YUMI DEADLY CENTRE
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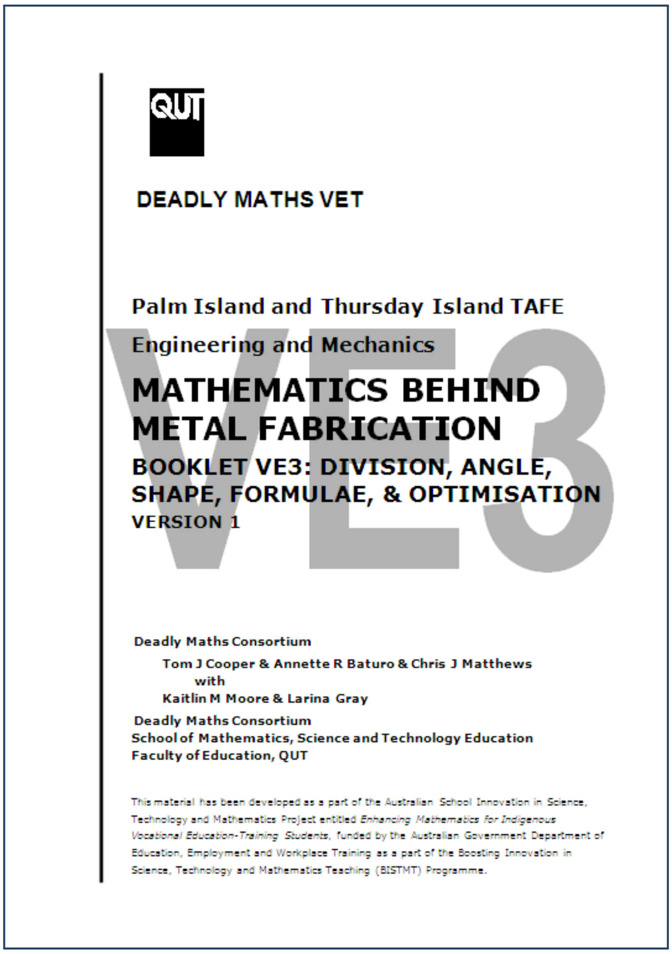
Engineering and Mechanics

Mathematics behind Metal Fabrication

Booklet VE3: Division, Angle, Shape, Formulae, and Optimisation


DEADLY MATHS VET

Palm Island and Thursday Island TAFE

Engineering and Mechanics

MATHEMATICS BEHIND METAL FABRICATION

BOOKLET VE3: DIVISION, ANGLE, SHAPE, FORMULAE, & OPTIMISATION

VERSION 1

Deadly Maths Consortium

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with
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Deadly Maths Consortium
School of Mathematics, Science and Technology Education
Faculty of Education, QUT

This material has been developed as a part of the Australian School Innovation in Science, Technology and Mathematics Project entitled *Enhancing Mathematics for Indigenous Vocational Education-Training Students*, funded by the Australian Government Department of Education, Employment and Workplace Training as a part of the Boosting Innovation in Science, Technology and Mathematics Teaching (BISTMT) Programme.

YuMi Deadly Maths
Past Project Resource

Acknowledgement

We acknowledge the traditional owners and custodians of the lands in which the mathematics ideas for this resource were developed, refined and presented in professional development sessions.

YuMi Deadly Centre

The YuMi Deadly Centre is a Research Centre within the Faculty of Education at Queensland University of Technology which aims to improve the mathematics learning, employment and life chances of Aboriginal and Torres Strait Islander and low socio-economic status students at early childhood, primary and secondary levels, in vocational education and training courses, and through a focus on community within schools and neighbourhoods. It grew out of a group that, at the time of this booklet, was called "Deadly Maths".

"YuMi" is a Torres Strait Islander word meaning "you and me" but is used here with permission from the Torres Strait Islanders' Regional Education Council to mean working together as a community for the betterment of education for all. "Deadly" is an Aboriginal word used widely across Australia to mean smart in terms of being the best one can be in learning and life.

YuMi Deadly Centre's motif was developed by Blacklines to depict learning, empowerment, and growth within country/community. The three key elements are the individual (represented by the inner seed), the community (represented by the leaf), and the journey/pathway of learning (represented by the curved line which winds around and up through the leaf). As such, the motif illustrates the YuMi Deadly Centre's vision: *Growing community through education*.

More information about the YuMi Deadly Centre can be found at http://ydc.qut.edu.au and staff can be contacted at ydc@qut.edu.au.

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Queensland University of Technology

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DIVISION, ANGLE, SHAPE, FORMULAE,

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08/05/09

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THIS BOOKLET

This booklet (Booklet VE3) was the third booklet produced as material to support Indigenous students completing certificates associated with engineering and work readiness at the TAFE campuses on Thursday Island (run by Tropical North Queensland TAFE) and Palm Island (Barrier Reef Institute of TAFE). It has been developed for teachers and students as part of the ASISTM Project, *Enhancing Mathematics for Indigenous Vocational Education-Training Students.* The project has been studying better ways to teach mathematics to Indigenous VET students at Tagai College (Thursday Island campus), Tropical North Queensland Institute of TAFE (Thursday Island Campus), Northern Peninsula Area College (Bamaga campus), Barrier Reef Institute of TAFE/Kirwan SHS (Palm Island campus), Shalom Christian College (Townsville), and Wadja Wadja High School (Woorabinda).

At the date of this publication, the Deadly Maths VET books produced are:

VB1: Mathematics behind whole-number place value and operations
Booklet 1: Using bundling sticks, MAB and money

VB2: Mathematics behind whole-number numeration and operations
Booklet 2: Using 99 boards, number lines, arrays, and multiplicative structure

VC1: Mathematics behind dome constructions using Earthbags
Booklet 1: Circles, area, volume and domes

VC2: Mathematics behind dome constructions using Earthbags
Booklet 2: Rate, ratio, speed and mixes

VC3: Mathematics behind construction in Horticulture
Booklet 3: Angle, area, shape and optimisation

VE1: Mathematics behind small engine repair and maintenance
Booklet 1: Number systems, metric and Imperial units, and formulae

VE2: Mathematics behind small engine repair and maintenance
Booklet 2: Rate, ratio, time, fuel, gearing and compression

VE3: Mathematics behind metal fabrication
Booklet 3: Division, angle, shape, formulae and optimisation

VM1: Mathematics behind handling small boats/ships
Booklet 1: Angle, distance, direction and navigation

VM2: Mathematics behind handling small boats/ships
Booklet 2: Rate, ratio, speed, fuel and tides

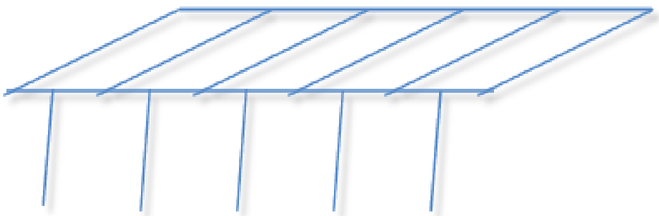
VM3: Mathematics behind modelling marine environments
Booklet 3: Percentage, coverage and box models

VR1: Mathematics behind handling money
Booklet 1: Whole-number and decimal numeration, operations and computation

CONTENTS

1. UPRIGHTS, CROSSPIECES AND DIVISION

Many constructions in steel have posts and rails, particularly for high storage. For example:

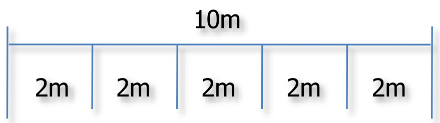


The mathematics underlying this situation is, in the first instance, measurement, ensuring uprights and crosspieces are the same length and are placed in the correct positions. However, students had difficulty with a second problem of even placing; that is, placing the crosspieces and uprights equally spaced. This leads to two problems: (1) working out what the even spacing is; and (2) working out how many uprights or crosspieces are needed.

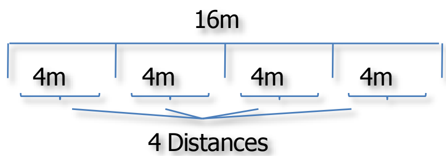
1.1 Mathematics for uprights, crosspieces and division

The mathematics for even placing is division +1. This is because the division is not done by the posts or rails but by the distance between them. And the number of distances = the number of posts/rails −1. Let us look at two examples.

- Suppose the length is 10m and the distance between posts is 2m. Then the number of distances is 10m ÷ 2m = 5. Five distances means 6 posts as in the diagram below.



- Suppose the front rail is 16m and we have 5 posts, how far apart do we place them? Now 5 posts means 4 distances as in the diagram below.



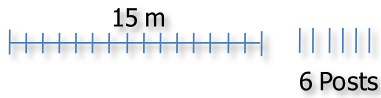
Thus, the length of each distance is 16m ÷ 4 = 4m.

1.2 Teaching uprights, crosspieces and division

We can teach the mathematics in Section 1.1 by acting out the building of posts and rails in a similar manner to the two examples above. However, we would advocate using a more guided discovery approach as in the following steps.

(1) Place posts evenly, when given posts

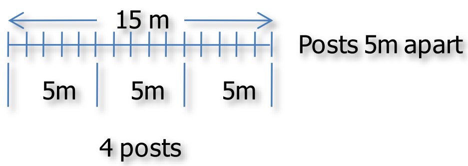
Use models or virtual materials to place posts evenly and work out the distance between posts. This can be done by trial and error, that is, by giving a distance and some posts and letting the students place by trial and error. For example:



Students will place the posts by trial and error and see that every 3m is what is possible for even placing.

(2) Place posts evenly, when given distance

Use models or virtual materials to place posts so they are an even distance apart and work out how many posts. For example:



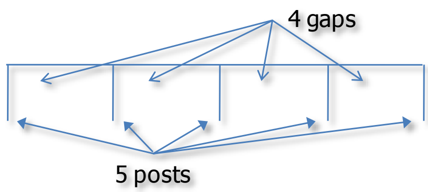
Students will place the posts by placing them 5m apart and then getting the number of posts at the end.

(3) Develop the idea that the number of posts is 1 more than the number of distances

Do a few of the examples from Step (1) and (2). Record the posts and distances – use a table as follows.

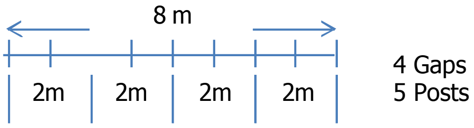
PROBLEM	Number of posts	Number of distances/gaps	Length of gap
1) 12m rail, posts 3m apart. How many posts? How many distances/gaps?	5	4	3m
2) 12m rail, posts 4m apart. How many posts? How many distances/gaps?	4	3	4m
3) 6m rail, 3 posts. How many distances/ gaps between posts? How far apart?	3	2	3m
4) 6m rail, 4 posts. How many distances/ gaps between posts? How far apart?	4	3	2m

Look at the pattern. Discuss the results and encourage the students to see that the number of gaps is 1 less than the number of posts. Draw an example. Discuss how you need posts on either end.



(4) Discover that number of gaps × length of gap = length of rail

Widen the recording of examples. For each example, make a drawing and record results on a table where the number of columns has been increased to include the length of the rail.

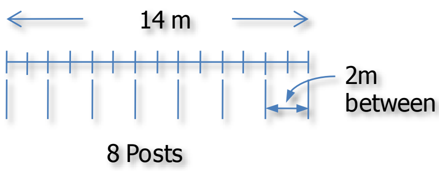


PROBLEM	Number of posts	Number of gaps	Length of gap	Length of rail
1) 8m rail, 2m gap. and so on...	5	4	2m	8m

Discuss the results and encourage students to see that the number of gaps × length of gap = length of rail – from both table and drawing.

(5) Show that number of posts = length of rail ÷ length of gap + 1 and that length of gap = length of rail ÷ (number of posts – 1)

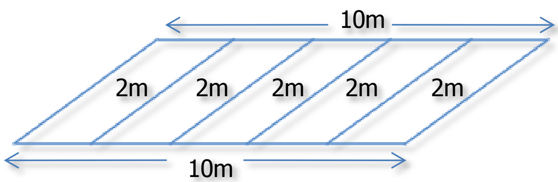
This can be argued from combining steps (3) and (4) and also from an example. For example, length of rail = 14m, 8 posts, how far apart (length of gap)?



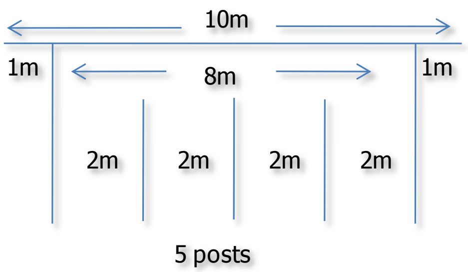
Take 1 from posts = 7. Then 14m ÷ 7 = 2m gap.

(6) Extend idea to other situations

(a) *Close by*: a rack is 10m wide. How far apart if there are 6 rails?



