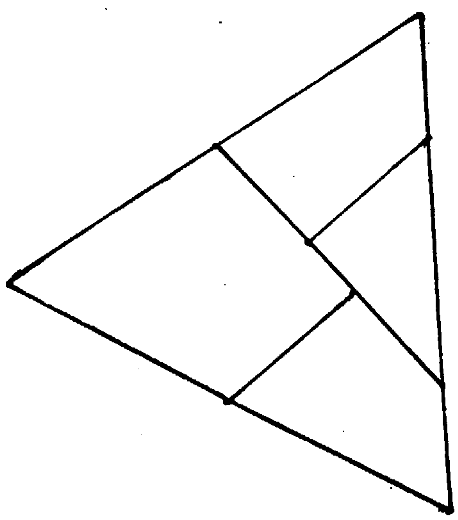
Construct three one cut dissections of a square from your own imagination.

3 (1) Cut the following along dotted lines and form a square from the pieces.

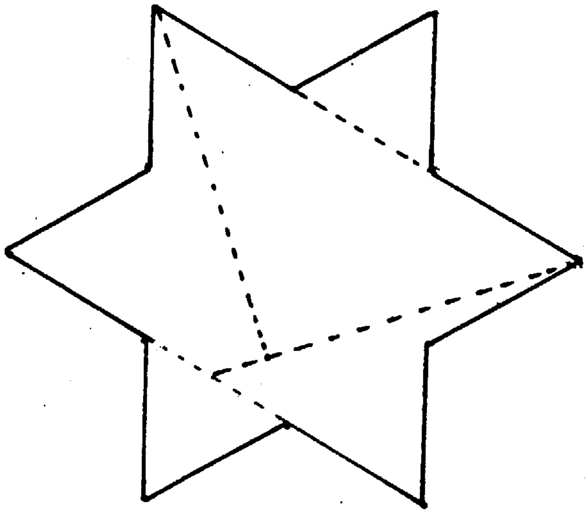
(a)



(b)



(c)

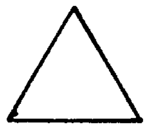


(2) Make such a multiple cut dissection of a square by cutting a square and reforming it.

4. Dissections can also be made from shapes other than a square. They are easy to do.

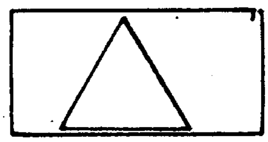
<u>*Type A*</u>

(a) *Take a shape - say a triangle*



(b) *Cut the shape into pieces*

(c) *Make a template of the original shape, e.g.*



(d) *Give the pieces and the template to children and direct them to cover the template with the pieces.*

<u>*Type B*</u>

(a) *Take a shape - say a circle.*

(b) *Cut into two pieces (or maybe three)*

(c) *Reform the two (or three) pieces into another shape*

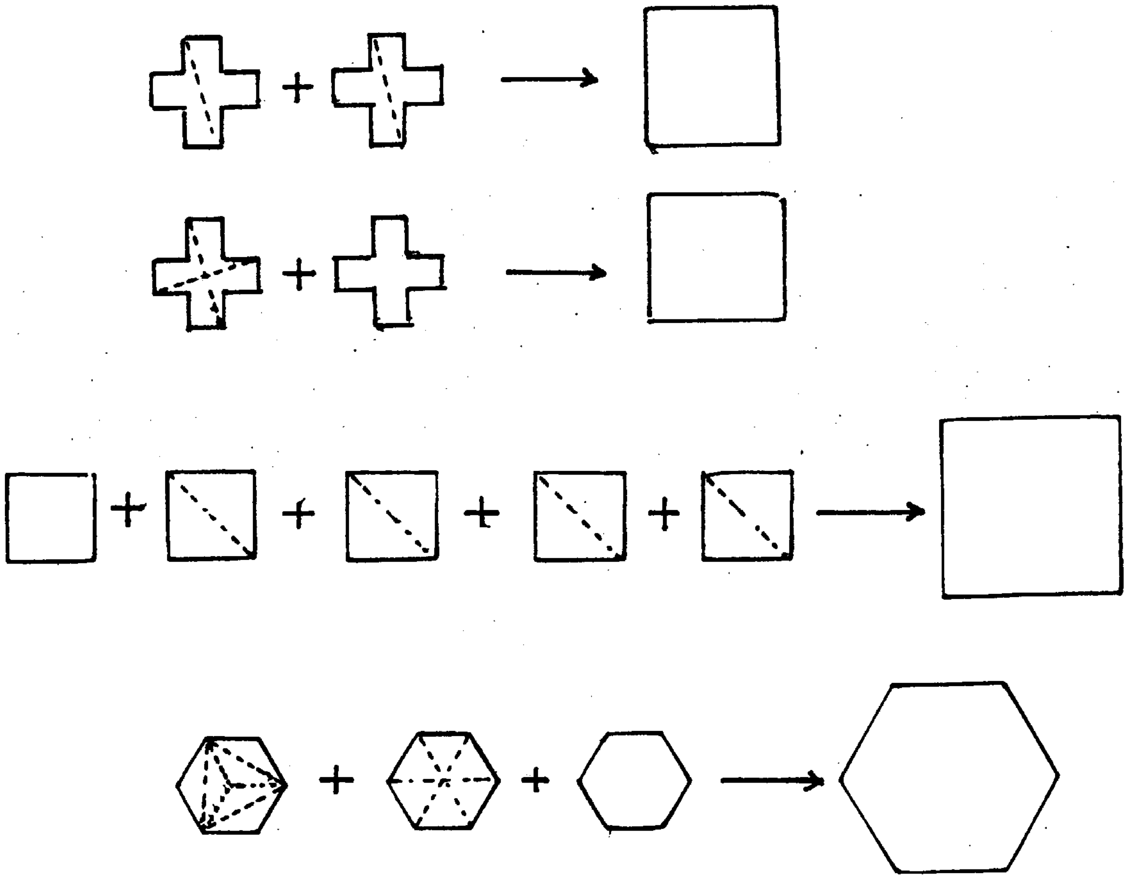
(d) *Make a copy of this new shape and give to children. Direct them to make one cut (or two) so that the pieces can be reformed into the original shape.*

(1) Construct a type A dissection from an equilateral triangle.
(2) Construct a type A dissection from a circle.
(3) Construct a type B dissection from a rectangle.
(4) Construct a type B dissection from a hexagon.

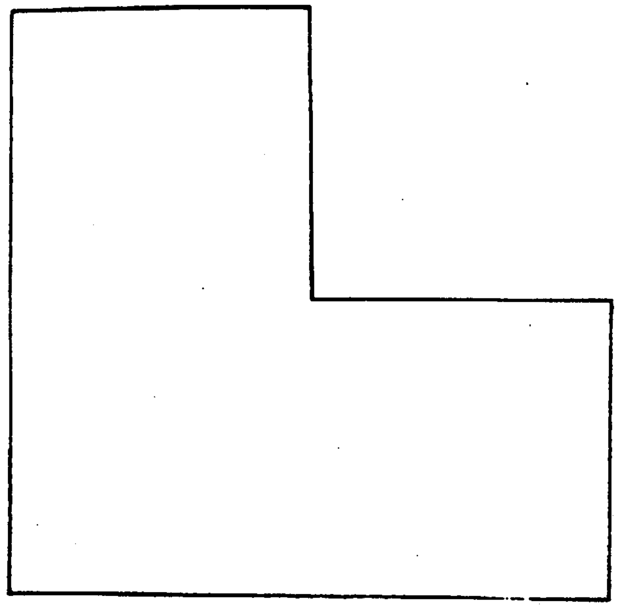
5. Dissections can be made from grid paper.
(1) Obtain 2cm square and triangular grid paper.
(2) Cut out an 8 x 5 rectangle from the 2cm square grid paper and a large rhombus from the triangular grid paper.

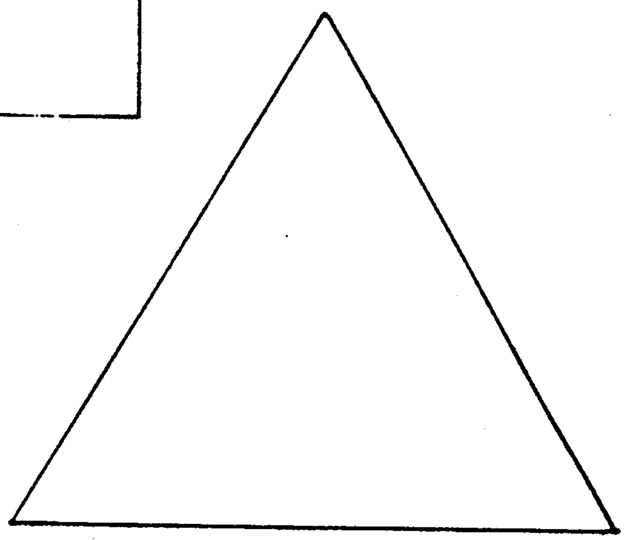
(3) Cut each of the rectangle and the rhombus into five pieces - ensuring that cuts are made along grid lines.
(4) Give these pieces to a friend to reform into the rectangle and the rhombus.

6. Here are some more dissections - try them.



7. An extension of dissections is to make the cutting or dissecting the problem. For example:
   Make copies of the following figures
   (1) Try to cut each of the figures into two congruent parts.
   (2) Try to cut each of the figures into three congruent parts.
   (3) What about 4 parts? 5 parts?





Teaching Hints:
For small children the shapes can have pictures from magazines stuck over them before they are dissected. In this way, the picture will help the children reform the pieces. It will be like a jigsaw puzzle.

The dissections need not be restricted to two dimensional shapes. There are many puzzles that can be bought which are made of solid pieces that have to be formed into solid shapes. These are excellent value because they reflect visualising in the real 3-D world.

This chapter gives some examples of dissections. There are many more. Teachers can collect a resource of them. But this is not necessary. They can be so easily constructed when needed by reversing the procedure of solving them as is described in this unit. Children can make them for each other.

The value of dissections in the classroom is manifold. Following are some reasons why they should be included in primary mathematics.

The tangrams mentioned in these reasons is covered in the next unit.

(1) Engaging in puzzles of this type is sometimes frustrating but nevertheless enjoyable; Mathematics might be seen by children to be more than just memorising facts, working out "sums".

(2) Puzzles of this type can help with the appreciation of geometric forms; consider the simple dissection puzzles at the beginning of this section.

(3) Most of the puzzles can be adapted to the maturity of the child. Consider the Tangram puzzles. With young children the lines for dissecting the square could be provided on stencil and their task would be simply to cut out the seven tans. The pieces would then have to be fitted onto completed puzzles, i.e., a simple shape fitting exercise. At a later stage children could be encouraged to construct the dissection lines using standard ruler and compass constructions. The puzzles would then appear only in outline form.

(4) As preshadowed in the previous paragraph the completion of the pieces for puzzles does provide interesting applications for standard construction techniques, the practice of measurement skills.

(5) Many of the puzzles provide scope for developing the child's problem solving ability.

(6) Awareness of dissection puzzles could be used to encourage the child's creativity. Consider the simplicity of the very first dissection puzzle described; children without much difficulty could invent similar puzzles. Children could be encouraged to create and name their own designs using Tangrams - a class could construct, using a suggested theme, a wall mural for its room to emulate "The Orchestra" (see next unit

(7) Activities with puzzles provide an appropriate medium for the use of geometric language.

(8) Puzzles can be used to develop other mathematical concepts, such as area.

UNIT 18: TANGRAMS AND OTHER SPECIAL DISSECTIONS

Focus:
In this unit we discuss dissections which have many applications in puzzles, e.g. tangrams and some cubes.

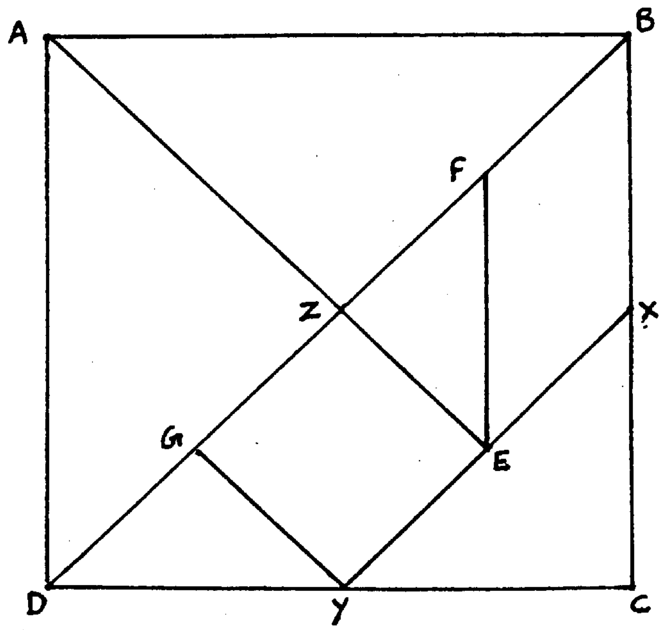
Background:
Tangrams.

The name tangram refers to the dissection of a square into seven pieces. The resultant pieces for the basis of many intriguing puzzles.

Tangrams are Chinese in origin and appear to be about 4000 years old. There has been claims that they were invented by a Chinaman named Tan but this appears to be mythical. The name probably comes from the Chinese T'an (to extend), or the Cantonese t'ang (Chinese), added to the European ending "gram". The term was most likely coined in the USA between 1847 and 1864.

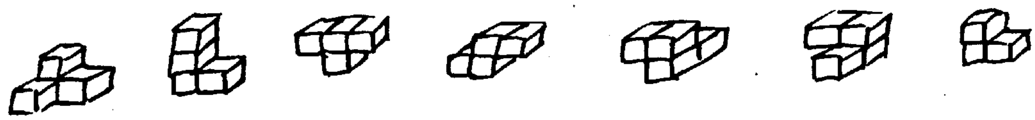
The square is dissected in the following way:

(i) Draw a square ABCD of side length 8 centimetres, say.
(ii) Join B to D.
(iii) Locate the midpoints of BC and DC, and label these X and Y respectively.
(iv) Join X to Y
(v) Join A to Z, the midpoint of DB and extent AZ to meet XY at E.
(vi) From E draw a line parallel to BC meeting BD at F.
(vii) From Y draw a line perpendicular to BD meeting it at G.



Soma Cubes

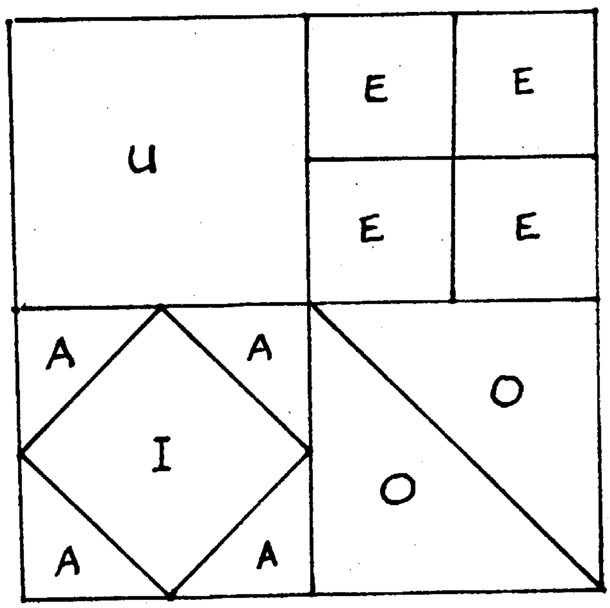
A soma cube is a dissection of a 3 x 3 cube composed of pieces as below (six made out of 4 cubes and one out of 3 cubes):



It can be made out of cubes stuck together or bought already made.

5 piece dissection.

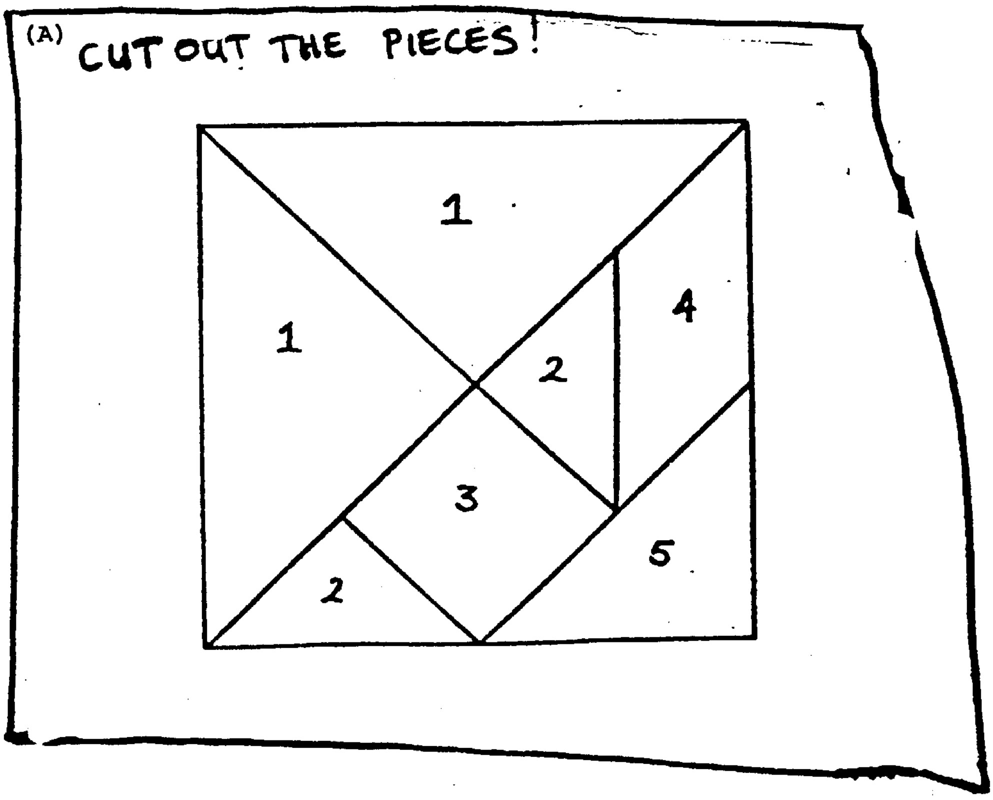
A square can be cut differently to the tangram to form another dissection that is the basis of many puzzles.

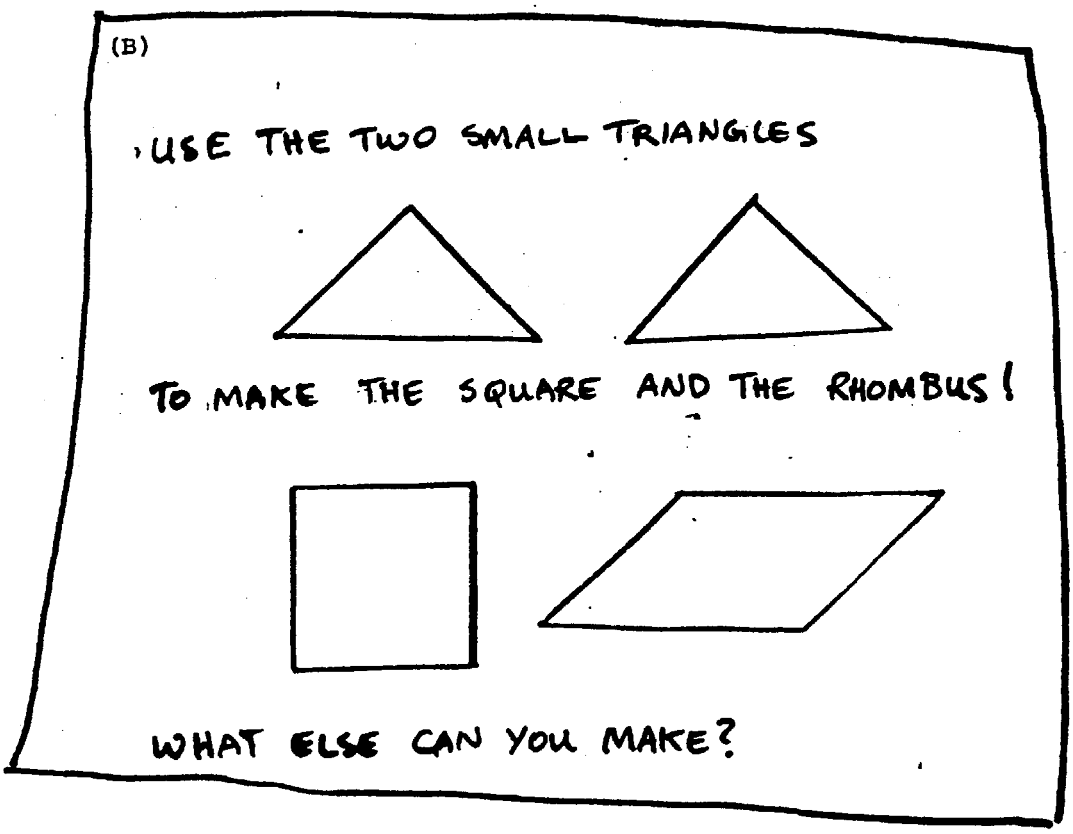


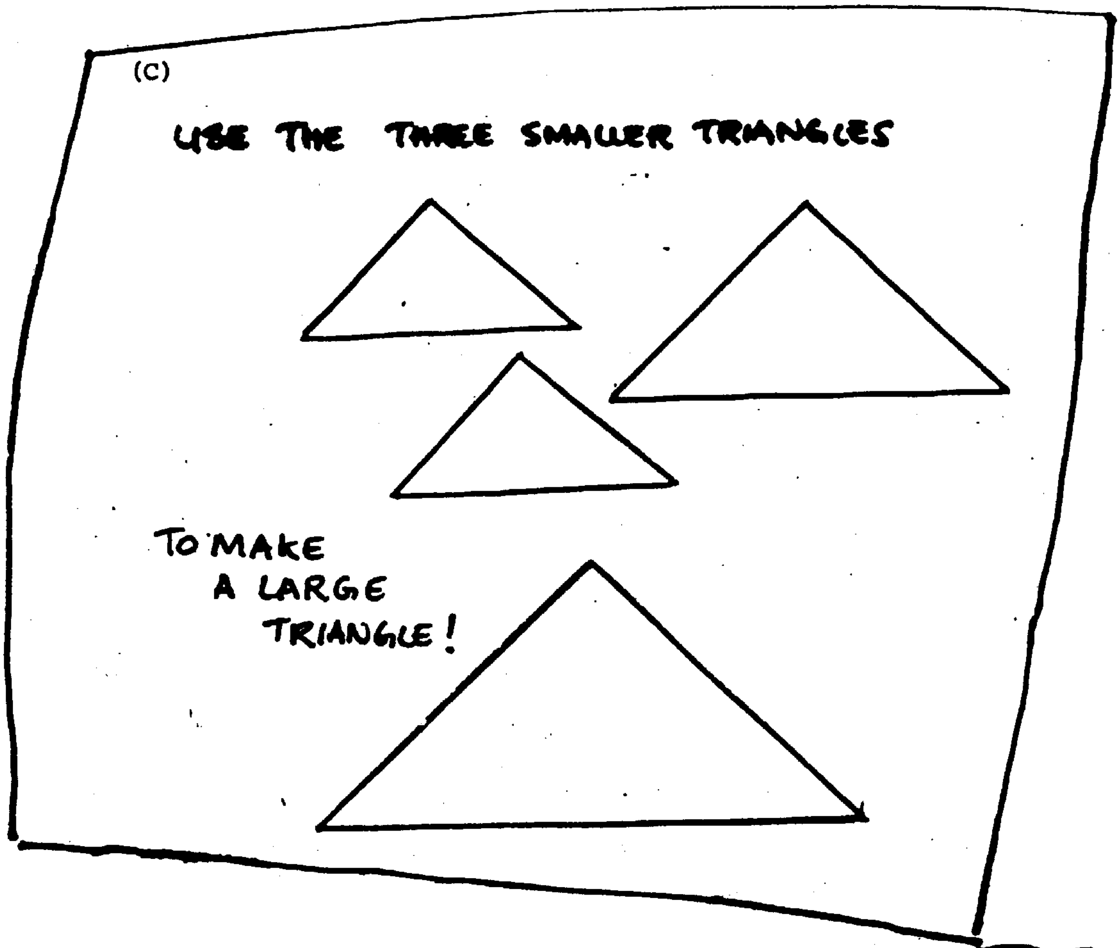
Materials:
Scissors, paper, cardboard, glue, ruler.
Cardboard, wooden or plastic copies of tangrams, soma cubes and 5 piece dissections.

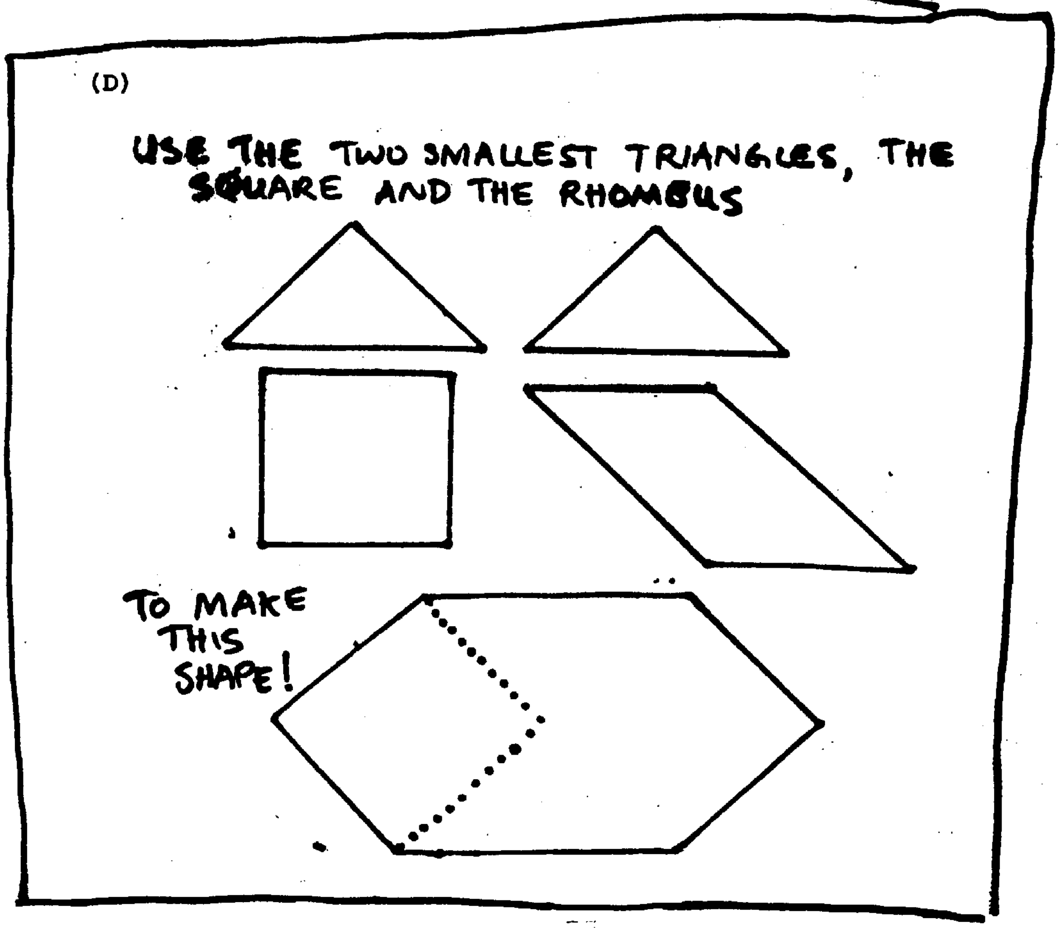
Activities:

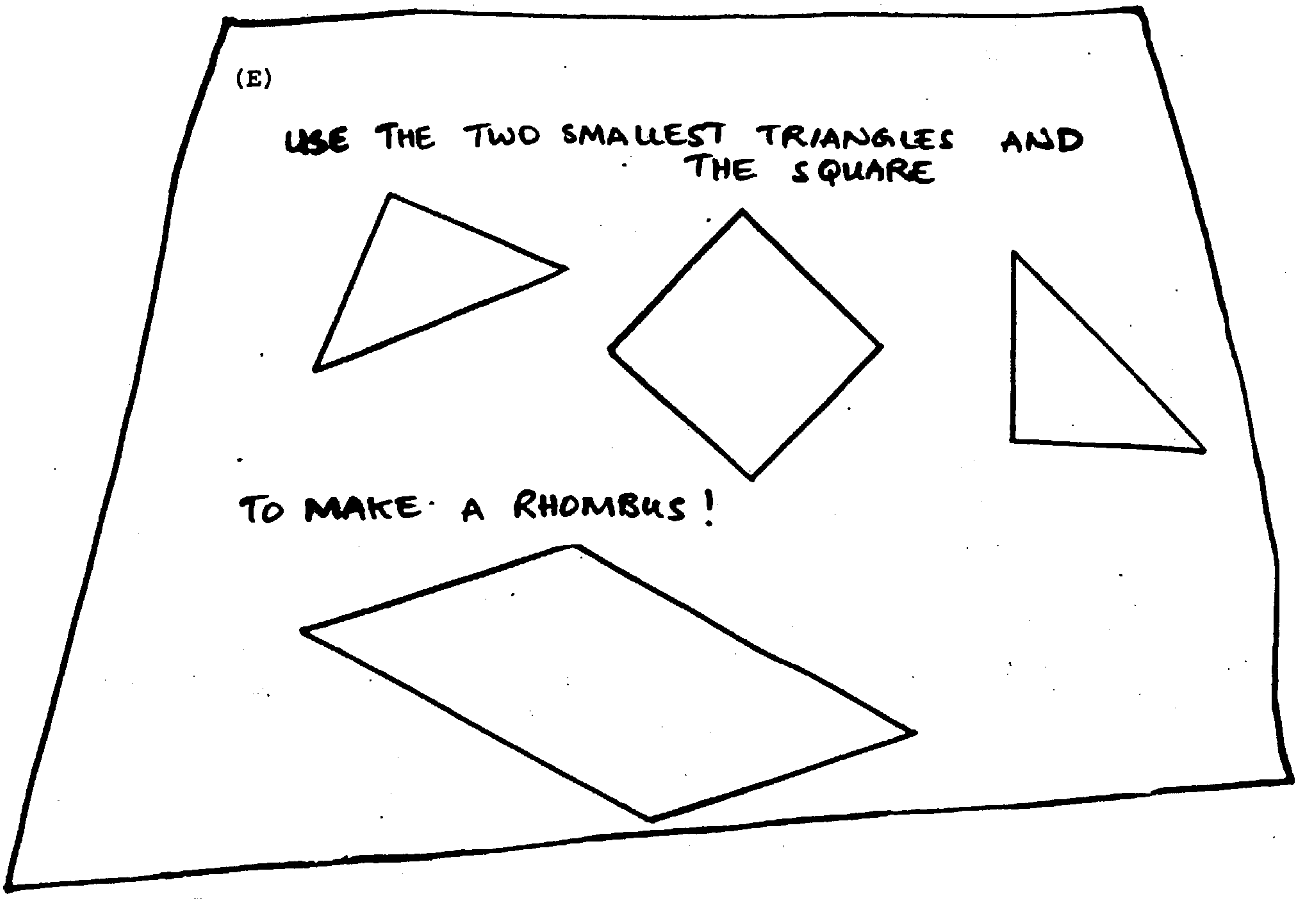
1. Complete the following tangram cards.

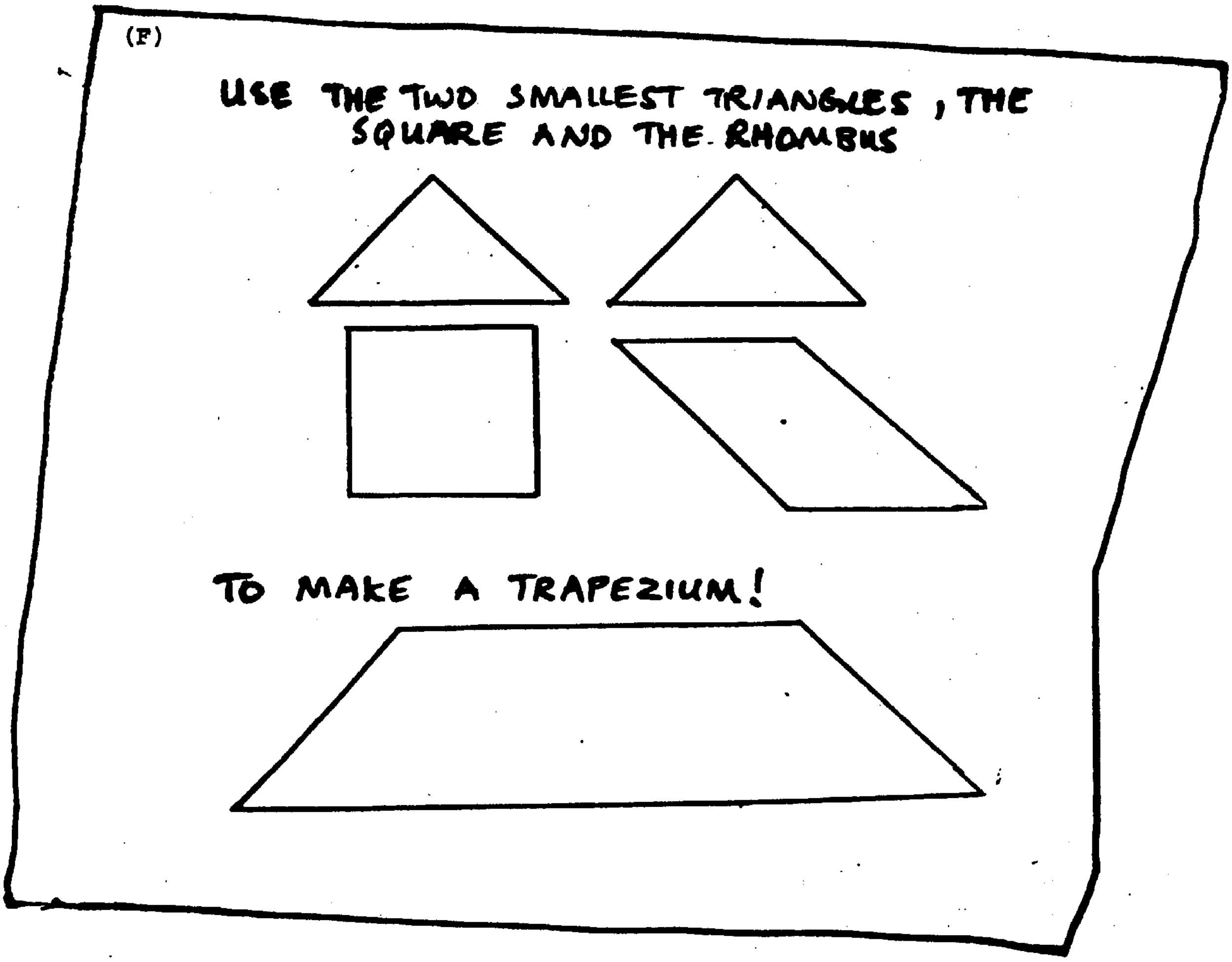


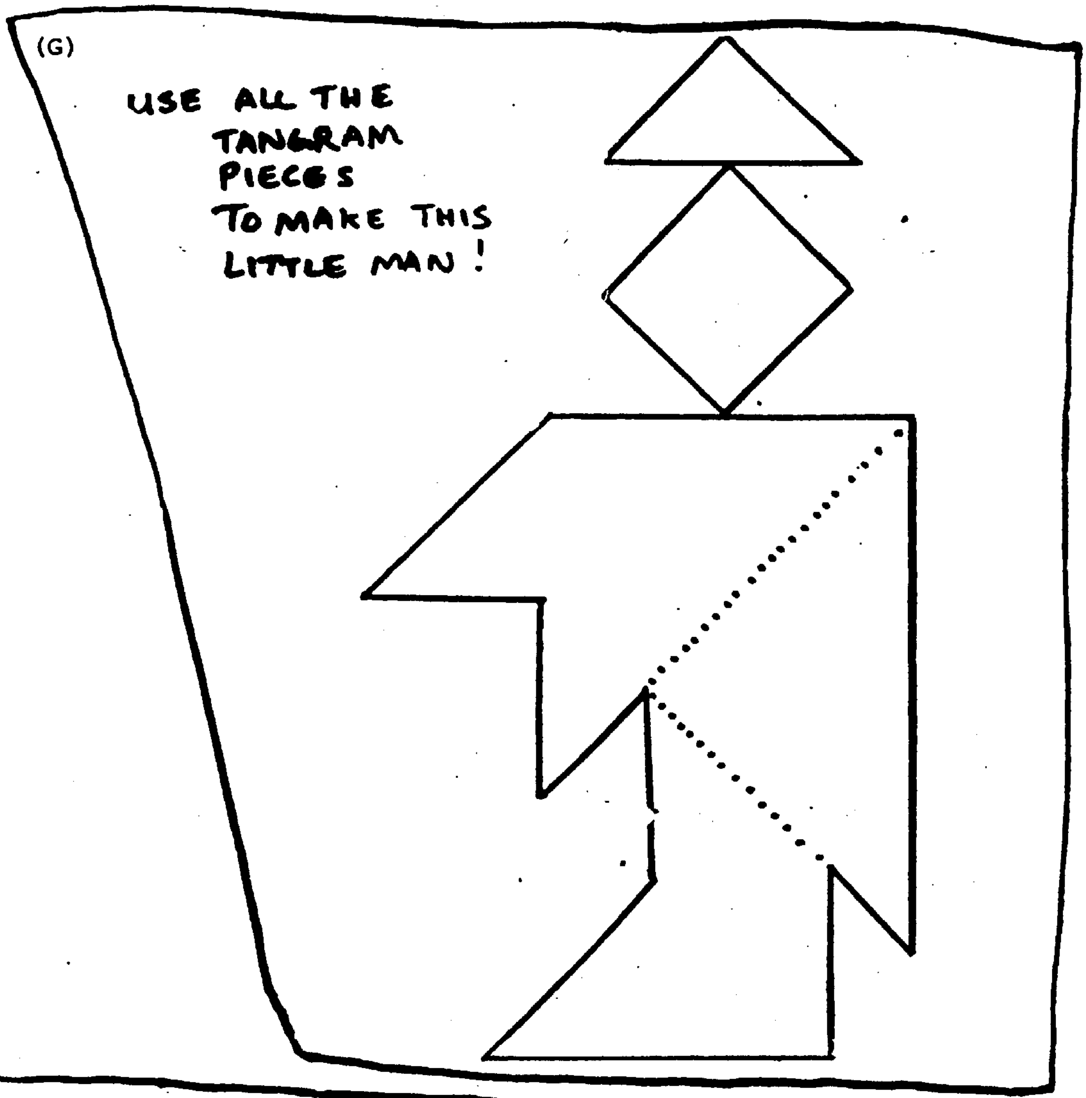


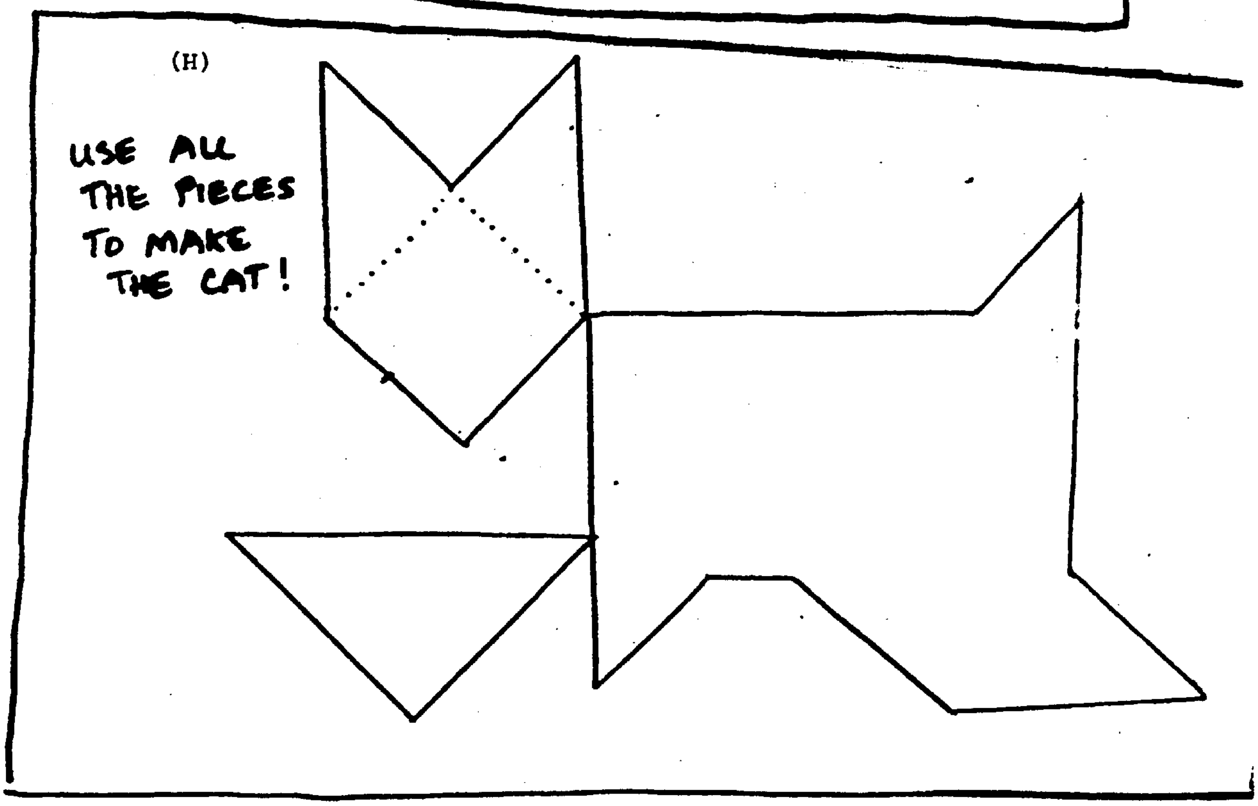

(C)
USE THE THREE SMALLER TRIANGLES
TO MAKE
A LARGE
TRIANGLE!


