A spatio-temporal stick-breaking process

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Introduction

Spatial data are becoming more and more popular!

With the term “spatial data”, we mean any type of data that directly or indirectly references a specific geographical area or location.

Spatial data is data that describes a surface, usually through variables on a two-dimensional plane.

Spatial Modelling

Consider a random field $\{Y(s) : s \in \mathscr{D}\}$
for $\mathscr{D} \subseteq \mathbb{R}^2$,
and let $s = (s_1, \ldots, s_n)$ be a set of distinct locations in $\mathscr{D}$, such that $s_i = (s_{i,1}, s_{i,2})$ for $i = 1, \ldots, n$.

$y(s)$ is characterized by

$$y(s) = \theta(s) + x(s)^T \beta + \varepsilon(s)$$

- $\theta(s)$ is a spatial effect
- $x(s)$ is a column-vector of p covariates
- β is a column-vector of p regression coefficients
- $\varepsilon(s) \sim \mathscr{N}(0, \sigma_\varepsilon^2)$

In the following, we assume covariates are not available, then

$$y(s) = \theta(s) + \varepsilon(s)$$

Kriging

A common approach to problems of modelling surfaces is kriging [Stein, 2012], which interpolates and smooths spatial data.

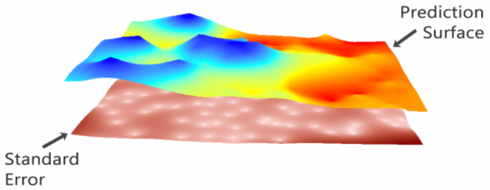


Figure: Courtesy of `gisgeography.com`

Assumptions:

- Gaussianity
- stationarity
- data cannot have trends

Kriging

In kriging, prediction $y(s^*)$ at a spatial location s^* is obtained as a weighted average, where observations at closer locations are given higher weights than observations at distant locations.

$$\theta(s) \sim N(\mu, \Sigma)$$

where $\Sigma_{ij} = \mathbb{C}\text{ov}(\theta(s_i), \theta(s_j))$ [Cressie (2015)]

Spatio-temporal processes

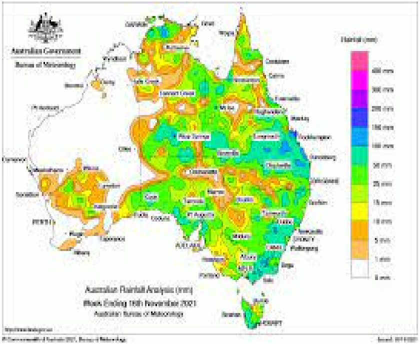
The concept of spatial process can be extended to the spatio-temporal case

$$y(s,t) = \{y(s,t), (s,t) \in \mathscr{D} \times \mathscr{T} \subset \mathbb{R}^d \times \mathbb{R}\}$$

The kriging model can be extended by defining a spatio-temporal covariance function given as $\mathbb{C}\text{ov}(\theta(s,t), \theta(s',t'))$ [Gelfand et al. (2010)],[Cressie and Wikle (2015)].
Often space and time are considered separable [Finkenstadt et al. (2006)]:

$$\mathbb{C}\text{ov}(\theta(s,t), \theta(s',t')) = \sigma^2 C_1(d(s,s')) C_2(|t-t'|)$$

Examples of non-separable covariances: [Cressie and Huang (1999)],[Gneiting (2002)]



A first relaxation of the Gaussianity assumption may be represented by finite mixture models; see [Fernandéz and Green (2002), Neelon et al. (2014) and Hossain et al. (2008)]

However, finite mixture models assume it is possible to consistently select the number of components, which may be difficult in presence of covariates.

Let's go nonparametric!

A random distribution F is distributed according to a Dirichlet process (DP) if its marginal distributions are Dirichlet distributed [Ferguson, 1973].

Taking F_0 to be a distribution on Θ, and $\alpha > 0$, consider $A_1, \dots, A_r \in \Theta$; then $F \sim DP(\alpha, F_0)$, if

$$(F(A_1), \dots, F(A_r)) \sim Dir(\alpha F_0(A_1), \dots, \alpha F_0(A_r)).$$

Sethuraman's representation

[Sethuraman, 1994] shows that $F \sim DP(\alpha, F_0)$ if

$$V_k \sim Beta(1,\alpha) \qquad k = 1, \ldots$$
$$\theta_k \sim F_0$$
$$\pi_1 = V_1 \qquad \text{and} \qquad \pi_k = V_k \prod_{j=1}^{k-1} (1 - V_j) \quad \text{for } k = 2, \ldots$$
$$F = \sum_{k=1}^{\infty} \pi_k \delta_{\theta_k},$$

where

- $\delta(\theta)$ is a Dirac mass
- V_k and θ_k are independent for $k = 1, 2, \ldots$

This can be generalised to $V_k \sim Beta(a_k, b_k)$.

Spatio-Temporal SB

Suppose $y(s,t)$ can be represented as an infinite mixture model:

$$g(y(s,t)) = \sum_{k=1}^{\infty} \pi_k(s,t) g(y(s,t)|\theta_k)$$

where the mixing probability $\pi_k(s,t)$ is the probability that the observation at location s and time t belongs to component k.