(b) *A little problem solving*: a rail is 10m, the posts start 1m in from each side, they are 2m apart (2m gap). How many posts?



(c) *New vocations*: the problem (how many posts/how far apart) is the same in fencing, sides of walls, roof trusses, etc.

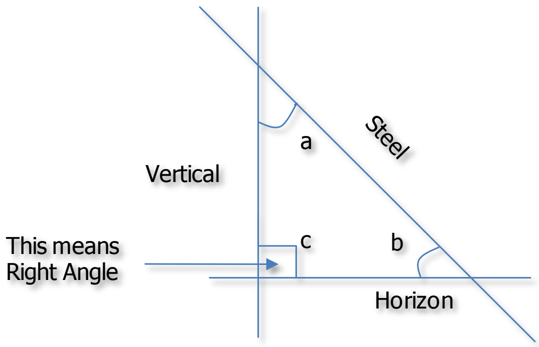
2. CONSTRUCTION AND ANGLES

Many constructions in steel are not always vertical and horizontal. For example, a saw-horse as below:

In making this neatly, steel is used which is rectangular. To end in a flat surface for the floor or as a support for a top, the steel rectangular length has to be cut so that it is horizontal (and sometimes vertical). What angle is the cut?

2.1 Mathematics for construction and angles

If we have a rectangular length at an angle, there is a relationship with how it cuts the horizontal and the vertical (as follows):



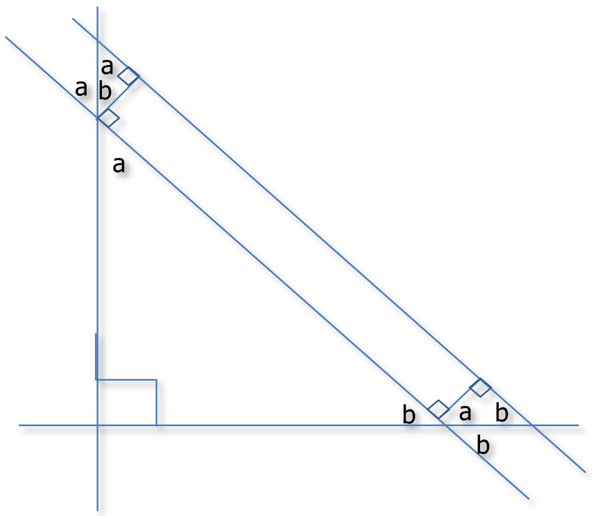
This is a right-angle triangle (i.e., c is 90°)

and $a + b + c = 180°$

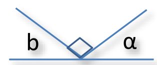
Therefore $a + b = 90°$ and

$b = 90° - a$

When a rectangle cuts the horizontal and the vertical, there are more relationships (as follows):



All the angles a and b are the same in the diagram. This is due to angles being on lines.

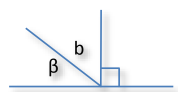


The three angles add to 180°,

so $b + 90 + \alpha = 180$, so $b + \alpha = 90$,

so $\alpha = a$

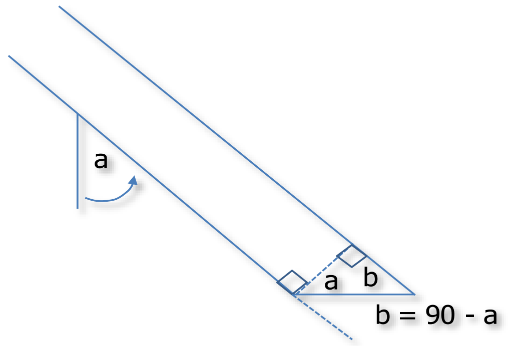
Similarly for this diagram:



$\beta + b + 90 = 180$, so $\beta + b = 90$,

so $\beta = a$

Thus, when we have a rectangular length at angle a to the vertical as below, the angle of the cut is 90−a (or b) or angle to right angle is a.

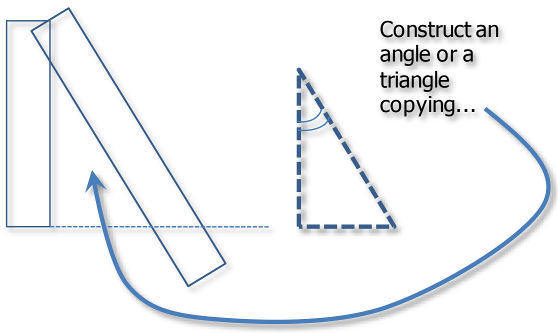


2.2 Teaching construction and angles

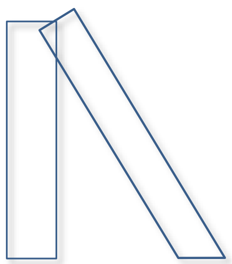
Relationships like the above can be shown logically but are best learnt through experience – by guided discovery. So act out the cuts with models (could be cardboard or even light steel but smaller in size).

(1) Trial and error

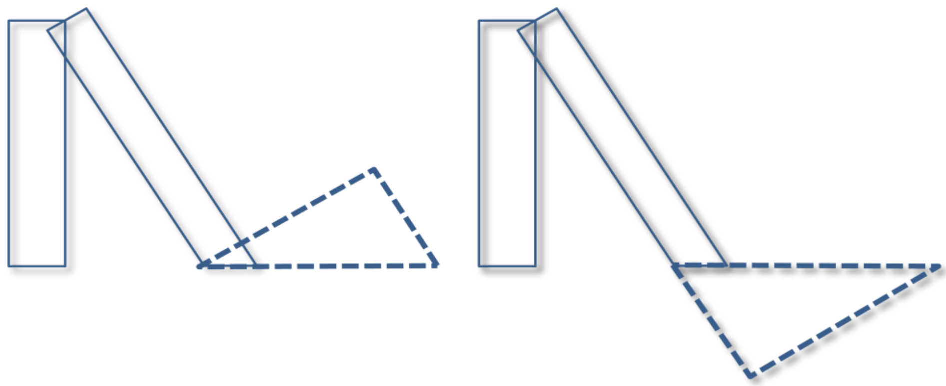
Try to find the angle that appears to work. Construct something like this:



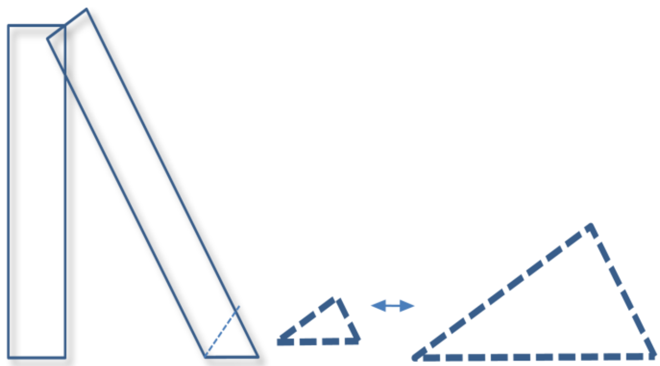
Cut the longer one until it looks flat:



See how it relates to the triangle:

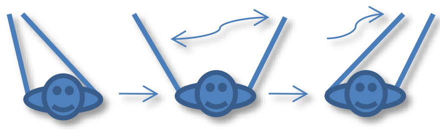


See how this small triangle relates to the "copy":

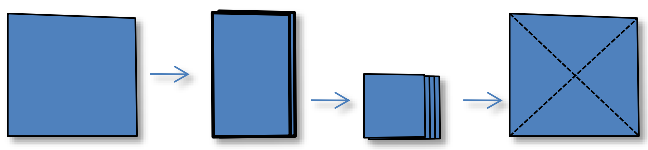


(2) Build up understandings

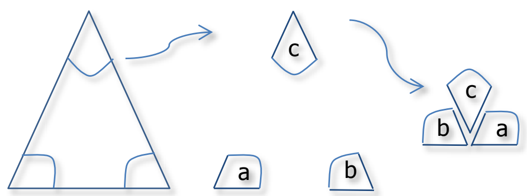
(a) Introduce angle as amount of turn using hands and body:



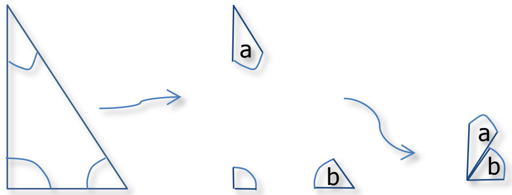
(b) Use paper folding to introduce right angle



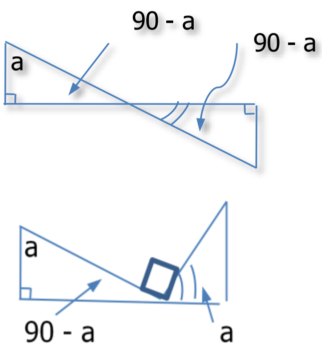
(c) Show angles of a triangle add to 180° (or straight line):



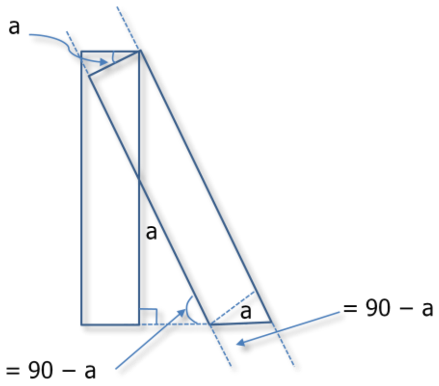
(d) Show two other angles of a right-angled triangle add to 90° or a right angle:



(e) Use triangles and lines to work out unknowns:

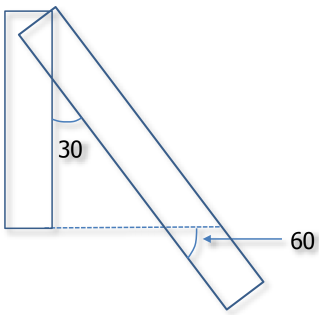


(f) Use this ability to explore angles:



(3) <u>Validation</u>

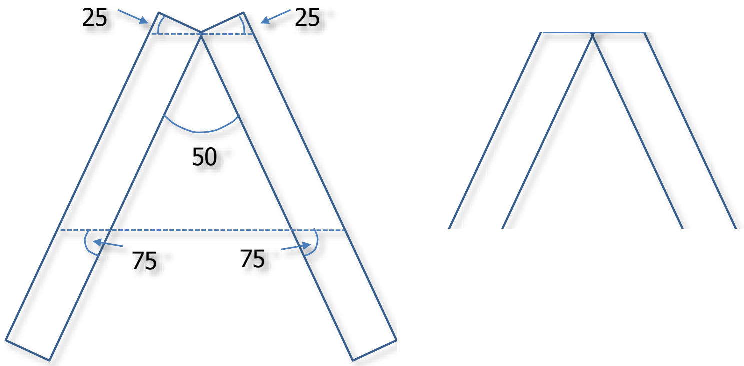
Get students to make things with rectangular lengths and check the angles work:



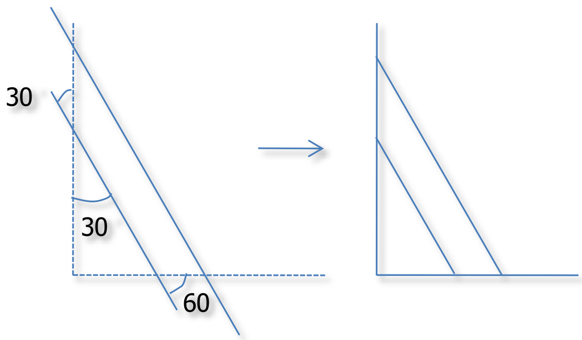
(4) <u>Extensions</u>

Expand to other things or ways of doing angles

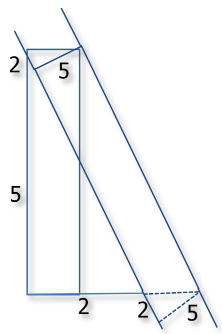
(a) Two angles



(b) Vertical angle



(c) Angle as a ratio



3. CONSTRUCTION STRENGTH AND BALANCE

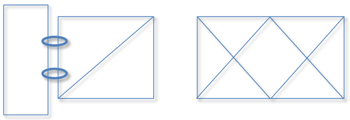
Many things are made out of metal because of its strength. But what makes shapes strong? And strength is not enough on its own, the result must balance so as not to fall over. One example of this is to make a support for fixing outboard motors.