(D)
USE THE TWO SMALLEST TRIANGLES, THE
SQUARE AND THE RHOMBUS
TO MAKE
THIS
SHAPE!


(E)
USE THE TWO SMALLEST TRIANGLES AND THE SQUARE
TO MAKE A RHOMBUS!

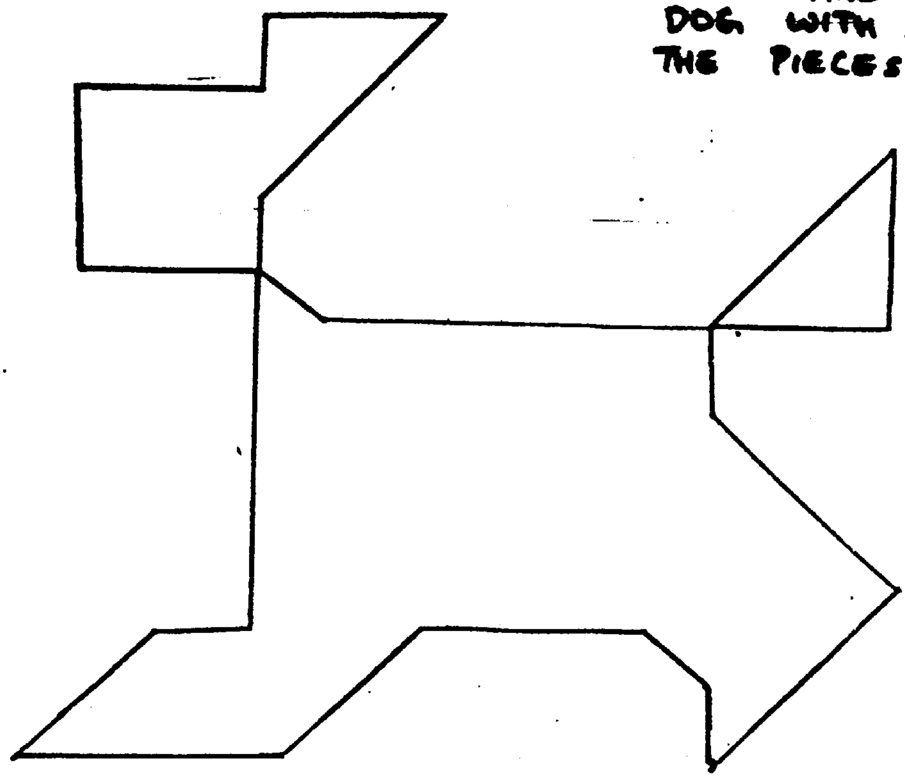

(F)
USE THE TWO SMALLEST TRIANGLES, THE SQUARE AND THE RHOMBUS
TO MAKE A TRAPEZIUM!


(G)
USE ALL THE
TANGRAM
PIECES
TO MAKE THIS
LITTLE MAN!


(H)
USE ALL
THE PIECES
TO MAKE
THE CAT!

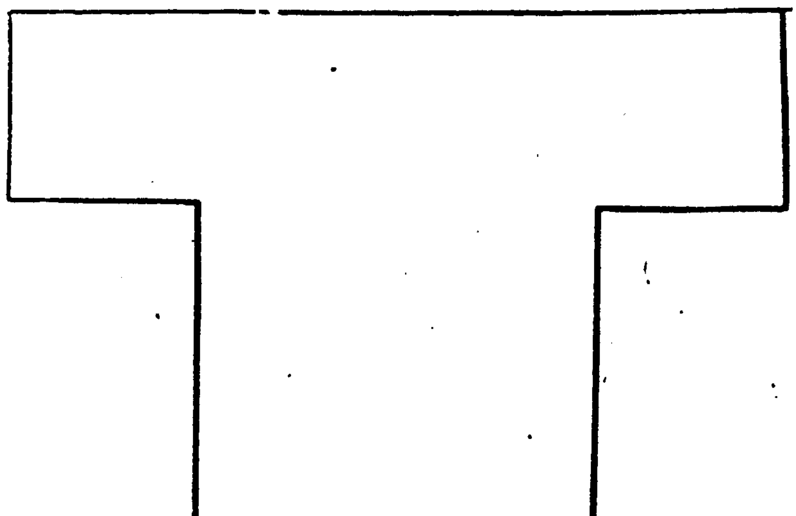
(I)

TRY TO MAKE THIS DOG WITH ALL THE PIECES!



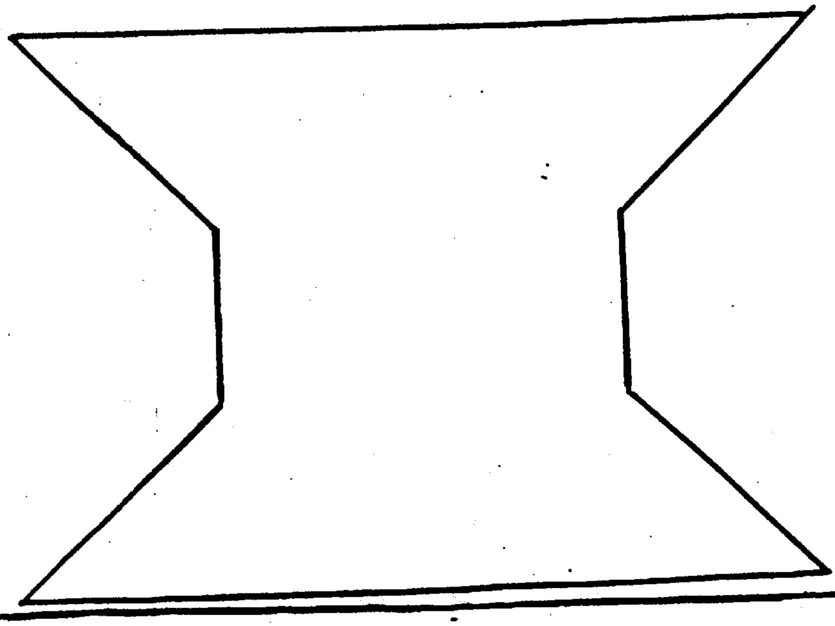
(J)

USE ALL THE PIECES TO MAKE A T!

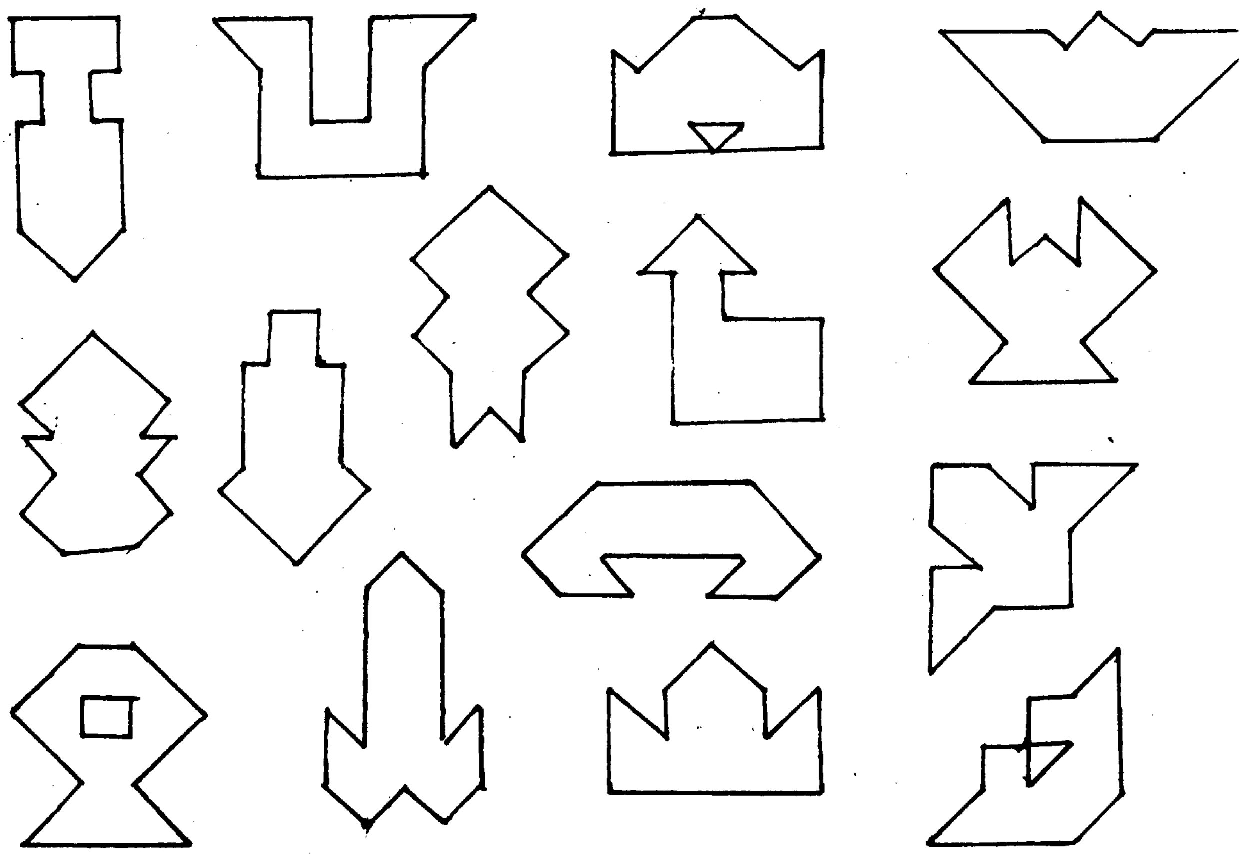


(K)

TRY TO MAKE THIS SHAPE WITH ALL THE PIECES!

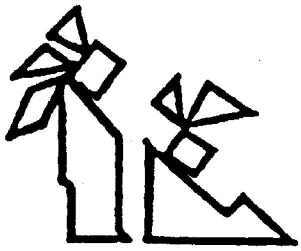


2. Here are some more shapes to form with the tangrams.

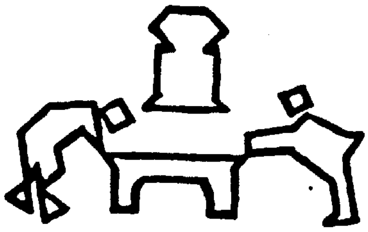


3. Interesting pictures can be formed by using more than one set of Tangrams. Each of the shapes in the following puzzles requires more than one set of tangrams.

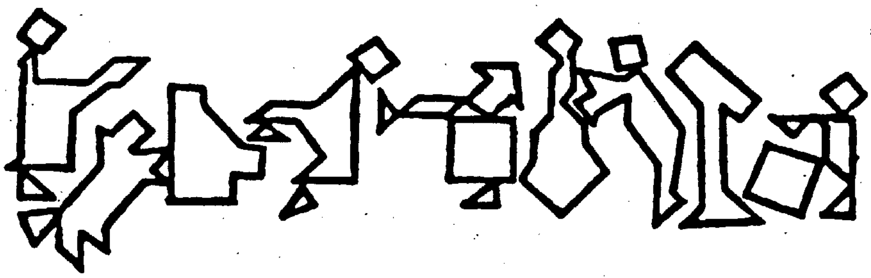
(a) "Indian with Squaw" (Sam Lloyd) - 2 sets.



(b) "A Game of Billiards" (H.E. Dudeney) - 4 sets.

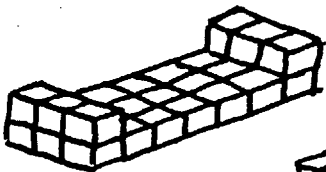


(c) "The Orchestra" (H.E. Dudeney) - 9 sets.

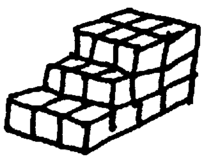


4. Obtain 22 cubes. Make up a set of seven soma cubes by sticking or taping the cubes together. Use your cubes to make

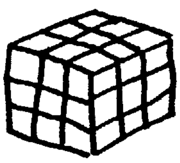
(a) the bed.



(b) the steps



(c) the cube



5. Make up a set of the 5 piece dissection. Use it to make the following:

   (a) A triangle, a square and parallelogram from two As
   (b) A triangle, a square, a rectangle and a six sided figure from four As.
   (c) A four sided, five sided and six sided figure from four Es
   (d) 6 different ways to use all the pieces to make the original square.

6. Puzzles for these dissections are easily made up. For example:
   (a) Take a set of the tangram pieces.
   (b) Assemble them to make a shape.
   (c) Draw around this shape on to paper to make a template.
   (d) Instruct children to assemble their tangram pieces to cover this template

   Make your own tangram puzzles. Do the same for the 5 piece dissection pieces.

Teaching Hints:
When constructing puzzles for children, teachers can ensure a reasonable gradation in difficulty by keeping the following suggestions of Mick Redden of Riverina- Murray IAE in mind.
(1) Task Imposed Difficulties

SOURCE OF DIFFICULTY	← LESS DIFFICULT		MORE DIFFICULT →
The number of pieces used in the puzzle	A small number of pieces (2 or 3)		A larger number of pieces (5 or more)
The complexity of the shape to be covered or constructed	A template outline giving clues to place -ment of pieces.	A template outline giving few clues	A misleading template outline
The type of movements required to make the shape	Only turning is required		Both turning and flipping is required.

(2) Teacher controlled difficulties

SOURCE OF DIFFICULTY	← LESS DIFFICULT		MORE DIFFICULT →
Shape outline	A full scale outline supplied	Outline drawn to scale	Outline suggested but not accurate
Clues given	Some pieces positioned in the outline	Verbal hints supplied	No help given
Presentation of the task	Task broken into simpler subtasks		Full task supplied only
Organization of the tasks	Similar tasks grouped together and similarities discussed	Similar tasks together, with little comment.	No organisation supplied

The shape puzzles herein have a wider use than just imagery. The tangrams can be used for area. The square is considered to be one unit. Then areas for shapes constructed can be found.

The 5 piece dissection puzzle can be used for logic. For example take one F, one I and one U and rename them Ali and Bet and Tim. Ali said: "I'm bigger than Tim". Bet said "I'm bigger than Ali". Who's who?

UNIT 19: TESSELLATION PUZZLES

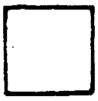
Focus:
This unit returns to the concept of tessellation for further examples of shape puzzles. In particular, the shape puzzles of pentaminoes, hexiamonds and McMahon Colour shapes are discussed.

Background:
Tessellating puzzles emerge from tessellations of a figure. All the possible shapes formed, when a given number of tessellating figures are joined on their edges, can be used for "jigsaw" type puzzles. Any tessellating figure will give rise to such a set of "jigsaw" pieces.

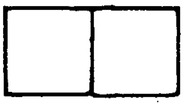
Polyominoes
Polyominoes belong to the branch of mathematics, combinational geometry, which deals with the ways in which geometrical shapes can be combined. Many design problems in practical engineering are combinational in nature. Yet it is a frequently neglected area in mathematics because it seems to have few general methods and because, in it, systematic rules have not replaced ingenuity.

Polyminoes are formed from squares placed together along their edges. Two polyminoes are considered to be the same if one can be formed from the other by a rotation (turning 90, 180 or 270 degrees) and/or a reflection (flipping over).

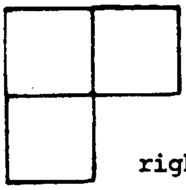
(1) Monomino: Single square



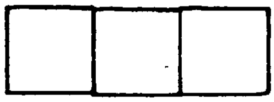
(2) Dominoes: Two squares - 2 types



(3) Trominoes: Three squares - 2 types

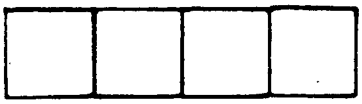


right tromino

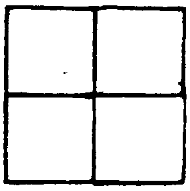


straight tromino

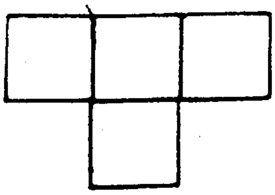
(4) Tetrominoes: Four squares - 5 types



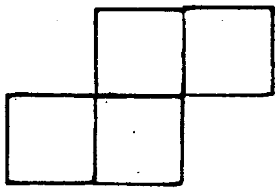
straight tetromino



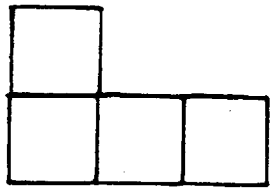
square tetromino



T tetromino

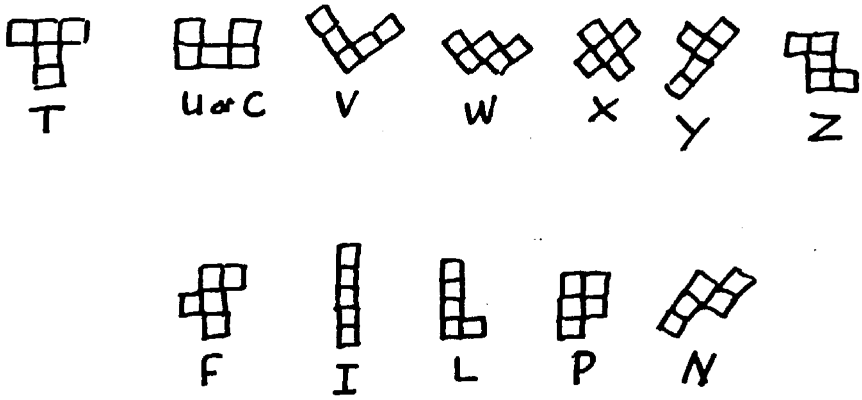


skew tetromino



L tetromino

(5) Pentominoes: Five squares - 12 types
- named by letters as below



Polyiamonds.

These are formed from triangles placed together along their edges.

1 moniamond 

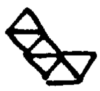
1 diamond 

1 triamond 

3 tetriamonds 

4 pentiamonds 

12 hexiamonds

Crook 

Signpost 

Hook 

Crown 

Bar 

Chevron 

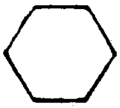
Snake  Butterfly 

Lobster  Hexagon 

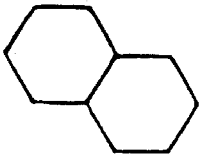
Yacht  Sphinx 

Hexagonal animals

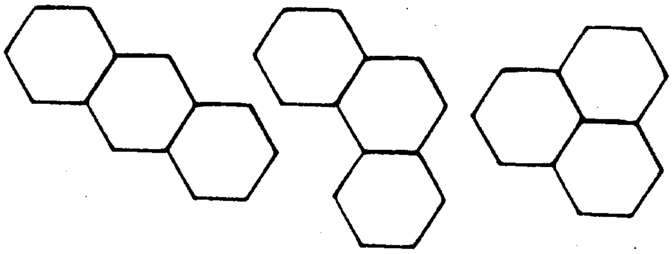
This is based on the hexagon tessellation



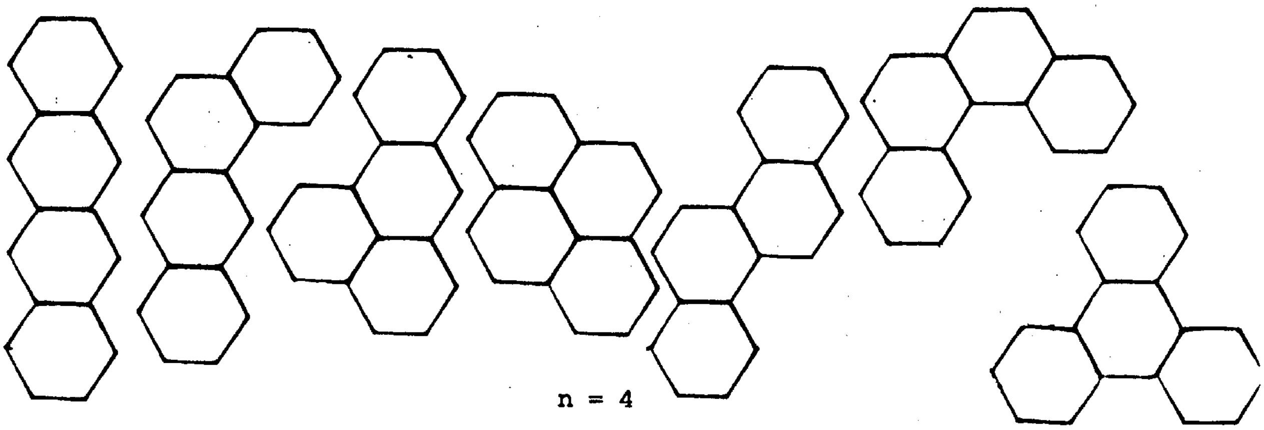
n = 1



n = 2



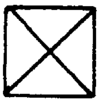
n = 3



n = 4

McMahon Colour shape

(1) McMahon three colour squares.
This is an activity based on squares with diagonals drawn in, e.g.



Each of the four triangles is coloured with one of three colours. A complete set of such squares which are different except for rotation is used to construct rectangles whose boundary is of one colour. Where two squares have a common edge (are adjacent) the triangles at this edge are the same colour.

Example

 Colour 1

 Colour 2

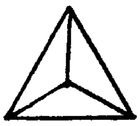
 Colour 3

Using these 3 colours, we can make 24 different 3 colour squares.
But be careful.
These are all the same 3 colour squares:

If we can turn one of our squares so it is the same as another square, it is <u>not</u> a different square.

(2) McMahon three colour triangles

This is an activity based on equilateral triangles divided into regions as shown below which are coloured using four colours (one extra than the squares).



McMahon 3-colour triangles are similar to McMahon 3-colour squares. They are equilateral triangles with lines drawn as below and coloured by one, two or three of four colours.

Example

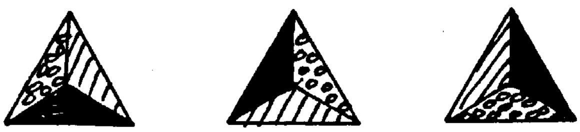
 Colour 1

 Colour 2

 Colour 3

 Colour 4 (not used in this case)

Once again there are 24 different triangles. Once again we have to be careful in colouring. Remember, all the following are the same triangle:

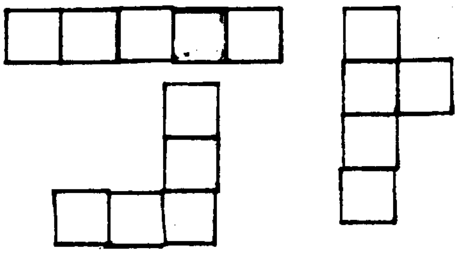


<u>Materials</u>:
Cardboard, paper, coloured textas, pens or pencils, scissors, glue
Plastic square counters
"Commercial" copies of pentominoes, hexiamonds and McMahon colour shapes.

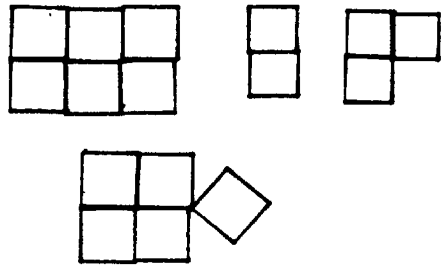
Activities:

1. A pentomino is a shape formed from 5 squares joined at edges. In fact a pentomino:
   (a) is in one piece;
   (b) can be formed by 5 squares together; and
   (c) have all touching squares with a common edge.

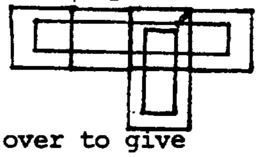
These are pentominoes



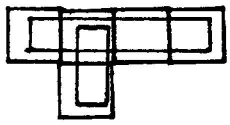
These are not



(1) Take 5 plastic squares and make a pentomino as shown below



(2) Turn the shape and flip it over to give



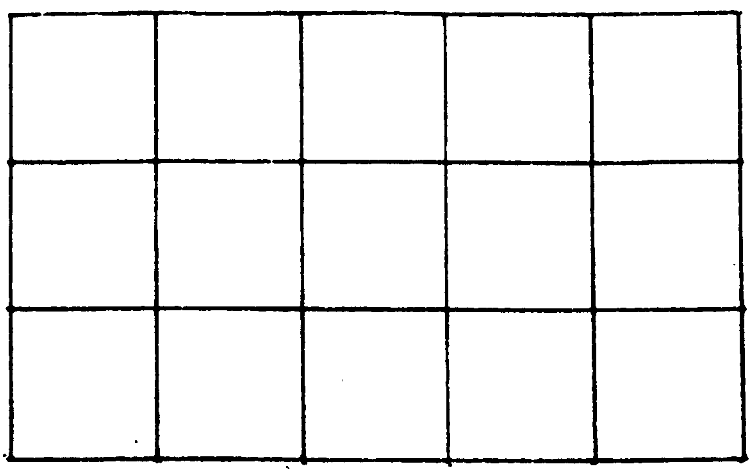
There is still the same pentomino

(3) Continue making shapes out of 5 squares until you have all twelve.

2. Use your pentominoes (or cardboard copies of those given at the end of this unit) to complete the puzzles below.

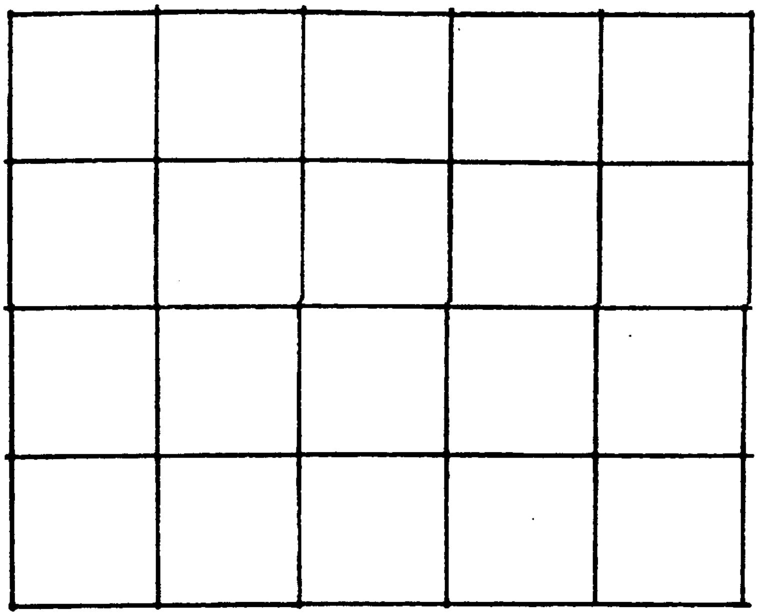
   (A) Make a 3 x 5 rectangle, using these sets of shapes:

   (i) C, N & P (ii) C, Y & P (iii) C, P & V.

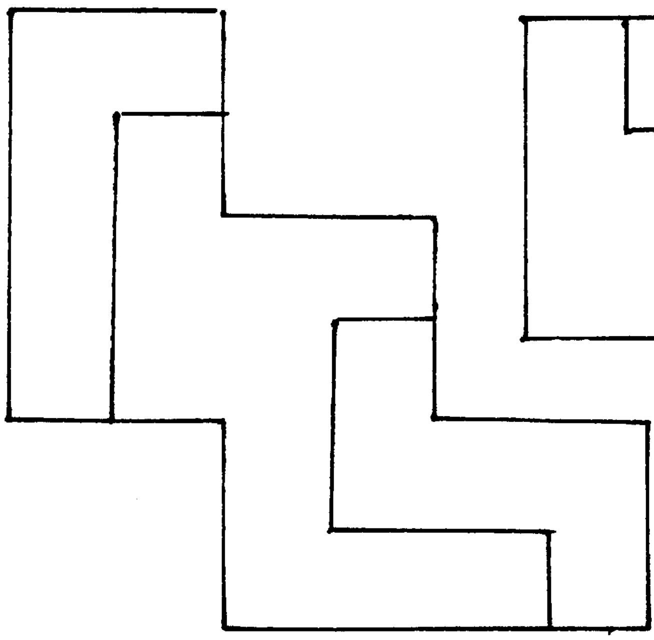


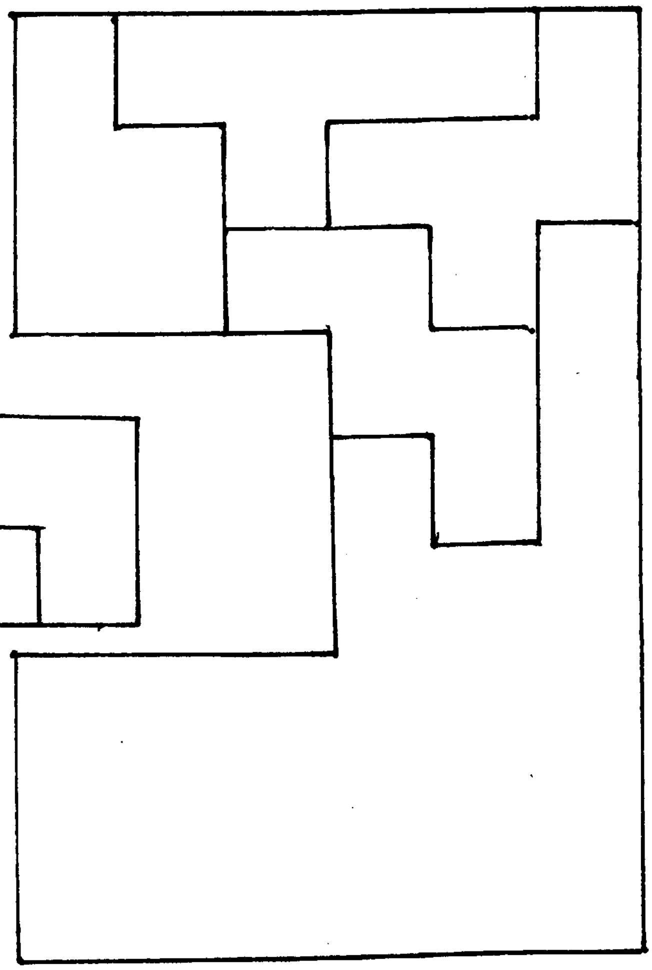
(B) Now try a 5 x 4 rectangle using these sets of shapes,

(i) Y, L, W & P (ii) P, N, Y & C (iii) V, T, W & P.

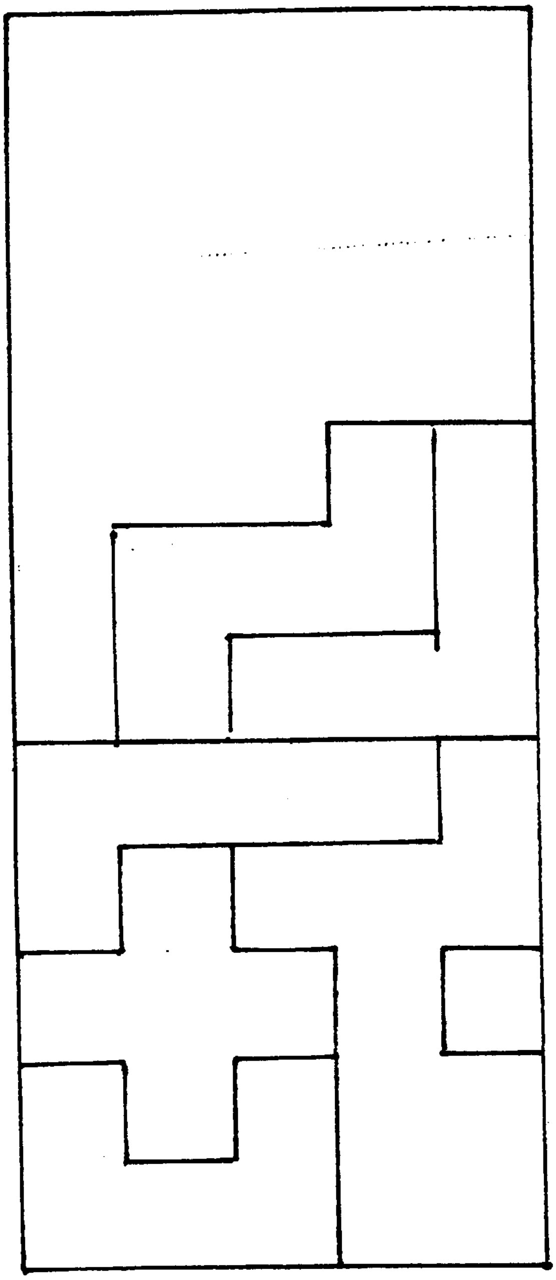


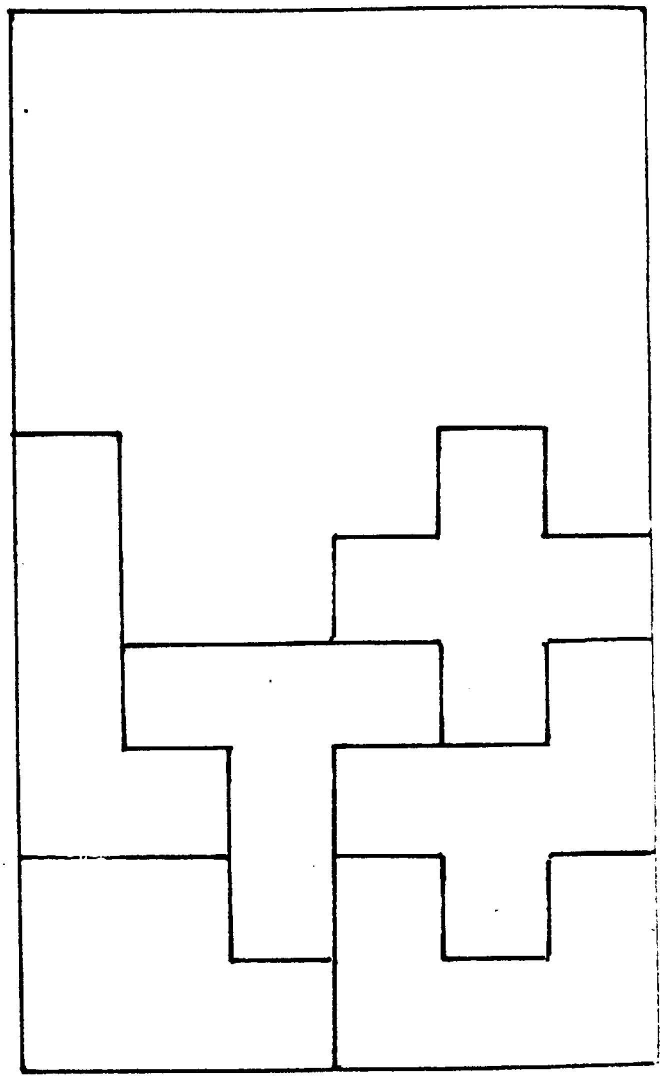
(C) Use 4 of the shapes to make the double W.
Use 9 of the shapes to make the triple C



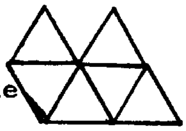


(D) Complete the following rectangles (use all the pieces)



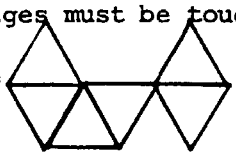


3. Hexiamonds are made from 6 triangles.

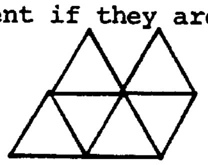


- their edges must be touching e.g. like

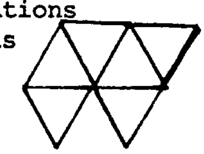
not like



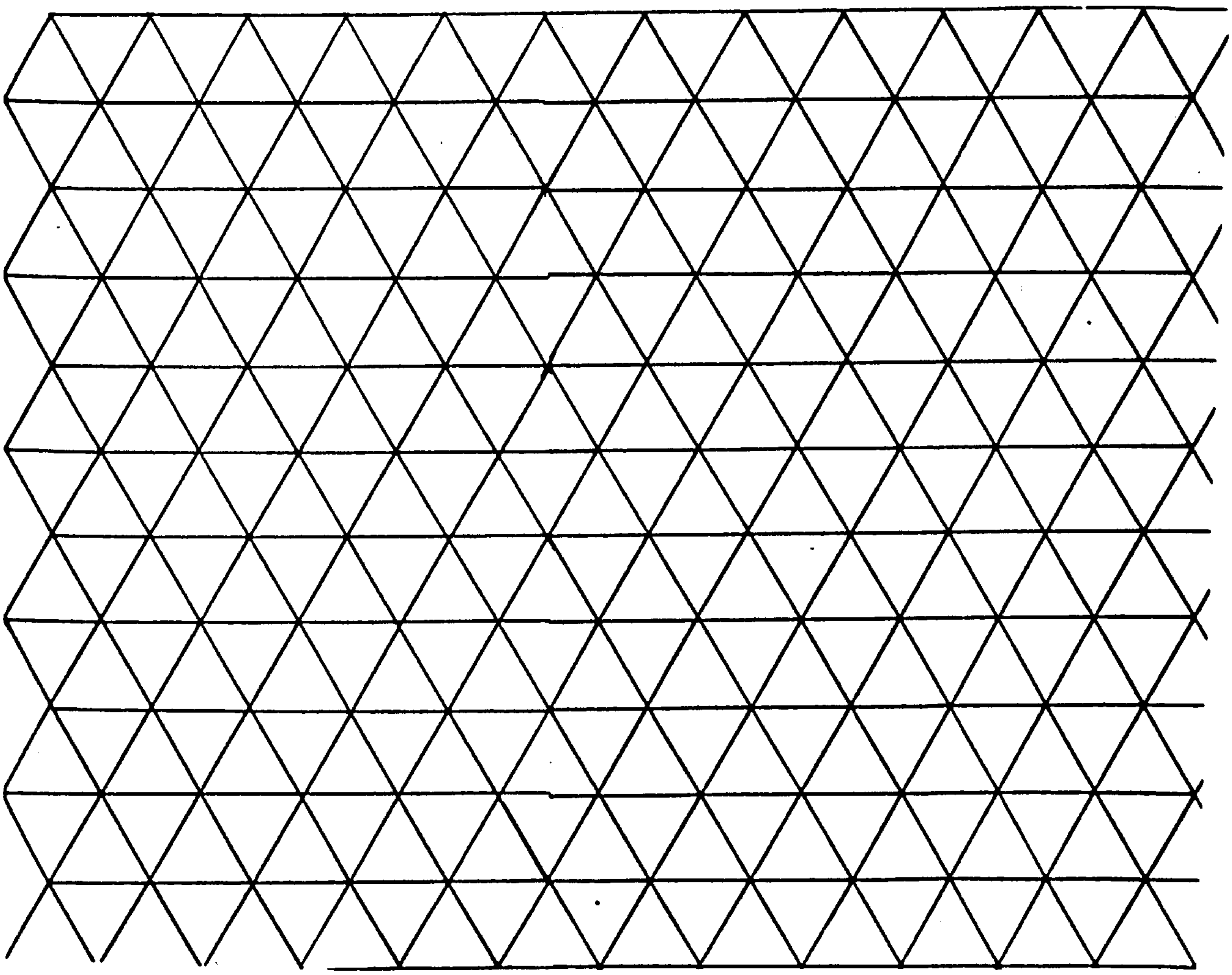
- they are not different if they are reflections or rotations of each other, e.g.



is the same as



Cut 6 equilateral triangles out of cardboard. Try to make the twelve hexiamonds. Record your answers on this graph paper.



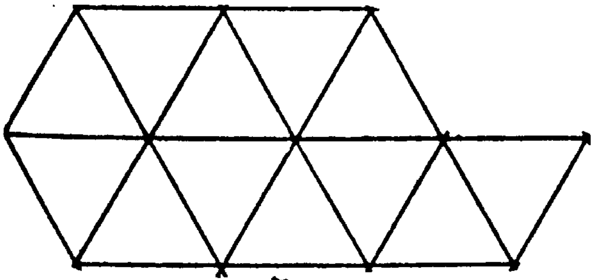
4. Use cardboard copies of the hexiamonds at the end of this book to complete the following puzzles.

A.

A.

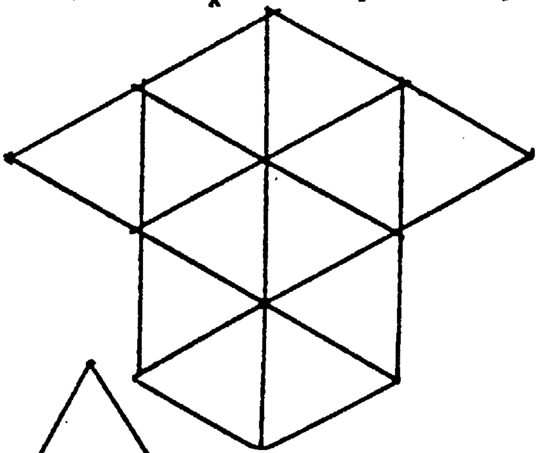
- Make this shape using 2 hexiamonds

Now make the shape again using any other 2 pieces.

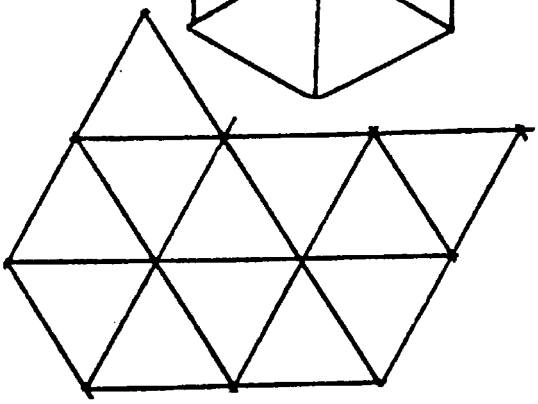


- Make this shape using 2 hexiamonds.

Make it again using any other 2 pieces.