3.1 Mathematics for construction strength and balance

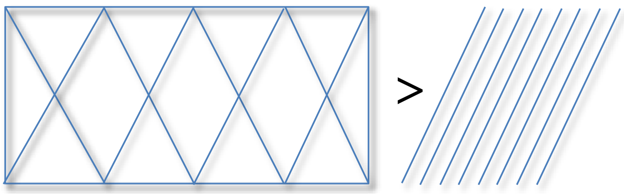
(1) Strength

A construction is strong if its edges will not bend and the construction does not lose its shape. For example:

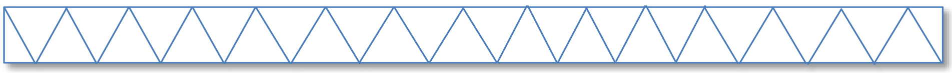
The method that brings strength is to build with triangles, for example:



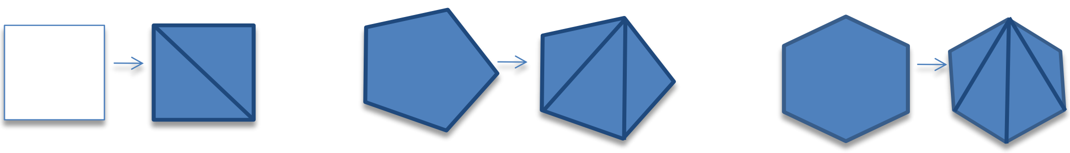
A construction made of triangles can be stronger than all the pieces side by side:



A "triangle" construction of thin weak wood can make a long strong beam, for example:

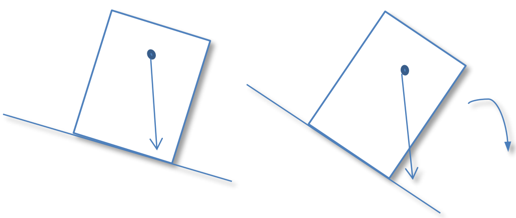


Thus, the following makes constructions strong:



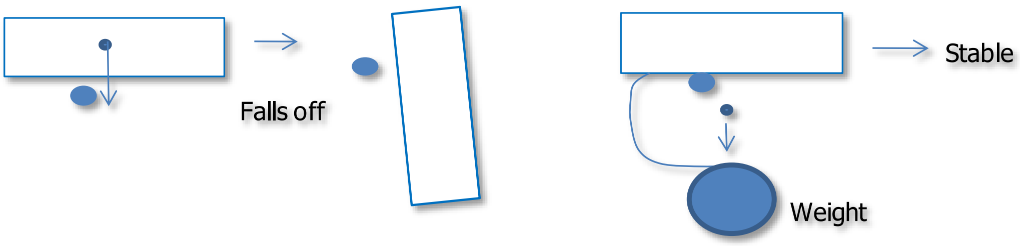
(2) Balance

A construction is balanced if its weight is over its supports.



The weight operates from a centre of gravity – the centre of the shape in terms of weight.

Interestingly, if the centre of gravity can be below the base of support then it stays in balance. For example:



3.2 Teaching construction strength and balance

(1) Trial and error

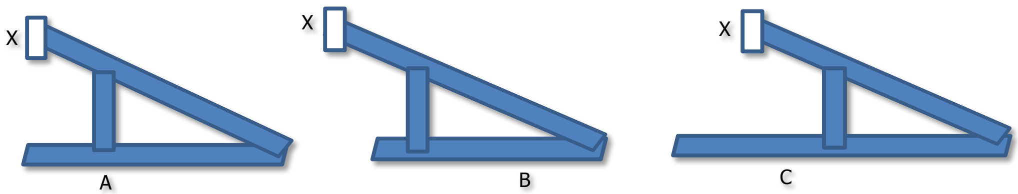
Construct simple constructions of steel joined at ends, for example:

A B C

Manipulate the shapes, which is rigid? Experience how to make them rigid by the addition of extra pieces.

(2) Validation

Make shapes based on triangle(s), for example:



A B C

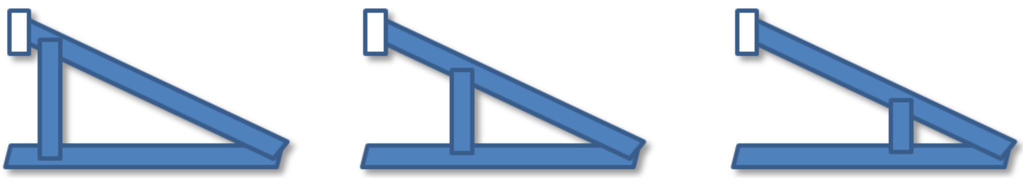
Notice how they are strong and rigid.

(3) Balance

Push down at point x. Do any of the constructions A, B or C unbalance? Which unbalances the easiest? What happens if a heavy outboard is attached at x? What would be the danger? Why? Which is the safest construction: A, B or C? Why?

(4) Experimentation

In point (2), all the triangles are the same size. Look at the three constructions below.

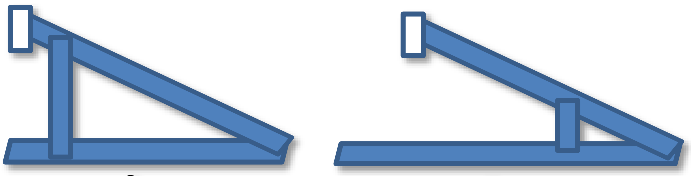


P Q R

Which is the strongest? Why?

Which is easier to tip over? Why?

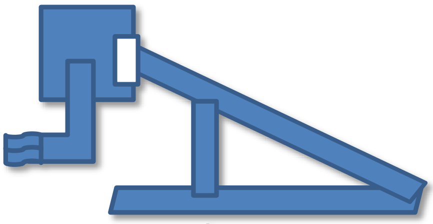
Which is the safest in terms of tipping over? S or T? Why?



S T

(5) <u>Making it safe</u>

In point (2), shape A and B could both over balance, with Shape B the most likely. Look at Shape A. Put an outboard on it.



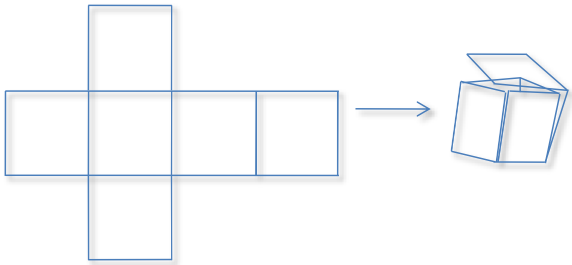
A

What would have to happen for the construction to tip over?

What change could be made to the shape to make sure it would never tip over?

4. FABRICATION, NETS AND SOLID SHAPES

One of the common constructions in fabrication is to make containers, for example, tanks. The simplest way to make a strong tank is to cut out a shape that folds into the tank, thus reducing the welding. For example:

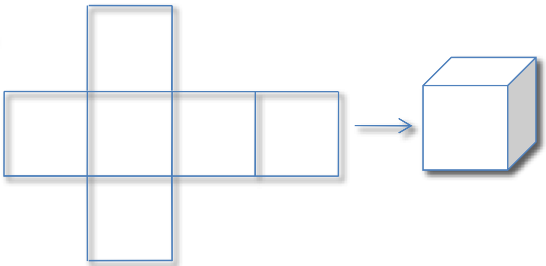


4.1 Mathematics for nets and solid shapes

(1) Definition

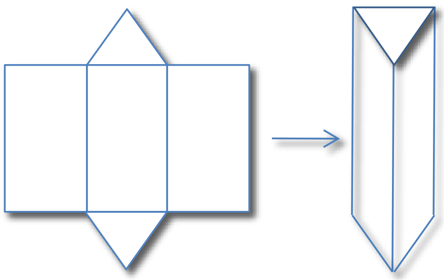
A net is a plane shape that can be folded to make a solid shape. For example:

(a)

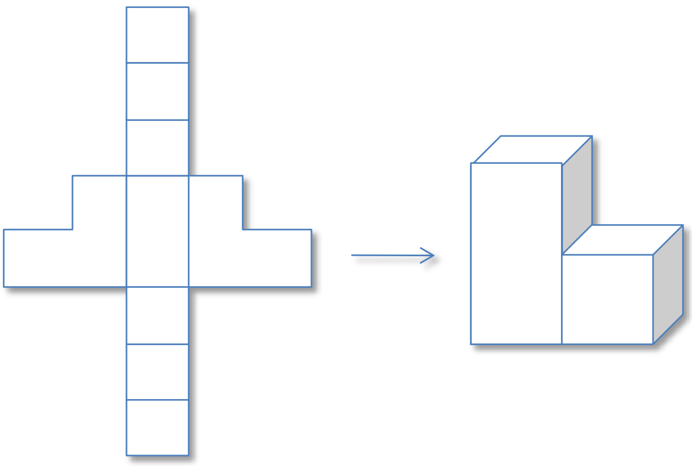


A simple way to make a net is to have sections that form the base of the solid and then other sections that go around filling in all the sides.

(b)

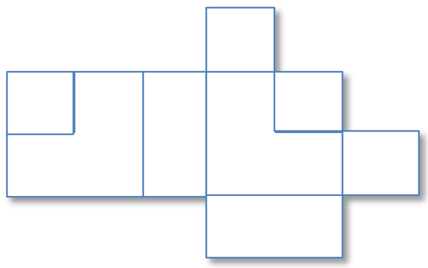


(c)



(2) Efficient nets

In point (1) (c) above, the net could be more efficient as follows:



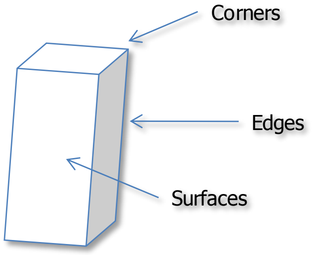
It is more efficient because it is more compact in the material it uses and is more likely to have less wastage.

(3) Mental rotation

Place patterns on faces of a cube. Where would they be put on a net so when folded it is the same as the cube? To work this out, fold the shape in the mind and rotate in the mind beside the solid shape.

(4) Euler's formula

Look at solid shapes. They have surfaces, edges and corners (vertices) as follows:



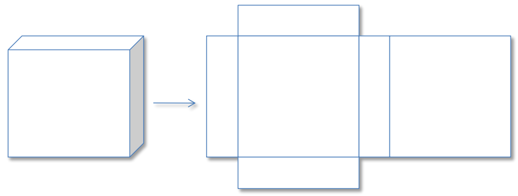
When you count all these three, it is always true that: edges + 2 = surfaces + corners

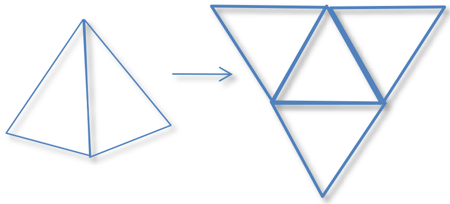
Thus, it is possible to make a solid shape for 4 surfaces, 5 corners and 7 edges but not for 4 surfaces, 5 corners and 6 edges.

4.2 Teaching nets and solid shapes

(1) Deconstructing

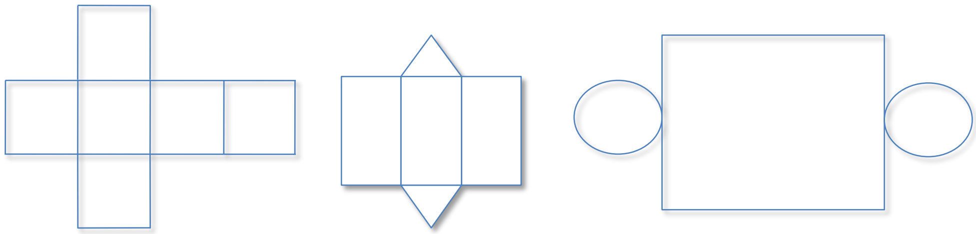
Take boxes and break them down into the plane shape they were folded from. Look at the relations between the plane shape and the solid shape. For example:





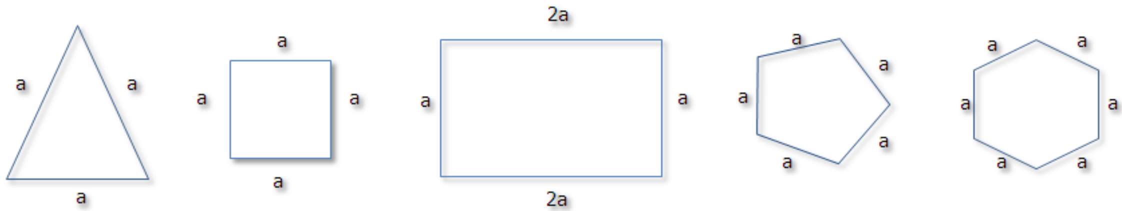
(2) Constructions

Find nets, cut them out and see what shapes they make when folded. The following are examples of nets:



(3) <u>Experimentation</u>

Get a set of "Polydrons" (a set of plane shapes that click together) or make the following shapes out of cardboard as a substitute for polydrons.



Investigate possible shapes that can be made from these faces. For example, what shape can you make from 4 triangles; 8 triangles; 6 squares; 2 squares and 4 rectangles; and so on.

(4) <u>Euler's formula</u>

When you construct a shape, put its dimensions in a table:

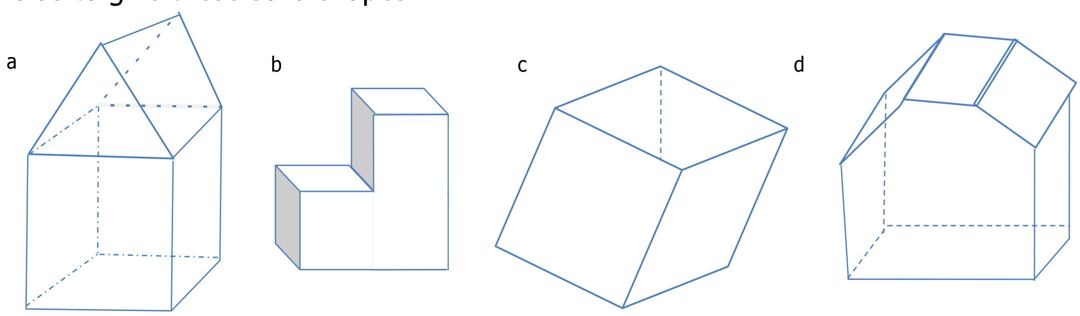
Shape	No. of Surfaces	No. of Edges	No. of Corners
	4	6	4
	6	12	8

Look down the table. There is a formula that relates surfaces, edges and corners. It is called Euler's formula. What is it?

How could we use the formula to help us work out what solid shapes can be made from surfaces?

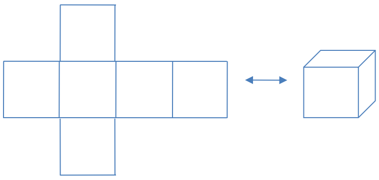
(5) <u>Complex shapes</u>

Move onto identifying nets for more complex shapes. For example, what plane shape folds to give these solid shapes?



(6) Building shapes

Identify a net for a shape, e.g.

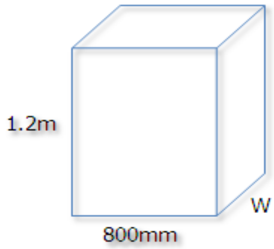


Place patterns on all sides of the solid. Then determine where patterns have to go on the net so it folds into the same thing.

Note: Where would you put numbers 1 to 6 on a net for a cube to make a dice?

5. FORMULAE

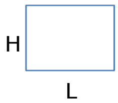
Once the net for a container (e.g. a tank) has been determined, then the dimensions of the container are determined by using formulae, e.g.



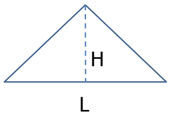
What width (*W*) will make the volume 1.5 m^3?

5.1 Mathematics for formulae

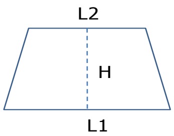
(1) <u>Area</u>

Area of a rectangle 

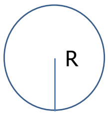
is $L \times H$

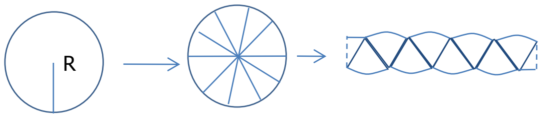
Area of a triangle 

is $\frac{L \times H}{2}$

Area of a trapezium 

is $\frac{L1+L2}{2} \times H$

Area of a circle 

is πr^2 

(2) <u>Volume</u>

Volume of a prism/cylinder is Area of base $\times H$

Volume of a pyramid/cone is $\frac{1}{3}$ Area of base $\times H$

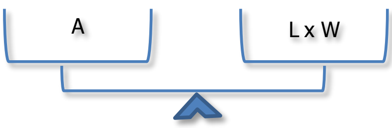
Volume of a sphere is $\frac{4}{3} \pi r^3$

(3) <u>Litres and length</u>

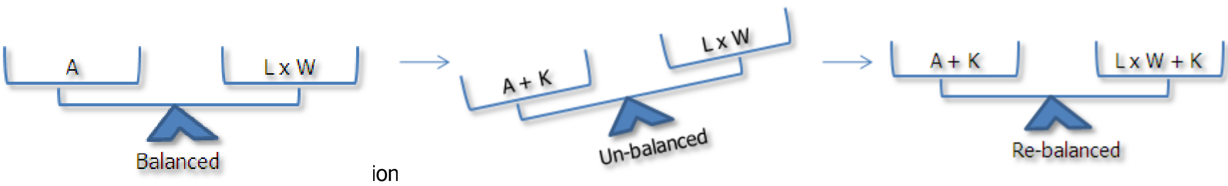
One litre is 100 mm × 100 mm × 100 m or 0.1 × 0.1 × 0.1 = 0.001 m^3

(4) <u>Balance</u>

Formulae are equations; equations balance, that is LHS balances RHS



If something is done to one side to unbalance it, it can be rebalanced by doing the same to the other side. For example:



This can be represented with formulae, e.g.,

$$A = L \times W \quad \rightarrow \quad A + K \neq L \times W \quad \rightarrow \quad A + K = L \times W + K$$

(5) Changing the subject

In $A = L \times W$, A is the subject. What if we wanted to know, say, L? We have to make L the subject, that is, to be on its own on one side of the equation. How can we make L on its own?

$$L \times W \quad \rightarrow \quad L \times W \div W \quad \rightarrow \quad L$$

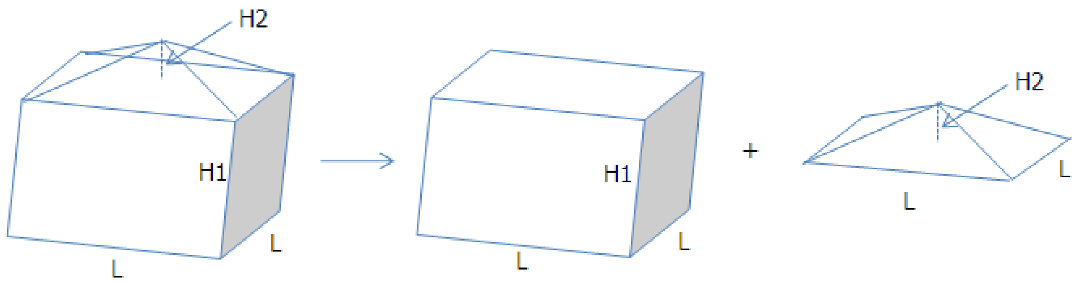
However, $\div W$ unbalances so it needs to be done on the other side as well.

$$L \times W = A \quad \rightarrow \quad L \times W \div W = A \div W \quad \rightarrow \quad L = \frac{A}{W}$$

In other words, moving an operation means the opposite operation on the other side.

(6) Complex volumes

If a construction is made up of more than one shape then break it into parts and add, for example:

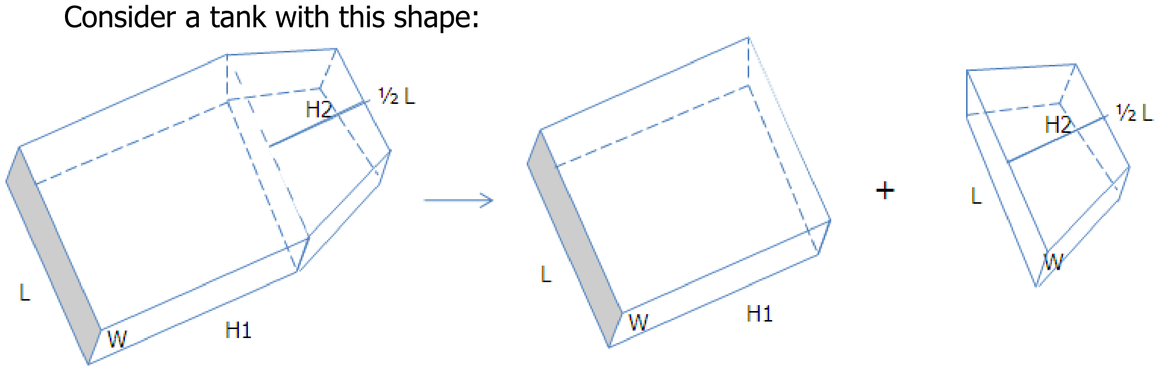


Volume $= L \times L \times H1 + \frac{1}{3} \times L \times L \times H2$

$= L^2\,(H1 + \frac{1}{3}\,H2)$

(7) Changing subject in complex volume

Consider a tank with this shape:



This means that: $V\,(tank) = L \times W \times H1 + \frac{L + \frac{1}{2}L}{2} \times W \times H2$

$= L \times W \times H1 + \frac{3}{4}L \times W \times H2$

$$= L \times W(H1 + \tfrac{3}{4}H2)$$

What if *V* has to equal 200 Litres, that is, 0.2 m^3, and *L* = 0.5 m, *H1* = 0.575 m, and *H2* = 0.30 m. Then:

$$0.2 = 0.5 \times W \times \left(0.575 + \tfrac{3}{4} \times 0.3\right)$$

$= 0.5 \times W \times (0.575 + 0.225)$ (using a calculator)

$= 0.5 \times W \times 0.8$ (using a calculator)

$= 0.4 \times W$

$W = \frac{0.2}{0.4} = 0.5\ m\ or\ 500\ mm$ (using a calculator)

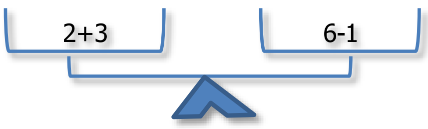
5.2 Teaching formulae

See Appendix for how to teach formulae.

5.3 Teaching changing subject in formula

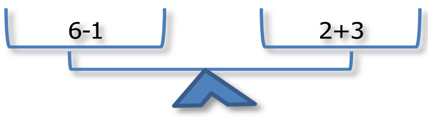
(1) Balance

Introduce equations as balance, e.g.,



$2 + 3 = 6 - 1$

Show order can be reversed, e.g.,



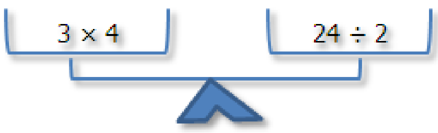
$6 - 1 = 2 + 3$

Start with and relate equations to real world, e.g. 2 + 3 = 6 − 1 is, for example, *2 boys joined another 3 to play soccer giving them the same number as their opponents where there were 6 boys but 1 had to leave*.

(2) Balance rule

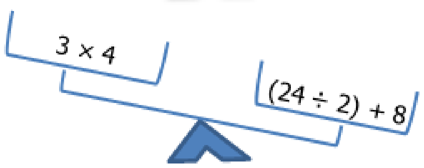
Show that acting on one side unbalances the formula and that balance can be restored by doing the same to the other side, e.g.

Balance



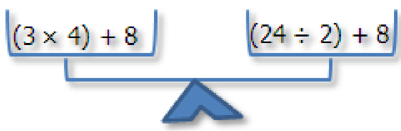
$3 \times 4 = 24 \div 2$

Unbalance



$3 \times 4 \neq (24 \div 2) + 8$

Rebalance

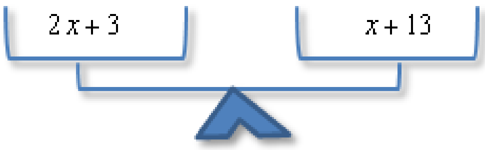


$(3 \times 4) + 8 = (24 \div 2) + 8$

Note: the brackets are unnecessary under BODMAS convention but have been put in to help understanding.

(3) Unknowns

Show that equations can still be represented by balance if they have an unknown. *There were 2 boxes of tiles and 3 loose tiles in one ute and a box of tiles and 13 loose tiles in the other ute. If both the utes had the same number of tiles, how many tiles in a box?*

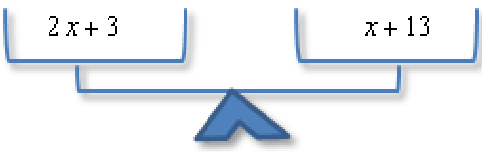


$$2x + 3 = x + 13$$

(4) Getting x alone

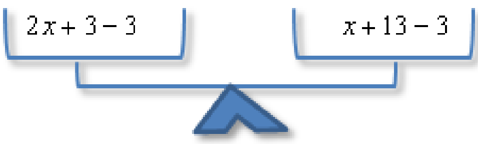
Use the balance rule to get x alone on one side.

Balance



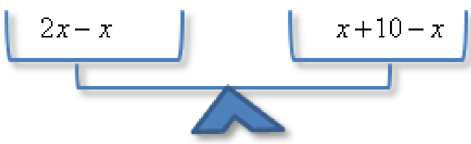
$$2x + 3 = x + 13$$

Unbalance & rebalance



$$2x + 3 - 3 = x + 13 - 3$$
$$2x = x + 10$$

Unbalance & rebalance



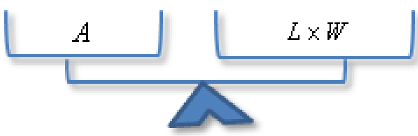
$$2x - x = x + 10 - x$$
$$x = 10$$

So there are 10 tiles in a box.