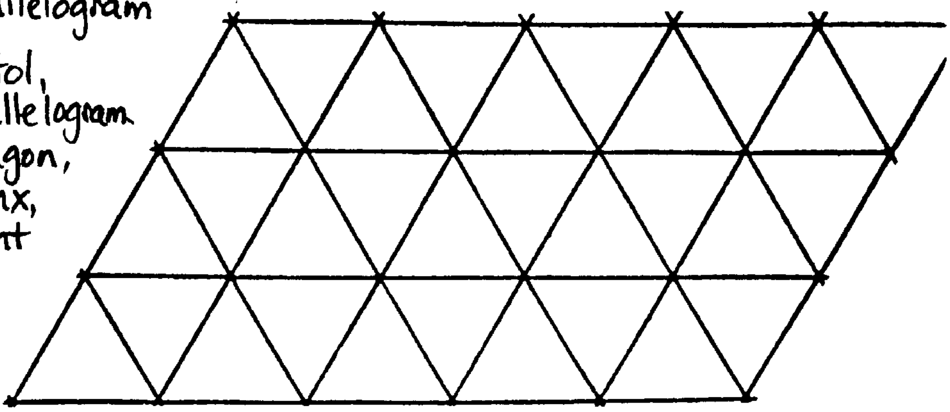


- Do the same for this shape.



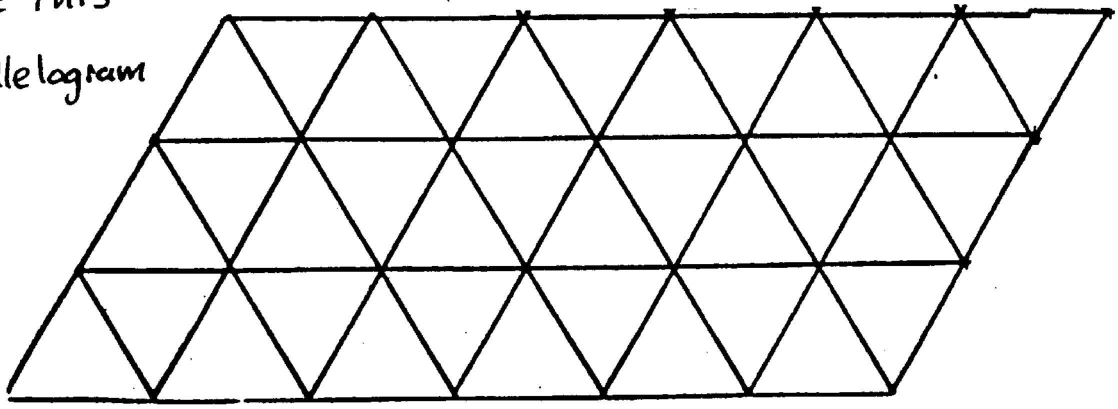
B. Make the 3x5 parallelogram below using

- yacht, sphinx, crown, club, parallelogram
- pistol, parallelogram, hexagon, sphinx, yacht



C.

- Now use 6 hexiamonds to make this 3x6 parallelogram

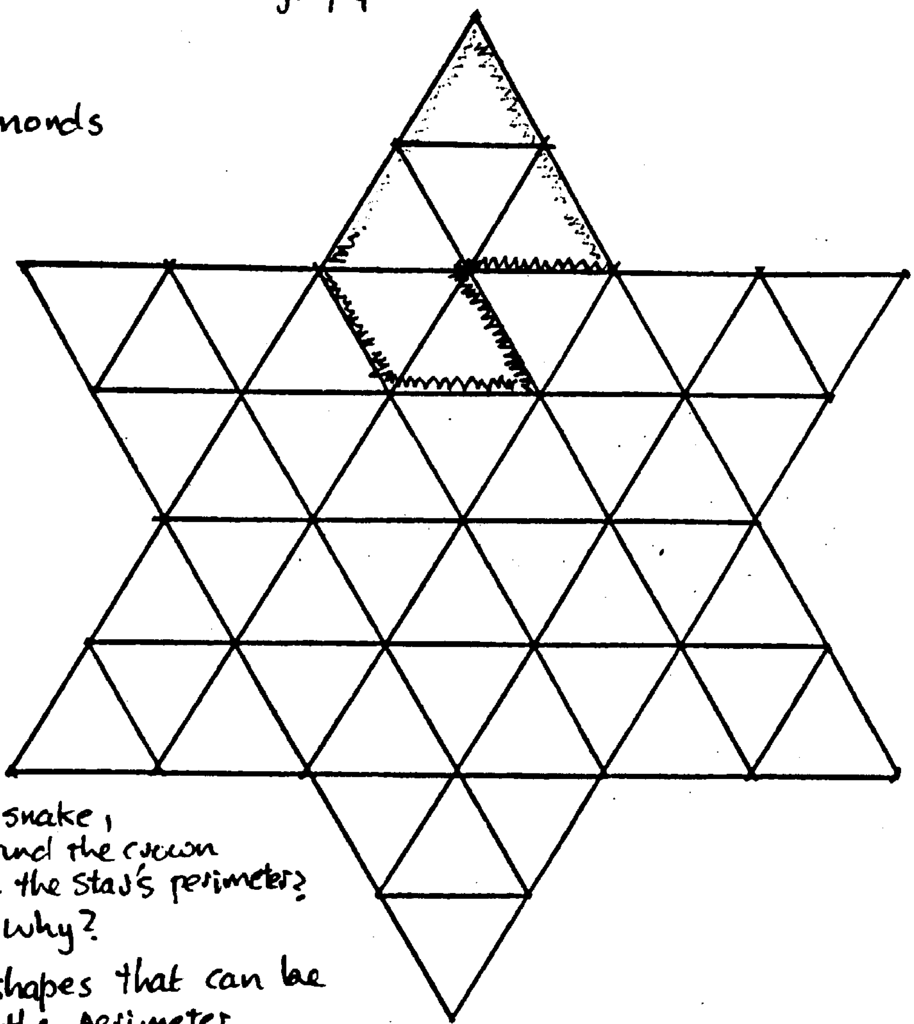


Record your answer on triangle paper.

D.

- Fit 8 hexiamonds together to make the figure below.

  As a clue, we show where the yacht goes.

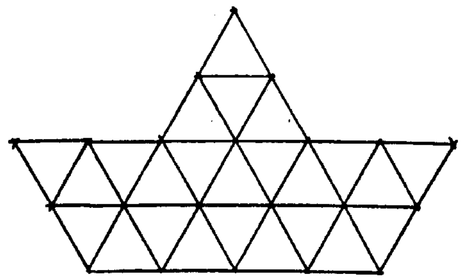


- Record your answer on triangle paper
- Answer these questions

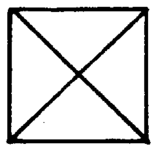
  (1) Could the snake, hexagon and the crown ever be on the star's perimeter? (Try it!) Why?

  (2) List the shapes that can be used on the perimeter

D, Use the sphinx, the arrow, the parallelogram and the yacht to make a double-sized crown.



5. McMahon 3 colour squares

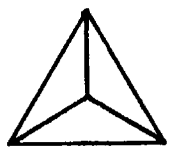


(1) Choose 3 colours
(2) Cut out 24 squares 5 cm wide and 5cm. long
(3) Draw in the diagonals on each square.
(4) Using one, two or all of your chosen colours, colour each of the 24 squares so it is different from the others. BE CAREFUL! Check before you ruin a square.
(5) Put your 24 squares together so they form a 6 x 4 rectangle obeying the following rules:

RULE 1: The boundary of the square must be the same colour.

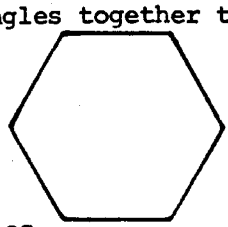
RULE 2. When two squares touch, the triangles at the common edge must be the same colour.

6. McMahon 3 colour triangles.

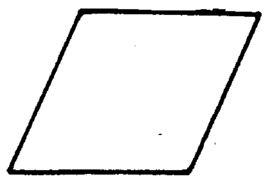


(1) Choose 4 colours
(2) Cut out 24 equilateral triangles of side 5 cm
(3) Draw in the lines in each triangle as above
(4) Using one, two, three or all of your chosen colours, colour each of the 24 triangles differently. BE CAREFUL!
(5) Put your 24 triangles together to form

(a) a hexagon



(b) a rhombus

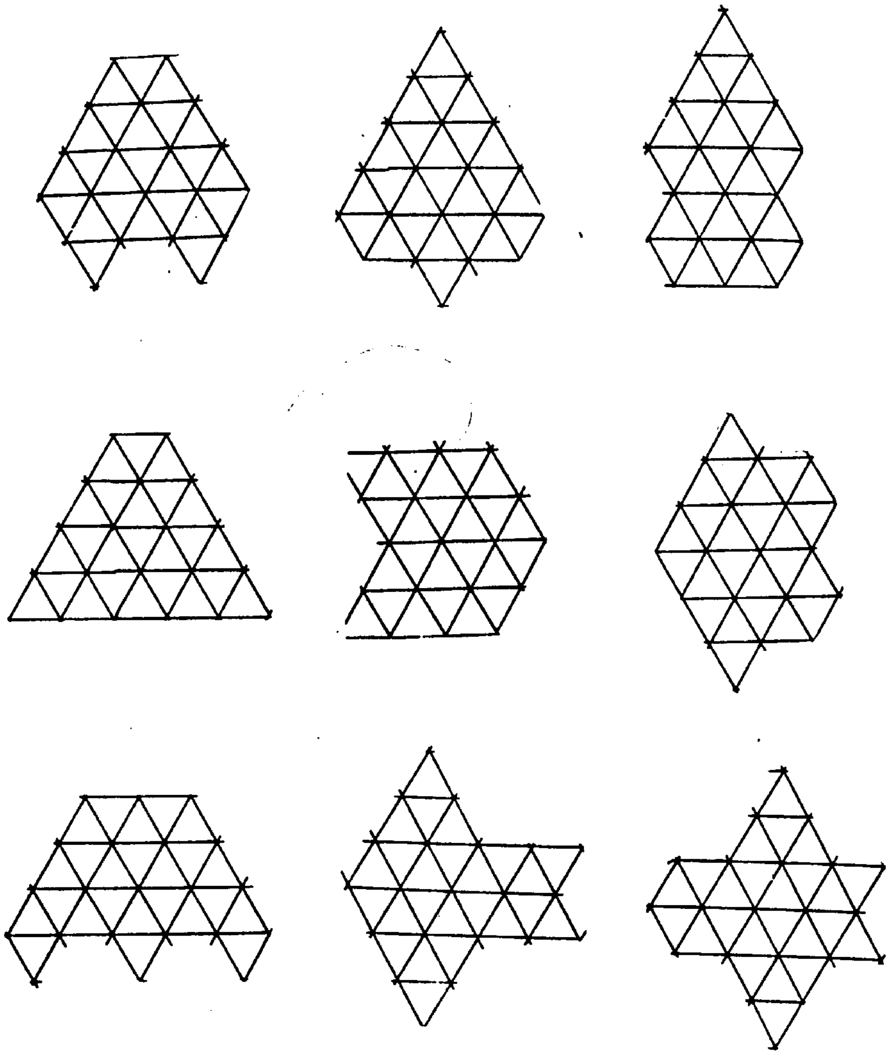


Obey the following rules.

RULE 1: The boundary of the shape must be the same colour.

RULE 2: When two triangles touch, the smaller triangles at this common edge must be the same colour.

(6) Make the following using the same rules.



(7) Make a puzzle for the seven n = 4 hexagonal animals.

<u>Teaching Hints</u>:

It is extremely easy to set children shape puzzles. Just obtain (say) 6 pentominoes and assemble them on graph paper. Then draw around the resulting shape and hand this outline plus the pentominoes to a child. The child has to reassemble the pentominoes to fit inside the outline. Hexiamonds can be done the same way (but use triangular grid paper). Making shape puzzles is of course, only one of the possible pentominoes activities. Here is a list of possibilities.

(1) Construct them: Children can be given the task of actually finding or constructing polyminoes. This may be done by joining squares or by shading in sections of square grid paper (or on geoboards). You must ensure they understand that the squares are to be joined the same way postage stamps are (along edges).

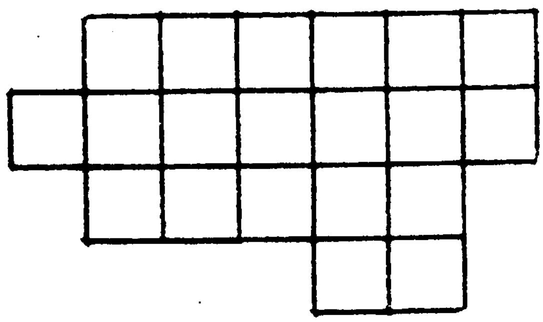
(2) Tiling patterns: Putting all or some of the polyominoes together as for jigsaw puzzles, to cover grids or make shapes.

Examples:

(a) Show that you cannot form a rectangle with all the tetrominoes. Form a rectangle with 5 tetrominoes and one pentomino.

(b) Make a 5 x 3 rectangle using the following pentomino pieces: (i) U, N, P (ii) U.Y, P (iii) U, P, V.

(c) Fit the pentomino pieces F, I, T, W together to make the figure below:-.



(d) Construct a 5 x 4 rectangle using pentomino pieces Y, L, W, P.

(e) Make 5 x 5 squares using pentomino pieces

(i) L, W, P, Z, Y

(ii) L, F, P, X, U.

(f) Form a 6 x 5 rectangle from 6 pentominoes.

(g) Form a 3 x 20, 4 x 15, 5 x 12 and 6 x 10 rectangle with all the 12 pentominoes.

(h) Form a 8 x 8 square with 4 squares uncovered at the centre, using the 12 pentominoes.

(3) Excluding polyominoes: This is opposite to the above - what must be done to keep a polyomino <u>off</u> an 8 x 8 checkerboard.
<u>Example</u>: For each of the 12 pentominoes, what is the least number of monominoes that can be placed on an 8 x 8 checkerboard so that the given pentomino can no longer be fitted to the board? This can be done by getting an 8 x 8 grid and shading individual squares until the given pentomino can't be put on it.

(4) Area and Perimeter: Example: Complete the following tables:
What do you notice about the relationship of perimeters to areas?

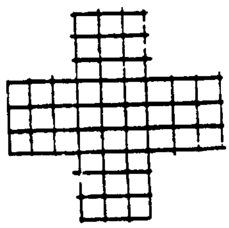
Pentomino	L	N	P	Y	F	I	T	W	C	V	X	Z
No. of Units in Perimeter												
No. of square units in area												
No of axes of symmetry												

(5) 3 - D constructions: Making polyominoes out of cubes and using them to construct 3 - D shapes.
<u>Example</u>: Construct a 3 x 4 x 5 solid from the 12 solid pentominoes.

<u>Note</u>: These solid polyominoes can be used with <u>soma cubes</u> (see the Martin Gardner Pelican Book <u>More Mathematical Puzzles and Diversions</u> for more on soma cubes).

(6) Counting: Enumerating all possibilities
<u>Example</u>: How many inequivalent (distinctly different) ways can the 3 monimoes be placed on a 3 x 3 grid.

(7) Back tracking: Proving that a construction or pattern is impossible by exhaustively testing all possibilities in a logical way.
<u>Example</u>: The 12 distinct pentominoes cannot cover the following cross.

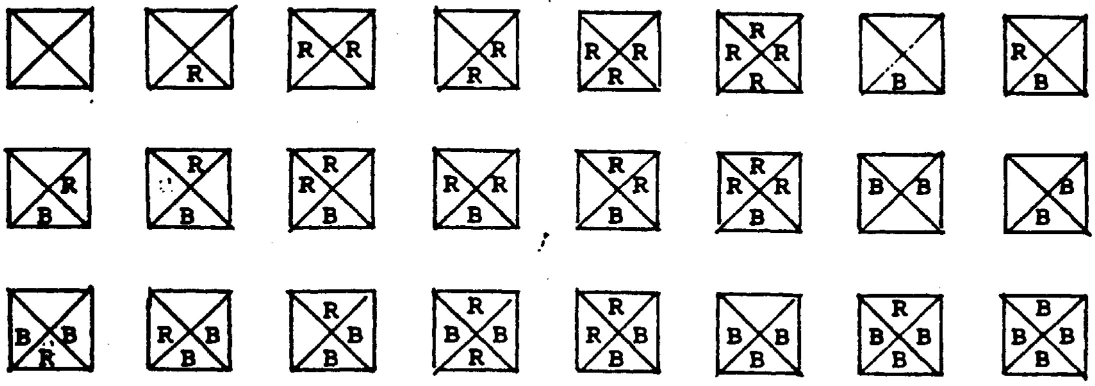


<u>Note</u>: This is difficult for primary students.

(8) "One Sides" Polymominoes: Working with only one side of the polyominoes (increase the number of polyominoes by adding all other sides which cannot be made by rotating any of the "one sides").

Note: As well as a variety of activities for any shape, puzzle, teachers should use a variety of puzzles. The McMahon 3 colour shapes are an unusual form of puzzle. Here is some more information on them.

McMahon Three-Colour Squares can be best constructed on 5 cm x 5 cm. squares. There are 24 different coloured squares as below, where colours are red (R); blue (B) and white :



Separate the squares into sets :

(1) the 3 squares showing only one colour;

(2) the 12 squares showing two colours
- 6 squares with three like-coloured triangles
- 3 squares with two pairs of like-coloured opposite triangles
- 3 squares with two pairs of like-coloured adjacent triangles

(3) the 9 squares showing all three colours
- 3 squares with one pair of like-coloured opposite triangles
- 6 squares with one pair of like-coloured adjacent triangles

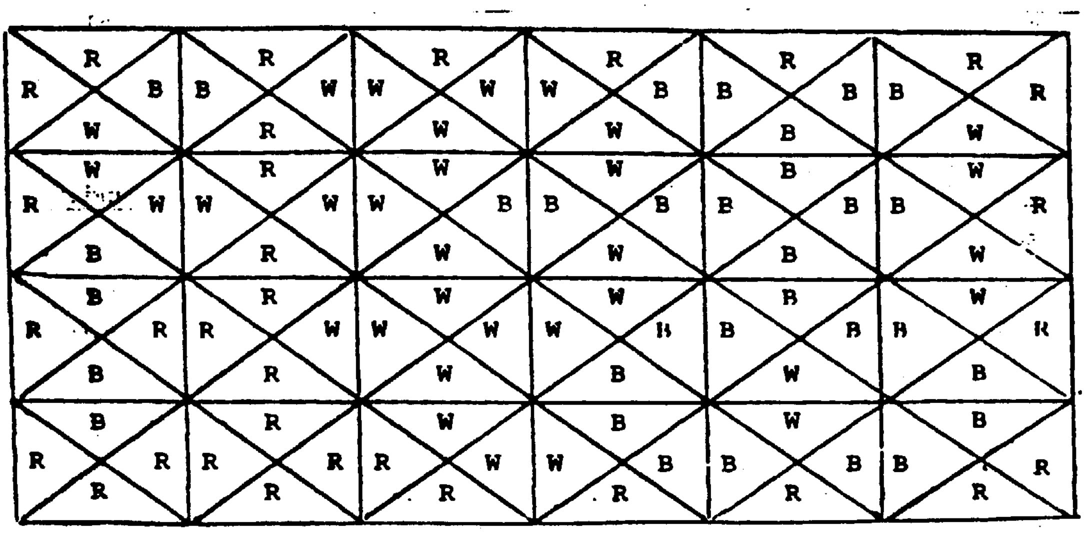
Note that 24 squares give 96 triangles and that there are 32 triangles of each colour (CHECK)

To form a 6 x 4 rectangle such that

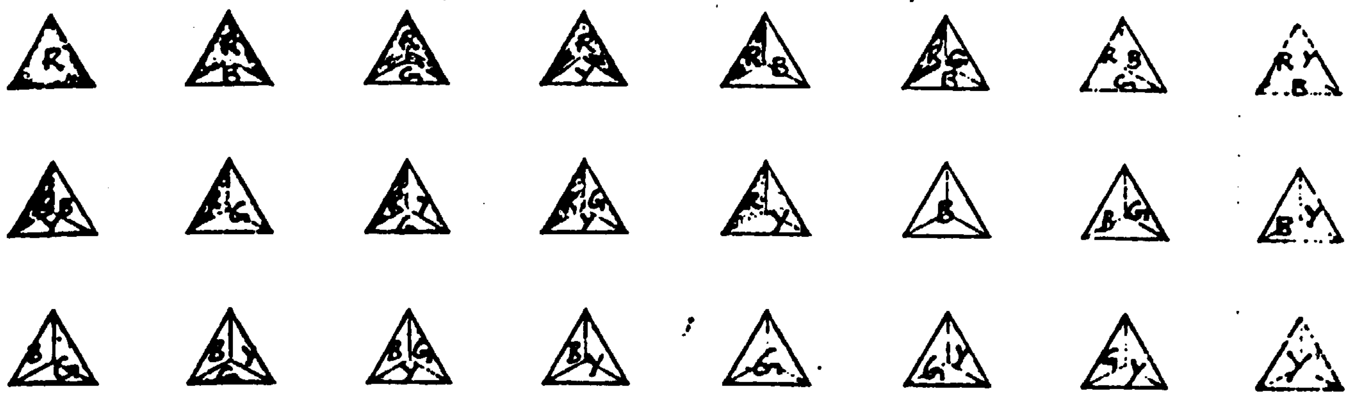
(i) All triangles forming the boundary are to be of one colour

and (ii) Where two squares have a common edge, the triangles at this edge must be of the same colour.

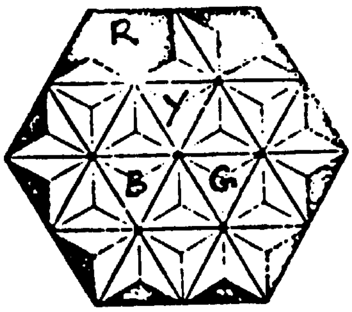
One possible solution is as follows :

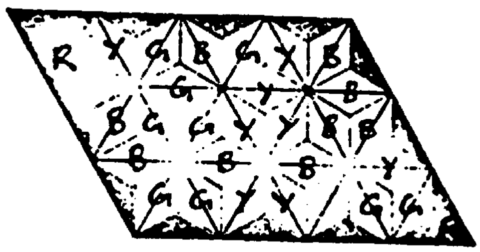


There are 24 coloured triangles which are different omitting rotations.
These are (where colours are red (R); green (G); ~~and~~ yellow (Y) and blue (B))



Possible solutions are :





UNIT 20: VISUAL IMAGERY

Focus:
In this unit we look particularly at the development of visual imagery.

Background:
Visual imagery is the ability to turn, flip and move images of shapes in the mind. It is the ability to mentally view from another perspective. It is the ability to mentally transform a shape, to mentally superimpose two shapes and look for similarities.

In particular there are four activities that attract our attention:

(1) fitting shapes into spaces - selecting from amongst alternatives to fill a gap;

(2) perception - drawing what a scene would look like from a different direction;

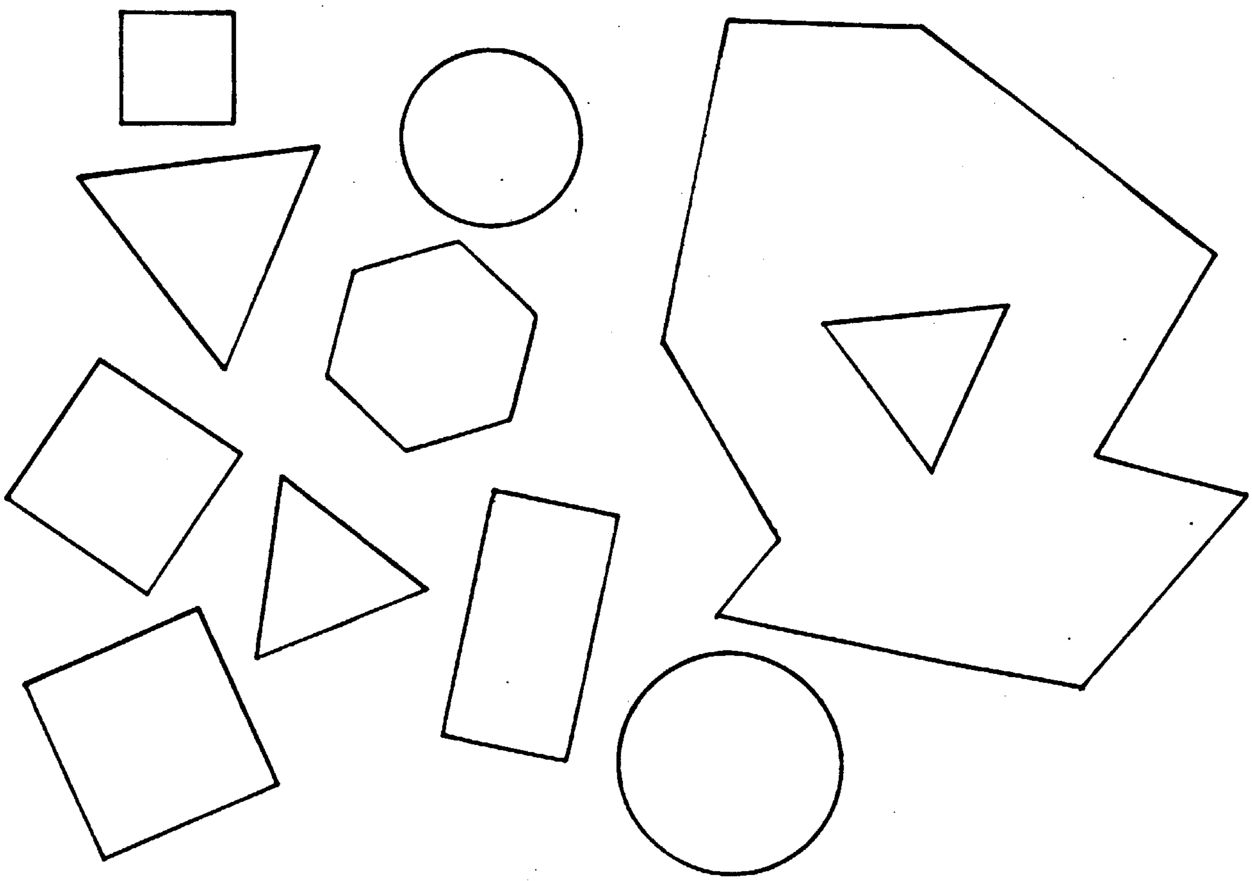
(3) similarity and difference - detecting which amongst alternatives is the same as a starting figure and selecting two similar shapes from amongst a collection; and

(4) fitting shapes together - see earlier units in this chapter.

Materials: Pen, paper, pentominoes, dice, cardboard, glue, triangular grid paper.

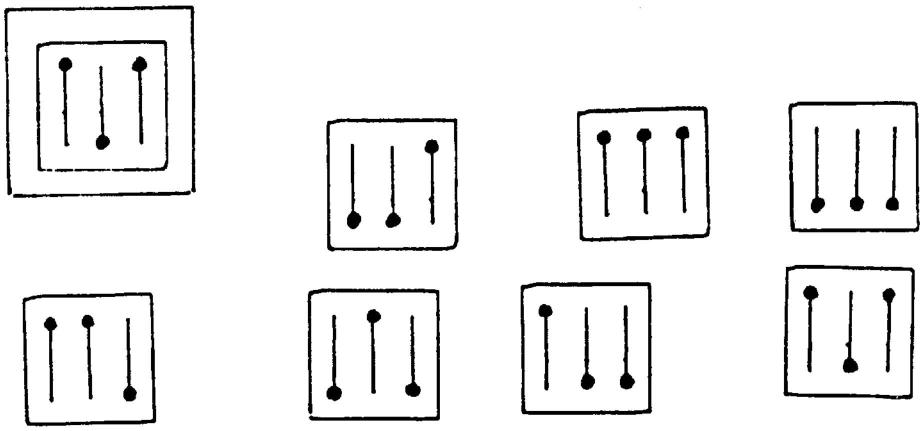
Activities:

1. Select which of the following shapes will fit into the given shape.

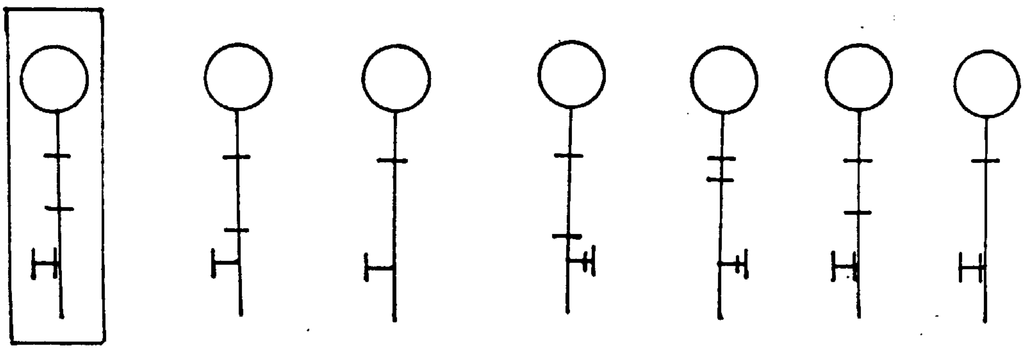


2. Complete the following activities:-

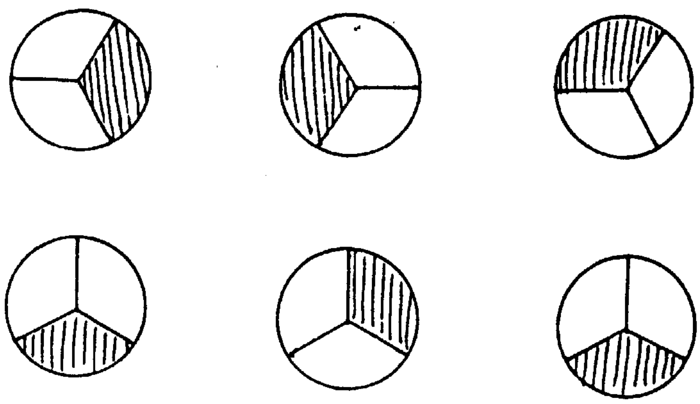
(1) Find the design that is <u>exactly</u> like the one in the box.
Draw an X on it.



(2) Find the key that is <u>exactly</u> like the one in the box.
Draw an x on it.



(3) Find the <u>two</u> figures in the same position. Draw an X on them.

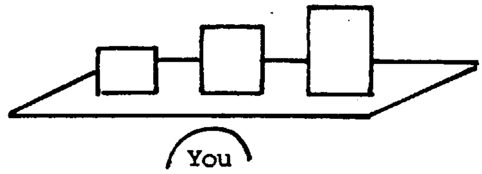


3. (1) Place a pencil in front of you so you view it sideways. Draw the pencil.

(2) Draw how the pencil will look from position A.

A.

(3) Place containers as shown



B.

Draw what B would see.

4. Games : Games are an excellent way to develop imagery. Play the following:

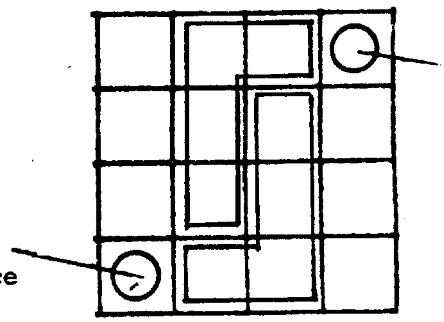
THE L-GAME

A game for 2 players.

Materials: a tiny board of sixteen squares (4 x 4) two L shaped pieces (cover 4 squares) two neutral discs.

Rules of Play:

Each player has an L shaped piece which covers four squares. The starting positions as shown below.

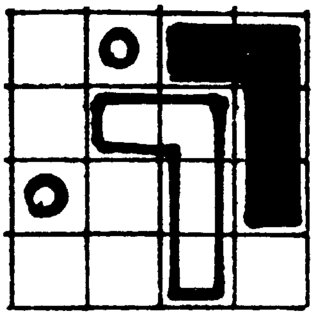


In his turn a player moves to any unoccupied position what-so-ever on the board. It is enough that only one of the squares covered by his L-piece has altered. The L-piece may be turned right over, moved across other pieces and treated in any way a player likes provided it is replaced on an unoccupied set of four squares that are not exactly the same as the ones it covered before.

When a player has moved his L-piece he may, if he wishes, move __either__ one of the small neutral pieces to any new position (i.e. any unoccupied square). He does not have to move a neutral piece. The L-shape must always be moved __before__ the neutral piece.

The object or the game is for a player to so arrange the playing pieces so that his opponent cannot move his L-piece.

e.g. the blank L has won below



Reference: Games and Puzzles, No. 30, No. 74.

PENT UP

"Pent Up" is a strategy game for two or three players. It uses the twelve pentominoes and the 8-by-8 playing field on the next page. The 12 pentominoes are at the end of this book.

Rules for Pent Up

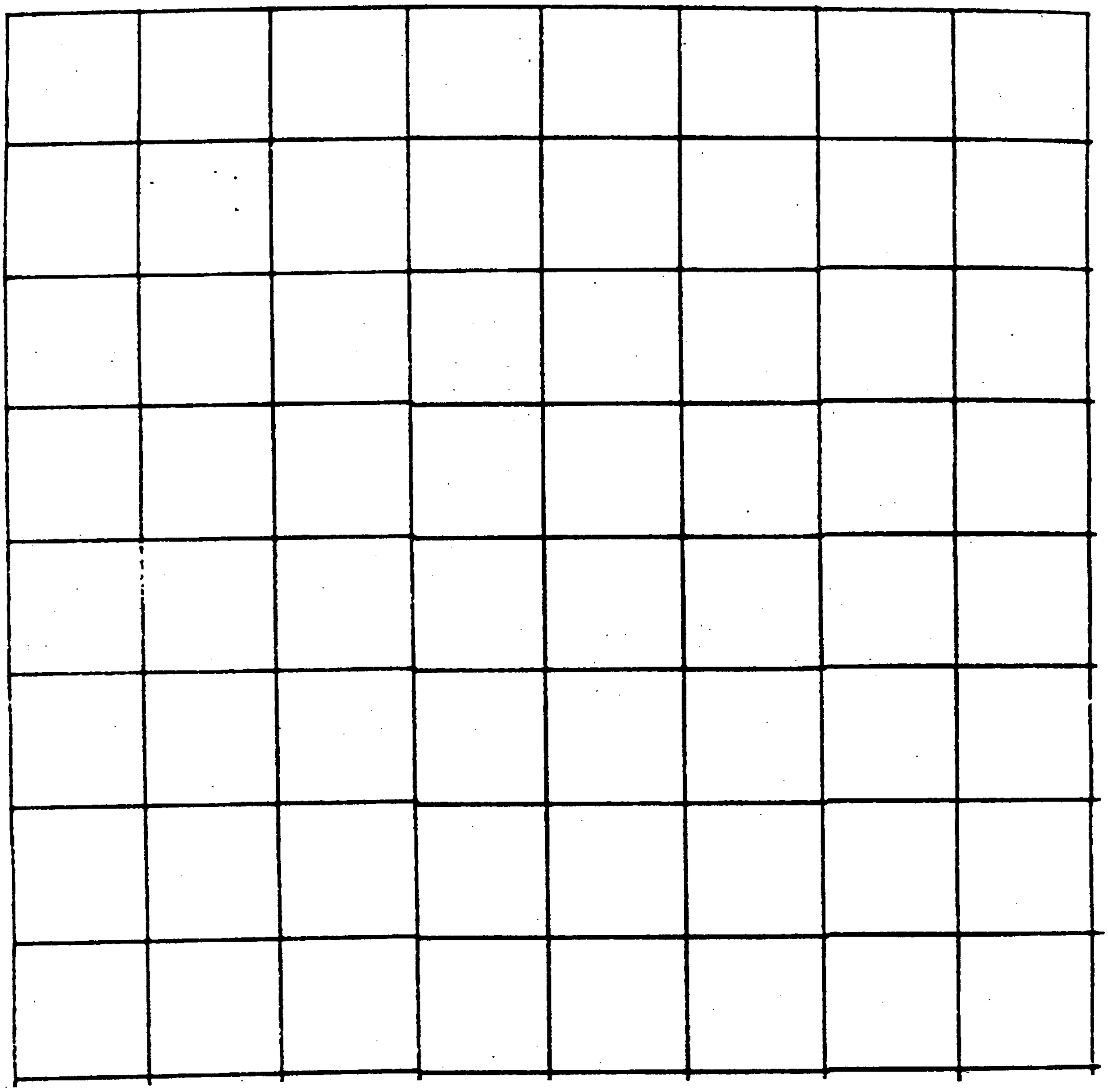
1. *Place the set of twelve pentominoes in a pile near the players.*

2. *Players take turns. In his turn, a player chooses a pentomino from the draw pile and places it so that it covers five squares on the playing field. (Pieces can be "dealt" to each player at start.)*

3. *Any pentomino may be placed either side up.*

4. *The last player who can play a pentomino is the winner.*

It is emportant though that the experiences do move to mental ones as the years of primary schooling go by. Lower grade children will act out with material - will use trial and error. Upper grade children should be doing their trials in their head.



Teaching Hints:

It is important not to stress correct answers here, but to offer the experiences. If children have difficulties, find ways to enable them to act out the situation concretely.

It is important, though, that the experiences become more abstract as the years of primary schooling go by. Lower grade children will act out with material - will use trial and error. Upper grade children should be doing their trials in their head.

CHAPTER SIX: POSITION.

In the overview to this book, we argued for basing our considerations of geometry on shape, size and position. Up to now we have had a large focus on shape and size but little on position. This chapter exists to restore the balance.

Traditionally, position in geometry has been restricted to coordinates - the denoting of position by an ordered set of numbers. This formal definition of position which is of great importance to science and mathematics, is covered in unit 21 where we look at how we can develop this skill in children. But as must now be very evident, this book takes a much wider view of geometry than that traditionally taken. We see geometry as encompassing any ability that enables children to understand, operate in and modify their environment.

Therefore we would add three areas to position beyond coordinates: following road maps; finding position and direction through the bush; and representing and drawing neighbourhoods. Hence we have added three more units to this chapter. Unit 22 which looks at networks; Unit 23 which covers compass bearings and orienteering; and Unit 24 which covers scale drawings.

UNIT 21: COORDINATES.

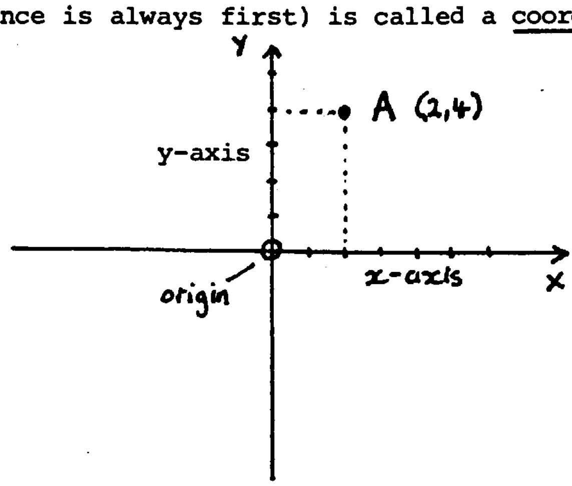
Focus:
This unit focusses on the development of understanding of coordinates in children, beginning with the language of position, moving onto position by letter and number and finally ending with the ordered pair of numbers.

Background:
Within a plane, position can be determined relative to a starting point by two numbers. There are commonly two ways of doing this

(1) Cartesian coordinates.
Here there is a starting point, called the origin, and two directions at right angles to each other from this origin, called the x-axis and y-axis. Position in the plane, relative to the origin, is given by two numbers which denote the distance in the x-direction from the origin and the distance in the y-direction from the origin. This ordered pair of numbers (the x-direction distance is always first) is called a coordinate pair of numbers.

For example:

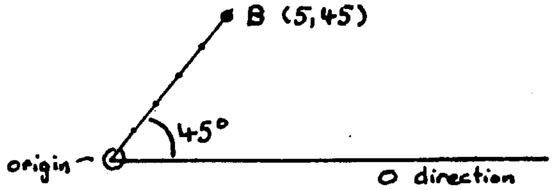
A is at coordinate position (2,4)



Note: The four sectors the graph is divided into by the two axis are called quadrants.

(2) Polar Coordinates.
Here there is a starting point and a zero direction from it. Position in the plane, relative to this origin, is given by the direction (an angle relative to this zero direction) to the position and the distance along this direction from the position to the origin. These coordinates are also an ordered pair with the distance first.
For example:

B is at
polar coordinate
position (5,45°)

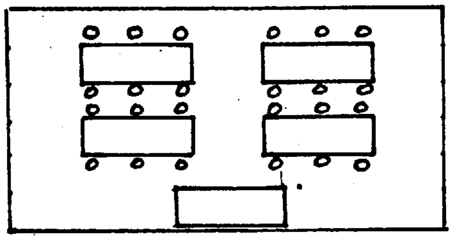


Note: Polar coordinates are not generally part of the primary syllabus and will not be covered in the activities below. But they are used informally in Unit 23 where compass bearings and pacing is used to find positions.

Materials: Graph paper, ruler, pen, paper.

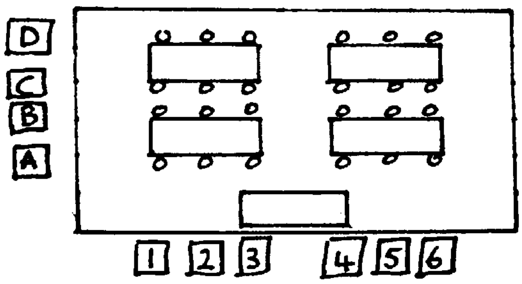
Activities:

1. Organise your room so that it has coordinates
   (1) Organise the seats (or tables) so that they are, as far as possible, in rows and columns, e.g.



   (2) Place large letters from A onwards (until the number of rows are exhausted) and large numbers from 1 onwards (until the number in each row is exhausted) on to pieces of A4 paper.

   (3) Stick the numbers on the walls (front and one side so they coincide with each row and column), e.g.



2. Play the "find your seat" game:

(1) As you enter the room, get your seat ticket from the instructor.

(2) Find the row you are to sit in by looking at the letter on the ticket (e.g. row A, B, C, etc.). The rows are marked by letters along the side of the room.