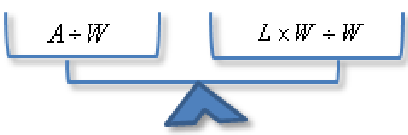
(5) Applying to formula

Area of rectangle is $A = L \times W$. What is L?

Balance



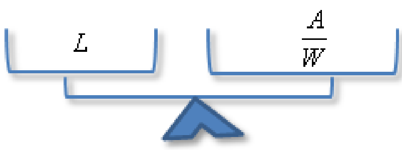
$$A = L \times W$$



$$A \div W = L \times W \div W$$

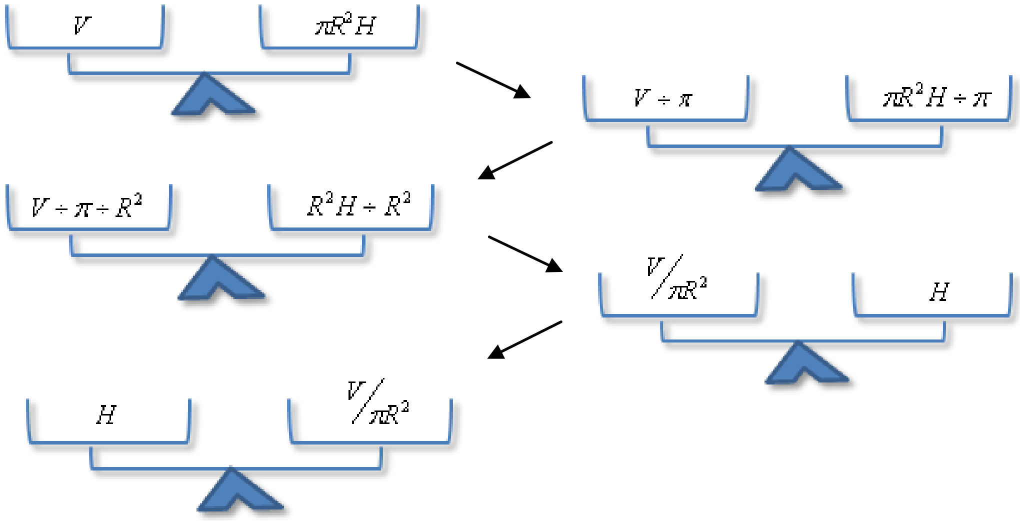
Unbalance & re balance to get L

$$A \div W = L$$



so, $L = \dfrac{A}{W}$

Volume of cylinder is $V = \pi r^2 H$, what is H?

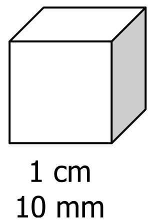


Action	**LHS**	**RHS**
Balance	V	$\pi R^2 H$
Unbalance and rebalance to get H	$V \div \pi$ $V/_{\pi}$	$\pi R^2 H \div \pi$ $R^2 H$
Unbalance and rebalance to get H	$V/_{\pi} \div R^2$ $V/_{\pi R^2}$	$R^2 H \div R^2$ H
Final position	H	$V/_{\pi R^2}$

APPENDIX:
PERIMETER, AREA, AND VOLUME FORMULAE

Small engines are described in terms of cubic centimetres of their cylinders, although length measurement is in terms of millimetres. We use the relationship 1000 mm³ = 1 cm³ which comes from the volume formula: $V = l \times w \times h$ where l is the length, w is the width, and h is the height.

Consider the cube at right with side lengths of 1 cm, which is equivalent to 10 mm. Then, in both units, the volume is:



$V = l \times w \times h$ $V = 1\text{ cm} \times 1\text{ cm} \times 1\text{ cm}$ $V = 1\text{ cm}^3$	$V = l \times w \times h$ $V = 10\text{ mm} \times 10\text{ mm} \times 10\text{ mm}$ $V = 1000\text{ mm}^3$

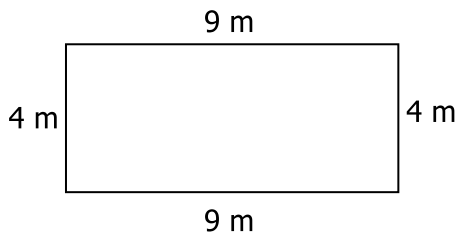
Hence, 1000 mm³ = 1 cm³.

Of course, cylinders are not rectangular prisms. The formula for the volume of a cylinder is $\pi r^2 h$ where r is the radius of the circular base and h is the height of the cylinder. However, in order to understand this formula, a prior knowledge of formula in perimeter (circumference), area, and volume of rectangles and circles is required.

A.1 Perimeter

The **perimeter** of a shape is the total distance around it. In other words, the perimeter is the distance over which a fence or wall would need to be erected to fully enclose the shape. For most shapes, calculating the perimeter is easy, because the only operation involved is addition.

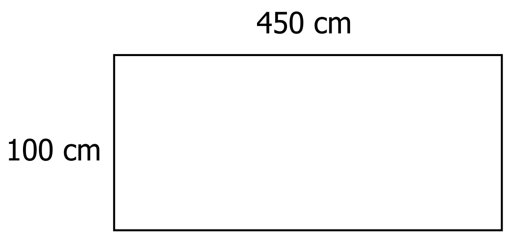
For example, have a look at the rectangle at right.



The total distance around the rectangle is simply the result of adding all of the sides together:

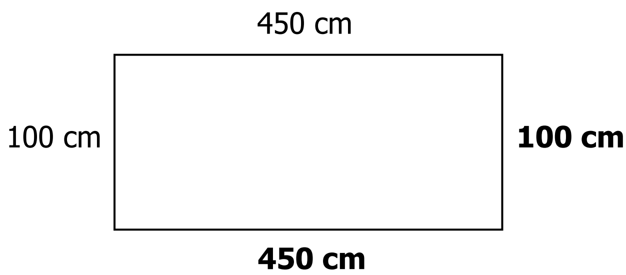
9 m + 4 m + 9 m + 4 m = <u>26 m</u>.

Remember that a rectangle will always have opposite sides that are of the same length. So, if given a rectangle with the lengths of only two sides marked, we automatically know the lengths of the opposite sides.



For example, have a look at the rectangle at right. We know what the lengths of the opposite sides are: the side length opposite the 450 cm must be 450 cm, and the side length opposite the 100 cm must be 100 cm.

So, we can add these lengths to the drawing so that we do not forget what they are.



Now, the total distance of this rectangle is simply:

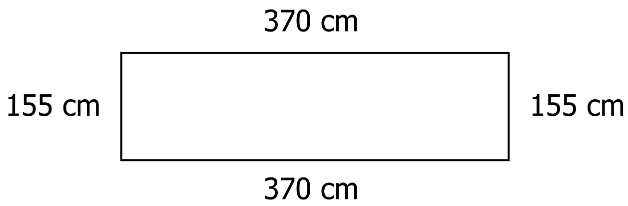
450 cm + 100 cm + 450 cm + 100 cm = <u>1100 cm</u>.

Note that, because you can add numbers in whatever order you like, you can start adding from any side of the rectangle that you choose.

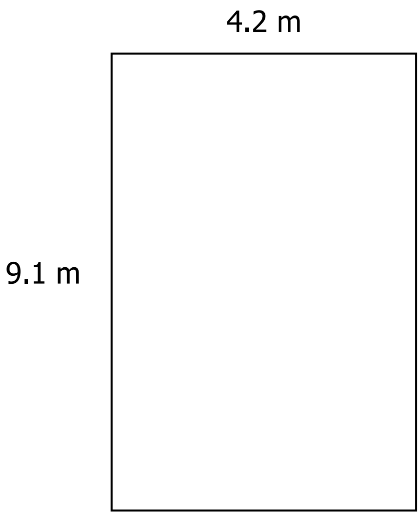
<u>Activity A.1a</u>

Find the perimeter of each of the following rectangles. Make sure that all of your answers include the correct units of length (e.g., m for metres, cm for centimetres).

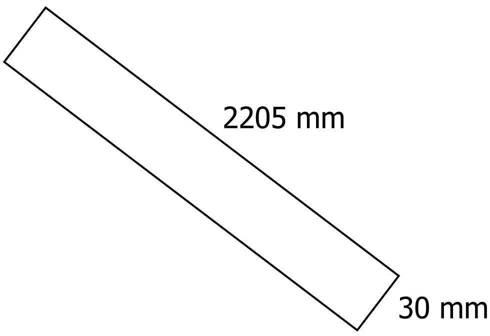
(1)



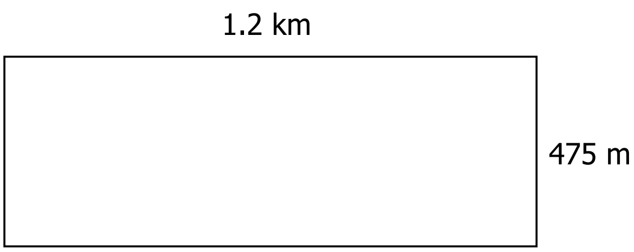
(2)



(3)



(4)

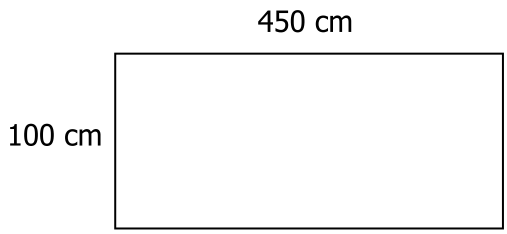


Remember that, in order to be able to add quantities together, they **must** be in the same units.

Because the lengths of opposite sides of a rectangle are always the same, we can find a quicker way to calculate the perimeter of rectangles.

Have another look at the rectangle at right.

Recall that the solution to finding the perimeter of this rectangle was:



450 cm + 100 cm + 450 cm + 100 cm = 1100 cm.

If we rearrange the order in which we add the side lengths of the rectangle, we get:

450 cm + 450 cm + 100 cm + 100 cm = 1100 cm.

Remember that we can add numbers in whatever order we like and we get the same answer.

Now, if we group our addition like this,

(450 cm + 450 cm) + (100 cm + 100 cm)

= 900 cm + 200 cm,

= 1100 cm

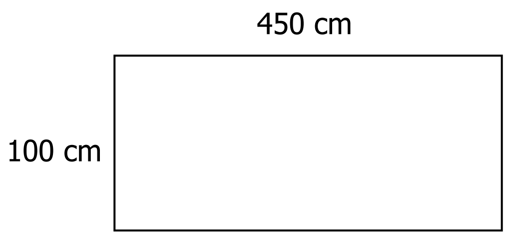
then we can see that the result of adding each group is the same as doubling each length and then adding them together.

So, we have now found another way to calculate the perimeter, P, of rectangles:

$$P = 2 \times a + 2 \times b$$

where a and b are the lengths of the rectangle.

Back to this rectangle again. If we use the formula, then we see that a = 450 cm and b = 100 cm (or a = 100 cm and b = 450 cm, because the order of addition does not matter). So,



$$P = 2 \times a + 2 \times b$$

$$P = 2 \times 450 + 2 \times 100$$

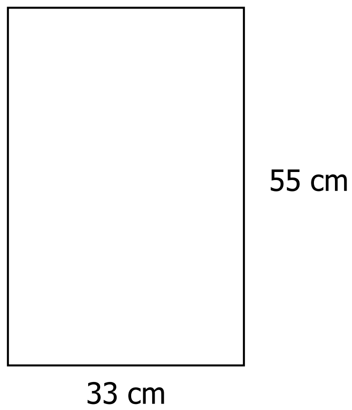
$$P = 900 + 200$$

$$P = 1100 \text{ cm.}$$

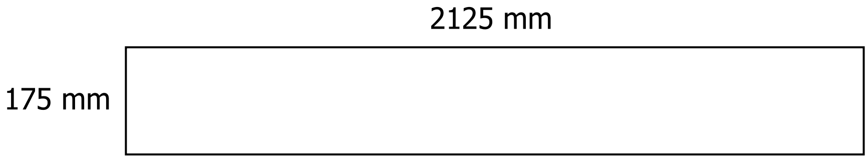
Activity A.1b

Use the new perimeter formula to find the perimeter of each of the following rectangles. Remember to show all of your working, as in the example on the previous page.

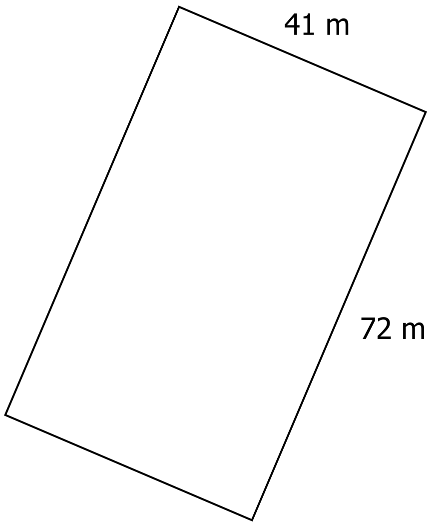
(1)



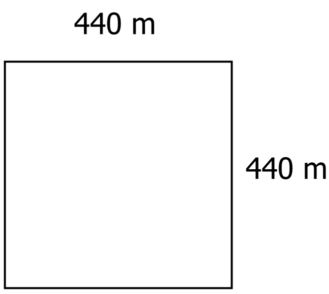
(2)



(3)



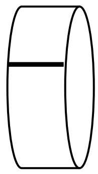
(4)



The perimeter of a circle has a special name, the **circumference**. The circumference of a circle is still the distance around the circle or the distance over which a fence would need to be erected to fully enclose the circle. The reason why a special name is used for a circle is because calculating the circumference of a circle requires a special formula. Let's find this formula.

Activity A.1c

To find the formula for calculating the circumference of a circle, follow the procedure below:



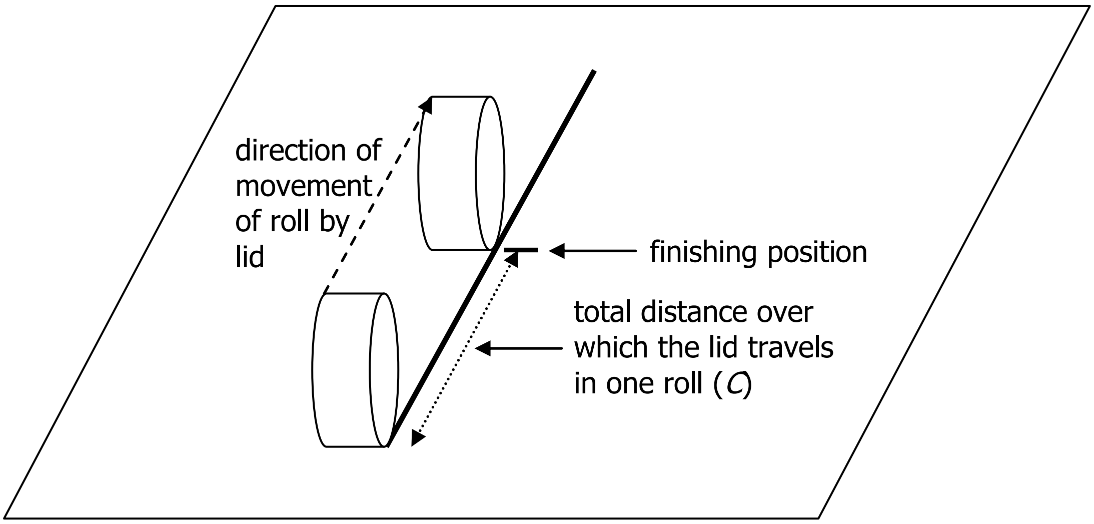
(1) Find a sheet of paper, three different-sized lids, a ruler, and a pen. Use a ruler to draw a straight line across a sheet of paper.

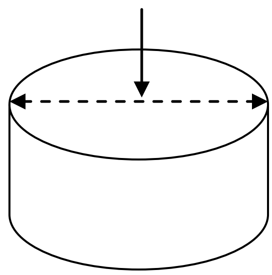
(2) Label each lid **small**, **medium**, and **large**, according to their relative sizes.

(3) Select one lid, and mark a line on it in one place on its side (as shown in the figure at right). Turn the lid on its side, and position the lid at one end of the line you drew on the paper so that the line you drew on the lid touches the paper. Roll the lid one time along the paper. Mark the position where it stopped on the paper.

(4) Measure this distance, and call it C (see figure below). Make sure you label the lid for which C is calculated.

(5) Repeat Steps 3–8 for each of the remaining lids.





(6) Select one lid, and place it on its top or bottom.

(7) Measure the distance directly across the centre of the lid from edge to edge (as shown in the figure at right), and call it d. Make sure you label the lid for which d is calculated.

(8) Repeat Steps (6)–(7) for each of the remaining lids.

(9) Write the data you collected in Steps (4) and (7) the table below (remember to include the units of length that you are using) and then divide C by d for the third column:

Lid	C	d	C ÷ d
Small			
Medium			
Large			

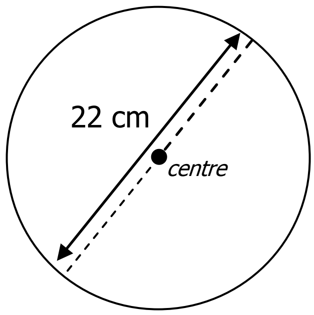
(10) What did you get in the third column for each of the lids?

You should have observed that the value for $C \div d$ is equal to 3-and-a-bit. This value represents the ratio of the circumference (C) of a circle to the **diameter** (d) of the circle. A better approximation of this ratio is known to be 3.14 and is more formally referred to as **pi** and is symbolised using the Greek letter π.

So, we have developed the formula $C \div d = \pi$. Rearranging the formula, we obtain $C = \pi \times d$. Hence, the formula for calculating the circumference of a circle is $C = \pi \times d$. That is, to find the circumference of a circle, we multiply the diameter of the circle by π.

By the way, your calculator may have a π button. If so, then you can use this button to calculate the circumference of a circle. If not, then you can just use 3.14 in place of π.

Have a look at the circle at right.



The diameter (d) is 22 cm. To find the circumference of the circle, we use the formula:

$$C = \pi \times d$$

$$C = \pi \times 22 \text{ cm}$$

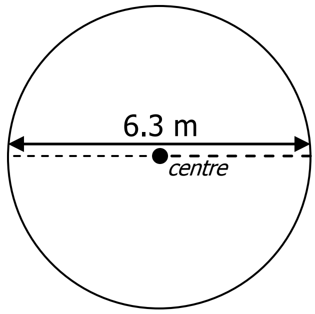
$$C = \underline{69.1 \text{ cm}}$$

So, the circumference of the circle is 69.1 cm.

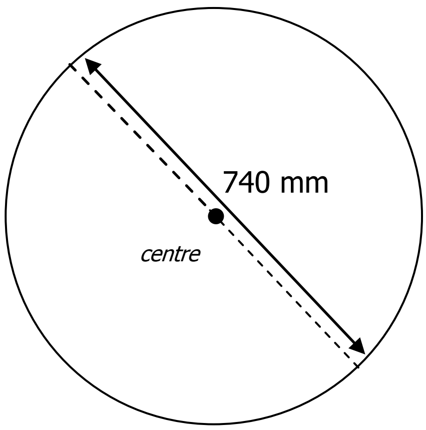
<u>Activity A.1d</u>

Find the circumference of each of the following circles:

(1)

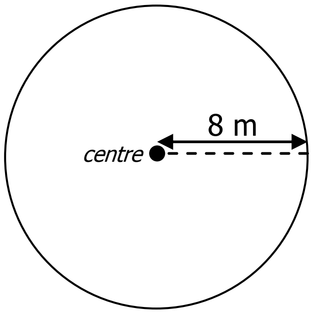


(2)



The **radius** (r) of a circle is the distance from the centre of the circle to any point on the edge of the circle. Because the diameter is the distance across the circle, the radius is equal to half the diameter. That is, $r = d \div 2$ which implies that $d = 2 \times r$. If we substitute $2 \times r$ for d in the formula for finding the circumference of a circle, we obtain $C = \pi \times (2 \times r)$. Because we can multiply numbers in any order we like, let us write the formula is this way $C = 2 \times \pi \times r$.

Have a look at the circle at right.



The radius is 8 m. To find the circumference of the circle, we use the formula

$$C = 2 \times \pi \times r$$

$$C = 2 \times \pi \times 8 \text{ m}$$

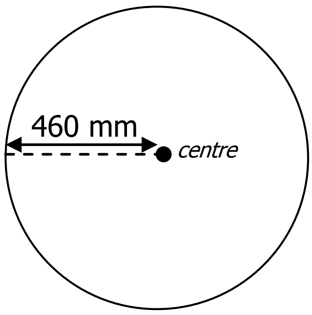
$$C = \underline{50.3 \text{ m}}$$

So, the circumference of the circle is 50.3 m.

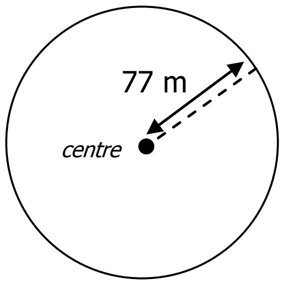
Activity A.1e

Find the circumference of each of the following circles:

(1)

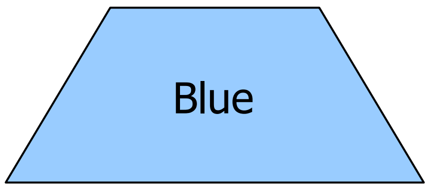


(2)



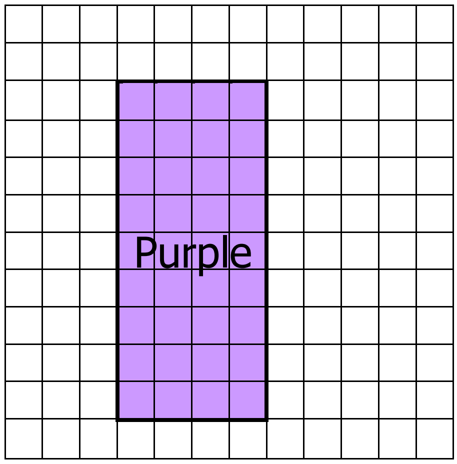
A.2 Area

The **area** of a two-dimensional shape is a measure of the size of the surface contained inside the shape. For the shape at right, the area is the blue shaded part.



Finding the area of a shape depends on the shape itself. Different shapes have different formulae for finding the area.

Have a look at the rectangle on the grid at right.

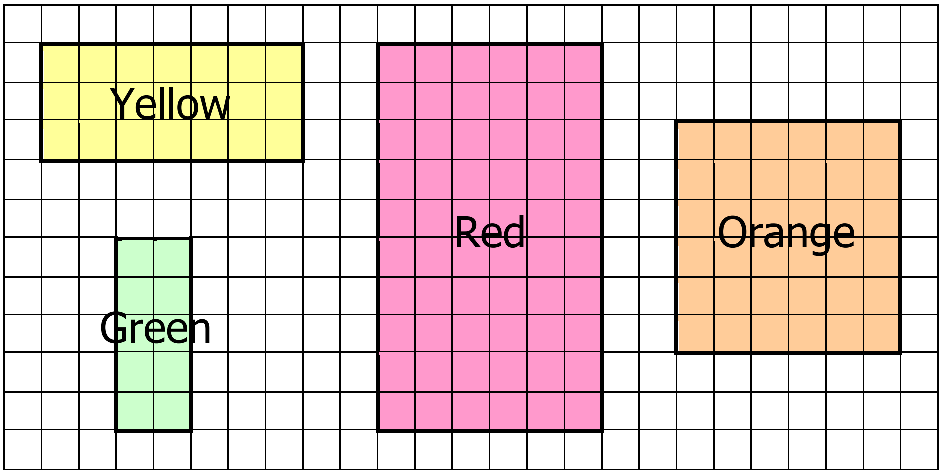


The area is the shaded purple region. The shaded purple region contains a total of 36 squares. Therefore, the area of the shaded purple region is equal to 36 squares.

Note that the rectangle is 4 squares wide and 9 squares long. Is there an operation that can be used to get 36 from 4 and 9?

Activity A.2a

(1) Find the area of each of the following rectangles by counting the number of squares inside it.



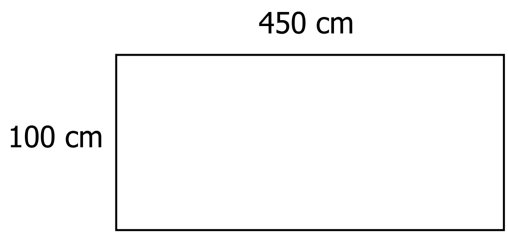
(2) Complete the table below with the length (a), width (b), and area (A) of each of the rectangles in Question 1.

Rectangle	Length (a)	Width (b)	Area (A)
Yellow			
Green			
Red			
Orange			

(3) What is the relationship among length (a), width (b), and Area (A) for each rectangle?