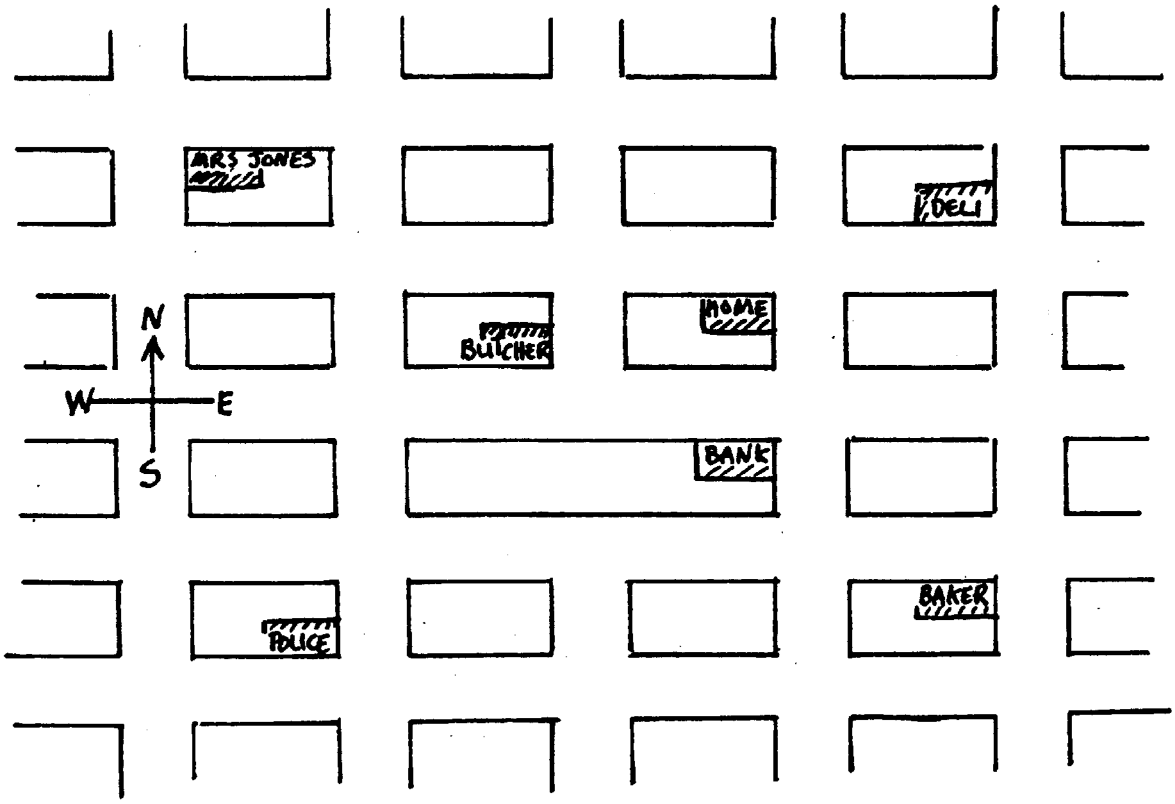
(3) Find your seat in that row by looking at the number on your ticket (e.g. row A6, B4, etc.). The seats are marked by numbers along the front wall. Sit in this seat.

(4) Find the row letter and seat number of those sitting in front, behind and beside you. What do you notice about their letters and numbers?

(5) What seat letter and number should your instructor have?

3. Why do we ask questions like 2 (5) above?

4. Obtain a road map of the local neighbourhood and ask other members of the group questions as for the map below.



(1) If you start from home and north is as shown, how do you get (in terms of blocks north, east, south and west) to:-

(a) Mrs. JONES

(b) BAKER

(c) POLICE STATION

(2) If I walk 3 blocks east and 3 blocks north, where did I start and where did I end?

(3) Fred went 1 block west and ended at the bank. He went 3 blocks west and 2 blocks north and ended at Mrs. Jones. What is his starting point?

5. Blocks, north, east, south and west can be used to introduce coordinates. So can latitude and longitude. What else in the world around us can be used to introduce coordinates?

6. The concept of origin can be introduced by activities such as 3 (3). What other questioning can be used to develop the notion of origin?

7. Children can draw pictures by placing points.

(1) Obtain a piece of graph paper. Take the origin or starting point at the bottom left hand corner.

(2) Walk in the following points and then join them in order

(1,1), (2,2), (2,11), (3,13), (4,14), (5,13),
(6,11), (6,2), (7,1), (1,1)

Of what do you get a picture?

(3) On another piece of graph paper, create a picture (or words of your own choosing.
List coordinate points that would create this picture (or word).
Give this list to your neighbour to plot and join.

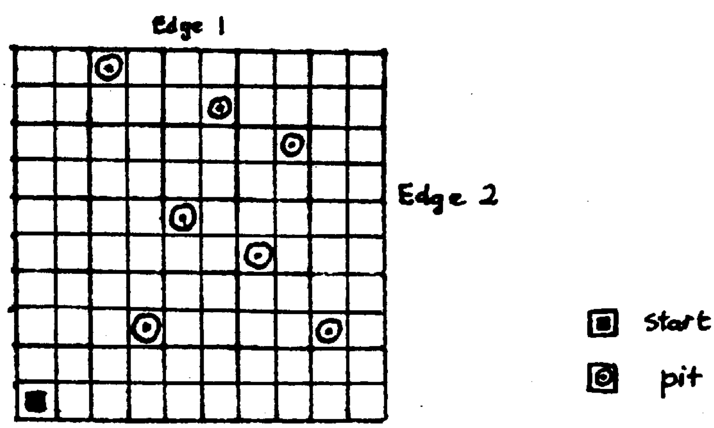
8. Games are particularly useful in coordinates. Play the following:

(1) *Bear Pits*

Materials: *A grid or a geoboard*

A die or spinner marked 1, 1, 2, 2, 3, 3

Instructions: *Each player throws or spins twice. The first turn determines the moves made across and the second upwards. If the player lands in a "pit" he returns to "start". The winner is the first to reach Edge 1 and 2.*

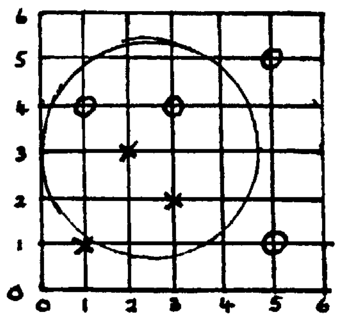


(2) *Most in the Circle*

Materials: Two dice (1-6), with a grid marked as below

Instructions: Each player throws the dice, and marks the coordinates scored (decide which is the"across" die and which is the "up" die).

After each person has thrown 6 times, find out who has the most inside the circle.



CO-ORDINATES

1. *Battleships*

Year 4 level and up. 2 or more players.

Purpose: Practice in locating points on grid.

Material: Each player has a single sheet of paper with two grids on it.

He spots his own battleships on one grid, and records his shots at his opponent's forces on the other.

Instructions:

1. Each player has three battleships. A battleship consists of three adjacent (horizontally, vertically, diagonally, or on a corner) points.
2. Players take turns in firing volleys of three shots by naming an ordered pair of numbers.

Variation:

Rather than using battleships, have a "Fishing Rodeo", Wales might be five connected locations, sharkes four points, and so on to whitebait, which are single points.

Tic-Tac-Toe

Year 4 level and above.

Purpose: Introduction to concept of ordered pair used to name points. Best played with whole class, divided into two teams.

Materials: Overhead projector, prepared and divided into four quadrants. Horizontal axis labelled with △ vertical axis with □

Recorder: Students records in two column on board, numbers given to her by other students.

	△	□
X	2	3
O	4	1
X	3	0

Instructions:

1. *In order to win you must get five X or* **O** *in a row.*

2. *In order to tell me where to place your X or O you must give me two numbers. Each number must be equal to or less than 10. Teacher places X or O correctly on grid as called, being careful not to count by pointing to the lines.*

3. *Once you say a number, you cannot change your mind. If number already given, or not within the limits, lose a turn.*

4. *You are not allowed to help your team members.*

In order to force the game out of the first quadrant, the teacher may reduce the limits to numbers less than or equal to five. The game will statemate in the first quadrant. The teacher can urge children to tell her numbers in the other quadrantsby thinking of other kinds of numbers they know.

3. *Hide-a-Region*

(with variations, year one upwards)

Can be played with two or more players.

Purpose: Practice in locating points on grid, and in concept of area of a region.

Materials: Graph paper or grid.

Instructions:

One group decides on location of a square region on grid, with vertices at ordered pairs of whole numbers. Other group tries to locate region by calling out ordered pairs, while the group that has hidden the region calles "Inside" or "Outside" in response to each trial. Region is located when the team has named all vertices. Number of guesses is tallied.

Variations:

1. *Hide a rectangular region, and nominate its area in advance.*

2. *The team that hides the region could specify that the boundaries are excluded. If a point on the boundary is named, it is called "Outside". Here the vertices only determine the region.*

3. *Negative numbers and all four quadrants can be used.*

4. Vertices at points at ordered pairs of fractional numbers.

5. Hide-a-Name. Make the name hidden by one of the children. (Best played with teacher and whole class).

4. The Racetrack Game

In the Racetrack game, the cars are represented by points on a grid, the track itself having been drawn on the grid. Adopt the normal convention of:

(i) lines running from top to bottom of the grid are vertical lines

(ii) lines running from left to right of the grid are horizontal lines

(iii) the positive vertical direction is from bottom to top.

(iv) the positive horizontal direction is from left to right.

Racetrack is played by two to four drivers who move alternately.

Rules

Rule 1 - Starting

(a) Drivers toss a dice, with the highest score starting first. Other drivers follow in turn

(b) Each driver can start his car from any of the points on the starting line.

Rule 2 - Moves

(a) A move has two components:

  (i) A move over a whole number of units on the grid in the horizontal direction; and

  (ii) A move over a whole number of units on the grid in the vertical direction.

(b) A driver moves his car by selecting its new position on the grid, and joinging the new and old positions by a straight line segment.

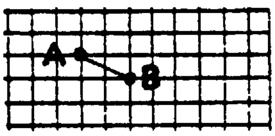
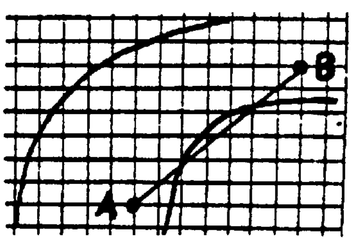


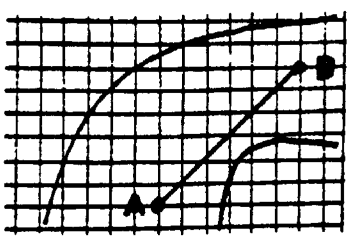
Figure 1

The car has moved from point A to point B - a move of two units in the horizontal direction and - 1 units in the vertical direction.

(c) The line segment connecting the newgrid point and the previous one must be entirely within the track.



This is not permitted



This is permitted.

(d) On each move, a car may move horizontally only the same number of units, or one more, or one less than the number of units it moved horizontally on the last move.
The same rule holds for vertical moves.
Look at Figure 1. The next move of that car must be 1, 2 or 3 units horizontally and - 2, - 1 or 0 units vertically. No other move is permissible.

(e) For the first move of each car, assume that its last move was 0 units horizontally and 0 units vertically.

Rule 3 - Crashes

(a) Two cars may not occupy the same grid point

(b) A car's path may not pass through a grid point occupied by another car.

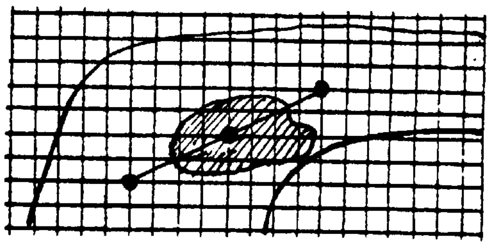
A car which cannot avoid crashing into another car or leaving the track is out of the race. The other drivers may proceed.

Rule 4 - Oil slicks

(Some of the racetracks have oil-slicks marked on them. These are marked as in the diagram below)

A car that passes through or stops in an oil slick must make the same move as its previous move.

e.g.



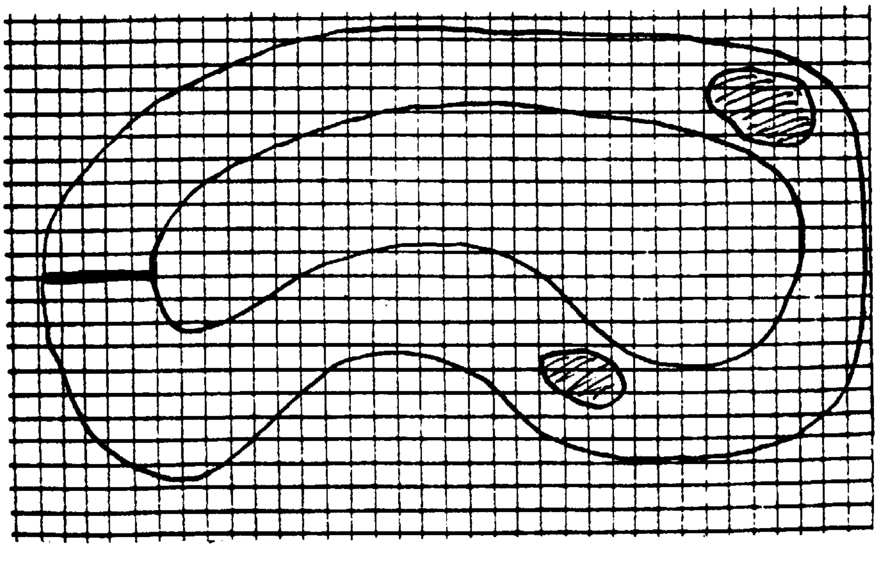
Rule 5 - Finishing

The first car to cross the finishing line wins the race regardless of which car started first).

<u>Variation:</u> Lap record

Draw a race track and have a competition on who can get around it in the least moves.

Example of a track



Teaching Hints

Children can be introduced to coordinates by blocks east, south, north and west, letters and numbers and pairs of numbers. A particularly interesting way is by seat letters and numbers - the "find your seat" game.

The"find your seat game" can be varied by

(a) magical mystery tour" - the ticket contains a set of coordinates at which objects are hidden;
and
(b) "mystery word game" - the ticket contains a set of coordinates at which letters are found (the letters making up a word).

Coordinates can be further developed by road maps and maps of the world (latitude and longitude). Special emphasis needs to be given to the concept of origin.

Later activities should focus on ordered pairs of numbers and using activities where pictures are plotted and games for interest and motivation. Scale drawings, enlargements and reductions can also reinforce coordinates.

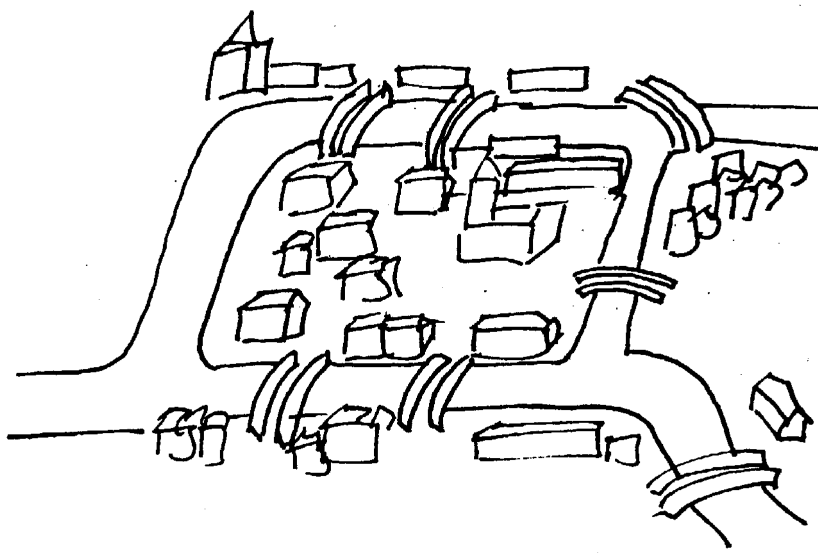
Transferring pictures from a small grid to a large grid (or vice versa) is also good consolidation of co-ordinates. The concepts learned in this enlarging (contracting) process can be extended to scale drawings of buildings, parks, classroom, school year, etc., where ruler and protractor place on graph paper what compass and pacing find in the world.

UNIT 22: NETWORKS

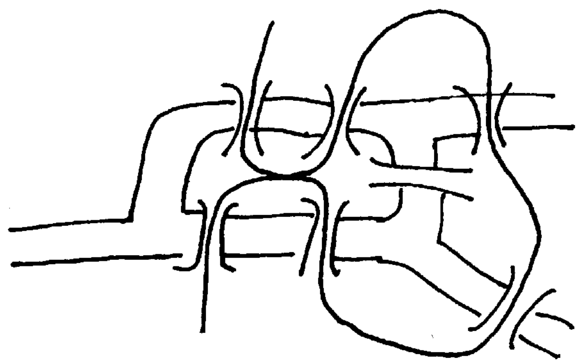
Focus:
Cities and major roads, towns and country roads, telephone lines, sewerage pipes, etc., can all be represented by networks of lines and points. This unit looks at what of this important concept can be tackled by primary children.

Background:
One of the most famous problems in topology is about seven bridges in a small city, called Konigsberg, in old Germany. The centre of this city was on an island in the middle of a river. The island was connected by four bridges to the banks of a river and by a fifth bridge to another island, which was joined to the rest of the city by two more bridges.

The people of Lonisgberg wondered: is it possible to travel through the city and cross each of the seven bridges without recrossing any one of them?



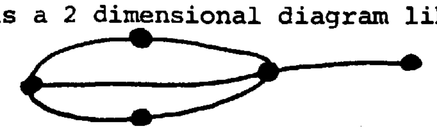
Whenever anyone tried it they ended up skipping a bridge or else crossing one more than once.



A famous Swiss mathematician Euler solved this problem by modelling the city with a network. A network is a diagram of lines and dots. Euler represented each area of land separated from the others (by the river) with a dot and each bridge joining two areas of land by a line (joining the dots).

Euler now had a pen and paper model of his problem. But of course the problem had changed in that different words had to be used to express it. There are now lines instead of bridges and dots instead of areas of land - and one can't walk around dots and lines as one can around a city. The Konigsburg problem becomes tracing over the lines and dots with retracing any line or lifting the pen.

In summary, a network is a 2 dimensional diagram like a road map e.g.



In a network we call

(1) the dot •, a node or corner (or town)

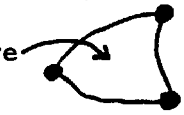
(2) the line joining two nodes  an arc

(3) a node with 1, 3, 5, 7, ...arcs coming from it e.g.

an odd node

(4) a node with 2, 4, 6, 8, ..., e.g. , an even node arcs coming from it

(5) an area which is enclosed by a section of a network, e.g.

here  , a region

(We should note that the area outside the network is also counted as a region, e.g.

has 3 regions).

Network theory is very important in modern mathematics - it is used to plan road system, bus timetables, freeways, telephone networks, computer design, city planning, trade, etc.

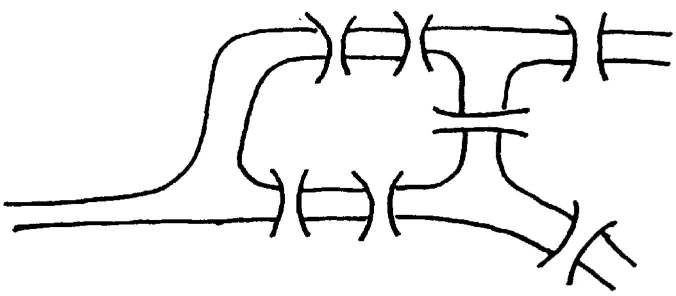
<u>Materials</u>: Tracing paper, pen, pencil, paper

<u>Activities</u>:

1. Work through the following series of Activities.

A. <u>Modelling the problem</u>

(1) Give yourself 3 minutes and try to see if you can walk around the city of Konigsburg crossing each bridge once only. Do you think it's possible.



(2) Draw a pen and paper model of the Konisgburg problem by representing each area of the city with a dot and each bridge by a line joining the dots.

(a) First work out how many areas of land there are in Konigsburg. (BE CAREFUL!)

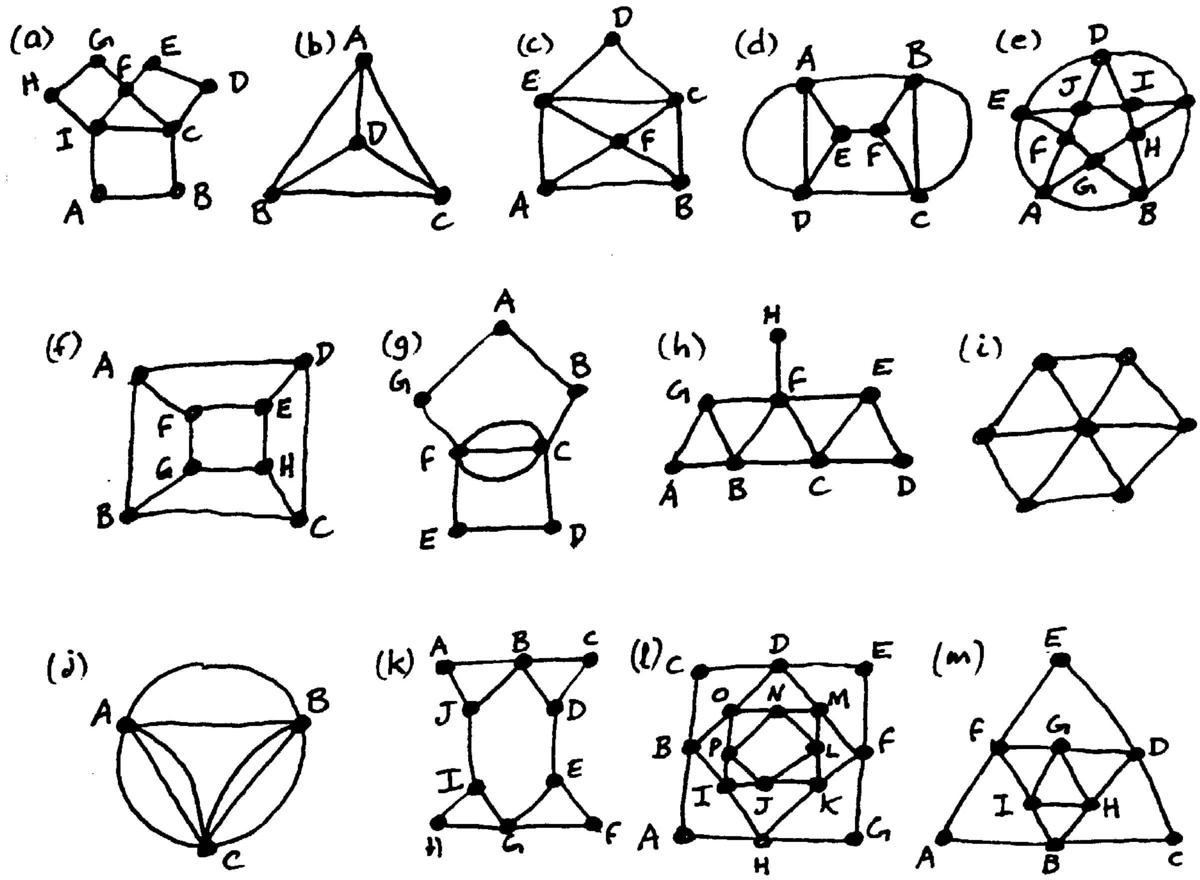
(b) Place a dot in the centre of each area of land and join two dots with a line for each bridge joining the areas of land.

Check your diagram with your lecturer.

Note: These diagrams are called networks, the dots are corners and the lines are arcs.

(3) Now reword the bridges of Konigsburg problem for networks. (Remember: the original problem is in terms of walking around a city - what can we replace "walking" with when we have dots and lines instead of a city?).

8. Working on the Model - exploring the problem



(1) For the 13 networks above use tracing paper to determine which are travellable and which are not. You will find 8 are travellable. Mark the corners at which you start and finish for these 8.

(2) A corner is of odd degree when it has an odd number of arcs connected to it (and of even degree when it has an even number of arcs connected to it). Mark all the corners in all the 13 networks 0 for odd (if they are of odd degree) or E for even.

C. Building a theory from the model

(1) Complete a table as below for the 13 networks.

Network	Total number of corners	The number of corners of even degree	The number of corners of odd degree	Can the network be travelled? YES/NO?
1	9	9	0	Yes
1	4	0	4	No

(2) Looking at the table, can you see a pattern for determining whether a network is travellable.
Write down your rule!

(3) If you are having trouble finding a rule, answering these questions may help:

(a) Do you think a network can always be travelled if it has more even than odd corners?

(b) Is it always true that a network cannot be travelled if it has more odd than even corners?

(c) Can a network be travelled if all its corners are even?

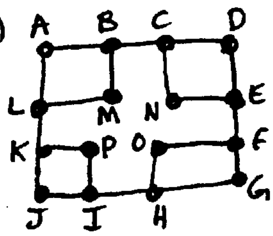
(d) Can a network be travelled if it was only 2 odd corners?

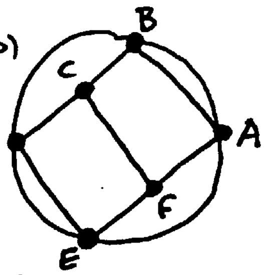
(e) Can a network be travelled if it has more than 2 odd corners?

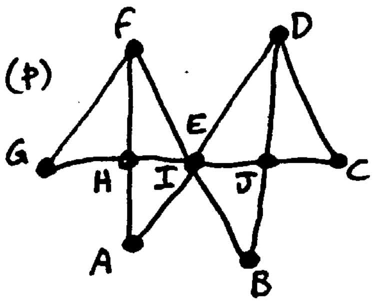
(f) Can a network have an odd number of odd corners?

D. Checking the Theory

(1) Use your rule to predict whether the following networks are travellable or not -

(n)


(o)


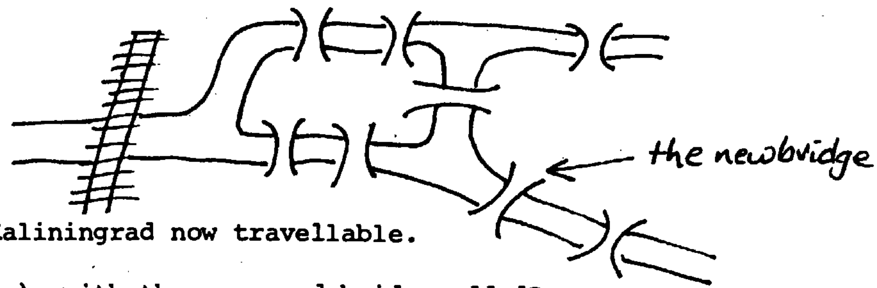
(p)


(2) Use tracing paper to check your predictions.

E. Applying the theory back to the original problem

(1) Look at the network for Konigsburg (see 1 (2) above). Is it travellable? Why or why not? Can you therefore walk around Konigsburg and cross each bridge once?

(2) Konigsburg is now Kaliningrad and has 2 bridges as below:



Is Kaliningrad now travellable.

(a) with the new road bridge added?

(b) with both the new road bridge and the new rail bridge added?

F. Deducing the theory

Look again at the Bridges of Konigsburg problem. Ignore what we've done before and make a new start! Suppose we drew a large network on the ground and walked around it. If it was travellable, we would start at a corner, walk along lines (arcs) once, visiting corners (maybe more than once) and finish at a corner (either our starting corner or another corner). As we walked around we would go to and leave corners. Can we use deductive thinking here to find the solution (without doing the explorations using tracing paper?)

Hint: As we go to and leave a corner, this means that the corner has to have 2 arcs attached to it. The number of arcs attached is the degree, so what is the degree of a corner where we don't end or start - we just walk through (maybe more than once?). And what (therefore) has to be the degree of the start and finish corners (remember: there's two cases - where we start and finish at the same corner and when we start at one corner and finish at a different corner?)

G. Drawing Networks

Draw a network which can be travelled and which has

(a) 3 even and 2 odd corners

(b) 3 corners and 7 arcs

(c) 2 even and 3 odd corners

(d) 3 corners of order 3, 4 and 5

(e) all odd corners

(f) 3 corners all of even order.

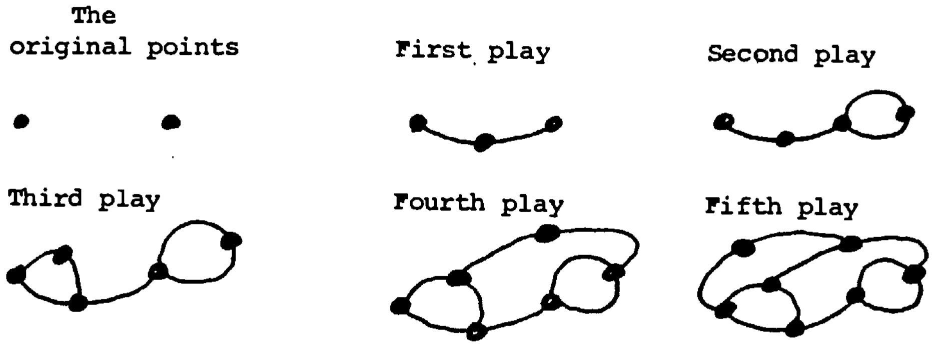
H. Game - "Sprouts"

An interesting new game invented by two mathematicians at Cambridge University of England is called "Sprouts". The game is played between two people and involves drawing a network.

First, several points, which serve as the original corners of the network, are marked on a piece of paper. The players then take turns drawing arcs, following these rules.

(a) Each arc must join two corners or else one corner to itself.
(b) When an arc is drawn, a new corner must be chosen somewhere on it.
(c) No arc may cross itself, cross another arc, or pass through any corner.
(d) No corner may have a degree of more than 3.

The last person able to play wins the game. Here is a diagram showing one series of plays in a game that starts with 2 points.



(1) Can you see why the game ends in the 5th move.

(2) Play a few games of sprouts starting with a differing number of points or corners so that you are able to answer these questions:

(i) when you start with 2 corners, what is the maximum number of plays? What about if you start with 3 or 4 corners?

(ii) Can a game be finished in less than this maximum number of plays? Give an example. What is possible?

(iii) Can you write down a rule connecting the number of starting corners (C) to the maximum number of plays (P)?

2. The above activity was suitable for a trainee or practicing teacher. Which parts of it are suitable for children? When would they be suitable?

3. Modify ~~part~~ C above to be more suitable for primary children.

4. Let us now consider the role of deductive proof.

In solving the Bridges of Konigsburg problems we have collected evidence (drawing the tables), noticed patterns and formed conclusions on the basis of these patterns. In science this would be called "experimenting"; in mathematics it is called reasoning inductively.

In diagrammatic form we could say that inductive thinking involves.

Collecting evidence (through exploration) $\longrightarrow$ Noticing patterns (often through recording results on a table of some kind) $\longrightarrow$ Forming conclusions (by making an "educated" guess)

It should be noticed that the conclusion has been generated (not illustrated) by the collection of the evidence (from the exploration).

Inductive reasoning is tremendously important but has two basic weaknesses - we may not have collected all the evidence and we may be fooled by what we think we see. For this reason, it can not prove our conclusion.

How do we overcome these deficiencies of inductive reasoning? How do we "prove" things to be so? The answer is to become "logical". We work out the facts we accept as true and then combine them in a logical way to get conclusions. These conclusions are then "proved" - we can guarantee them to be true.

So we have deductive reasoning. It is powerful and often quick. But it is also often divorced from reality and difficult to grasp. Consider the following:

In summary, deductive reasoning could be diagrammatically represented as:

Assembling the facts → Combining these facts logically → Arriving at a conclusion (which has been proved true)

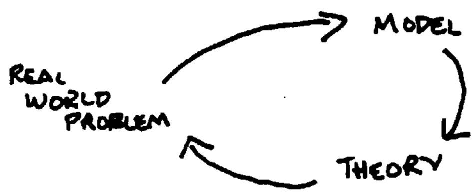
It should be noticed that this involves "abstract" thought.

(1) Which parts A to H of the activity in 1 above were inductive? Which were deductive?
(2) Which is suitable for primary children? Why?

5. The activity in 1 above is also interesting for another reason.

A powerful problem solving technique is exemplified by how the Bridges of Konigsburg problem was solved. Here the real world problem was modelled by a pen and paper substitute and this model was used to develop a theory which was then used to solve the real world problem. A point to stress, is that the pictorial model (networks) of the real world problem (Konigsburg) required a rewriting of the problem - from "can you walk around the city crossing each bridge only once" to "can we travel the network without retracing or lifting our pen".

Diagrammatically this technique could be represented:



(1) Should this Konigsburg problem be used to make children aware of this modelling technique?

(2) Is this a suitable role for geometry investigations, to make children aware of general problem solving processes?

Teaching Hints:

Analysing what has happened in the network activity gives support to the recommendation in the overview of this book to teach actively or "inductively".

Historically mathematics advanced

INDUCTIVE ⟵ TO ⟶ DEDUCTIVE

Most mathematical ideas are generated inductively from seeing pattern in explorations and real world situations, but they are proved deductively from the facts by logic.

Mathematics educators argue that mathematics should be presented (particularly) in primary) inductively because

- in harmony with developmental level
- children can see why
- children learn how to find answers as well as learning the answers.
- induction is the most powerful growth force in mathematics.

The inductive and practical aspects of networks can be used with children. Children enjoy travellability. For example, it is a common recreation problem.

Try as far as possible to connect the work on networks to networks in the world around us, e.g. roads. Networks is a very important practical application of mathematics. It is an excellent topic for showing relevant applications of mathematical thinking.

A teacher in England used to give her children a map of the local area in which they lived and ask them to act as an invading army. Which bridges would they blow up to make the greatest disruption in traffic movement? Can you think of problems like this?

There are many problems that can be set from networks. We list some of the important ones below, giving an idea of how they can be set out for children and some suggestions for questions to ask.

(1) Travellability

A network is travellable or traceable (this is sometimes referred to as traversable) if it can be traced without lifting the pen and without retracing any arc. Some can be travelled from any corner, others can only be travelled if you start from certain corners. Starting and ending corners should be marked.

A rule for determining whether a network is travellable or not, can be found from considering the odd or even degree of its corners. To find it, a collection of networks are traced to determine whether they are travellable or not and then the number of even and odd degree corners is determined. In other words, each corner is considered in turn, the number of arcs coming from it is counted and the corner is labelled E (even) and O (odd). Then the number of even degree corners and odd degree corners is counted. For example, network (a) has 9 corners of even degree and 0 corners of odd degree. When this has been done, the results can be tabulated as follows:

Network	Total number of corners even degree	The number of corners of even degree	The number of corners of odd degree	Can the network be travelled? YES/NO

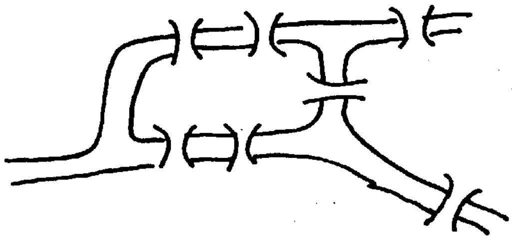
From this table a rule for connecting travellability to degree of corners can be found, i.e. a network is travellable when it has zero or 2 odd corners.

Note: You should check your rule using further examples.

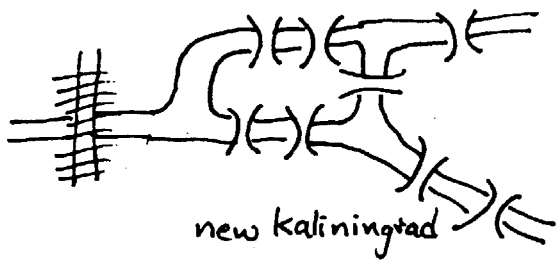
(2) The Konigsburg problem

We can use the results from (1) above to solve the Konisburg problem. We simply translate Konigsburg to a network, see if the network is travellable and this will determine whether we can walk around Konigsburg and cross each bridge only once. The network for Konisburg is as on right. It has 4 odd corners and is not travellable.

Konigsberg is now Kaliningrad and has 1 new bridge (and a rail bridge) as below:



old konigsbug

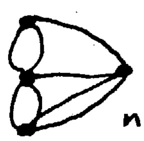


new kaliningrad

Can we show that a walk over the Konigsberg bridges is now possible by modelling the bridges with a network and looking at the corners?

What happens if we add in the new rail bridge?

The resulting networks are



new bridge added

rail bridge added

Both have 2 odd corners and are travellable.

(3) Game-sprouts

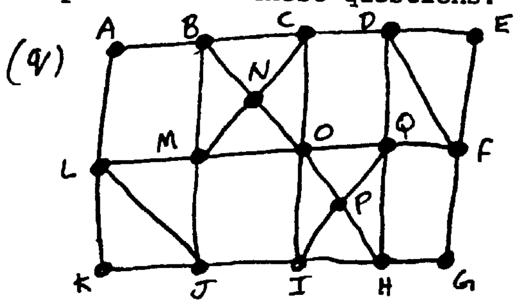
The game of sprouts described in the activities earlier always ends in a maximum of 3N-1 plays (where N is the number of starting corners).

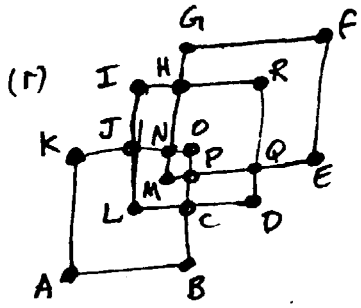
(4) Start and finish:

Look at those of the networks we have studied which are travellable. Where did we start and finish? Could we have started and finished else where? What is the order of the start and finish points? Does this matter?

Can you write down a rule for travellable networks which determines where you start and finish? (Hint: consider those networks with no odd corners separately from those with 2 odd corners).

It will help to answer these questions.





(a) Is it possible to travel network (a) in a single trip? At which worners must the trip begin?

(b) Is it possible to travel network (a) on a single trip? Can it be started at any corner? Test this by travelling it on tracing paper starting at a random selection of corners.

The result that can be discovered by children is that a network of no odd corners is travellable starting at any corner but we must finish at that corner as well. For a network of 2 odd corners, we must start at one odd corner and finish at the other.

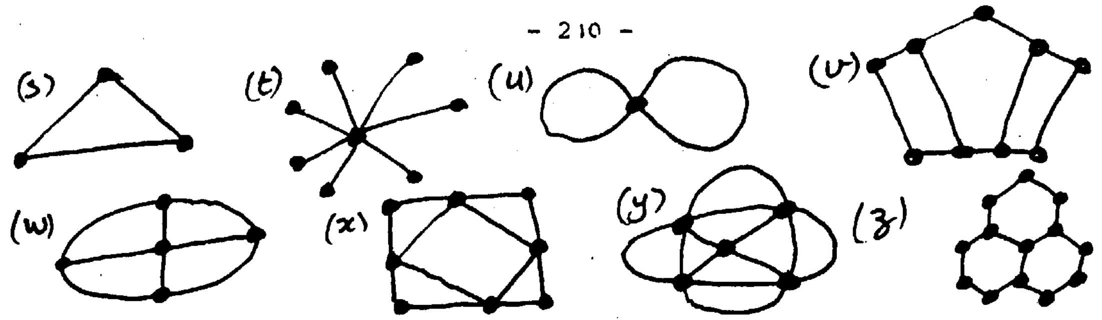
(5) Euler's formula:

For a network (as on right), we can count corners, arcs and regions (remember the outside one). The network on right has 6 arcs, 4 corners and 4 regions.

There is a simple relationship between the number of regions of a network and the number of corners and arcs that it contains. To find out what it is, count the numbers of regions, corners, and arcs in each of the following networks.

Record your results in a table like the one below:

Network	No. of regions	No. of corners	No. of arcs
1	2	3	3
2			
etc.			

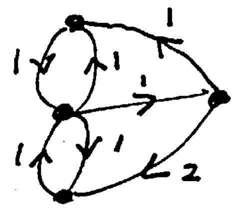


Can you write a formula relating the number of regions, R, in a network, the number of corners, C, and the number of arcs, A? (Hint: If you are having trouble, add a 5th column to your table above - number of corners plus number of regions).

The resulting formula (called Euler's formula) is R + C = A + 2.

(6) How many trips:

We can look again at non travellable networks. For example, the bridges of Konigsburg network on right. We can't travel this in one trip, but we can in two - with one lift of the pen (as shown).



(a) Can you find a rule for determining how many trips is needed to travel a network? (Hint: look at the number of odd corners).

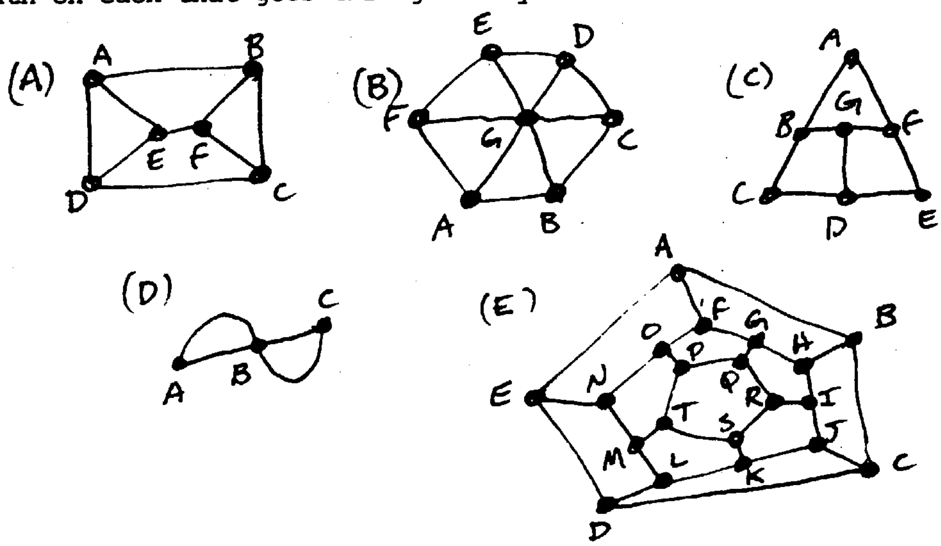
(b) If a network of one million corners has half its corners of odd degree, how many trips would be required to travel it?