From the previous activity, you should have realised that, instead of counting the squares inside a rectangle, we can find the rectangle's area (A) by multiplying its length (a) by its width (b). Hence,

$$A = a \times b.$$



Have a look at the rectangle at right. We can see that a = 100 cm and b = 450 cm (or a = 450 cm and b = 100 cm, because the order of multiplication does not matter). So,

$$A = a \times b$$

$$A = 100 \text{ cm} \times 450 \text{ cm}$$

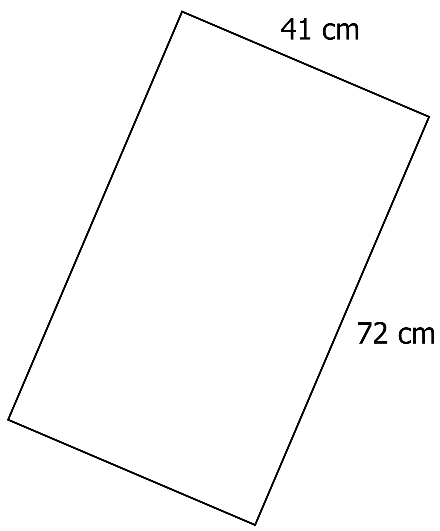
$$A = \underline{45\,000 \text{ cm}^2}.$$

That means that the area of the rectangle is 45 000 cm^2. But, what does the ^{2} mean in cm^2? It means "squared." That is, cm^2 stands for "centimetres squared" or "square centimetres". Because area is a measure of the dimensions of two sides, its units must be squared. So, when finding the area of a shape, remember to put in the squared symbol ^{2} (e.g. mm^2, cm^2, m^2, km^2).

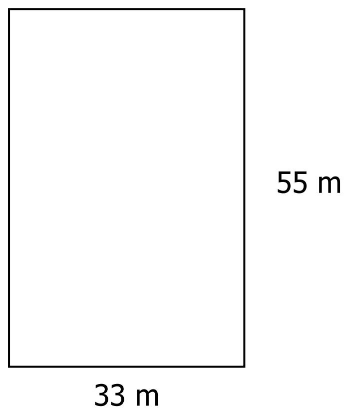
Activity A.2b

Find the area of each of the following rectangles by using the formula $A = a \times b$. Remember that your answer will be in square units (e.g. mm^2, cm^2, m^2, km^2).

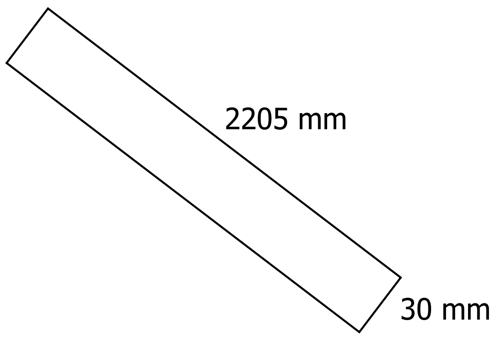
(1)



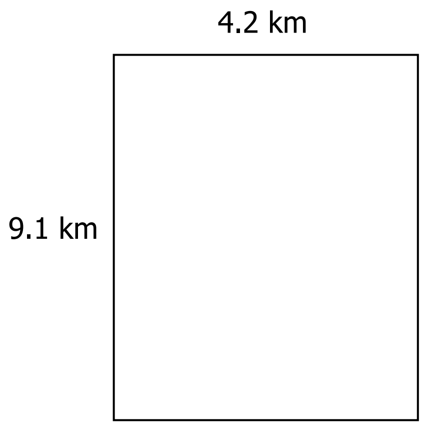
(2)



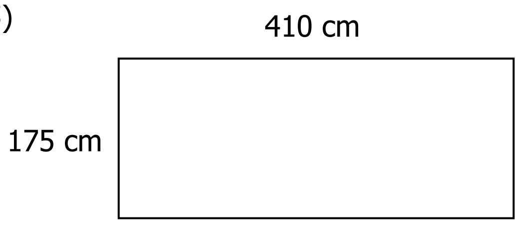
(3)



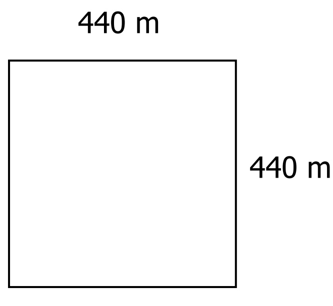
(4)



(5)



(6)



Now, we will investigate how to find the area of a circle.

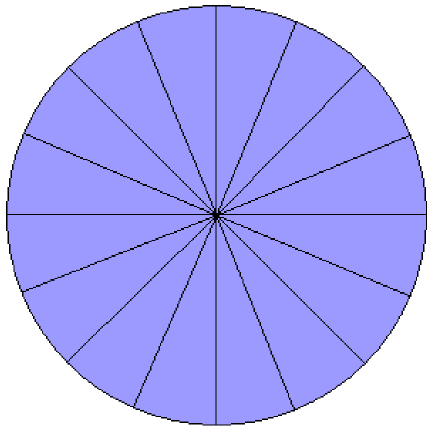
Activity A.2c

(1) Find a piece of paper, a large lid (at least 12 cm in diameter), a ruler, and scissors.

(2) Measure the diameter of the lid, and calculate the radius.

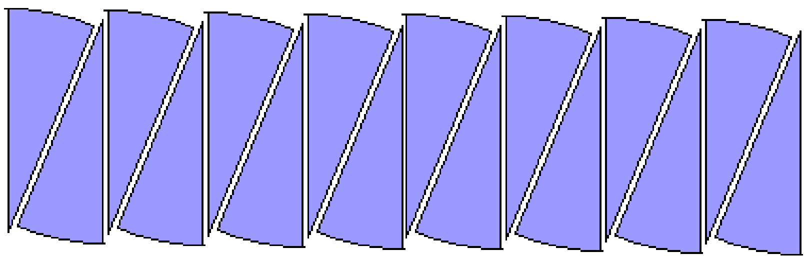
(3) Place the lid firmly on the piece of the paper, and draw a circle around it.

(4) Draw lines across the circle to form 16 sectors that are approximately equal in area, as shown in the figure at right.

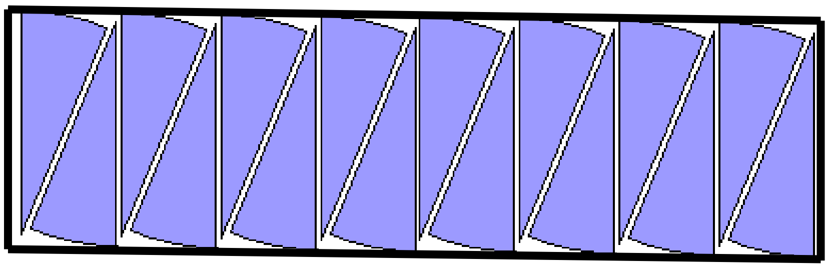


(5) Cut out the 16 sectors, as shown at right.

(6) Arrange the sectors in the configuration shown below.



(7) Note that this configuration looks very much like a rectangle.

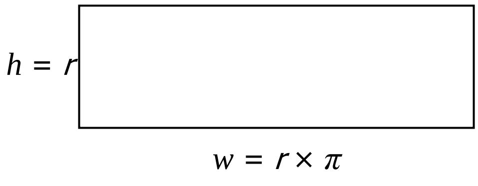


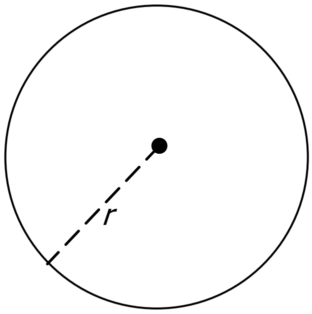
(8) Measure the height and width of the rectangle. Calculate the area of the rectangle.

(9) You should have recognised that the height of the rectangle is the same as the radius of the lid.

(10) Divide the width of the rectangle by the radius. Did you get an answer that is close to π? Remember that π is approximately equal to 3.14.

Since the height (h) of the rectangle is equal to the radius (r) of the circle and the width (w) of the rectangle is (approximately) equal to the radius (r) of the circle times π (see figures below), we can develop the formula for finding the area of a circle.





To calculate the area of the rectangle, you would have used $A = h \times w$. Since $h = r$ and $w = r \times \pi$, we can substitute

$$A = h \times w$$

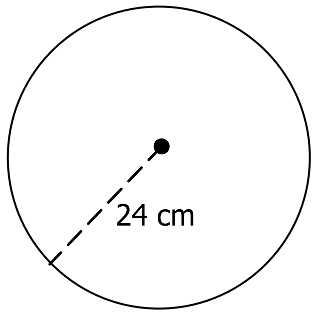
$$A = r \times r \times \pi$$

$$A = r^2 \times \pi.$$

Remember that $r^2 = r \times r$.

So, we have developed the formula for finding the area of circle, $A = r^2 \times \pi$ where r is the radius of the circle. Given the circle's radius, all we have to do is multiply the radius by itself and then multiply the result by π.

Have a look at the circle at right. We can see that $r = 24$ cm.
So, to calculate the area of the circle, we have



$$A = r^2 \times \pi$$

$$A = (24 \text{ cm})^2 \times \pi = 24 \text{ cm} \times 24 \text{ cm} \times \pi$$

$$A = \underline{1809.6 \text{ cm}^2}$$

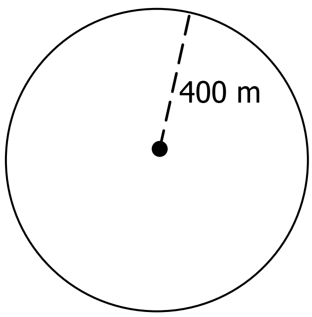
That is, the area of the circle is approximately equal to 1809.6 cm².

Remember that, if given the diameter of a circle instead, you must divide it by 2 to find the radius before using the formula for calculating the area of a circle.

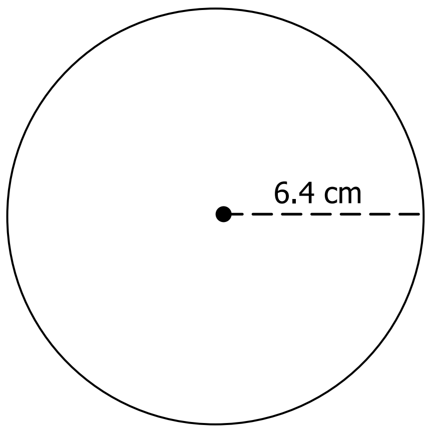
Activity A.2d

Find the area of each of the following circles.

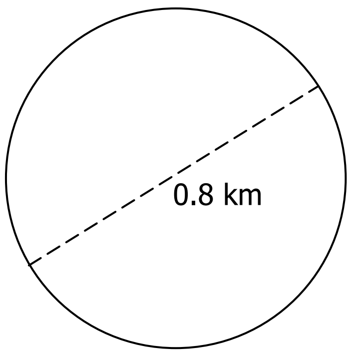
(1)



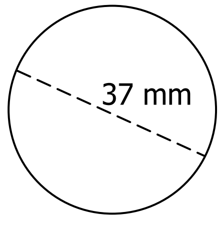
(2)



(3)



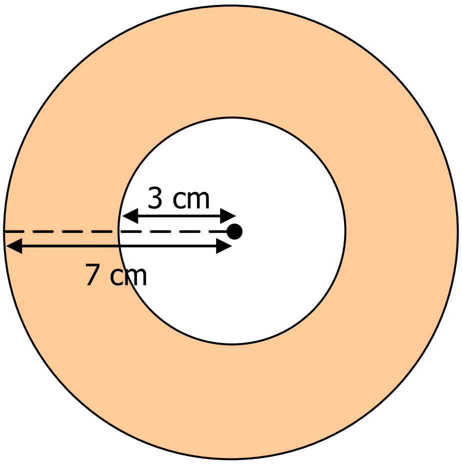
(4)



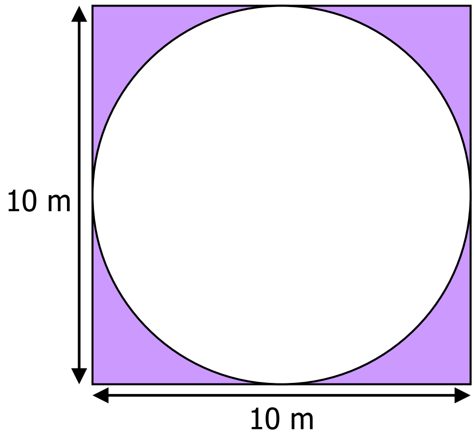
Activity A.2e

Find the area of the shaded region in each of the following figures.

(1)

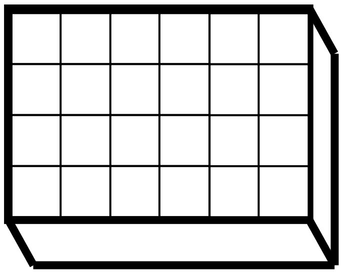


(2)

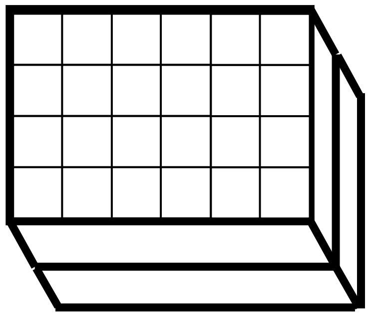


A.3 Volume

A **rectangular prism** is the formal name for a rectangular box. A rectangular prism looks like the figure at right when shown in two dimensions. The volume of the rectangular prism is the amount of space enclosed inside it.