The result that can be found by children is that any network with N corners of odd degree can be travelled in N/2 trips (Note - N must be 2 or more)

(7) The one visit network:

Of special interest telephone companies, truck operators, etc. are networks that can be travelled AND in such a way that each corner is only passed through once. So now not only can each arc be traced only once but each corner (except first and last) need only be visited once. What type of networks are these? What arcs have to be removed to make this so? Can removing arc allow this to happen? What type of network can't be made travellable in this way (what type can?).

Place a sheet of tracing paper over these networks and, if possible, draw a path on each that goes through every corner once and ends where it starts.



UNIT 23: COMPASS BEARINGS AND ORIENTEERING

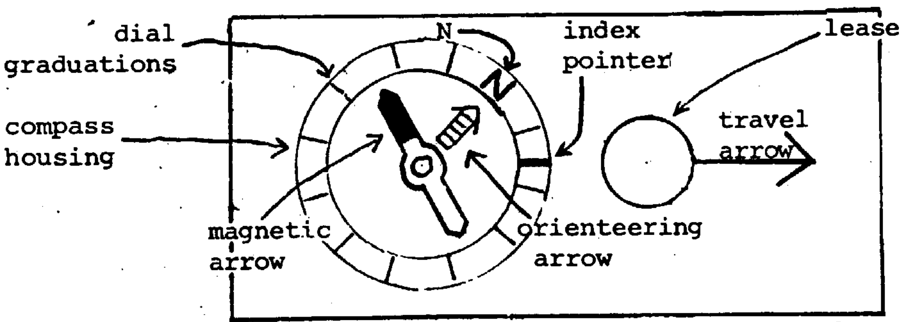
Focus:
Being able to find your way in the bush while, e.g. bushwalking or cross country skiing, is an important life and recreation skill and also a lot of fun. This unit focusses on how to use compasses to develop this skill in children.

Background:
Orienteering originated in Sweden some sixty years ago. Orienteering involves the use of a map and a compass to find reference points, which have been marked to define a predetermined route of travel. Individuals or small groups travel the course, completing it as accurately as possible.

Some variations in orienteering activities exist. Orienteers may work for speed as well as accuracy in completing a course. Other exercises call for the plotting of reference points, given directions and distances. This last type of exercise is most suitable to beginners and to orienteering courses that are set up on school grounds. Instruction leading to this type of orienteering could begin in the classroom and then move to the outdoors.

"Mathematical" orienteering tends to focus on distance and angle rather than speed and map reading and tend to be for shorter distances than "ordinary" orienteering.

Basic skills
The most common compass for orienteering is one like below (this is made by Silva):

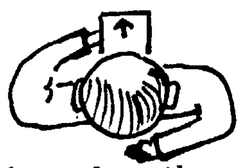


With this compass the following skills have to be developed before orienteering can be attempted.

(1) Facing a magnetic bearing e.g. 60^{o}

(i) Turn compass housing so 60^{o} on dial graduation is over index pointer.

(ii) Hold compass in your hand level enough to permit magnetic needle to swing freely. Also have travel arrow pointing straight ahead.

(iii) While holding compass like this 

turn yourself and compass so that the red north end of the magnetic needle points to the letter N (and is on top of orienteering arrow).

(iv) Look up in the direction of the travel arrow. This is 60^{o}.

(v) Note well. Keep iron and steel away from the compass. Hold your watch behind your back. Don't put it next to your steel buckle. Step away from steel uprights.

(2) Walking a bearing

(i) Look straight ahead (the farther the better) and choose a landmark in the direction you and the travel arrow are pointing.

(ii) Walk towards that landmark (it should not be necessary to check the compass as you walk).

(3) Taking a bearing

(i) Face the landmark

(ii) Hold the compass so that the travel arrow points to the landmark (and level enough so that the magnetic needle swings freely).

(iii) Turn the dial of the housing (without changing compass position) until the red end of the magnetic needle points to N.

(iv) Read the bearing off the dial at the index pointer.

(4) Pacing

(i) Mark off a 10 metre distance and pace this 3 times. Do not stride out - walk naturally.

(ii) Divide your total number of paces by 3 to get your average number of paces for 10 metres. (It is fun to repeat this for up hill and down hill to see how your pacing changes).

(iii) To pace off a distance (say 75 m) work out the number of 10m intervals in this distance (75m is 7½ ten metre intervals). Then either:

(a) multiply the number of intervals by your pacing average for 10 m (say 14 paces for 10m) and walk this number of paces. (In this case you would need to work 7½ x 14 = 105 paces to go 75m).

or (b) walk this number of paces. (In this case, you would walk 14 paces seven times and then an extra 7 paces for the half to go 75m).

Note: It is useful to restrict distances to multiples of 5.

Materials: Compass, graph paper, pen, pencil, paper, 30m tape.

Activities:

1. Compass pictures

(1) Obtain a piece of cm graph paper. Start near the bottom left corner.

Draw a line three squares north, continue two squares east, then four squares south, two squares northeast, three squares north, seven squares east, four squares southeast, five squares east, two squares northeast, one square west, and one square north.

(2) Obtain a piece of cm graph paper at least 17 cm across and 19cm up. Start at a grid point on the <u>bottom</u> line at least 10 cm from the left hand side of the graph paper and 7cm from the right hand side.

Imagine that each square has <u>sides</u> of 10m and diagonals of 14m.

To complete the picture, begin at the start point. Find point 1 and draw a line segment between the start point and point 1.

Then find point 2 and connect points 1 and 2. Continue this procedure until all the points have been located and all the segments have been drawn.

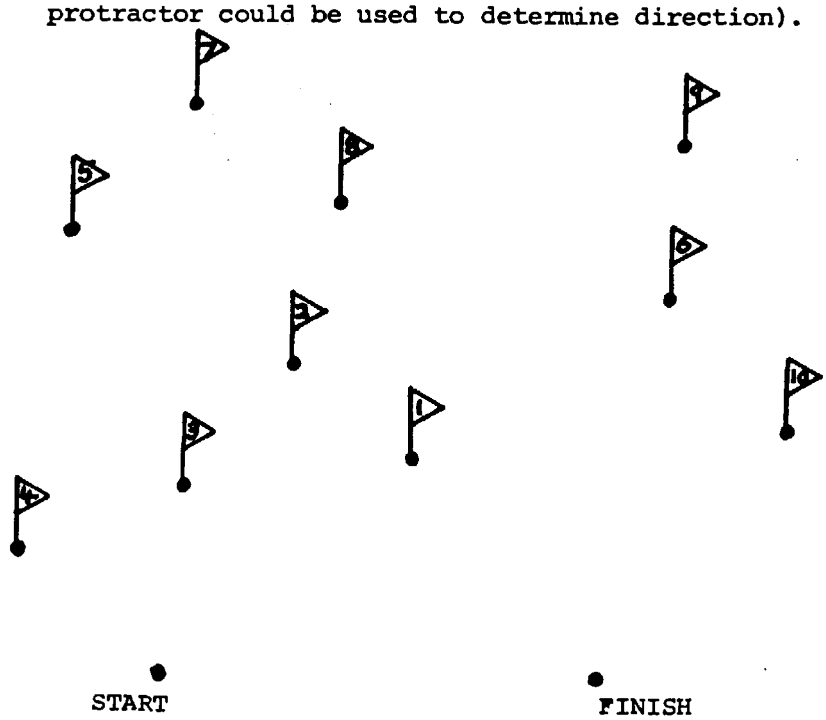
Read all distances in metres. North is up.

1. 40 N
2. 42 NE
3. 10 N
4. 42 NW
5. 10 W
6. 14 SW
7. 20 S
8. 28 NW
9. 10 W
10. 28 SW
11. 20 N
12. 14 SW
13. 10 S
14. 28 NW
15. 20 N
16. 84 NE
17. 20 E
18. 14 NE
19. 50 E
20. 42 SW
21. 56 SE
22. 50 S
23. 28 SE
24. 40 S
25. 42 NW
26. 56 SW

2. <u>Miniature orienteering</u>

(1) Distances are determined by using the scale one centimetre on the drawing is equal to ten metres on the orienteering course.

(2) Using a straight-edge, students connect the flags in order and compute the total distance from start to finish. A protractor could be used to determine direction).



3. Starting Activities in the great outdoors

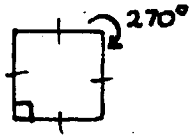
(1) Drop penny

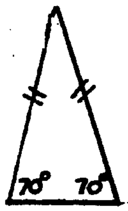
(a) Place a coin on the ground between your feet.
(b) Set your compass for an arbitrary bearing (you choose -say 40°)
(c) Face this bearing and walk 20 paces forward.
(d) Add 120° to this bearing. Set this new bearing (say 160°) on your compass, face it and walk it for another 20 paces).
(e) Add 120° again. Set this 3rd bearing (say 280°) on your compass, face it and walk it for a 3rd 20 paces.
(f) Your coin should be at your feet (or at least within a close distance). If this was not so, try again and/or approach an instructor.

(2) Constructing shapes

Use your compass and pacing to set out with rocks the outline of one of the following:

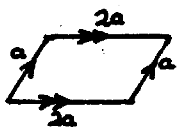
(a) Square





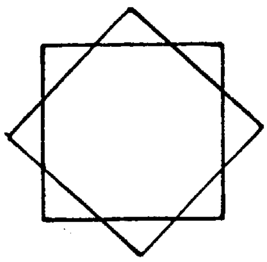
(b) isosceles triangle

(c) parallelogram



(d) Challenge: the following shape from 2 equal squares

You choose the size.

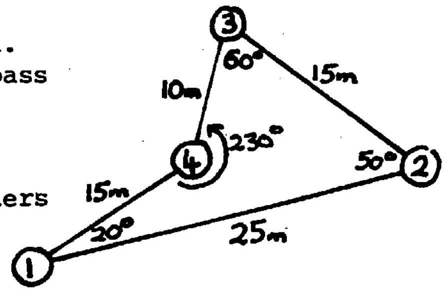


(3) Mapping

Numbered markers have been placed in an area. Your instructor will direct you. Using compass and pacing, draw a rough scale drawing of these markers showing

(a) distances between consecutive markers

(b) angles



An example of what you should end up with is on the right.

4. Orienteering Courses

(1) What's my word.

The letters to a word are cut our of cardboard. A starting position is found and the letters hidden (either jumbled or in order). Hiding places can be by placing under rocks or sticking on backs of walls or poles or hanging behind trees (any place where it can not be seen as the orienteerer walks towards it)

The orienteerers are given the starting point and the distances in m, and compass bearings to the first letter and from there to the next letter etc. A form can be used as below:

	Distance	Direction	Clue	Letter
Start to first letter	25m	320°	Look up	
First to second letter	70m	85°	Hanging around	

Once they have the letters, the orienteerers must rearrange these to get the word.

(a) See your instructor for directions

(b) Find the letters - make the word.

(2) Walking the track

Markers are placed on land marks such as trees, posts, rocks, etc. to form an orienteering course. These markers contain a letter and a mathematics symbol. They have been placed so they are not easily found and so that they can not be seen by the approaching orienteer.

Distances in metres and directions in degrees from one marker to the next is given on a table, e.g.

RED COURSE	DISTANCE m	DIRECTION °	CLUE	SYMBOL	DESCRIPTION OF WHERE MARKER FOUND
K → A	60	240	where the lizards are		
A → B	80	310	you'll have to look up		
B → C	20	224			
J → K	25	190			

Orienteers are shown to the starting point - this could be anywhere in course. From there they find the direction, with compass, to the next marker and pace off the required metres in that direction. When they reach the end of their pacing, they cast around for the marker, using the clue to help find it. Once they have found the symbol, they draw it on the table plus a brief description of where the marker was. (A group of orienteers could go around the course together).

Note: It is normal for the course to be circular - return to the start (but not necessary). Walking the course is a test of distance and direction (distances are small, there is no time limit, but you have to pace accurately in the correct direction). A scale drawing can be made of the track at the end.

(a) See your instructor for directions and table
(b) Walk the track, filling in the table.
(c) Draw a scale drawing of this orienteering track

(3) Making your own track

This requires you to reverse the procedure in (3) above. Find a starting point and place a marker. Look around for a likely landmark to place the next marks on. Record the direction to it. Then pace towards it and record the number of paces.

Sometimes we simply find a bearing and pace along it until we hit something interesting (e.g. a post or rock) and then stop, place the marker and record the bearing and distance.

(a) approach your instructor for suggestions for where you could make a track

(b) make your track, filling in direction and distances on a table as you go.

(c) Give your table to someone else to walk.

Teaching Hints:
The activities in this unit are a lot of fun. But they will require a lot of effort from you in preparing the courses or tracks.

Try to minimize the remedial work you will have to do on the day your children walk the course by:

(a) ensuring all have the skills of compass bearings and pacing; and
(b) doing the compass pictures,miniature orienteering, and starting activities before the courses.

It is a good idea to connect this unit into camps and other outdoor activities and into Physical Education. It is also a sport some of your children would like to go onto.

The extension of this mathematical orienteering is map reading, large scale orienteering and bush walking. If you use your compass to point the north arrow in your map in the actual north direction, the compass can be easily used to find a bearing from the map. In any case, estimates of direction can be quickly taken off maps to set compass bearings from.

There is a lot more than distance and bearing in reading maps. The level of the terrain is all important. If you are interested, you can always find someone in the local community who can instruct you or your children.

Special note: Much of the activity in this unit ends with a scale drawing, a skill which is discussed in the next unit.

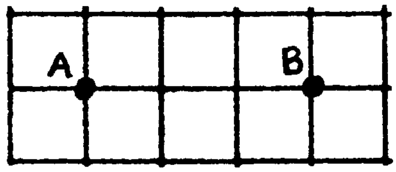
UNIT 24: SCALE DRAWING AND SURVEYING.

Focus:

In the previous units we have developed skills in interpreting network style maps and in following directions and distance instructions. Now we move onto constructing or drawing maps from information on direction and distance. This skill is dependent on knowledge of scale and is similar to some activity undertaken by surveyors. Hence this unit is titled scale drawing and surveying.

Background:

A tessellation of squares in graph paper divides the plane into an array (rows and columns) of coordinate points or lengths. On this graph paper we can draw copies of larger shapes by equating distances such as 5 or 10 metres or kilometres, with the length of the sides of the squares. For example, in the following 1 cm graph paper:



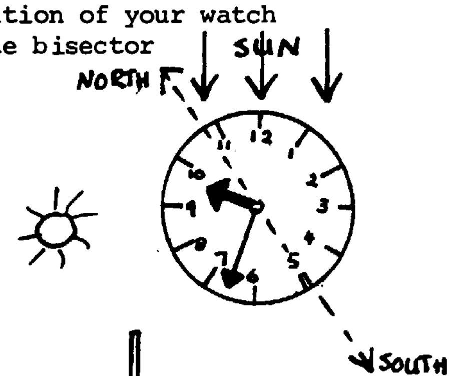
AB represents the distance from the gate to the classroom (a distance of 30 metres) by equating 1cm (the length of the side of each square) with 10m.

The equating of 1cm with 10m is called the scale of the drawing.

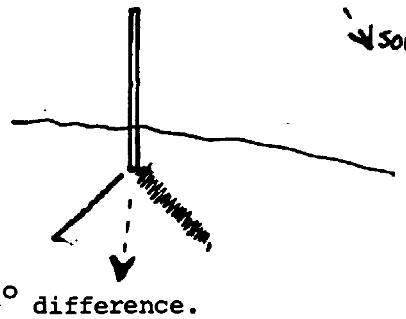
The most complex activity in scale drawing is surveying. The basic skills in this are given below.

(1) Finding North

(a) Using a watch, face the 12 o'clock position of your watch towards the sun as shown on right. The bisector of the angle between the 12 o'clock position and the hour hand will be approximately North - South as shown.



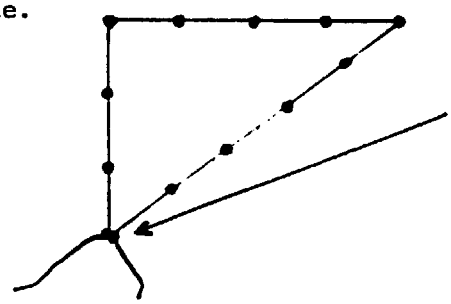
(b) Using shadows: when the shadow of an object is shortest, it points South (in Wagga). If this is difficult to judge, find 2 positions when the shadow of an upright stake has the same length (at a certain time in the morning, measure length and mark in shadow - wait until shadow same length again and mark in 2nd shadow). North-South is midway between these 2 positions.



(2) Right Angles

(a) Using a compass: 2 directions with 90° difference.

(b) Using string: Put 13 knots in a piece of string to make 12 equidistant segments. Hold as below.
The 3 4 5 triangle makes a right angle.



(3) Pacing

Measure out 10m. Pace it 3 times. Divide total number of paces by 3 to get your numbers of paces for 10 metres. Check this on sloping ground (see orienteering in Unit 23).

(4) Compass reading

See orienteering in Unit 23 for how to read silva compass. Basically you need to turn the compass until the magnetic needle points to N on the compass. Then the angle you wish to go is in the direction shown by the angle on dial.

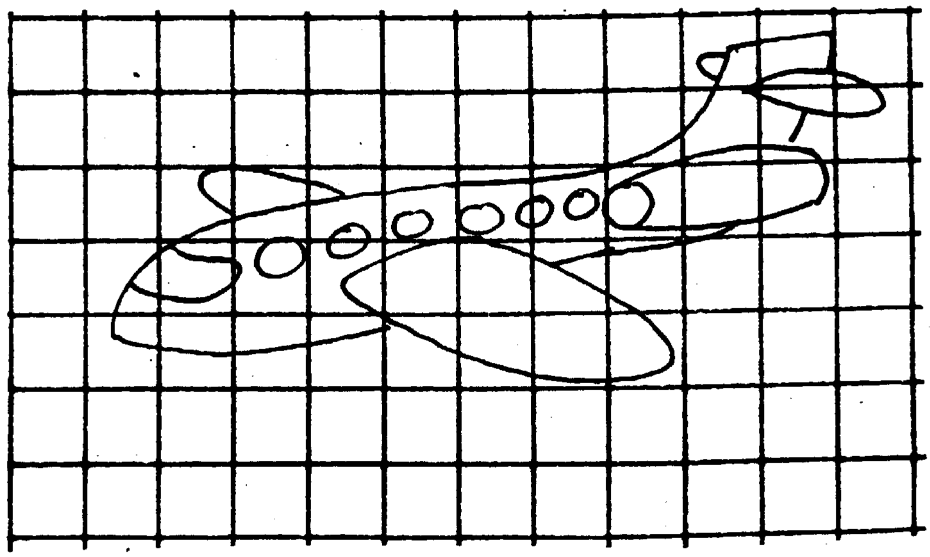
(5) Direction from maps

Ignore the magnetic needle. Align North on the dial with north on the map. Read the direction. For the silva compass, place the compass so that its edge is in the required direction. Turn the housing until north on dial is aligned with the direction of north on the map. The direction can be read from the compass. If the bearing is given, place it on the compass and then turn the compass until the norths aligned (remember - ignore the magnetic needle) - the compass direction given is the direction on the map of the bearing.

Materials: Pen, paper, graph paper, ruler, compass, clinometer, 30 m tape string, weight sticks mirror.

Activities:

1. Obtain a page of 2cm graph paper and make a large drawing of the following shape.



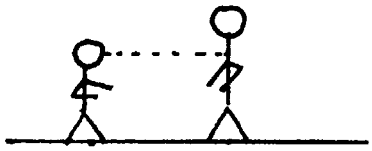
2. The measurement of a slope or incline is important to agriculture, geology, etc. It can also form the basis of cross section scale drawings and volume measurement. There are the following methods for determining incline.

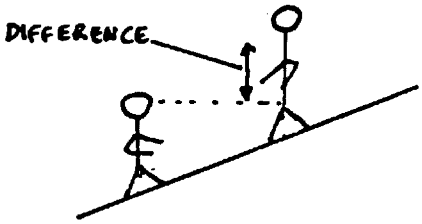
   (a) *Estimation*
   On flat ground, stand facing a friend. Stand naturally - look straight ahead. Note where on his clothing your eyes focus. Now go to incline. Stand as before - where do your eyes focus now. Measuring the difference gives an estimate of height difference.

   There are two possibilities: -

   - *you can stand a fixed distance away from your friend* and measure height difference

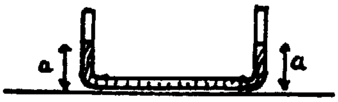
- you can move away from your friend until you are looking 1 metre say, below, focus point and measure distance between. Both will give inclination.

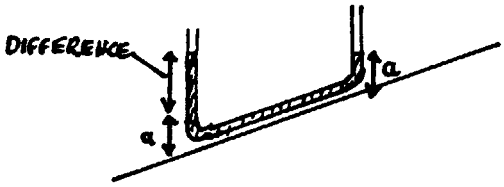




(b) Plastic hose

A clear plastic hose 3/4 filled with water can be carried with ends corked. When ends held up and uncorked, the water levels at each end are the same height. Moved to slope, will show height difference

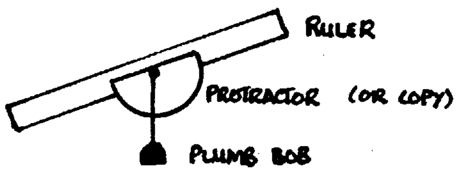




(c) Clinometer

These can be made by copying a protractor on to cardboard (or attaching a protractor to a ruler) and then attaching a plumb bob,

e.g.



Sighting up or down slope with protractor will give inclination (by how much plumb bob deviates from 90^{o}).

The ability to measure incline can be used in the following applications:

(a) Shapes of hollows or creek beds.

Take a point on either side of creek. Mark this distance AB into 2 metre or 5 metre or... (it depends on what you want to do) intervals.

At the end of each interval take a height difference reading from A. You will end up with "co-ordinates" (e.g.✱(4, 2) giving a distance from A and depth difference.

(i) Make a scale drawing of this creek bed as follows:
Distance AB and height difference from B to A enable you to draw in line AB

(ii) The "coordinates" will then enable you to mark in the crosses above. Use ruler to measure along the 2m (or 5m or whatever) intervals along line AB. Then use protractor and ruler to measure down (or up) the depth difference at the end of each interval and mark with a cross as above.

(iii) The creek bed is then a smooth curve through these crosses.

(b) The volume of water flowing along a creek.

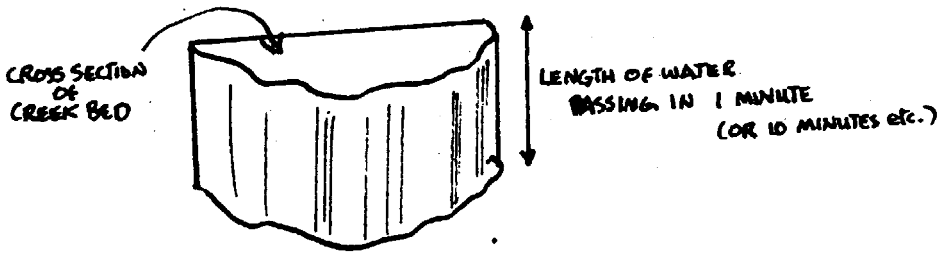
(i) Use the method above to draw a scale drawing of the bed of the creek.

(ii) If this scale drawing is on graph paper, it will enable you to calculate the cross sectional area of the water flowing by.

(iii) By placing sticks in the creek at various points across it and measuring how long they take to go (say) 5m (or 2m or whatever), the speed of the rivers can be calculated (or the distance the water travels in 1 minute).

(iv) By multiplying the speed by the cross sectional area the volume of water flowing down the creek can be calculated.

<u>Note:</u> *If the calculations above are a problem, make a model out of cardboard (using say 1cm for 1m) as follows:*



Then this model can be filled with water and the volume in ml or cm^3 measured. This is then an estimate of the volume of water passing along the creek in the 1 minute (or 10 minutes) in m^3.

The volume of earth to level a sloping piece of land may also be calculated in a similar fashion.

(1) Approach your instructor for directions on an incline or hollow that can be measured.
(2) Use the three approaches to measure incline.
(3) Draw a scale drawing of the slope or hollow.

3. <u>Height measurement</u>

There are eight ways to measure height. They are given below for the example of a tree.
(a) Simple methods.

WAY ONE: FELLING THE TREE

This is the method used by loggers and naturalists to estimate the height of a tree. It uses 2 people, a pencil or a stick. It's simple and easy to use.

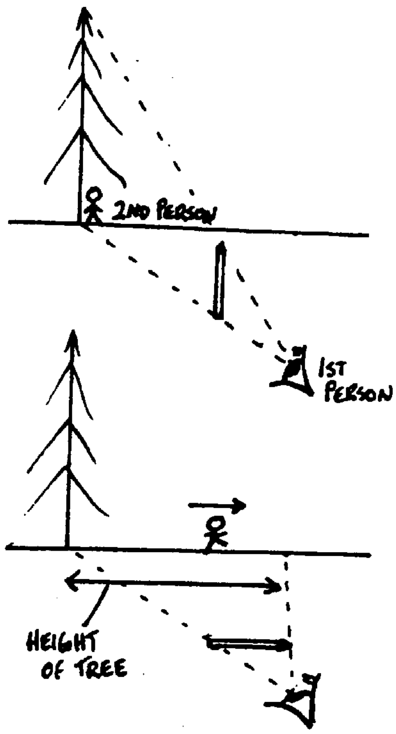
The pencil is held in front of one person so that its top and bottom coincide with the top and bottom of the tree when viewed with one eye.

Ensuring it is held at the same distance in front of the eye, the pencil is turned to a flat position. It's end still coincides with the bottom of the tree.

A second person stands beside the tree and then walks sideways away from the tree.

When this second person coincides with the top end of the pencil, the 1st person stops her/him. The height of the tree is then the distance from this person to the base of the tree.

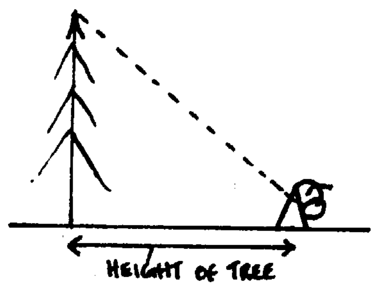
You virtually "felled" the tree.



WAY TWO: INDIAN METHOD

This is the method used by Indians in Canada. Youturn away from the tree, bend over and look through your legs. Then walk away from or back towards the tree until you can <u>just</u> see its top. Then, if your children are average, their distance from the tree equals its height.

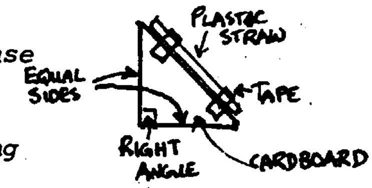
If they're not average, take the mean of your class' positions.



WAY THREE: RIGHT ANGLED ISOSCELES TRIANGLE

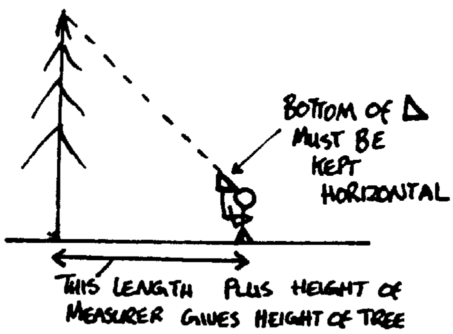
This uses the property of a right angled isosceles triangle that it's two non-hypothenuse sides are equal.

A right angled triangle, with straw for looking through, is constructed as on right.



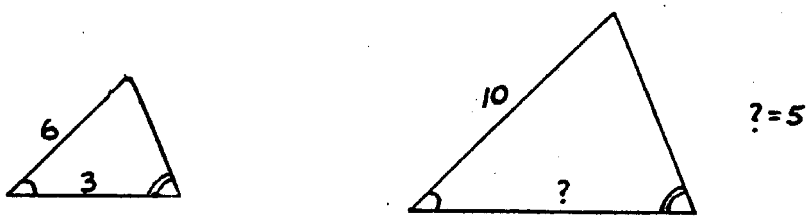
The child moves until he can see the top of the tree through the straw. The bottom of the triangle must be kept <u>horizontal.</u>

At this position, the distance of the tree equals the height of the tree <u>above</u> the measurer (because a right angled triangle has been formed). Add the height of the measurer and we have the height of the tree.



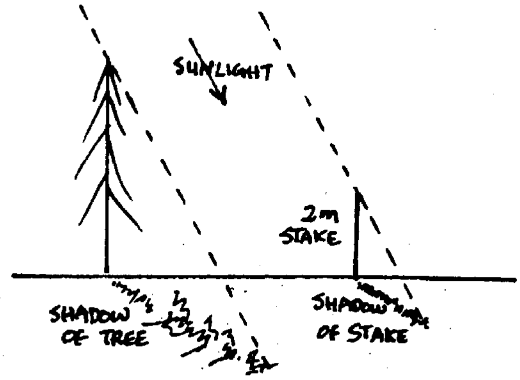
(b) Similar triangles

If two triangles are similar (same shape but different size) then their sides are in ratio.



Two triangles are similar if two angles are the same.

WAY FOUR: SHADOWS



A stake of fixed height (say 2m) is placed in the ground beside the tree.

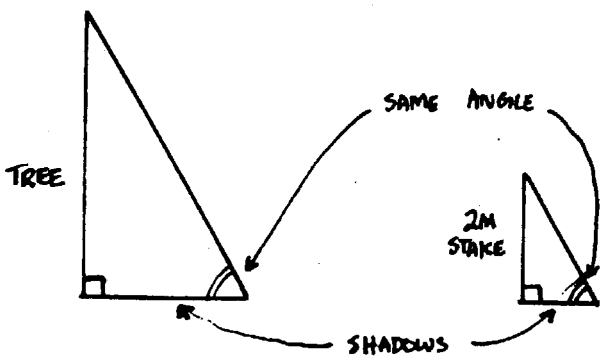
Because sunlight hits tree and stake at same angle, similar triangles are formed.

Then shadows and heights are in the same ratio.

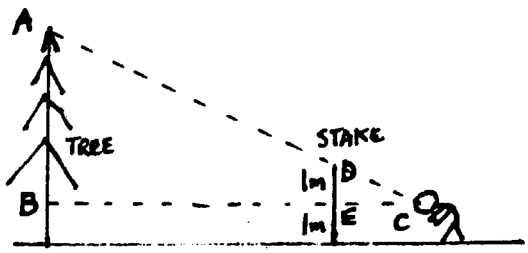
$$\frac{\textit{length of shadow of tree}}{\textit{length of shadow of stake}} = \frac{\textit{height of tree}}{\textit{height of stake}}$$

$$\textit{height of tree} = 2 \times \frac{\textit{length of shadow of tree}}{\textit{length of shadow of stake}}$$

Measuring the lengths of shadows gives the height of tree.



WAY FIVE: SIGHTING ALONG A STAKE



A 2m stake is placed on the ground with a mark at 1m. (or a 1m stake is placed on top of a table.

The child keeps eyes 1m above ground and moves to where the top of the tree can be seen beside the top of the stake.

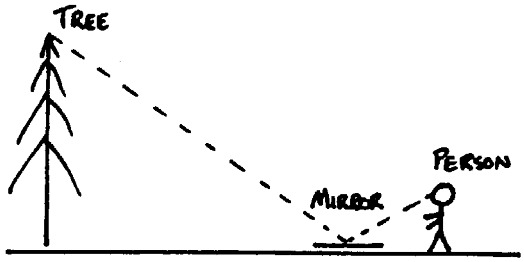
This forms two similar triangles. The distance from measurer to tree and stake is measured and the height worked out as below

$$\frac{AB}{DE} = \frac{AB}{1} = \frac{\textit{distance from measurer to tree (CB)}}{\textit{distance from measurer to stake (CE)}}$$

$$\text{Height of tree} = AB + 1m$$

$$= \frac{\text{distance from measurer to tree}}{\text{distance from measurer to stake}} + 1$$

WAY SIX: MIRROR



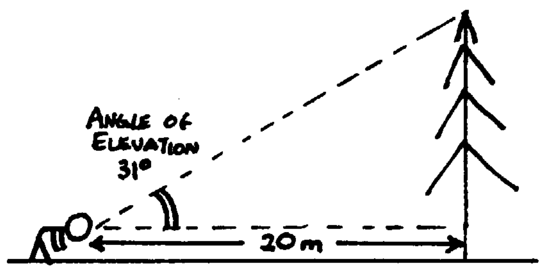
Place a mirror on ground and, standing straight, move to where the top of the tree can be seen in the mirror.

This forms two similar triangles. The distance from mirror image point to measurer and to tree are measured and the height worked as below.

$$\frac{\text{Height of tree}}{\text{height of measurer}} = \frac{\text{Distance mirror to tree}}{\text{Distance mirror to measurer}}$$

$$\text{height of tree} = \frac{\text{Distance mirror to tree x height of measurer}}{\text{Distance mirror to measurer.}}$$

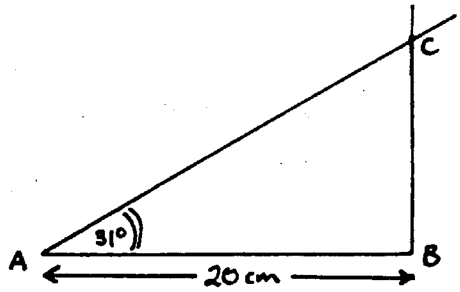
(c) Scale Drawings



If we know the angle of elevation to the top of the tree, we can find its height by scale drawing.

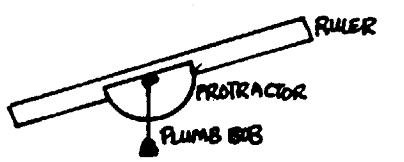
For example suppose we are 20m from the base of the tree and the angle of elevation is 31°.

Then we can use graph paper and a scale of 1m - 1cm as shown

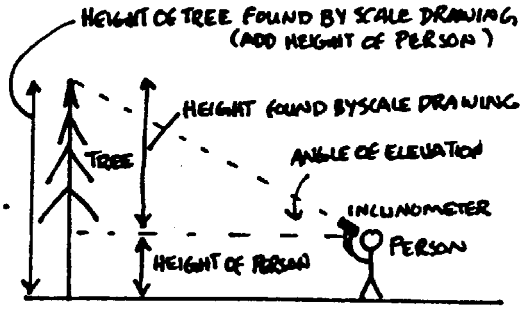


- draw a line (horizontal) of 20cm (AB)
- from one end construct a line at angle 31° and at the other a perpendicular
- where these two lines intersect (Point C) signifies the top of the tree
- BC (in cm) is the height of the tree (in m). above the height of the measurer.

WAY SEVEN: INCLINOMETER



Construct an inclinometer as shown on right. Have a child stand a known distance (say 20m or 12m) from the base of the tree and sight along the ruler to the top of the tree.

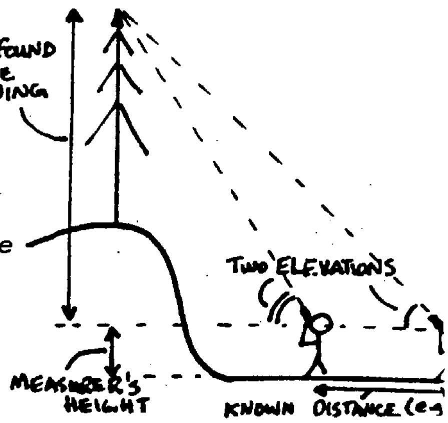


Have another child read the value shown by the plumb line. The difference between this reading and 90° is the angle of elevation.

Calculate the height by scale drawing. Remember to add the height of the measurer.

WAY EIGHT: INACCESSIBLE OBJECT

Sometimes we can not work out the horizontal distance to the base of the tree or wish to find its height above another level (like the sea - this is how men on a ship calculate the height of a mountain). In these cases, take 2 readings of the inclinometer, say 20m apart in line towards the object. Scale drawing will give the height of the object don't forget to add the measurer's height).



(1) Ask your instructor for some heights to measure

(2) Use methods one, two, three, seven and eight, drawing scale drawings where appropriate.

4. Surveying. If in the area to be surveyed, there are positions (e.g. trees, mountains, rocky outcrops, towers, hills) from which all the other positions and landmarks can be seen, then these can act as reference points from which the survey map can be drawn. There need be only 2 such points. Sometimes reference points are not available or the object to be surveyed (like a creek) is too long and wobbly to be placed by triangulation. At these times a straight line beside the object can be paced out with a compass and used to place the object in the survey map. This technique is called offsetting and is used whenever a boundary is non-straight.

The following is a description of these skills

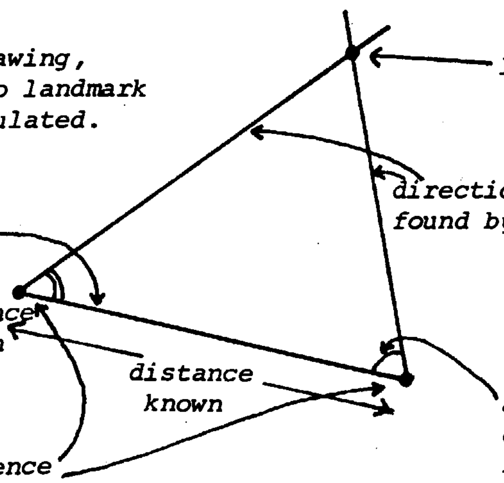
(i) Angle and Distance

The angle to and distance between reference points can be found by compass bearings and pacing.

(ii) Triangulation

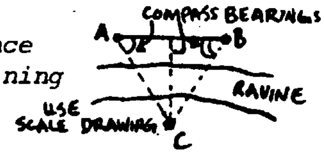
If terrain makes pacing difficult then the position of a landmark can be placed by triangulation from two reference points (the distance between which is known). To triangulate, compass bearings of reference points to each other and the landmark are taken. From these a scale drawing of the resulting triangle between the landmark and the 2 reference points is drawn.

From the drawing, distances to landmark can be calculated.

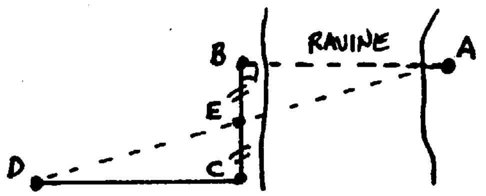


(iii) Distance between inaccessible points, e.g. width of a ravine

(a) Triangulation and finding perpendicular distance from landmark (one side of ravine) to line joining 2 reference points on other side of ravine.



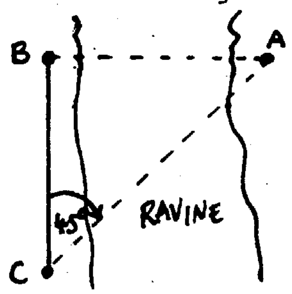
(b)



Find 2 points A and B across from each other on sides of ravine. With compass pace right angles to C.

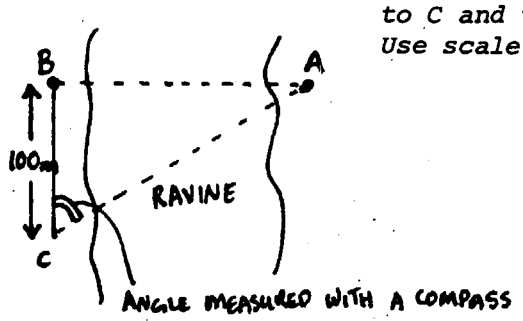
Stake or person left at E (mid point of BC). Another person walks at right angles away from ravine. When he is in line with E and A, distance DC = distance AB. Distance DC can be paced.

(c) Isosceles right angle triangle:



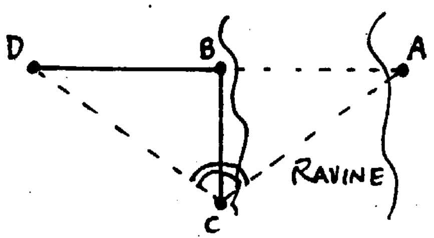
Points A and B found on opposite sides of ravine. Using compass, pace off at right angles from B. When compass shows bearings to B and A 45° different, triangle ABC is a right angled isosceles triangle and distance AB = distance BC (can be paced).

(d)



As in (c) above but pace out 100m to C and take bearing to A and B. Use scale drawing to find distance AB.

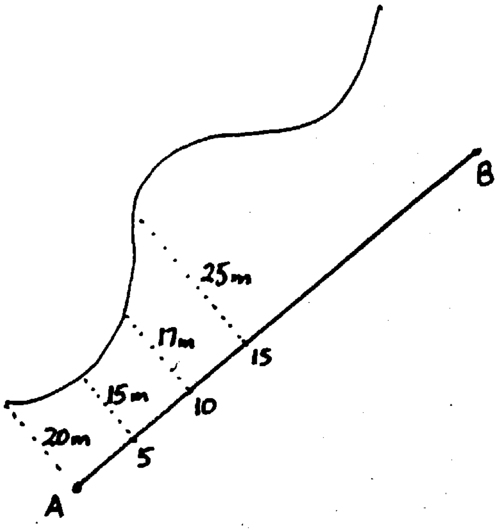
(e)



Points A and B as in (c) above. Pace (any reasonable distance) right angles to C. Use compass to find angle BCA. Have someone continue walking along AB in straight line. When (by compass) angle DCB = angle BCA, then distance AB = distance BD (can be paced).

(iv) Surveying from reference lines.

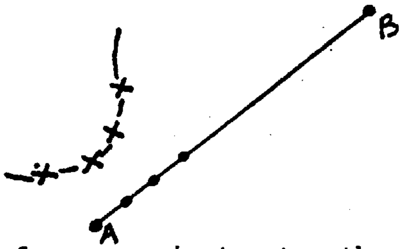
For example; suppose we have a creek as shown:-



(a) *Line AB is marked in and every 5 or 10 metres along AB (depends on curvature of creek) a right angle is made and the distance to the creek paced. (These are called <u>offsets</u>.)*

(b) *This gives a list of "co-ordinates" e.g. (0,20),(5,15) (10,17) and so on. These co-ordinates give distance (from A) to where offset is taken and distance away from AB to creek (length of offset).*

(c) *Using its starting point and bearing, the line AB is placed on our map and the 5, 10, 15, etc. metre points marked in. Using protractor and ruler the offset <u>end points</u> are marked in*



(d) *The creek is drawn in as a smooth curve through these marks.*

(1) Obtain from your instructor the description of an area to be surveyed.

(2) Use the skills above to determine the positions of major points in this area and the edges of the area.

(3) Draw a scale drawing of this area.

<u>Teaching Hints</u>

Once again the teaching role is to balance skill development with outdoor activity to, e.g., measure height or survey a section of the local environment. When the actual surveying or height measurements are being done, the teacher must ensure there is adequate time to complete all that has to be done. It is good to integrate these activities with camp or day camp activities.

For children to complete the surveying requires the application of many mathematical and psycho-motor skills. For example, in surveying all the folllwing are necessary.

- measuring distances
- reading compasses
- pacing
- scaling distances for the graph
- recording information
- making decisions about what to do first

Children may need a lot of support to do it well.

Furthermore children may need more supervision than can be given by one teacher as they spread out over the playground to do their measurements.

CHAPTER SEVEN: SIMILARITY, CONGRUENCE AND TRANSFORMATIONS

Two of the major upper primary concepts in geometry are similarity and congruence. These have been traditionally dealt with in terms of the properties of congruent and similar shapes.

In this chapter we wish to take a more active view of them and look at them in terms of things that do not change when certain transformations occur. As such similarity can be seen as part of projective geometry and congruence as part of euclidean geometry.

Hence unit 25 looks at the three types of transformation: topological, projective and euclidean and what is invariant under these changes. Unit 26 follows this by describing a series of interesting activities that emerge from these transformations.

Units 27 and 28 then look at similarity and congruence. We end the chapter by describing, in Unit 29, a very interesting application of euclidean transformation to art - modulo art posters.

UNIT 25: UNDERSTANDING TRANSFORMATIONS

Focus:

Transformations are of three types: topological, projective and euclidean. In this unit we investigate what is involved in each, looking particularly at what remains the same during these changes.

Background:

Topological change is change through twisting, bending, stretching and moulding with no breaking, tearing, punching holes or joining together allowed. During a topological change, the following stay the same (are invariant)

- closed shapes stay closed (and open stay open)
- things inside closed shapes stay inside (and same for outside)
- order along a line remains the same
- simple (non intersecting) shapes stay simple
- the number of holes in a shape remains the same

and the following change

- length
- ratio of length
- straightness
- parallelness
- measure of angles
- number of sides and angles
- size
- shape.

Projective change is that which occurs between a shape and its shadow - the change between reality and a drawing, between what we are looking at and what we see. There are two types of projective change - that from diverging light (e.g., a torch or a candle) and that from parallel light (e.g., sunlight). During a projective change, the following stay the same (are invariant)

- closed and simple
- insides and outsides
- number of holes
- order along a line
- straightness
- parallelness (parallel light only)
- number of sides and angles

and the following change

- length
- shape
- size
- measure of angles
- parallelness (divergent light only)
- ratio of sides.

Euclidean change is restricted to slides (translations), flips (reflections) and turns (rotations). During an euclidean change, the following stay the same (are invariant).

- closed, simple, insides and outsides
- number of holes and order along a line
- straightness, parallelness
- number of sides and angles
- measure of angles and ratio of sides
- length
- shape and size

and the following change

- position
- orientation.

Materials:

Pen, paper, graph paper, mira, tracing paper, cardboard or wood, torches or candles, sunlight.

Activities:

1. We can model topological transformations with string, rubber bands, rubber sheeting, clay, plasticine by stretching or twisting without tearing and by moulding without joining or compressing.

   Which of the before/after pictures below represents a topological transformation? (If you're not sure, get out some string, rubber bands or clay and see if you can do it.)

If you think the change is <u>not</u> topological, then write down why.

(a) before after

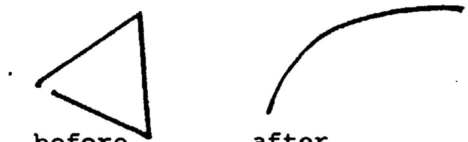
(b) before after

(c) before after

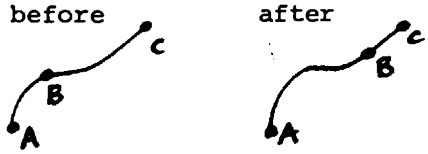
(d) before after

(e) before after

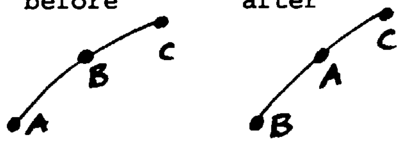
(f) before after



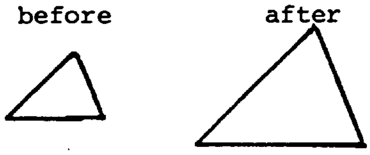
(g) before after



(h) before after



(i) before after



(j) before after

(k) before after

(l) before after

2. If we can change one object into another topologically, we call the two objects <u>topologically equivalent</u>.

(1) Which of the following is topologically equivalent to a sausage? Which are topologically equivalent to a doughnut?

ball	flower vase (without handles)	cheddar cheese
plate	shirt	swiss cheese
pipe	loaf of bread	bottle
spoon	shovel	sieve

(2) Divide the capital letters of the alphabet up into groups so that every member of a group is topologically equivalent to the other members (to do this list all letters topologically equivalent to the letter A, then B, then C, then D and then E. Any letters left over? Do any of these go together? And so on).

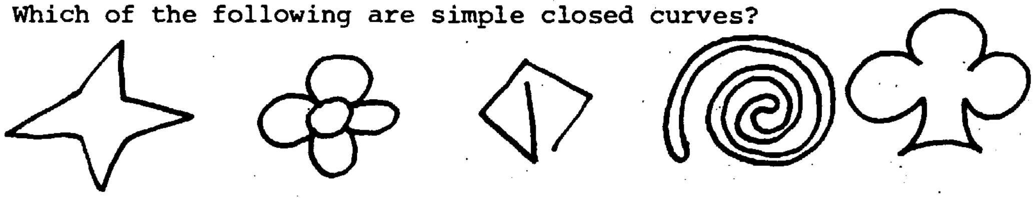
3. Insides and outsides

(1) All these curves can be got from a circle topologically.

All these curves can not:

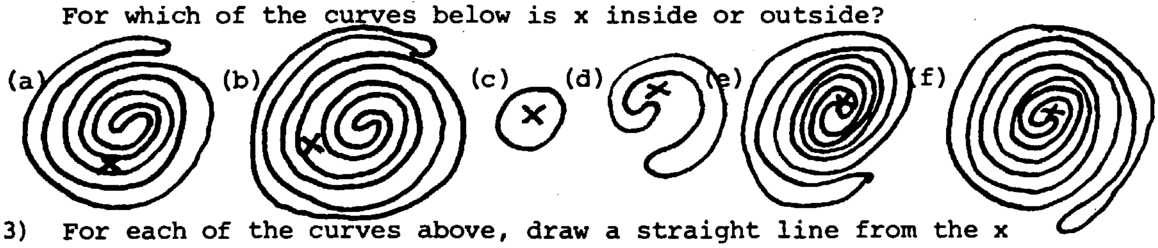
We call curves that are topologically equivalent to a circle <u>simple closed curves</u>.

Which of the following are simple closed curves?



(2) Circles have <u>one</u> <u>inside</u> and <u>one</u> <u>outside</u>, they separate the plane into these <u>two</u> regions. So do <u>all</u> simple closed curves (this is the Jordan Curve Theorem!)

For which of the curves below is x inside or outside?



(3) For each of the curves above, draw a straight line from the x in any direction out of the curve.

e.g.

There is a simple rule for telling whether the x is <u>inside</u> or <u>outside</u> by counting how many times this line crosses the curve.

(a) Draw up a table of results to help you find this rule!

(b) What is the rule?

4. Topological Invariants

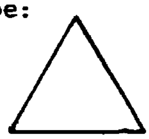
What properties below stay the same (i.e., do not change - are invariant) during topological changes:

(a) Having one hole.
(b) Being 2m long.
(c) Being straight.
(d) Being inside or outside.
(e) Being closed.
(f) Being a polygon.
(g) Having sides in ratio.
(h) Having an angle of 30°.
(i) Being close together.
(j) Having a certain order on a curve.
(k) Consisting of 10 points only.
(l) Not being a simple closed curve.

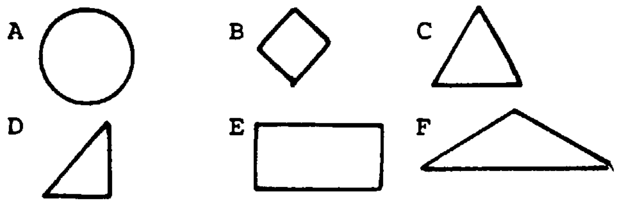
5. We can model projective transformations by casting shadows from a projector, torch, candle or overhead ((point source or divergent light) or in sunlight (parallel light). We can change the shape of the shadow by moving the shape and/or the screen.

Cut the shapes below out of cardboard, and see which of the shadows can be cast in divergent or parallel light.

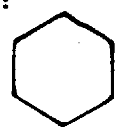
(a) shape:



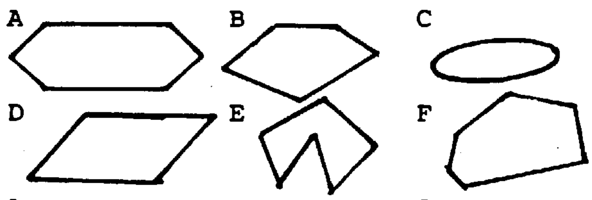
shadow:



(b) shape:

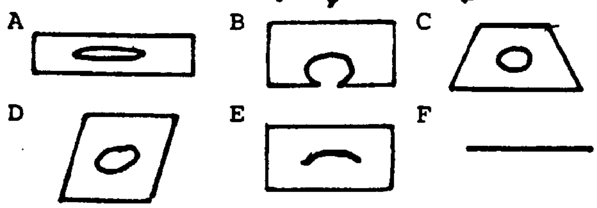


shadow:



(c) shape:

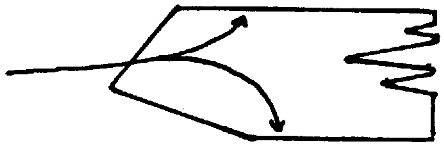
shadow:



6. Divergent vs parallel light.

A shadow of a square from a divergent light can be quite different. Cut out a polygon from cardboard as below (with hole and notches):

parallel sides



Cast it's shadow in divergent and then sunlight. Move screen and polygon. Check shadow (measure if you have to) to answer the following questions.

(1) For divergent light

(a) Does the number of sides (and number of angles) stay the same?

(b) Does the length of sides stay the same?

(c) Do the angle measurements stay the same?

(d) Do the stright lines stay straight?

(e) Do the parallel lines stay parallel?

(f) Do the lengths of side stay in ratio?

(g) Does the order of the notches stay the same?

(h) Does the hole stay in the centre?

(2) For sunlight

(a) Is anything different to divergent light?

(b) List those properties that are the same and those that differ?

7. Projective Invariants

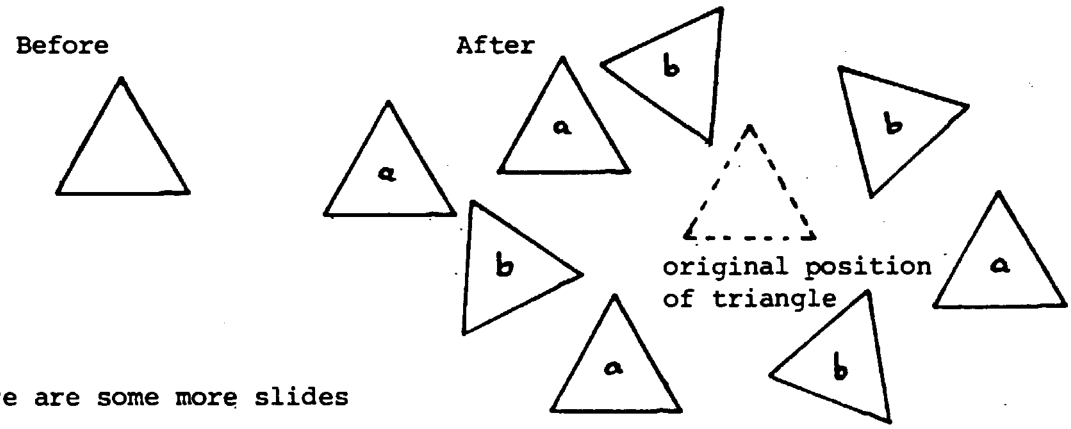
What properties below stay the same (i.e., do not change - are invariant) for

(1) divergent light
(2) parallel light

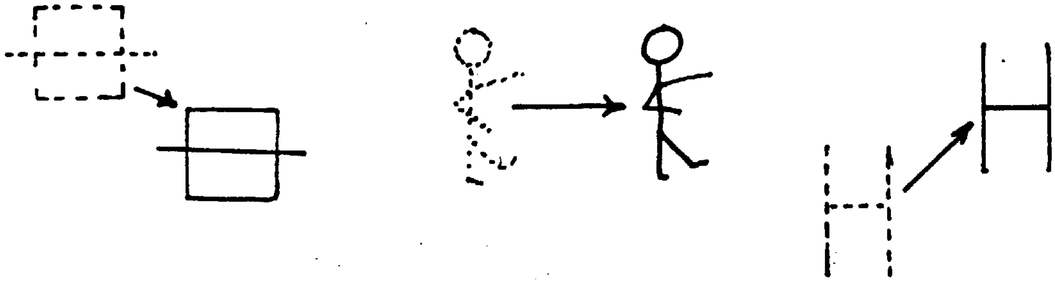
(a) Length of sides	(b) Ratio of sides
(c) Number of sides	(d) Number of angles
(e) Measure of angles	(f) Straightness
(g) Parallelness	(h) Order of notches
(i) Position of hole	(j) Size
(k) Shape	

8. Slides

(1) The triangles (a) below are the result of slides.
The triangles (b) are not.



Here are some more slides



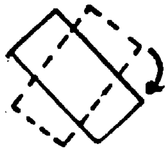
Describe in your own words what a slide is.

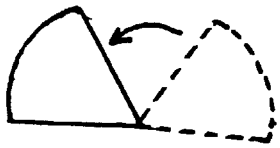
(2) Slides can best be practiced on graph paper. The original figure is drawn in on the graph paper (can be done with a dotted line to show it is the original shape). The slide is represented by an arrow giving the length the direction of the slide. The task of the student is to draw (as accurately as possible) the resulting shape after the slide has occurred.

Complete the slide exercises at the end of this unit.

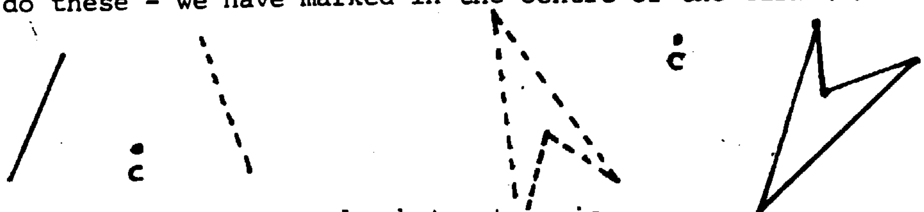
9. Turns

(1) The following changes result from turns (the dotted shape is before and the firm line is after).



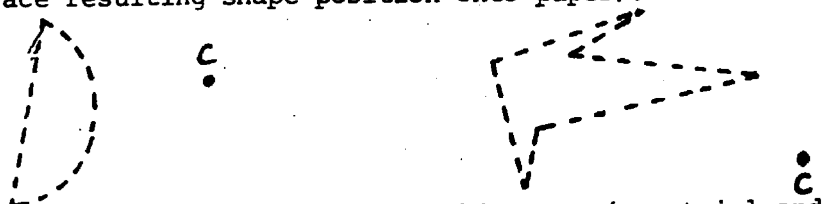


So do these - we have marked in the centre of the turn (C).



Describe in your own words what a turn is.

(2) Turn these shapes through an angle of your own choosing (place tracing paper down, trace shape, hold pen at centre C, turn and trace resulting shape position onto paper).



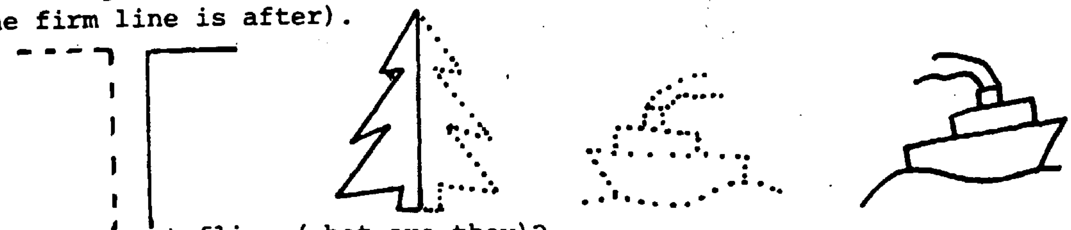
(3) See if you can find the centre of this turn (use trial and error).

(4) There is a procedure for finding centres of turn. You fold one point of the before shape to the same point on the after shape and crease the fold line. Repeat this for a second point and where the 2 fold lines intersect is the centre.

Use this procedure to find the centre of this turn (trace it first onto tracing paper).

10. Flips

(1) The changes result from flips (again the dotted shape is before and the firm line is after).

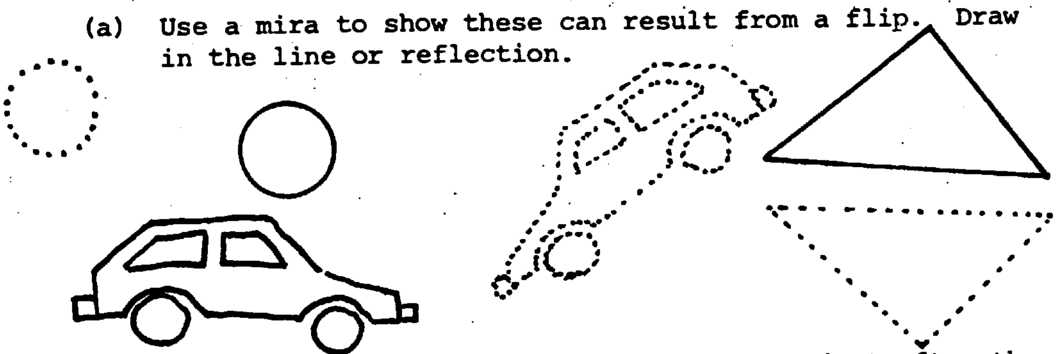


These are not flips (what are they)?

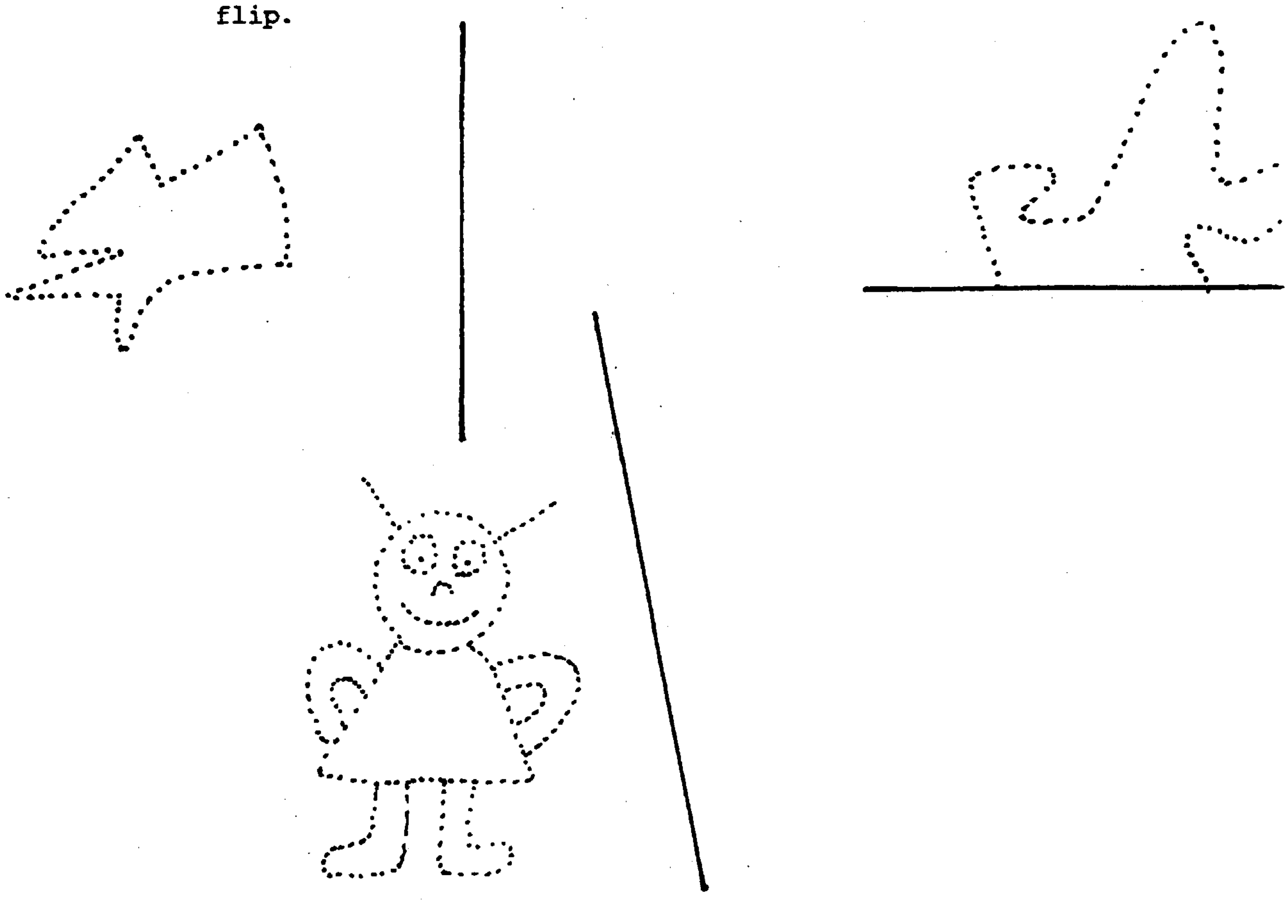
Describe in your own words what a flip is.

(2) Flips can be practiced using paper folding or mira mirrors.

(a) Use a mira to show these can result from a flip. Draw in the line or reflection.



(b) Use your mira to draw the image of the object after the flip.



11. Euclidean Invariants

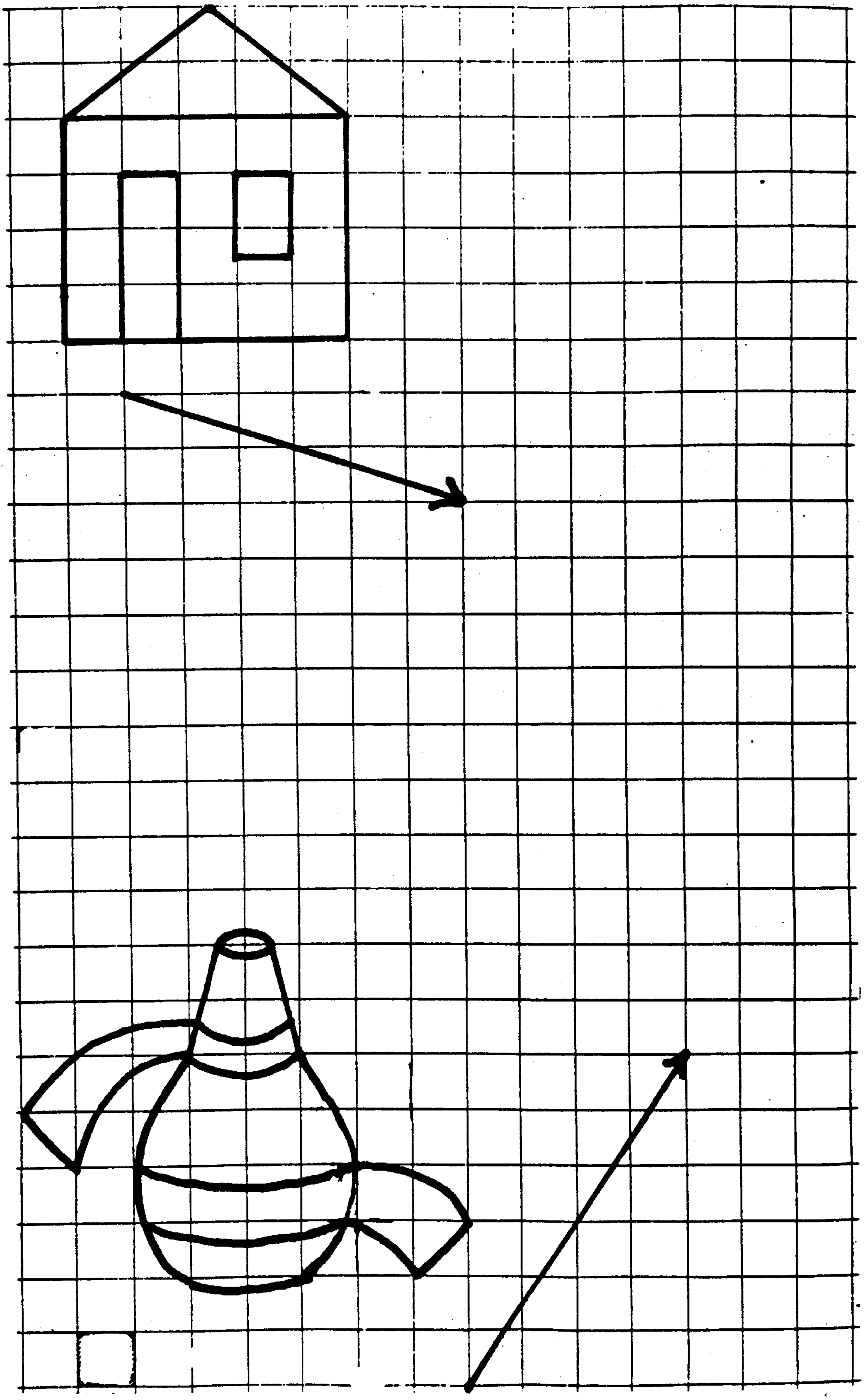
Tick the boxes in the table for which the property holds true.

PROPERTY	SLIDE	FLIP	TURN
Moves every point in the plane			
Moves every point in the plan except one			
Moves every point in the plane except a line			
Moves points along concentric arcs through an equal angle			
Moves points equal distance along parallel lines			
Moves points along parallel lines but not necessarily the same distance			
Does not change size			
Does not change shape			
Changes left right orientation			
Produces a congruent shape			

Teaching Hints:

The activities in this unit should be given only with great care to children. They were designed so that you the reader, through doing them, will become more able to comprehend the three different transformations. Only those which in your professional opinion could be completed by your class (for their benefit) should be used.

Slide Exercises



UNIT 26: TRANSFORMATION ACTIVITIES

Focus:

The three transformations (topological, projective and euclidean) give rise to many unique and interesting activities. This unit exists to display some of them.

Background:

The activities, except for perspective drawings, come from toppological and euclidean change. There is a plethora of such activities in enrichment and puzzle books.

Please note that, from previous units, networks is a topological activity and shape puzzles and tessellations are a part of euclidean change.

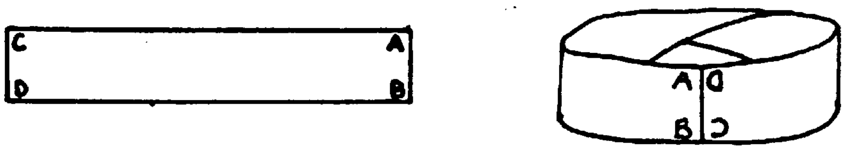
Materials:

Pen, paper, graph paper, coloured pencils or pens, mira.

Activities:

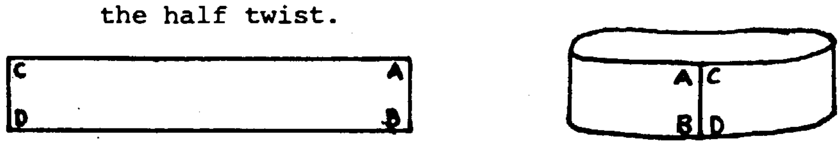
1. Moebius Strip and Caliph's Puzzle (a topological activity)

   To make a <u>moebius strip</u>, cut a piece of paper about 3cm by 3cm give the strip a half twist, and tape the ends together as below.



moebius strip

   To make an ordinary circular strip or cylinder tape the ends without the half twist.

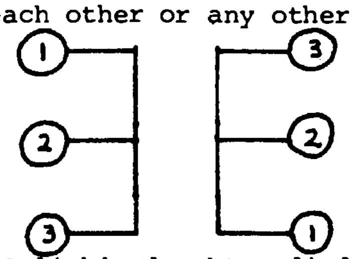


cylinder

   (1) How many surfaces has the cylinder? The Moebius strip? (E.g., if you wanted to paint the cylinder you would find an inside and an outside. How many colours would you need if each surface was to be painted a different colour? How many colours would you need for the Moebius strip?)

   (2) Cut the cylinder lengthwise along a line in the centre of the strip. What happens?
   Cut the Moebius strip similarly. What happens? Repeat this for cylinder and Moebius strip but lengthwise along a line 1/3 of the way in from the edge. What do you get?

(3) Caliph's puzzle: To select a husband for his daughter the Caliph declared that anyone who could solve the following puzzle will be able to marry his daughter. (Heaven knows what would have happened if a woman solved the puzzle!) The puzzle was to connect like numbers in the figure below with lines that do not intersect each other or any other line.



It is said that the Caliph's daughter died unmarried .. What do you think? If the puzzle was drawn on a Moebius strip, would it make a difference?
(Hint: Remember a Moebius strip is only a point thick and you can approach the puzzle from both sides.)

2. The four colour problem (a topological activity)

This famous problem is based around how many colours are needed to colour in a map of a country so that each state is a different colour from every other it shares a border with. Four colours are needed for examples like

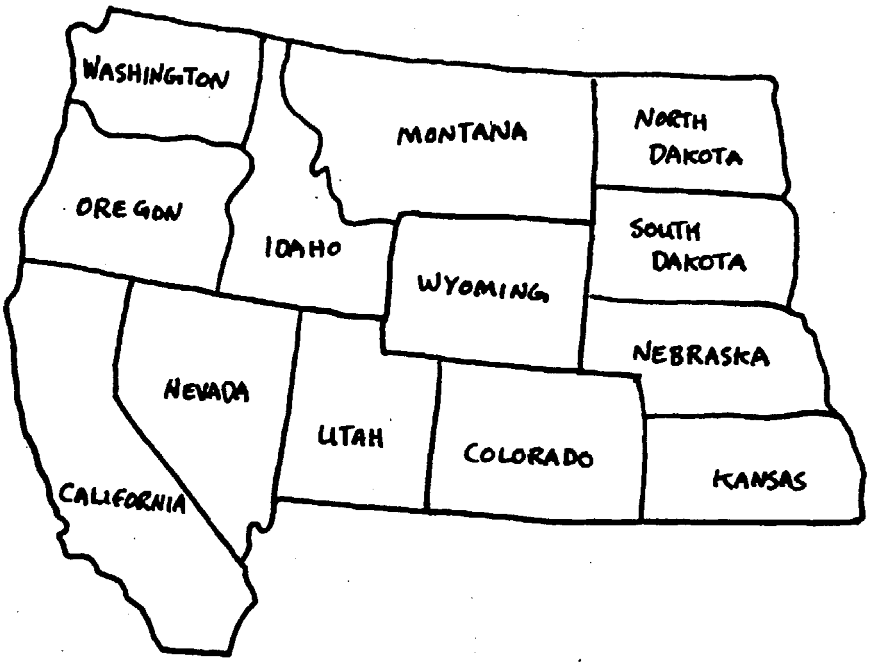
(1) Find some maps and colour the states so that no two states with a common border are the same colour. Can you use only 4 colours? (You should not need 5).

(2) Game: Polychrome

Number of players: 2 Number of colours: 1, 2, or 3 (you choose)

Playing Board: A copy of a map (e.g., a larger version of the one below or one out of an atlas or make up your own).

Rules:
* Players choose the number of colours to be used in game and determine (flip a coin) who goes first.
* Players, in turn, choose any of the colours and shade any region (or state) in the map. (The colours are not assigned to particular players. Any player can use any colour in any order and over and over again if wished.)
* Players can not shade adjacent countries the same colour.
* The winner is the player who is able to shade the last country or region - i.e., who is able to stymie his or her opponent.



Questions: Suppose in a game of polychrome, two colours were chosen and the game went:

	Player A	Player B
1st play	Chooses green and colours Idaho	
2nd play		Chooses red and colours Washington
3rd play	Chooses green and colours North Dakota	
4th play		Chooses green and colours Kansas
5th play	Chooses red and colours California	

Player B can now win if he chooses the correct colour and uses it to colour the correct state.

(1) What colour must he choose (red or green)?

(2) What state must be coloured?

3. Topological games

Play the following two games.

(a) <u>Dotty</u>

2 players. *Gameboard consisting of a 4 x 4 array of dots.*

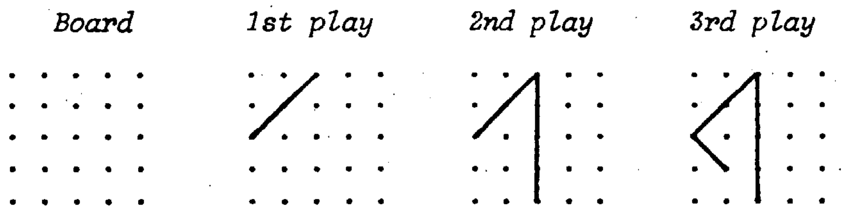
Rules: * *Take turns connecting as many dots as you wish with a* <u>*straight line*</u> *which starts from one of the ends of the previous lines.*

* *The resulting path of straight lines must not have branches, must not have gaps and must not be branched.*

* *No dot can be visited twice.*

* *The winner is the player who draws the last permissible line.*

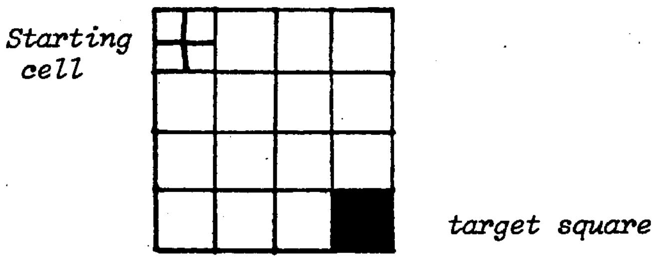
Examples of the first plays of a game:



(b) <u>Black</u>

2 players. *Gameboard consisting of a 4 x 4, or 5 x 5 or .... up to a 8 x 8 grid of squares (e.g., 2cm graph paper).*

Rules: * *Starting position as below*



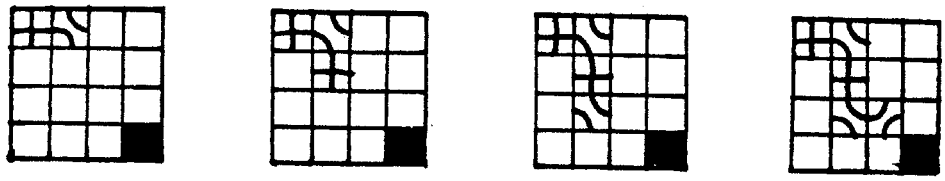
* *Players in turn mark in their choice of one of the three drawings below in a square next to where a drawing already is and so that there is an unbroken line from the starting cell*

* *A square can not be marked so that this line is broken, or so that the line is taken to the edge of the grid.*

* *The winner is the last player to make a correct play or the player who joins the unbroken line to the target square.*

Examples of the first plays of a game:

1st play *2nd play* *3rd play* *4th play*

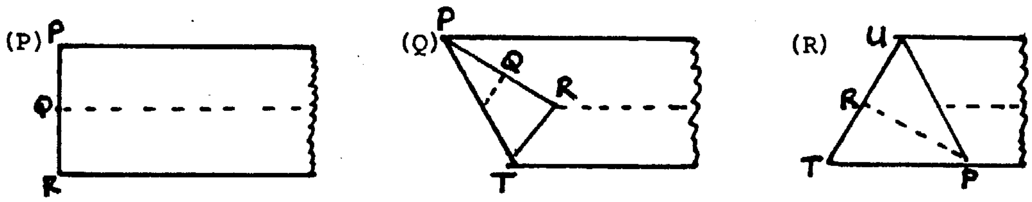


4. Flexagons (a topological activity)

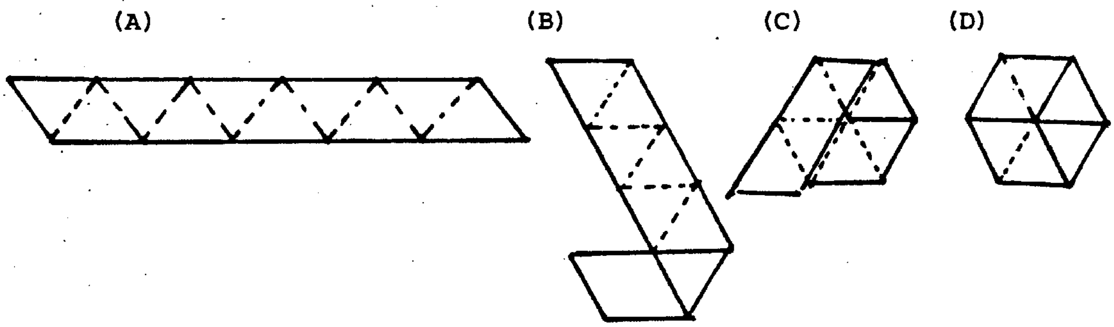
These are paper polygons that are made from straight or crooked strips of paper which have the remarkable property of changing their faces when they are "flexed".

(1) Construct a hexaflexagon as below:

Cut a strip of paper which is at least 6 times its width in length. Then fold this strip to locate the centre line QS (P). Then fold the strip so that R falls on this centre line (Q). Then fold again so that P lies on opposite edge (R). Cut along UP.



Then mark the strip of paper by folding into 10 equilateral triangles (A). The strip is folded backward along the line AB and turned over (B). It is then folded backward along the line CD and the next to the last triangle and glued to the other side of the first.



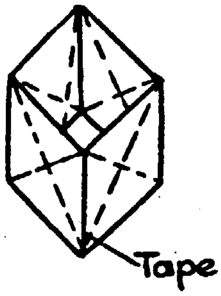
(2) Pinch 2 adjacent triangles together and push the opposite corner of the hexagon towards the centre. The hexagon will open out again, like a budding flower and show a new face.

(3) Colour the opposite faces of the hexagon different colours.

(4) Fold and open the hexagon and observe the new faces formed. How many different designs are possible? Draw and label them.
HINT: Number the equilateral triangles in (A) (both sides) from 1 to 18 (don't number the two triangular faces which are glued together) and use this in your investigation.

(5) Flexatube - another type of flexing paper model.

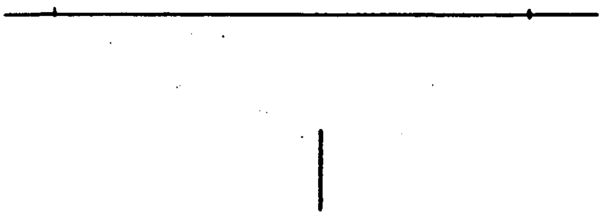
One form is made from a strip of 4 squares, each of which is ruled into 4 triangles. Crease back and forth on all lines and tape the ends to form a tube. The problem is to turn the tube inside out.



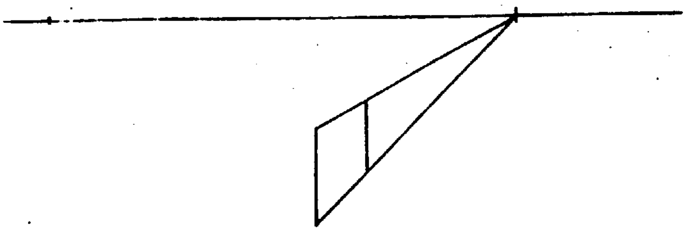
5. Perspective Drawings (a projective activity)

We will now draw a cuboid in perspective with only 2 horizontal points of perspective.

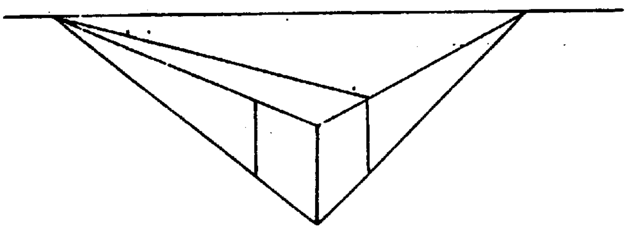
(1) First draw a horizon, and mark 2 vanishing points or points of perspective on it. Draw the front edge of the cuboid.



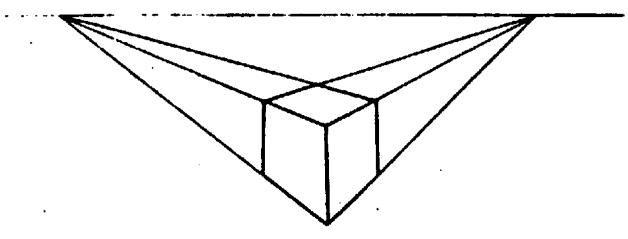
(2) Then draw lines towards one vanishing point, and put in a vertical line to complete one face. Draw all the lines lightly.