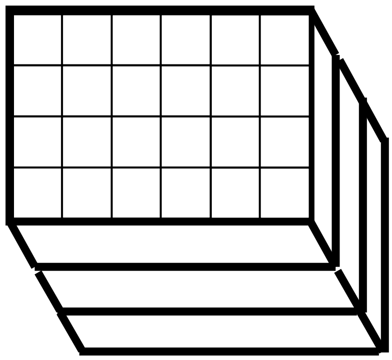


Have a look at the block at right. Note that the face of the block is 6 units wide and 4 units long. Therefore, the area of the face of the block is 6 units × 4 units = 24 square units. Note that the block is 1 unit height. Therefore, the volume of the block is 24 cubic units. That is, a measure of the amount of space inside the block is 24 cubic units.



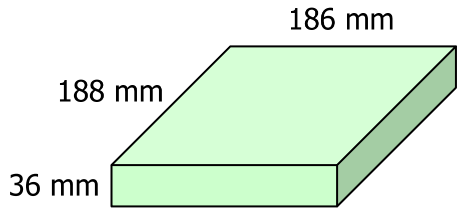
If we put another block of the same dimensions against the first one, as shown at right, then you will see that the prism now has a height of 2 units. The volume, therefore, is twice that of the first block. So, the volume of this prism is 48 cubic units.

If we add a third block to the prism from before, then the prism will now be 3 units high (see the figure below right). Therefore, the volume of the prism will be 24 × 3 = 72 cubic units.



Hence, the volume (V) of a rectangular prism is equal to the area (A) of the face of the prism times the height (c) of the prism. Since the area (A) of the face is simply length (a) times width (b), we have the formula for calculating the volume of a rectangular prism: $V = a \times b \times c$. Because we are multiplying three quantities together, it makes sense that units of volume must be cubic (e.g. mm^3, cm^3, m^3, km^3).

Have a look at the rectangular prism at right.



We see that the a = 186 mm, b = 188 mm, and c = 36 mm. (Remember that, because we can multiply numbers in any order and get the same result, we can assign a, b, and c to any of the quantities and obtain the same result.) The volume of the rectangular prism is

$$V = a \times b \times c$$

$$V = 186 \text{ mm} \times 188 \text{ mm} \times 36 \text{ mm}$$

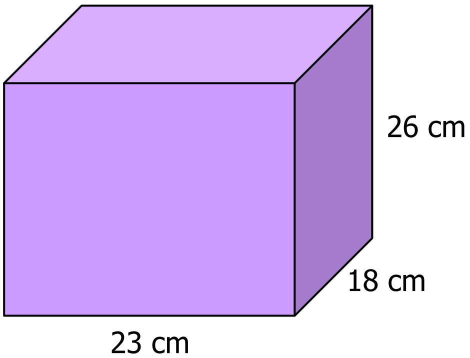
$$V = \underline{1\ 258\ 848 \text{ mm}^3}.$$

So, the volume of the rectangular prism is 1 258 848 mm^3, which is the measure of how much space is enclosed inside the rectangular prism.

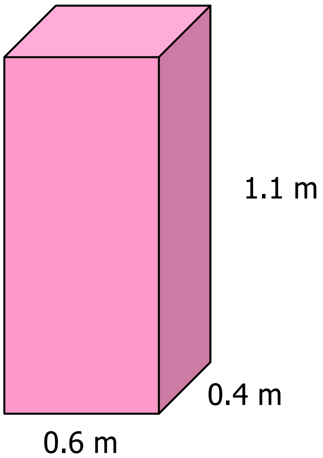
Activity A.3a

Find the volume of each of the following rectangular prisms.

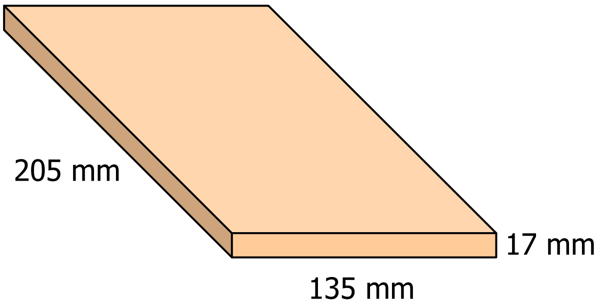
(1)



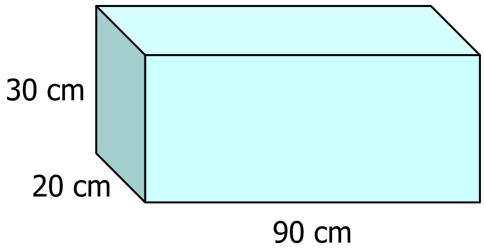
(2)



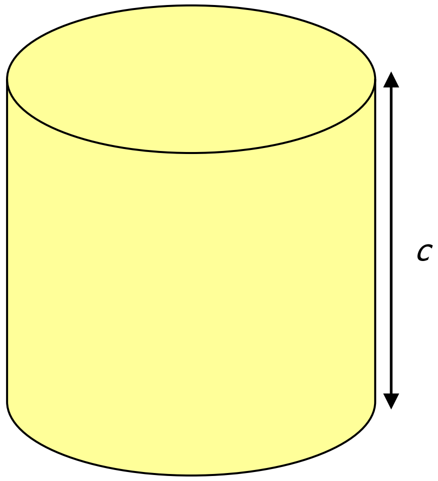
(3)



(4)



A **cylinder** is just like a prism, except that the shape on the base is a circle, as shown in the figure at right.



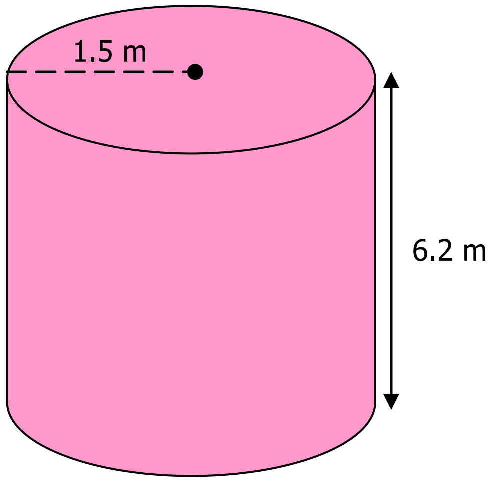
You should have recognised that finding the volume of a cylinder is the same as finding the volume of a prism. First, you find the area of the face. Then, you multiply the area by the length of the cylinder.

Since we have already learned that the area of a circle is $A = r^2 \times \pi$ where r is the radius of the circle, we can derive the formula for finding the volume of a cylinder:

$$V = A \times c$$

$$V = r^2 \times \pi \times c.$$

Have a look at the cylinder at right. We can see that the radius (r) is 1.5 m and the length (c) is 6.2 m. The volume of the cylinder is



$$V = r^2 \times \pi \times c$$

$$V = (1.5 \text{ m})^2 \times \pi \times 6.2 \text{ m}$$

$$V = 1.5 \text{ m} \times 1.5 \text{ m} \times \pi \times 6.2 \text{ m}$$

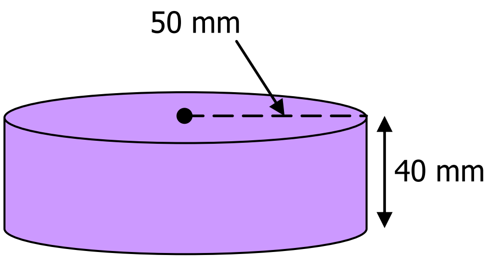
$$V = \underline{43.8 \text{ m}^3}.$$

So, the volume of the cylinder is 43.8 m³.

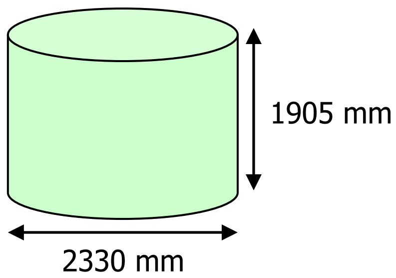
Activity A.3b

Find the volume of each of the following cylinders.

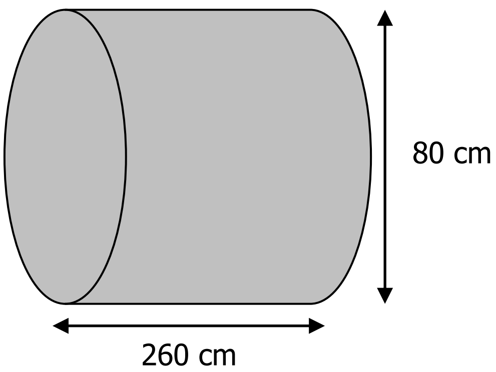
(1)



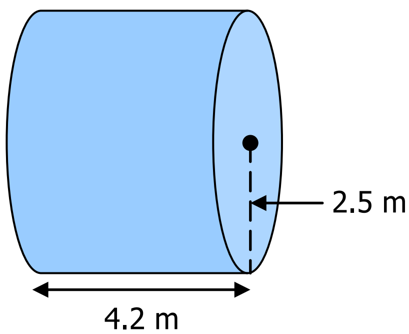
(2)



(3)



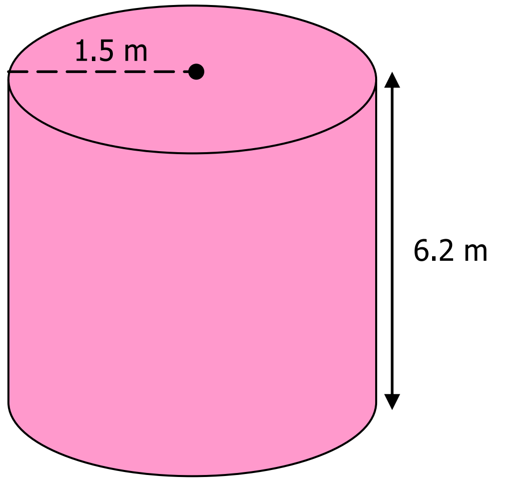
(4)



As we learned before, volume is the amount of space inside a prism or cylinder. Volume can also be viewed as the maximum amount of solid material that can be used to fill up the prism or cylinder. On the other hand, when we are interested in finding the amount of liquid that can fill up a prism or cylinder, we need to find the capacity of the prism or cylinder. **Capacity** is the volume of liquid.

Finding the capacity of a prism or cylinder is just like finding the volume. However, once we have the volume, we must convert it to capacity, which is usually measured in litres (L). The conversion is 1 m^3 = 1000 L.

Have another look at the cylinder at right from before.



We already know that the volume of this cylinder is 43.8 m^3. If we need to calculate the capacity (C) of this cylinder, then all we have to do is multiply the volume (V) by 1000 L:

$$C = V \times 1000$$

$$C = 43.8 \text{ m}^3 \times 1000$$

$$C = \underline{43\,800 \text{ L}}.$$

So, the capacity of the cylinder is 43 800 L. That is, a maximum of 43 800 L of liquid can fill up the cylinder.

The conversion factor of 1000 works only when you have the volume in m^3. If you have the volume in units other than m^3, then the conversion factor to capacity is different. The table below shows the conversion factors for different units of volume:

1 km^3 = 1 000 000 000 000 L

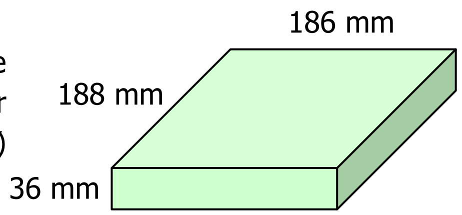
1 m^3 = 1000 L

1 cm^3 = 0.001 L = 1 mL

1 mm^3 = 0.000 001 L = 0.001 mL

We always want to make sure that we use the correct conversion factor given the units of volume that we have.

Have another look at the rectangular prism at right from before.



We already know that its volume is 1 258 848 mm^3. If we need to calculate the capacity (C) of the rectangular prism, then all we need to do is multiply the volume (V) by 0.000 001:

$$C = V \times 0.000\,001$$

C = 1 258 848 mm³ × 0.000 001

C = 1.26 L.

Alternatively, we could have multiplied the volume (V) by 0.001 in order to obtain the capacity in mL. In this case,

C = V × 0.001

C = 1 258 848 mm³ × 0.001

C = 1258.8 mL.

So, the capacity of the rectangular prism is 1.26 L or 1258.8 mL.

Activity A.3c

Find the capacity of each of the prisms and cylinders (in either mL or L) from Activities A.3a and A.3b.