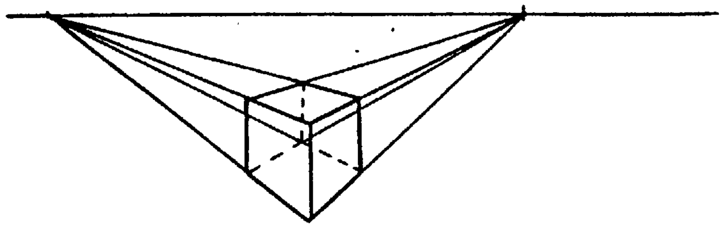
(3) Now draw the edges of the top and bottom faces towards the other vanishing point. Put in a vertical line to complete a second face.



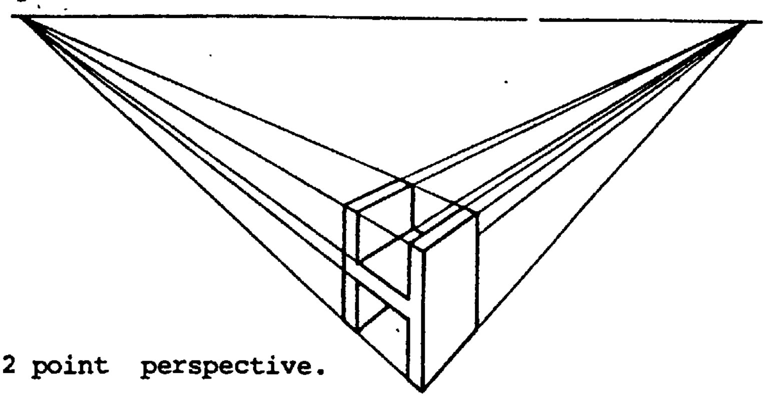
(4) Complete the diagram with a line to the first vanishing point, and draw the edges in with firm lines.



(5) If we want to include the hidden edges in our drawing we need to draw two further lines to the vanishing points.

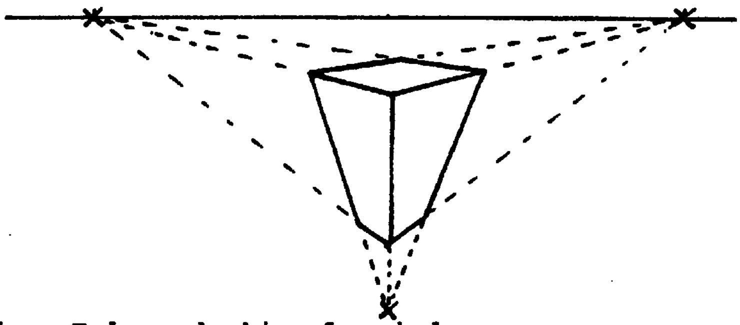


(6) A further example is this H

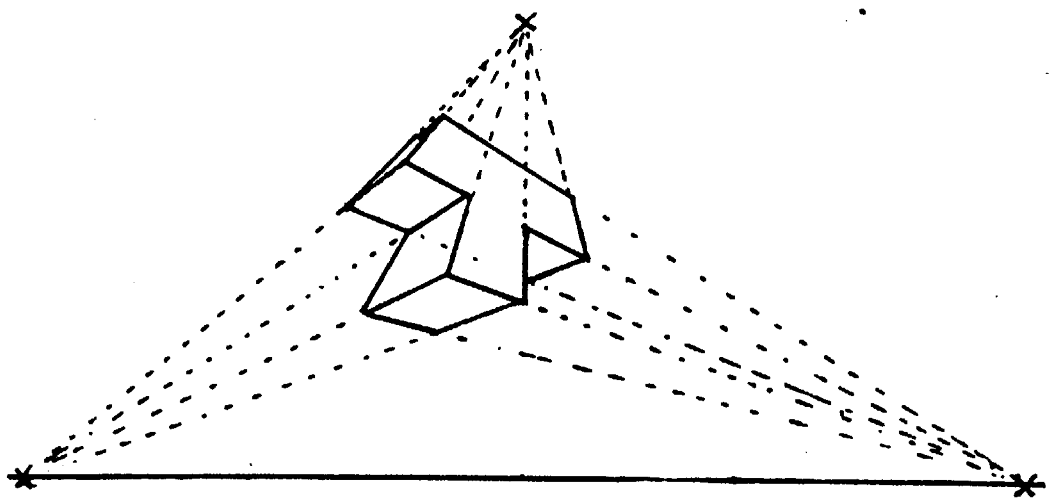


(7) Draw an N in 2 point perspective.

(8) For experts, a third point of perspective can be added vertically below the other two.

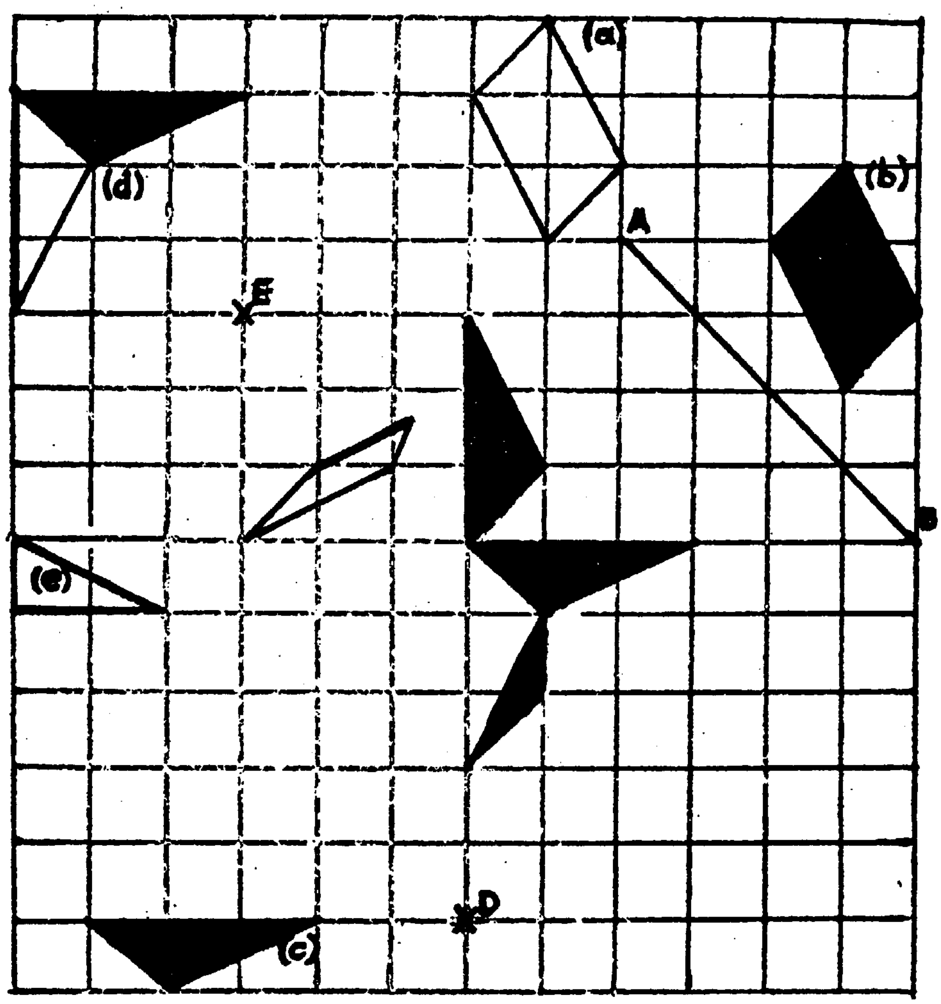


Here is a T drawn looking from below



(9) Draw a F in 3 point perspective looking from above and an L in 3 point perspective looking from below.

6. Slides, flips and turns (an euclidean activity)



(1) Make a copy of all the various bits and pieces above on a piece of centimetre graph paper, then carry out the following instructions.

   (i) Slide the parallelogram (a) 3cm towards the bottom of the paper.
   (ii) Reflect parallelogram (b) in the mirror line AB.
   (iii) Turn triangle (c) a quarter turn clockwise about the centre D.
   (iv) Turn shape (d) a half turn about point E.
   (v) Slide line (e) 3cm to the right.

(2) You can make up your own pictures by drawing a complete shape (very lightly) in the centre of the sheet, move pieces by slides, flips and turns, then write instructions for others to solve. Try it!

7. Using Mira flips to construct geometric figures (an euclidean activity)

(1) Stand a mira on paper. Mark point A on the mira line and point B in front of the mira. Mark B's image and label it C. Remove mira and joint A, B and C to make an isosceles triangle.

(2) Draw a line segment called PQ on paper. Use the mira to draw its perpendicular bisector M. Place pencil at Q and mira against the pencil point. Rotate the mira until the image of P is on M. Mark this point and call it R. Joint P, Q and R to make an equilateral triangle.

(3) Make points A and B infront of the mira. Mark in their images C and D. Join A, B, C and D to make an isosceles trapezium.

(4) Mark two points A and C on the mira line. Mark B in front of the mira and mark its image. Label it D. Join A, B, C and D to make a kite.

8. Try some of the position and movement puzzles at the end of this unit (euclidean activities mostly).

Teaching Hints:

Much of what is in this unit may be considered enrichment and so it is. But much (e.g., perspective drawings) has been at the basis of our societies image of itself. Do not dismiss it. It can be a source of amusement and discovery and a keyhole through which to look at our society.

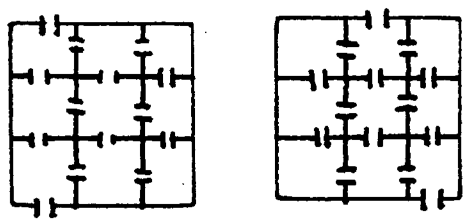
Find a history book and look up the history of, e.g., moebius strips, networks, perspective and the 4 colour problem.

Puzzles of Position and Movement:

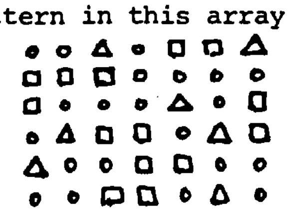
There are many examples of puzzles and problems with position and movement as their basis. We include a few below.

1. Plant 24 trees in such a manner that 28 rows of 4 trees are formed.

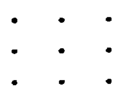
2. Can you walk through the houses on the right going from door A to door B and walking through each room once and only once.



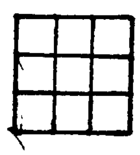
3. Find the pattern in this array (don't give up easily).



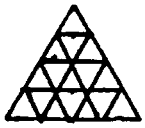
4. Draw 4 connected straight lines that pass through all 9 points.



5. How many squares?

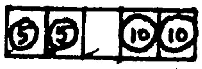


6. How many 10c pieces can you arrange so that each 10c piece touches all other 10c pieces.   e.g. 

7. How many triangles? 

8. Interchange adjacent coins to make the following changes.

HTHTH	to	HHHTT
HTHTHTHT	to	TTTHHHH

9. Change  to 

by moving 10c to right, 5c to left with jumps allowed over single coins to vacant space. Try 7 squares and three 10c pieces and three 5c pieces.

10. Arrange 4 coins on a 4 x 4 grid so that <u>no</u> 2 coins are in the same row or column or in the same diagonal line.

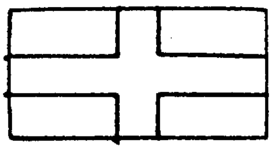
11. How many different ways can a die be marked with 1, 2, 3, 4, 5 and 6.

12. The big bear has the middle sized bowl of porridge, the middle bear the smallest bowl and the smallest bear the largest bowl. How can they get the right bowl by passing only one bowl at a time, passing the larger of 2 bowls held and not passing a bowl to a bear already holding a larger bowl?

13. 3 missionaries and 3 cannibals at a river have a boat to carry only 2 across. How can they ferry all 6 when the cannibals can never be allowed to outnumber the missionaries at any side.

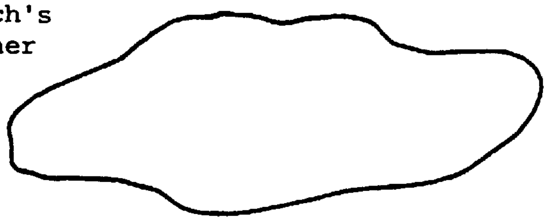
14. 4 soldiers, 2 boys at a river with a boat capable of holding only one soldier or 2 boys but not 2 soldiers or a soldier and a boy. How did they get across.

15. Three cloths each 1m x 1m. What is the largest <u>square</u> table you could cover? Overlaps are allowed by not cutting or folding or trickery.

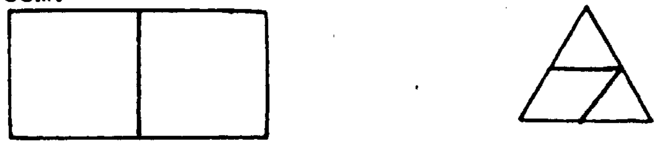
16. 

What thickness for a cross which is the same area as the background?

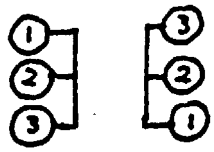
17. Can 4 sons divide this land so each's land has a boundary with every other son? Can 5 sons?



18. Can you start in a room, cross each wall once and return to your starting room.



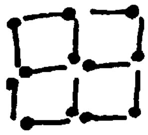
19. Caliph's puzzle: Can 1 be joined to 1, 2 to 2 and 3 to 3 without any lines crossing?



20. Matchstick puzzles

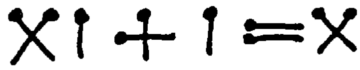
(1) Make 2 triangles with 5 matches.

(2) 12 matches: Rearrange by moving 3 matches to get 3 squares (same size).

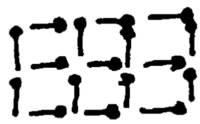


(3) Move 1 match to get a true equation.

(4) Correct this one without touching anything.



(5) 17 matches - remove 6 to leave 2 squares,
- remove 5 matches to leave 3 squares the same shape and size as the original 6.



(6) Remove 4 matches to leave only equilateral triangles.

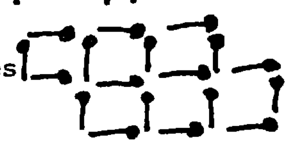
(7) Move 2 matches to form true equation.

(8) Make 5 triangles with 9 matches.

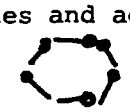
(9) Move 1 match to form a true equation.

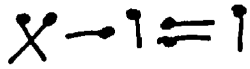
(10) Move 1 match to form a true equation

(11) The matches form 6 squares. Remove only 2 of these matches in order to form 4 squares of the same size.

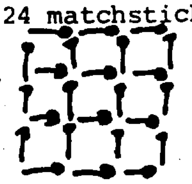


(12) Move 2 of these matches and add 1 more to form 2 diamonds.



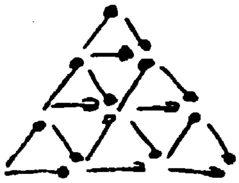
(13) Move 1 match to form a true equation. 

(14) Construct figure below using 24 matchsticks.

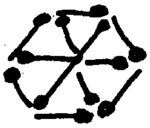


- (a) How many squares are in the given arrangement?
- (b) Remove 10 matchsticks from the design to leave 2 squares.
- (c) Remove 6 matchsticks from the design to leave 3 squares.
- (d) Take away 4 matchsticks to leave 5 squares.

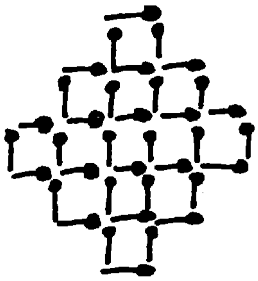
(15) Take 2 matchsticks away from the trianqle figure so that there are 5 triangles remaining.



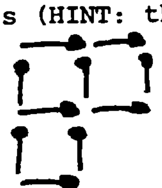
(16) Move 3 matchsticks so that 7 quadrilaterals are formed in figure.



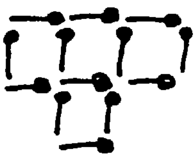
(17) Remove 4 matchsticks from the design in figure leaving 9 squares.



(18) The 10 matches form 3 squares. By moving only 2 matches, form 2 squares (HINT: the 2 squares are not the same size).



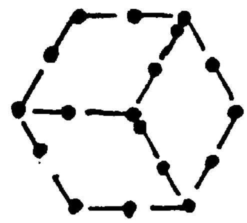
(19) The 12 matches form 3 squares - by moving 3 matches obtain 5 squares.



(20) The 5-match Puzzle.
We may arrange 5 matches to give an expression for 3 as follows:
Arrange 5 matches to show each of: 0. 12, 11, 4, and 1.

(21) Further match puzzles

(a) Use nine matches to make ten.
(b) Use six matches to make nothing.
(c) Use eleven matches to make nine.
(d) Use ten matches to make five.
(e) Form 17 matches into six squares. Take away 5 matches to leave only 3 squares of the same size.
(f) These 18 matches form a hexagon divided into congruent regions. Move just 4 matches to divide the hexagon into only two congruent regions.



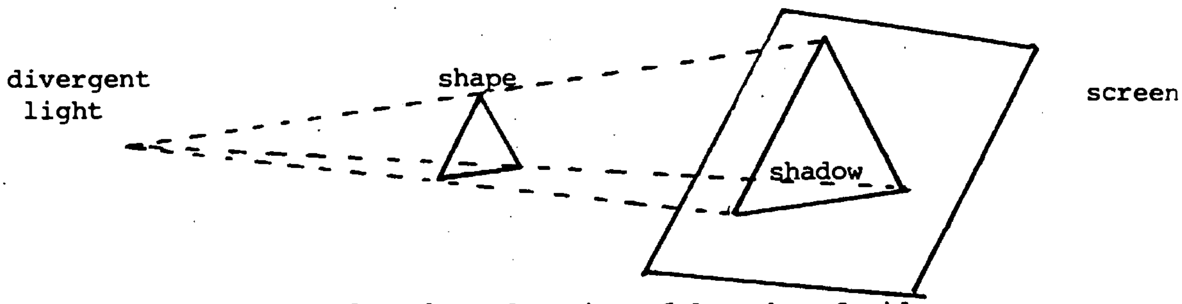
UNIT 27: SIMILARITY

Focus:

In this unit we discuss the concept of similarity and how it may be developed in children. Its application to two dimensional shape is also discussed.

Background:

Similarity is based on a particular projection - the projection in divergent light where shape and shadow (i.e., the screen) are held parallel, e.g.



for this projection, measure of angle and ratios of lengths of sides are unchanged.

Similarity projections are therefore enlargements or reductions - shape is not changed.

Two shapes are similar if one is such an enlargement of the other. It is sufficient for their corresponding angles to be the same or the lengths of corresponding sides to be in ratio for this to be so. For triangles, two are similar if:

(a) all lengths of corresponding sides are in the same ratio;
(b) all corresponding angles are the same;
(c) two angles are the same (hence the third is - add to 180°); or
(d) two sides are in ratio and the enclosed angle is equal.

Materials:

String, cardboard (or wooden) shapes, tape, hook, metre rule, pencil, coloured pen, protractor.

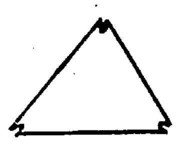
Activities:

1. Darken the room. Use a torch or candle or cast shadows of cardboard triangles, squares, rectangles and circles on a screen. Keep screen and shape parallel.

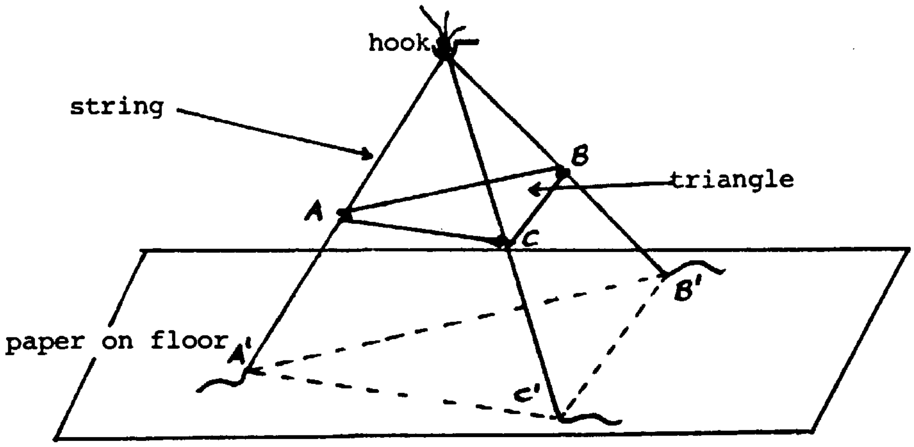
2. String projection.

   (1) Obtain three pieces of string, knot together at one end.

   (2) Cut a trianlge out of carboard with notches at each corner as below.



(3) Connect the knotted end of the strings to a hook or nail on the wall or on a whelf. Hold the notched triangle horizontally below this. Stretch the strings tightly through the notches. Place a large piece of paper on floor and hold the ends of the strings against it as below.



Note: The strings must be straight. The triangle must be held horizontal. The strings must pass through where the corners of the triangle would be. The best way to do this is for one person to hold the triangle flat and the others to hold the strings on the paper on the floor - then to slowly move hands so that string, staying straight, moves into the notches.

(4) When strings are straight and through the notches, mark on the paper on the floor where the string ends. Connect these points to form a triangle. This triangle is an enlargement of the notched triangle. It shows similarity.

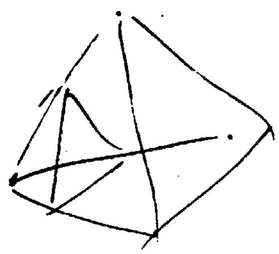
(5) To discover the properties of similar triangles, use a ruler and protractor to complete this table.

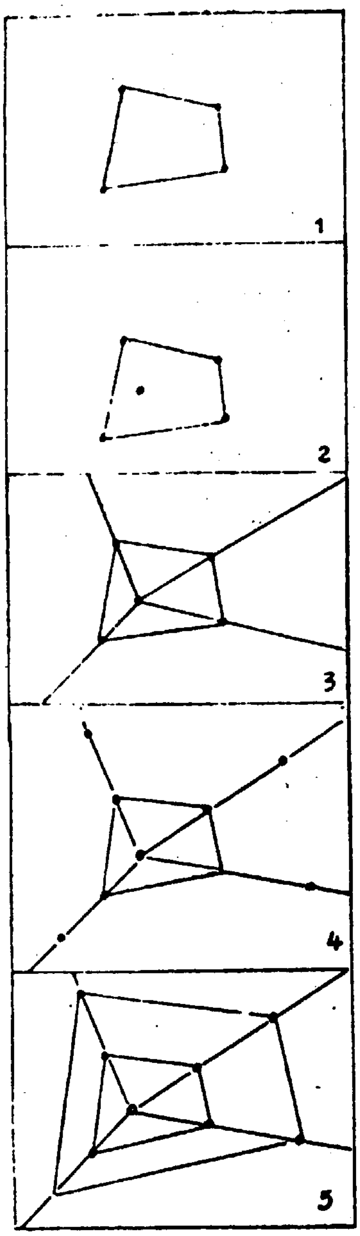
TRIANGLE ABC	TRIANGLE $A^1B^1C^1$	RATIO
AB =	A^1B^1 =	AB/A^1B^1 =
AC =	A^1C^1 =	AC/A^1C^1 =
BAC =	$B^1A^1C^1$ =	
ABC =	$A^1B^1C^1$ =	
ACB =	$A^1C^1B^1$ =	

(6) Repeat this activity for a quadrilateral (4 strings needed) and a pentagon (5 strings).

3. Enlarging with Pen and Paper

This method of deriving similar shapes is like part of a script for making a piece of animated film to illustrate enlargement. In the example below the shape is enlarged by a factor of 2.





The film starts with an irregular quadrilateral on a plain background.

A dot appears inside the quadrilateral and moves around in random fashion. It stops somewhere.

Lines grow out from the dot through each of the four vertices and disappear off the edge of the screen.

Dots leave each of the vertices and move along the coloured lines. Each dot moves a distance equal to the distance from the centre dot to each particular vertex. (Note: enlargements with factors of 3 or 4 are made by having a distance 2 or 3 times the distance to the centre dot.)

Four straight lines join up the four dots and the enlargement is complete.

Children can do examples where the dot in the second picture is outside the shape or lies on one of the edges of the shape or on one of the vertices. In these cases: the method as above will still produce a similar shape (but this similar shape will not be 'around' or 'enclose' the original shape, as it does above).

Complete, using this method,

(1) an enlargement of a triangle where the dot is inside the polygon but the enlargement is by a factor of three (3 times large).

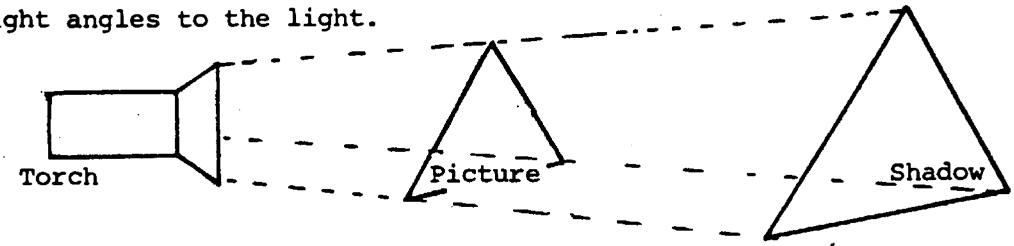
(2) an enlargement of a pentagon (by a factor of 2) where the dot is outside the pentagon.

4. Under the direction of your instructor, find the height of a tree or building outside using ways 4, 5 or 6 of unit 24, activity 3.

Teaching Hints:

Similarity means the same shape. Similarity should be understood in terms of processes that produce a same shape such as enlargement (or contraction), not in terms of equal angles and sides in ratio (which are properties to be later derived).

One activity to introduce similarity is to use torchlight in a darkened room and cast shadows with a figure held parallel to the screen, e.g., at right angles to the light.



If a wide variety of shapes is used, then a real feeling for similarity can develop.
(Note: A candle will do instead of a torch. And the torch works best when the reflecting mirror and focussing glass have been removed).

Another method is to use string as described in the activities. Again, if a wide variety of shapes is used, then a good grasp of similarity will come.

To develop properties of similar shapes (e.g., triangles), similar triangles can be labelled A, B, C, and A^I, B^I, C^I. The sides and angles can be measured. A table such as the one in the string projection activity can be filled in. If the children's work has been accurate enough, they will discover (for the triangle and any other polygon used) that shapes are similar if:

(i) angles are the same
(ii) corresponding sides are in the same ratio.

But the important thing is to give children variety of experiences in enlarging shapes (e.g., overhead projector) so that an intuitive grasp of similarity is achieved.

UNIT 28: CONGRUENCE

Focus:

In this unit we discuss how the concept of congruence can be developed in children looking particularly at its application to two dimensional shape.

Background:

Congruence can also be represented by a projection (a projection of parallel light where shape and shadow (i.e., the screen) are held parallel), but it is mainly a Euclidean concept. We say that, two shapes are congruent if they are the same size and shape, i.e., they differ only in position and orientation. This occurs when one shape can be changed to the other by flips, slides and turns only i.e., a copy of one shape can be moved and placed exactly on top of the other.

Congruence occurs when corresponding angles and corresponding sides are equal. For triangles, two are congruent if:

(1) all corresponding sides are equal;

(2) two sides are equal and the angle between them is the same; or

(3) two angles are equal and the side between them is the same length.

Materials:

Pen, paper, cardboard, scissors.

Activities:

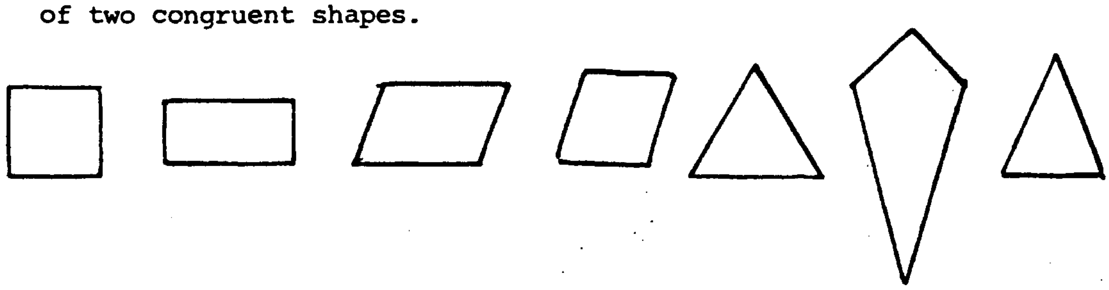
1. Draw a shape. Cut out a copy of it. Slide this shape, turn it and flip it. Draw around it. The result is congruent to the first shape. What differs?

2. An isosceles triangle has two congruent sides and two congruent angles. What is congruent about

   (a) a rectangle
   (b) a parallelogram
   (c) an equilateral triangle.

3. What properties of sides and angles are sufficient for two quadrilaterals to be congruent (Extend the list given for congruent triangles in background).

4. (1) Draw one straight line across each shape to show it is made up of two congruent shapes.



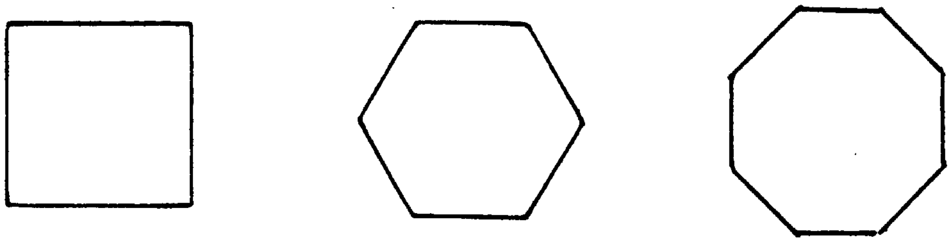
(2) Draw two straight lines in each shape to show it is made up of three congruent shapes.

(3) Draw the number of lines shown across these shapes from corner to corner to divide them into congruent parts.

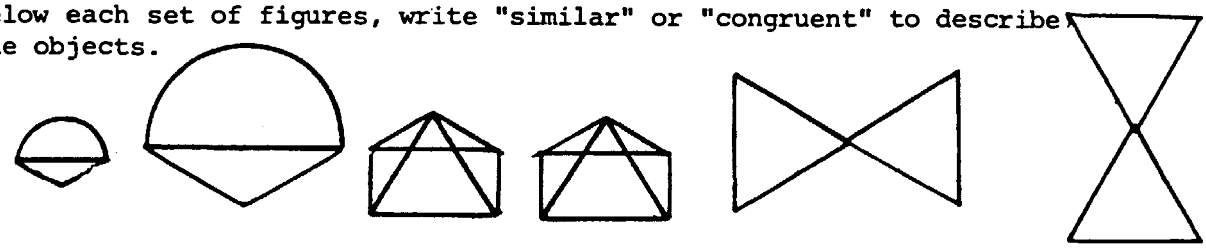
2 lines across and 4 parts the same

3 lines across and 6 parts the same

4 lines across and 8 parts the same



5. Below each set of figures, write "similar" or "congruent" to describe the objects.



6. (1) Construct two congruent shapes (draw around a cardboard copy of one shape to make the second). Ensure that the shapes differs by a rotation. Use a mira (or tracing paper) to shown that the first shape can be changed to the second by a series of reflections or flips.

(2) How many reflections did you have to do? (You should have used no more than 3).

7. (1) Draw a triangle. Rotate or turn this triangle through 60° and draw the result. Using a mira (or paper folding), show that two flips or reflections will change the first triangle to the second (will be equivalent to the turn).

(2) Check that the lines of reflection of the two flips pass through the centre of the turn.

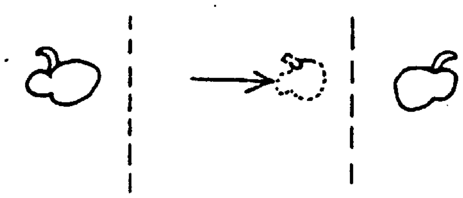
(3) Check that the angle between the two lines of reflection is 30°.

(4) Do you think that this relation between lines of reflection and centre and angle of turn holds for any turn?

<u>Teaching Hints</u>:

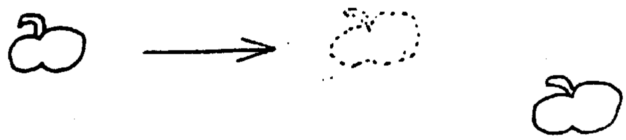
A <u>flip</u> is a reflection about a line, as in a mirror or mira.

e.g.



A slide is a change in position (movement) without change in orientation (translation is its other name).

e.g.



A turn is a rotation about a point (the centre).

e.g.

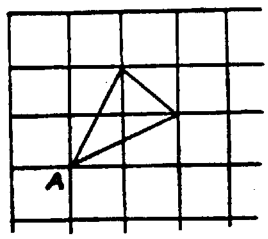
An object changed only by flips, slides and turns remains the same size and shape although its orientation and position may change. Two shapes that are the same size and shape are said to be congruent. Hence two objects can be said to be congruent when one is transformed to the other by a combination of flips, slides and turns.

Children can flip, slide and turn cardboard cutouts of shapes. They can use graph paper to flip, slide and turn shapes as below.

Example: (a) Cut out a piece of cm graph paper which is 10 x 10.

(b) Draw a triangle, similar to that shown in the top left hand corner of the piece of graph paper. Vertices should be on co-ordinate points.

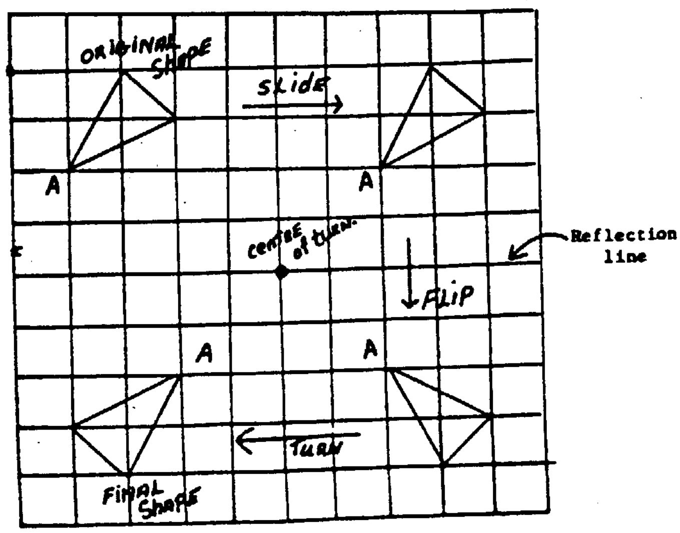
Label the lower corner A (The triangle does NOT have to be the same as that shown).



(c) Slide this shape 6cm to the right, similar to shown below.

(d) Reflect figure about reflection line, similar to shown on right.

(e) Using tracing paper, turn the shape into the vacant lower left corner, similar to shown on right.



(Place tracing paper over graph paper. Copy shape. Hold tracing paper at centre using pencil or compass point. Turn tracing paper clockwise. Retrace shape onto graph paper - press heavily on tracing paper drawing of shape to transfer a dent to the graph paper. This final shape does NOT have to have vertices at co-ordinate points).

The children can then discover that congruent figures have equal angles and equal length sides.

It should be noted that for congruence it is not encessary to flip, slide and turn the shapes accurately. As long as the child can see that a cutout of the first shape can be placed directly over (to match exactly) the second shape, this is a good enough concept (and test) of congruency.

The important thing, is once again to enable children to gain an intuitive understanding of the concept of congruence.

UNIT 29: MODULO ART

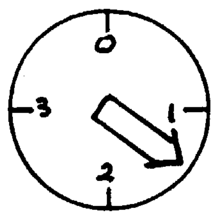
Focus:

An excellent application of flips, slides and turns is in the production of modulo art posters. This unit describes how to do these.

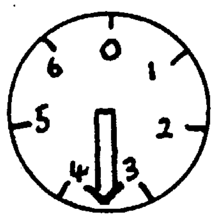
Background:

1. Modulo Arithmetic

This can be introduced to primary children by using clock faces (as below) with single hands. Numbers can be added by moving the hand spaces clockwise (and subtracted by anticlockwise). Numbers can be multiplied by moving the hand clockwise lots of spaces (and divided by seeing how many groups of movements anticlockwise will give zero). During these movements whenever the hand passes 0, the numbers begin again.



4 minute timer for modulo 4 arithmetic



7 minute timer for modulo 7 arithmetic

Any pair of integers is said to be congruent moduleo 5 if they differ by a multiple of 5. For example, 3 and 18 are congruent, modulo 5, because 3 and 18 differ by 15, which is a multiple of 5. We write this as

$$3 \equiv 18 \pmod 5$$

Examples: (i) $2 \equiv 9 \pmod 7$ since the difference between 2 and 9 is 7 and this is a multiple of 7.

(ii) $47 \equiv 11 \pmod 6$ since the difference between 47 and 11 is 36 and this is a multiple of 6.

(iii) $21 \equiv 21 \pmod{79}$ since the difference between 21 and 21 is 0 and this is a multiple of 79 (as $0 = 0 \times 79$).

The most important congruence for our purposes is the congruence (mod n) between a number and its remainder after division by n. For example:

$$21 \equiv 3 \pmod 6$$

Because 3 is the remainder after dividing 21 by 6.

Likewise, $27 \equiv 1 \pmod{13}$
and $28 \equiv 0 \pmod 7$

Using this congruence we can construct modulo addition tables and modulo multiplication tables.

Tables modulo 4 are given below.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Addition (mod 4)

x	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Multiplication (mod 4) with zero

x	1	2	3
1	1	2	3
2	2	0	2
3	3	2	1

Multiplication (mod 4) without zero

In these illustrations, the integer 1 appears in the fourth row of the grid of the addition (mod 4) table because

$$3 + 2 = 5 \equiv 1 \pmod 4$$

The integer 0 appears in the second row of the grid of the multiplication (mod 4) without zero table because

$$2 \times 2 = 4 \equiv 0 \pmod 4$$

Similarly for all other integers in the grid.

Special Note: If large multiplication tables (say mod 21) are constructed and squares containing particular numbers (say 1) are shaded, interesting patterns emerge.

2. **Generating artistic posters**

The technique for modulo art is to:

(a) begin with a modulo table as on right (addition (mod 4))

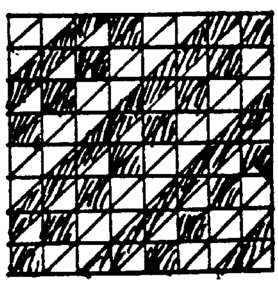
0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

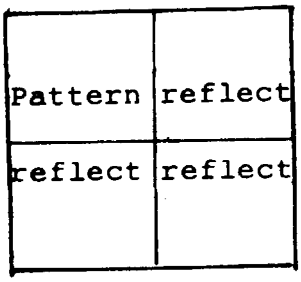
(b) replace the numbers with a simple design, shaded or coloured to taste; and

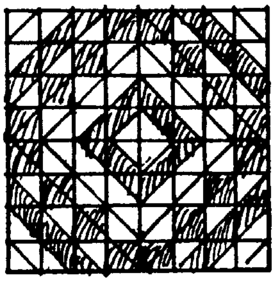
0 1 2 3

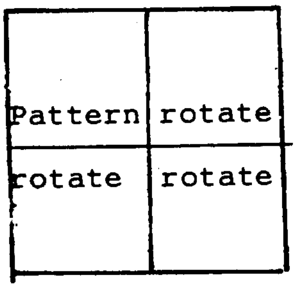
(c) then repeat, rotate or reflect the resulting pattern as below.

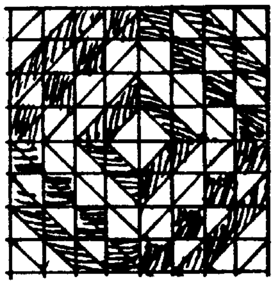
Pattern	repeat
repeat	repeat



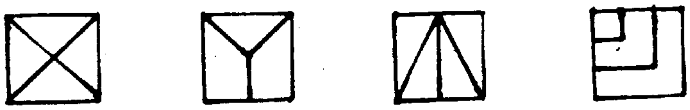




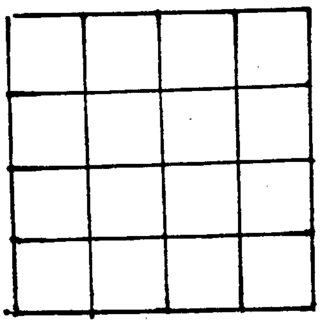




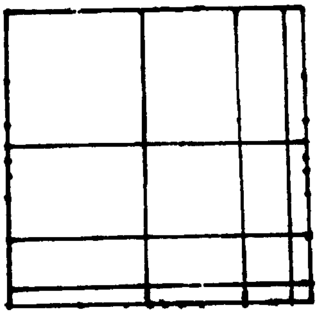
The designs can be anything imaginable

e.g. 

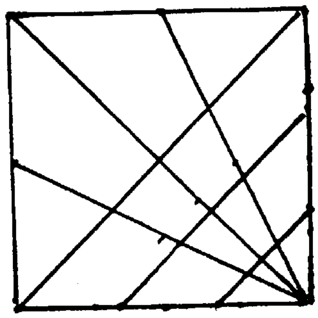
The grids can be any size (5 x 5 or 6 x 6 or whatever) and varied in shape too, e.g.



Standard



converging segment



kaleidoscopic

References:

Forseth S. and Troutman A., "Using mathematical structures to generate artistic designs", The Mathematics Teacher, May 1974, pp 393 - 397

Locke., "Residue designs" The Mathematics Teacher, March 1972, pp 260 - 263

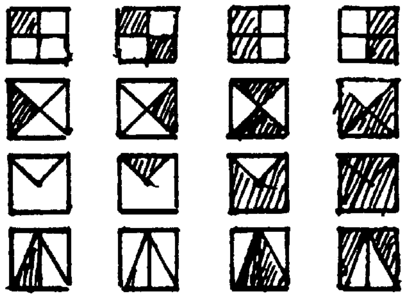
Cooper, T.J. and Watson A.J., "Modulo Art" Sigma, Nos. 90, 91, 1976

Materials: Pens, coloured pencils, ruler, tracing paper.

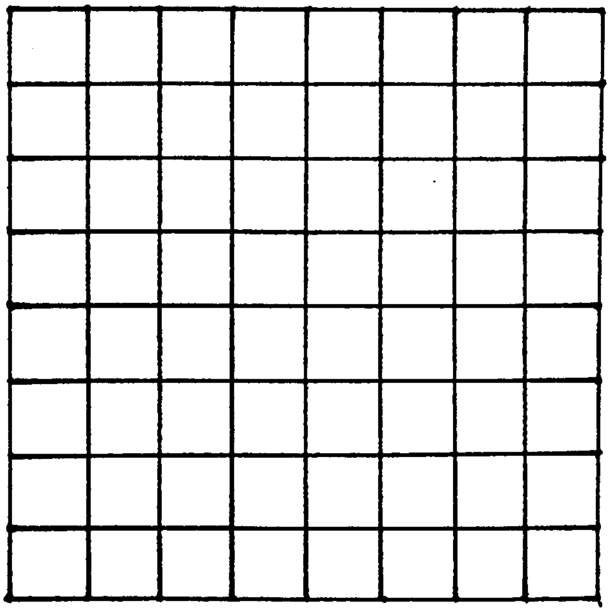
Activities:

1. Use the grid on the right and one of the basic designs on the right to make a poster. Colour as you see fit.

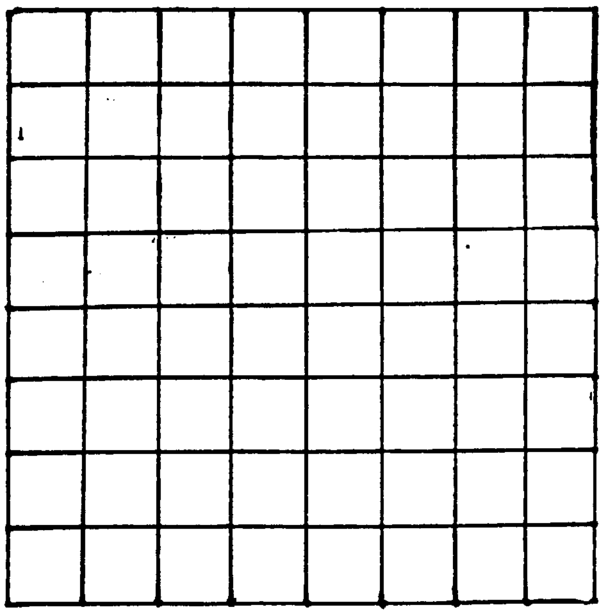
0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2



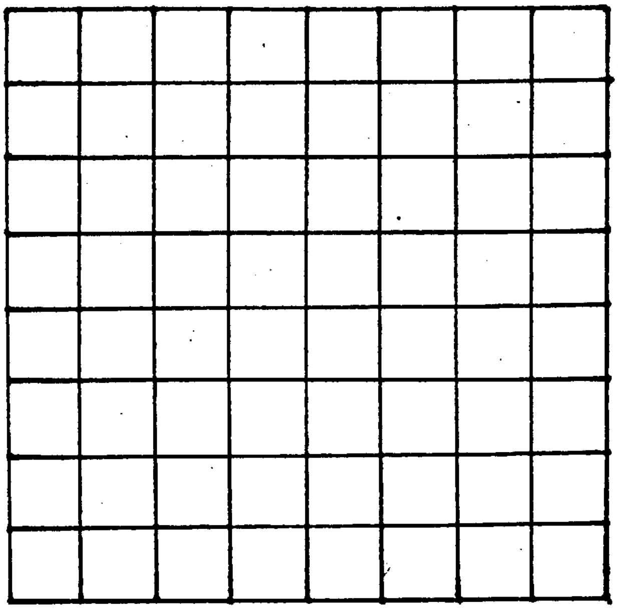
(a) a repeated poster



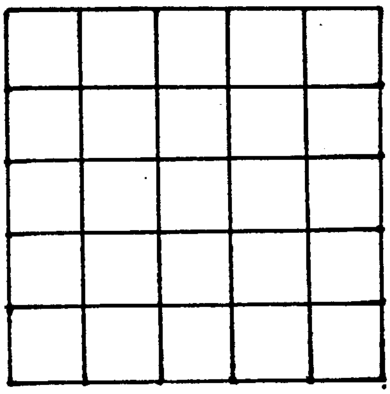
(b) reflected poster



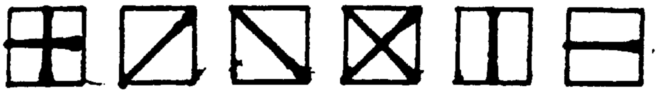
(c) rotated poster



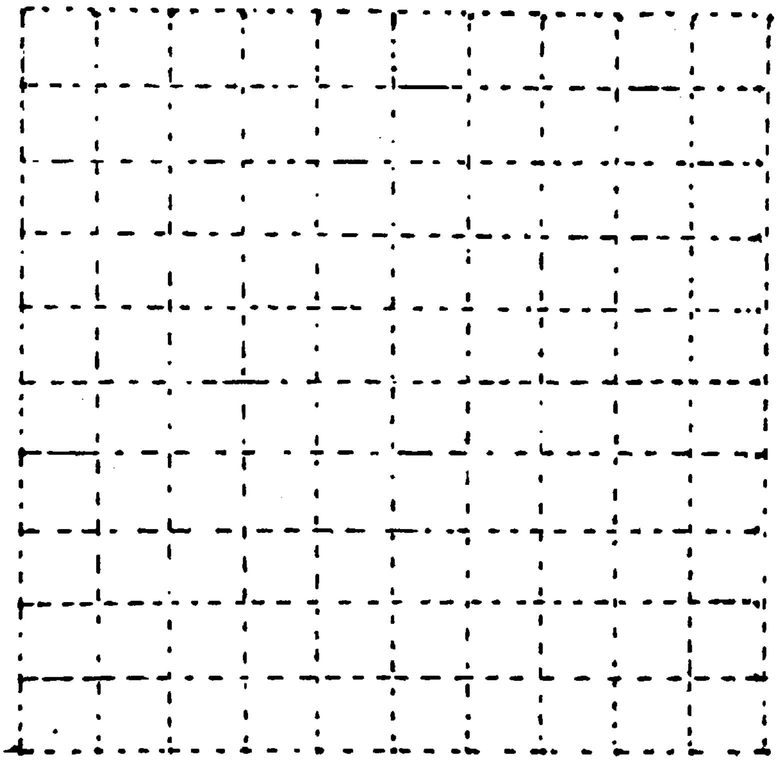
2. Construct a 5 x 5 grid from multiplication modulo 6 (without zero)



Use the following designs to make a reflected poster (use a bold bright Texta for the lines.)



0 1 2 3 4 5



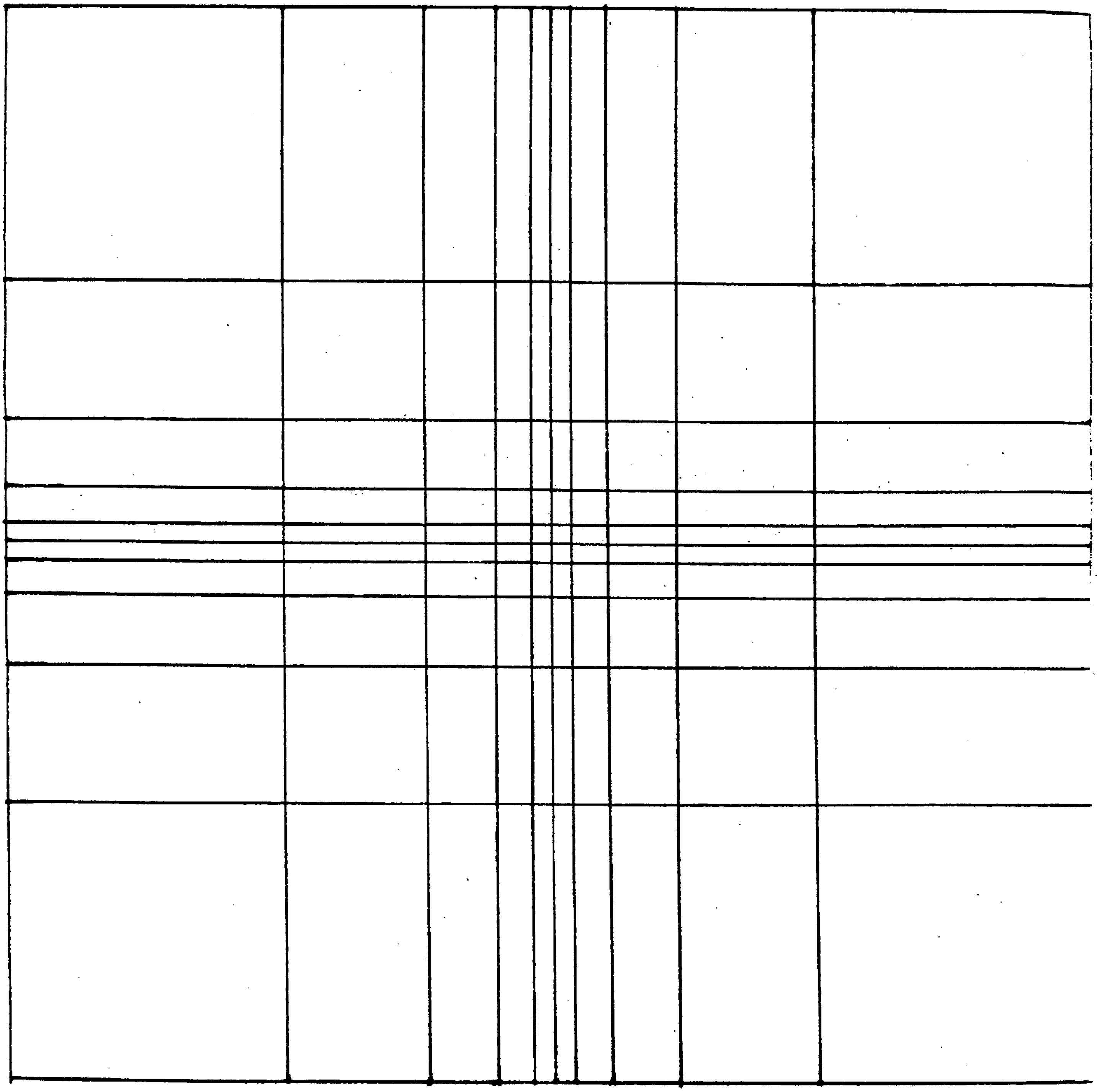
3. Construct a 5 x 5 converging segment poster and a 4 x 4 kaleidoscopic poster of your choice using the grids at the end of this unit.

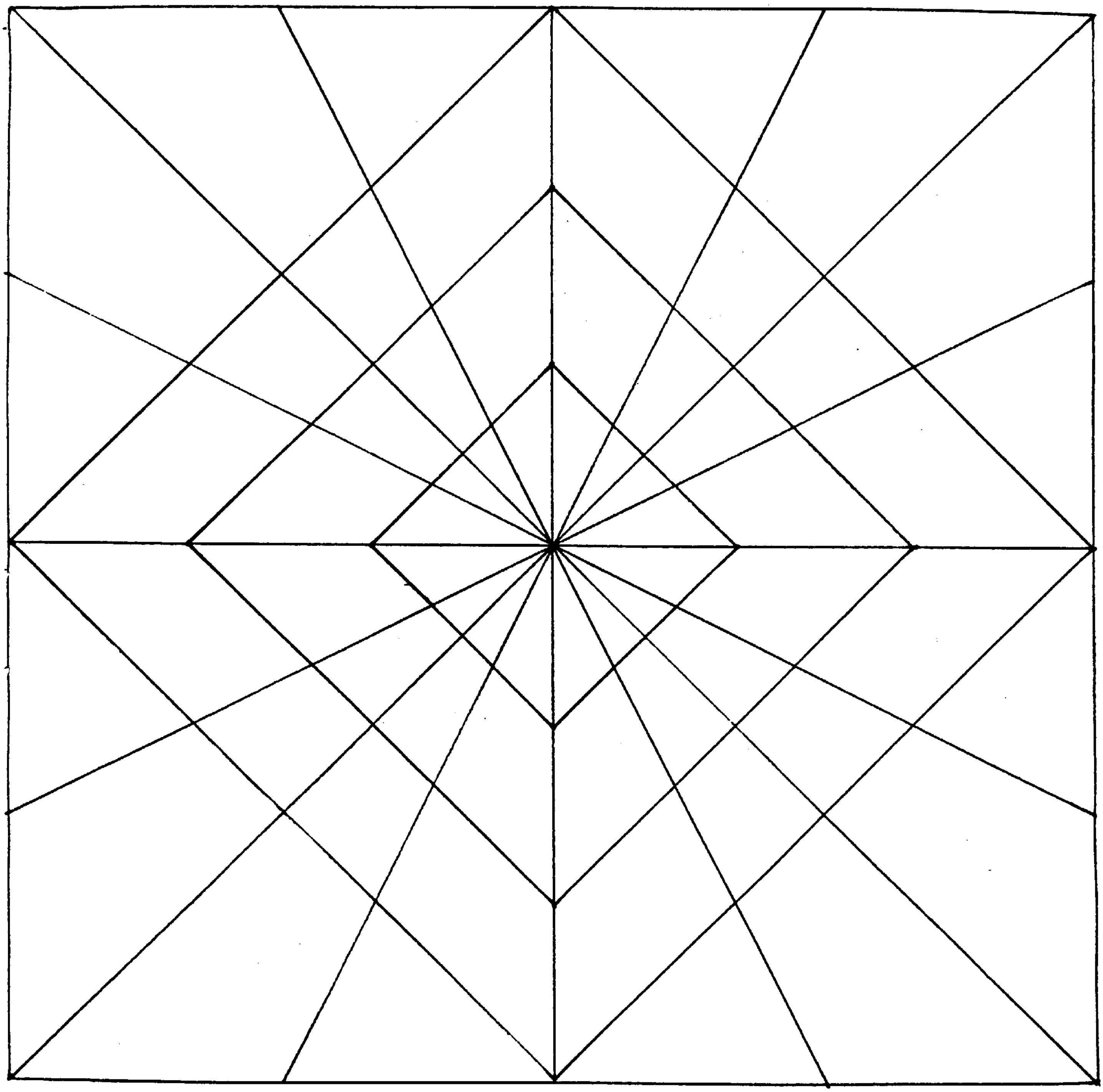
4. Construct a giant poster of your choice.

Teaching Hints:

Make a large poster a term project for your children. You will be amazed at what upper primary children can achieve?

Lower grade children can help you to make a "mural" type poster by sticking on, e.g., coloured paper under your direction.





CHAPTER EIGHT: SEQUENCING THE TEACHING OF GEOMETRY

In this chapter we consider the appropriate sequencing of geometric ideas. There is considerable disagreement in mathematics education over what should be taught in geometry and when it should be taught. This book contains a far larger range of teaching ideas than most syllabi. Both these situations make curriculum statements on geometry difficult and contentious. This chapter is therefore much more limited in scope than it should be. It is hoped that the final edition of this book will contain a much more detailed chapter on sequencing.

In unit 30, we discuss the four different approaches to teaching geometry described in the overview to this book. In unit 31, we look at how the activities in this book could be presented year by year to children using the curriculum framework from page 9 of the overview.

UNIT 30: TEACHING APPROACHES

Focus:
Many directions are possible in the development of geometric concepts. They may be abstracted from the environment or built up from other concepts. They can be integrated in with other topics. This unit focuses on how the various approaches to teaching geometry described in this book, the environmental, subconcept, thematic and transformational approaches, can be used to plan the teaching of geometry.

Background:
In pages 10 to 15 of the Overview to this book, we describe four approaches to teaching geometry.

(1) the environmental approach - intuitive experiences with shapes in the world around us leading to more formal development of language and properties (3-dimensional to 2-dimensional) and finally to applications back into the world;

(2) the subconcept approach - the building up of geometric ideas from their subconcepts and subprocesses (boundary and turn to line and angle, 2-dimensional to 3-dimensional);

(3) the thematic approach - organising learning around central themes that interconnect the various topics and concepts with an emphasis on exploration and investigation; and

(4) the transformational approach - developing geometrical ideas through three different foci on change, topological (geometry without both straightness or length), projective (geometry with straightness but no length) and enclidean (geometry with both straightness and length).

The transformational approach is further developed in chapter seven, particularly in units 25 and 26. The environmental approach was used in chapter two, unit 5 and the subconcept approach in chapter three unit 10.

Both the environmental and transformational approaches can be considered to start from the world around us. The environmental approach begins from objects in the world. We label these objects. We study their spatial properties. We learn how to move between them. As we develop more language and more understanding, our observations become deeper and more complete. We are able to abstract general rules and concepts from what we see. These rules and concepts enrich our perception and enable us to get more from our observations. For instance houses can be seen to be made up of squares, rectangles, triangles, cubes and pyramids. As we refine these concepts, we are able to put insights from them back into designing houses and analysing house construction. For example, the triangle of the roof becomes the triangle of material that gives the walls rigidity.

The transformational approach is similar to the environmental approach in that it focusses on the world but is different in that it looks a change. Topology is the change we see in life and growth. Topological change occurs as flowers grow and wither. It occurs as we move around, bending our legs and turning our bodies. It is the cooking of a cake and the inflation of a balloon. It is change that twists, stretches bends and distorts but does not cut or rupture or join. Projective geometry is how we see and draw our world. It is the 2-dimensional representation of the 3-dimensional world, the perspective from different directions. Euclidean is man made and technical. We see it when we drive a car, more furniture or follow the trajectory of a rocket. It focusses on angle and length, coordinates and congruence.

The subconcept and thematic approaches come from the mathematics of the geometry. In the subconcept approach, we break a concept or process down into parts in a formal manner and then look for the appropriate sequence to present these parts. The rectangle, for example, is seen as straight and parallel lines and right angles, as a boundary. A pyramid is seen as triangular faces and a base. Subconcept sequences, therefore, are often the antithesis of environmental sequences. Concepts are broken down in the environmental approach, from bigger to smaller. They are built up, smaller to bigger, in the subconcept approach.

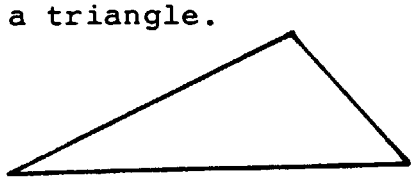
In the thematic approach, we look for similarities and connections across topics, concepts and processes. Unlike the subconcept approach, where we try to work "upwards", here we try to work "sideways". For example, how can shape, dissection tessellation and symmetry be integrated? The answer: "very easily"! We can use shapes to develop tessellation and symmetry. We can then use tessellation and symmetry properties to define the shapes. Dissections can be used to reinforce shape concepts and to lead to visual imagery. Geometry has a structure that allows for a lot of integration.

Materials:
Pen, paper

Activities:

1. Read about the 4 approaches to teaching geometry on pages 10 to 15 of this book and in the background of this unit.

2. This is a triangle.



The concept of a triangle can be introduced using the transformational approach by casting shadows from candlelight of a triangular piece of thin board onto a screen. The screen is turned to create different shadows. The shadows are drawn onto paper placed on the screen. All shadows will have triangular shape and a wide variety of triangles can be constructed as exemplars in this manner by the children themselves.

(1) Describe how you would introduce this concept using:

(a) the environmental approach;

and (b) the subconcept approach.

(2) Are these two programs mutually exclusive or can they complement each other?

3. The triangular tessellation leads to the hexiamond shape puzzles. List topics/activities from this book that could be part of a theme of triangles. What topics/activities from number, operations and measurement could be added to this list?

4. Describe someways in which we could integrate:

(1) symmetry with flips, slides and turns;

(2) orienteering with the study of 2-dimensional (plane) shapes;

(3) compass bearings and surveying with the development of angle and angle properties; and

(4) 3-dimensional (solid) constructions with shape puzzles.

5. There are in this book many explicit references to art and geometry. List as many activities as you can that combine geometry with:

(1) physical education; and

(2) the study of number and operations

6. Read units 25 and 26 of this book. Then designate the following activities as topological, projective or euclidean. Then order them in the sequence they should be given to children.

A. Investigating networks to discover Euler's formula.

B. Using a silva compass to walk an orienteering track.

C. Casting shadows with a rectangular piece of thin board in an attempt to form different shapes with non parallel sides.

D. Constructing a tetrahedron out of straws and string.

E. Drawing a square on rubber sheeting and stretching, bending and twisting the sheeting to form different shapes.

F. Drawing shapes similar to a pentagon using the pen and paper enlargement technique.

G. Using a large street plan to develop right angle turns left and right.

H. Testing shapes with a mira for line symmetry.

I. Moulding shapes out of plasticine and bending and reforming these shapes into new shapes.

Teaching Hints:

Although one of the four approaches described in this unit can be used almost exclusively, the most appropriate response is to be eclectic in their use, to use all methods where useful. The method chosen will therefore depend on the needs of the children. If, for example, the children have notions of square and triangle from seeing examples of them then it may be appropriate to more formally study them by looking at what they are composed of, e.g.straight and parallel lines, diagonals etc.

On the other hand, if the children have some formal study of such notions, it may be appropriate to place these notions back into the world by looking at solid shapes and classifying their surfaces or to build a period of study around a theme.

It is important to note that approaches are a framework for teachers not content for children. They provide a model from which content can be planned and sequenced. They should not be formally taught to children.

Integrating the approaches.

It is possible for all approaches to complement each other. One tantalizing way to do this is below.

(1) Use the transformational approach along with the framework from page 9 as the overall plan for teaching years 1 to 7. Focus in the early years on topological geometry moving onto projective until the middle primary school years. Middle to upper primary school years are then organised around euclidean geometry. Of course this is not prescriptive. Networks are a topological topic, but should be given to children in the middle and upper primary years.

(2) In the development of a topic more environmental to subconcept. Begin study, particularly in the early years, from the environment. When this is complete, add depth to the study by rebuilding the concept more formally from the important subconcepts.

Develop 2-dimensional shape informally from solids in the world (the environmental approach). Then rebuild the 3-dimensional shapes from the plane shapes more formally by focussing on combination of parts (the subconcept approach).

(3) When the concept is build through environmental and subconcept approaches it can be integrated into the scheme of geometry through themes. Themes provide variety and a chance to show connections.

(4) Use transformations as a wav of generating instances for a concept and for such invariance properties as similarity and congruence. Use the transformation idea with the notions of shape, that have been developed environmentally and through subconcepts and integrated through themes, to focus on puzzles, scale, outdoor geometry and visual imagery.

We will take up this integration in the next unit.

UNIT 31: A CURRICULUM FOR GEOMETRY

Focus:
The focus for this unit is the presentation and critique of a curriculum for geometry that involves the activities described within this book.

Background:
How then should we organise the sequencing of the activities within this book? The answer to this question will provide a framework for these activities. Because of the difficulties of sequencing geometry, such a curriculum will be contentious. Hence we have placed in this unit as a basis for discussion not only as a prescription for teaching.

The basis of this curriculum will be:

(1) the chapters as presented in this book;

(2) the framework on page 9, including the development

experiential (interactive with environment)
↓
informal (investigation of concepts and properties)
↓
formal (systematic study); and

(3) the integration of the approaches as described in the previous unit.

Within this curriculum, we will provide a balance of activities that

(1) interpret - read, understand and interpret visual representations and the vocabulary used in geometric work, graphs, charts and diagrams of all types; and

(2) process - process, manipulate and transform visual images and represent and translate relationships into visual terms.

It is our intention to provide a complete curriculum containing all the activities this book feels are important. As such it may contain many more activities than most other geometry curriculums.

Materials:

Pen, paper, text book series, departmental syllabi

Activities:

1. Read the proposed curriculum for geometry at the end of this unit.

   (1) What aspects of it are, in your opinion, contentious? Need changing? Need something added?

   (2) What is considered important for children in the early years? In the later years?

2. Compare the proposed curriculum with the departmental syllabus of your state.

   (1) Where are the major differences between the geometry sequence in the syllabus and in this unit?

   (2) In your opinion, what are the reasons this syllabus would have for its sequence?

3. Project One

Look at the geometry sequence in a text book series commonly used in local schools. Compare it with the department syllabus and this unit. From the result, construct a teaching sequence for geometry that you feel would be both practical and convenient for a school using that text book series as a basis for its teaching.

OR

Project Two

Choose a year level in primary school that interests you. Look at the geometry activities for that year in this unit, in the departmental syllabus and in text book series. As a result of this, prepare a sequence of geometry activities suitable for that year.

Teaching Hints:

There are many teaching possibilities for geometry that are appropriate for children at each year level. The important things are:

(1) to have a school plan to cover the major content;

(2) a rich series of activities that cover a wide variety of topics, teaching approaches and modes of learning;

(3) to ensure that there are activities that cover processing as well as activities that cover interpretation, and

(4) teach actively, with children using materials.

PROPOSED CURRICULUM FOR GEOMETRY

On the following 4 pages, we present a scope and sequence chart for geometry activities for years K to 7.

The geometry activities have been divided into 12 sections:

- Position/Coordinate
- Topology/Networks
- 3-d shape
- Line and Angle
- 2-d shape
- Symmetry
- Tessellations
- Dissections
- Outdoor activities
- Projections/Similarity
- Euclidean/Congruence
- Visual Imagery

Years K to 3 are on the left hand page with 4 to 7 on the right hand side. The units in this book which contain the activities referred to by that section are listed in the scope and sequence chart. Activities have been assigned to year levels. In many cases this is arbitrary although there is an overall plan to the resulting scope and sequence. Activities in any one year could certainly be repeated in the following year. In most cases this is assumed and not specifically stated.

TOPIC	PRESCHOOL/YEAR	YEAR 1	YEAR 2	YEAR 3
POSITION/ COORDINATES (Units 1 and 22)	. Exploring environment . Spatial language (sorting by position) . Position Games . Following instructions re movement	. Exploring environment . Spatial language . Rearranging objects/ hiding and finding . Drawing plans (home, classroom - informal)	. Exploring neighbour -hood . Maps and plans . Directions on road maps	. Simple coordinates (row D, seat 6)
TOPOLOGY/ NETWORKS (Units 2, 3, 22, 25 and 26)	. Proximity, separation, enclosure (closed/open, inside/ outside) . Spatial order (1st., 2nd., etc.) . Cooking/playdough/ plasticine	. Proximity etc continued . Rubber sheet geometry (balloons) . Growth and change (flowers/plants)	. Year 1 continued . Mazes	. Planning routes on maps . Finding shortest distance in environment and on maps.
THREE DIMENSIONAL SHAPE (Solids) (Units 1, 4, 5, 6 and 7)	. Sorting and classifying by shape and size . Constructing and building (blocks, logo etc.) . Carpentery, modelling, playing with toys . Sand and water play	. Comparing and ordering by shape and size . Constructing solid shapes from junk . Construction kits. . Link cubes, lego	. Investigating shape in the environment . Classification of shapes (rolls/does not roll) . Faces of shapes (Flat/curved, square/triangular)	. Environmental approach to teaching geometry continued. . Cutting open boxes . Cardboard construction.
LINE AND ANGLE (Units 2 and 8)	. Boundary - games (cat and mouse) - constructions (fencing something in) . Turn and direction - things that turn - pointing in a direction - straight and crooked.	. Boundary, turn and direction (clockwise/anti-clockwise right/left) . Closed/open, simple/non simple	. Concept of angle (paper folding) . Right angle (obtuse/acute) . Non standard units for angle . Straight and curved lines . North, South, East and West.	. Parallel lines (shadows) . Angle wheel and geoboard . Perpendicular . Reflex angles
TWO DIMENSIONAL SHAPE (Units 1, 3, 9, 10 and 11)	. Drawing and painting . Collage (cutting and pasting) . Recognition of simple shapes (sorting) . Natural shadows	. Rubber sheet geometry . Sorting, classifying, drawing, painting, collage . Surfaces of common shapes (top of tables) . Simple shapes	. Shapes in the environment . Shadows of shapes in torchlight . Introduction to geoboard (and dot paper)	. Environmental approach to teaching geometry . Beginnings of subconcept approach (line, angle, path, shape and region) - geoboard . Introduction to geostrips

TOPIC	YEAR 4	YEAR 5	YEAR 6	YEAR 7
POSITION/ COORDINATES	. Simple coordinates . Lattitude and Longitude	. Coordinates as as ordered pairs of numbers . Coordinate games (e.g. battleships)	. Coordinate . Graph paper activities	. Coordinate games and activities . Drawing plans to scale (formally)
TOPOLOGY/ NETWORKS	. Introducing networks . Travellability . Drawing networks	. Mobius strips . 4 colour problem . Topological puzzles	. Network activities Bridges of K Start and finish Euler's formulae etc.	. Topological invariance . Flexagons
THREE DIMENSIONAL SHAPE (SOLIDS)	. More formal construction of solids (straws/string, rubber bands/shapes) . Prisms, pyramids, polyhedra	. Properties of solids (relation to faces) . Cones, cylinders, spheres	. Eulor's formulae . Marshmallows, tooth -picks and geodesic domes . House construction	. Platonic solids . Cross sections . Systematic study of solids in the world (3-D tessellations, triangulation for strength, etc.)
LINE AND ANGLE	. Angle in degrees . Bisection of angles . Interior and exterior angles . Mira	. Angle sum properties . Angle properties of a circle (quadrants etc)	. Straight edge and compass constructions . Systematic study of point, line and plane . Curve stitching	. Linkages . Parallel line angle properties . Systematic study continued
TWO DIMENSIONAL SHAPE	. Subconcept approach continued . Polygons (triangles, squares, rectangles, parallelograms, etc.) . Classification by symmetry . Classification by angle size and parallelness	. Circle and disc properties . Diagonals (rigidity, triangulation - geo strips) . Diagonal properties . Similar shapes	. Systematic study of shape (polygons, circles) and relation to line and angle. . Classification by congruence of sides and angles (e.g. Isosceles triangle) . Congruent shapes	. Systematic study of plane shape in the world . Pythagoras' theorem

TOPIC	PRESCHOOL/YEAR K	YEAR 1	YEAR 2	YEAR 3
SYMMETRY (Units 12 and 13)	. Symmetry in the world (leaves, trees, houses, etc) - no formal use of language	. Early line symmetry . Ink blots, cutting folded shapes	. Early line and rotational symmetry . Patterns (cutting/pasting)	. Classifying shapes as having line and/or rotational symmetry. . Constructing line symmetric shapes
TESSELLATIONS (Units 14,15, 16 and 19)	. Mosaics with cutouts and templates . Covering surfaces	. Tiling (simple shapes) . Packing/stacking	. Investigating which shapes tessellate . Forming shapes from tessellating figures	. Two shape tessellations . Tessellating shape grid activities
DISSECTIONS UNITS (17 and 18)	. Jig saw type puzzles . Fitting shapes together and taking them apart . Dressing/undressing toys	. Jigsaw type puzzles (commercial and homemade)	. Tangrams (teacher led)	. 5 easy pieces . Cutting and reforming shapes
OUTDOOR ACTIVITIES (Units 23 and 24)	. Playing in play-ground . Climbing . Movement Games	. Shapes in the environment . Exploring school	. Exploring neighbourhood . Giving directions/ movement games	. Simple work with Silva compass . Simple height measurement
PROJECTIONS/ SIMILARITY (Units 1, 3, 25, 26 and 27)	. Drawing things - from memory - as seen	. Shadows in torchlight . Early perspective (drawings)	. Early perspective and shadows continued.	. Shadows in sunlight
EUCLIDEAN/ CONGRUENCE (Units 3, 25, 26 and 28)	. Building with blocks Jigsaws	. Informal collage by sliding, flipping, turning shapes . Building with blocks	. Informal flips, slides, turns in collage and building/ constructing continued	. Informal work continued . Paper folding techniques for turning and flipping.
VISUAL IMAGERY (Unit 20)	. Visual thinking - jigsaws - patterns - fitting shapes - sorting	. Visual thinking continued - which shape fits here. - shape patterns - classification by shape	. Patterning with shape . Finding the same shape . Tangrams . Simple geometry/ strategy games (noughts and crosses)	. 5 easy pieces puzzle . Simple games and puzzles (Find winning strategy) . Picking differences in similar shapes.

TOPIC	YEAR 4	YEAR 5	YEAR 6	YEAR 7
SYMMETRY	. Mira activities . Number of lines and number of rotations of symmetry . Classification of polygons	. Modification of symmetry . Frieze patterns	. Construction of symmetrical figures . Symmetry and art	. Symmetry rules (relation between line and rotational symmetry)
TESSELLATIONS	. Tessellating grid activities . Tessellating patterns (2 or more shapes)	. Angle sum properties of tessellating figures . Pentominoes . Games	. Hexiamonds . Tessellation and art	. Fabric design
DISSECTIONS	. Tangrams and 5 easy pieces . Block puzzles . Cutting and reforming shapes cut from grid paper	. Dissections of a square (and other shapes) . Constructing own dissections	. Soma cubes . Block puzzles & more complex tangram activities	. McMahon colour shapes
OUTDOOR ACTIVITIES	. Silvacompass - beginning activities with direction and distance . Height measurement - right angle triangle	. Simple orienteering . Inclinometer - measuring incline . Triangulation with Silva compass	. Orienteering and scale drawings . Height measurement - similarity and scale drawing	. Surveying and Mapping
PROJECTIONS SIMILARITY	. Photography	. Enlargements - string - shadows - pen and paper	. Similarity . Similarity properties . Scale drawings	. Perspective drawings . Projective invariance
EUCLIDEAN/ CONGRUENCE	. Mira activities (reflection)	. Systematic study of flips, slides and turns (reflections, translations and rotations) . Graph paper work	. Congruence . Congruence properties . Properties of reflection translation and rotation	. Modulo art . Interrelationships between reflection and rotation . Construction of plane shapes
VISUAL IMAGERY	. Games of alignment . Simple hunt games . Cutting and reforming shapes	. Dissections and pentomino type puzzles . Games tessellation - curvo, pentominogame.)	. Spatial puzzles and problems (e.g. Caliphs problem) . Games (Sprouts, Lgume)	. Perspective . Race track game

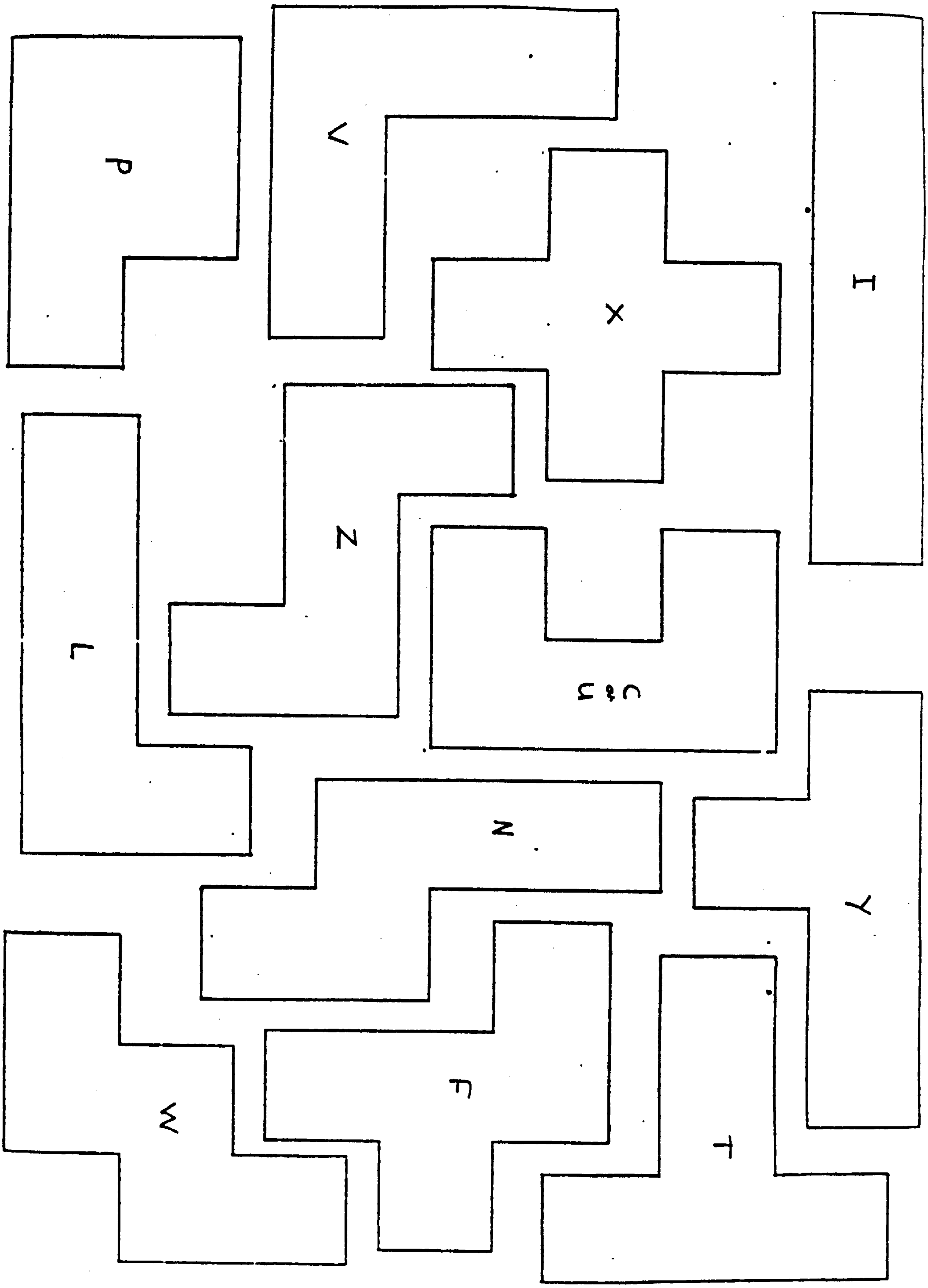
APPENDIX: TESSELLATION PUZZLES AND GRIDS

The 12 Pentominoes

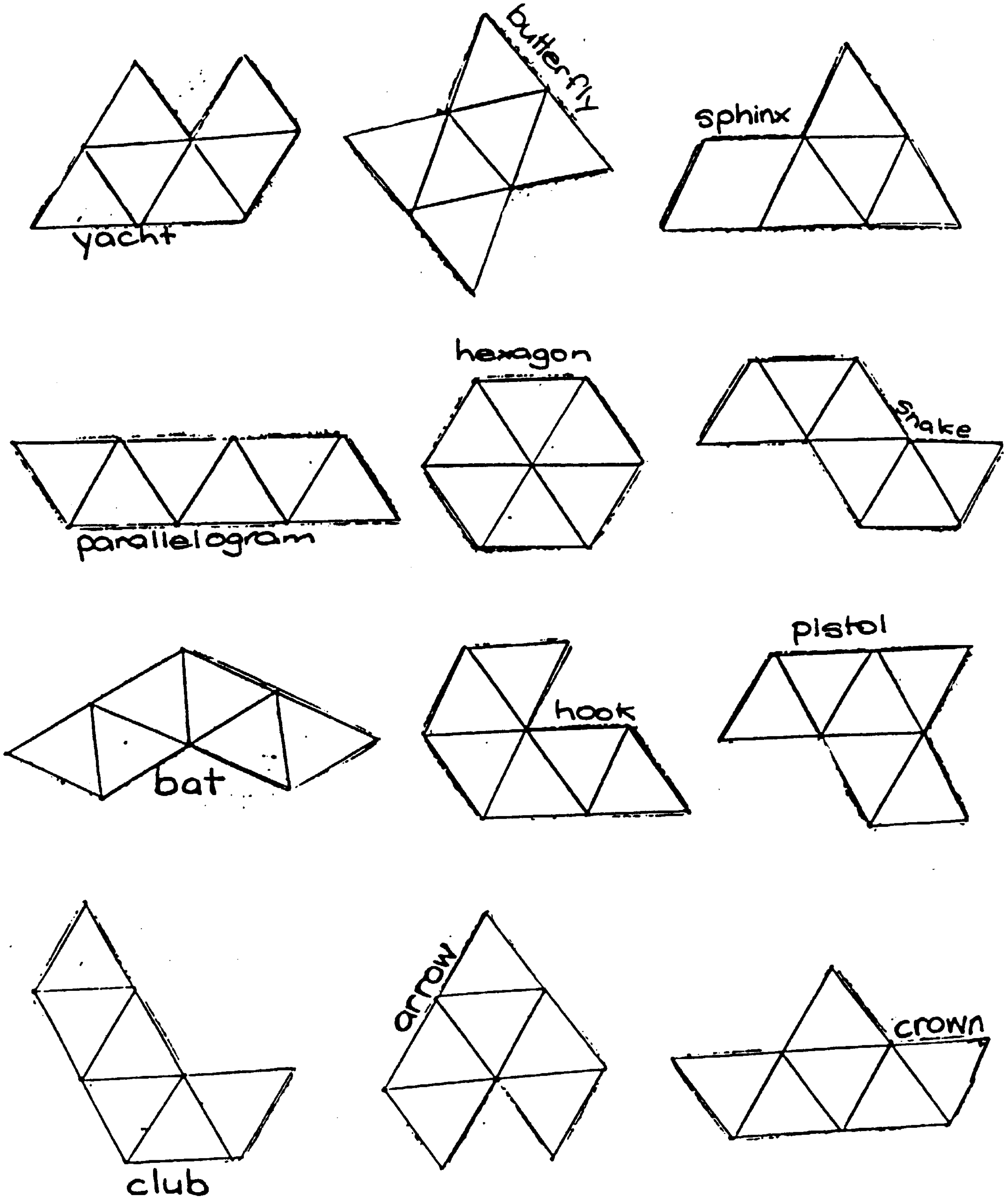
The 12 Hexiamonds

Grid paper of tessellating squares, triangles and hexagons

Dot paper



THE 12 PENTOMINOES



THE 12 HEXIAMONDS