The mixing weights are built similarly to the stick-breaking process:

$$F_{s,t} = \sum_{k=1}^{\infty} \pi_k(s,t)\delta_{\theta_k} \qquad s \in \mathscr{D}, t \in \mathscr{T} \qquad \text{where}$$

$$\pi_1(s,t) = V_1(s,t) \qquad \text{and} \qquad \pi_k(s,t) = V_k(s,t)\prod_{j=1}^{k-1}(1 - V_j(s,t))$$

$$V_k(s,t) = w_k(s, \psi_k, t, \zeta_k)V_k$$

$$V_k \sim Beta(a,b) \qquad \perp \qquad \theta_k \sim F_0 \qquad \text{for } k = 1,2,\ldots$$

Conditional covariance

The covariance between $y(s,t)$ and $y(s',t')$ conditional on the mixing weights but marginalised with respect to the atoms is

$$\mathbb{C}\text{ov}(y(s,t),y(s',t')|V,\psi,\zeta) = \sigma_\theta^2 \sum_{k=1}^{\infty} \left[w_k(s,\psi_k,t,\zeta_k) w_k(s',\psi_k,t',\zeta_k) V_k^2 \right.$$
$$\cdot \prod_{j<k} [1 - (w_j(s,\psi_j,t,\zeta_j) + w_j(s',\psi_j,t',\zeta_j)) V_j +$$
$$\left. + w_j(s,\psi_j,t,\zeta_j) w_j(s',\psi_j,t',\zeta_j) V_j^2] \right]$$

The covariance function

- depends on the choice of the kernels
- does not depend on the value of $y(s,t)$ and $y(s',t')$
- can be stationary or not

Marginal covariance

If we use $V_k \sim Be(a, b)$ for all k, then $\mathbb{E}[V_k]$ and $\mathbb{E}[V_k^2]$ have a fixed form, depending on a and b, and

$$\mathbb{C}\text{ov}(y(s,t), y(s',t')) = \sigma_\theta^2 \frac{h(s,s',t,t')}{2\left(1 + \frac{b}{a+1} - h(s,s',t,t')\right)}$$

where $h(s,s',t,t')$ depends on the expectation of

- $w_k(s, \psi_k, t, \zeta_k)$
- $w_k(s', \psi_k, t', \zeta_k)$
- $w_k(s, \psi_k, t, \zeta_k) w_k(s', \psi_k, t', \zeta_k)$

Separable kernels

It is possible to consider separable kernels:

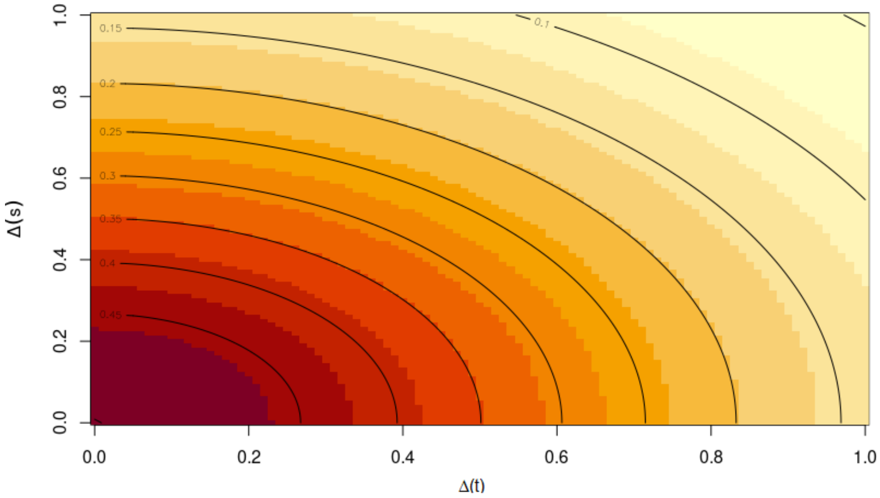
$$w_k(s,t,\psi,\zeta) = w_k(s,\psi)\cdot w_k(t,\zeta)$$

where ψ is a set of knots for the spatial domain and ζ is a set of knots for the temporal domain.
Then

$$h(s,s',t,t') = \frac{\int_\Psi w(s,\psi)w(s',\psi)p(\psi)d\psi}{\int_\Psi \int w(s,\psi)p(\psi)d\psi}\cdot\frac{\int_Z w(t,\zeta)w(t',\zeta)p(\zeta)d\zeta}{\int_Z w(t,\zeta)p(\zeta)d\zeta}.$$

While the kernel is separable, the covariance structure of the process is not separable.

Separable kernels



Non-separable kernels

Consider a non-separable kernel [Gneiting (2002)]

$$w(s,\psi,t,\zeta) = \frac{1}{\gamma|t-\zeta|+1}\exp\left(-\frac{(s_1-\psi_1)^2+(s_2-\psi_2)^2}{(\gamma|t-\zeta|+1)^{\lambda/2}}\right),$$

- γ is a nonnegative scaling parameter
- $\lambda \in [0,1]$ is a parameter controlling the space-time interaction
- if $\lambda = 0$, the kernel is separable; as $\lambda \to 1$, the space-time interaction increases and the spatial dependence at positive lags decreases more slowly

Non-separable kernels

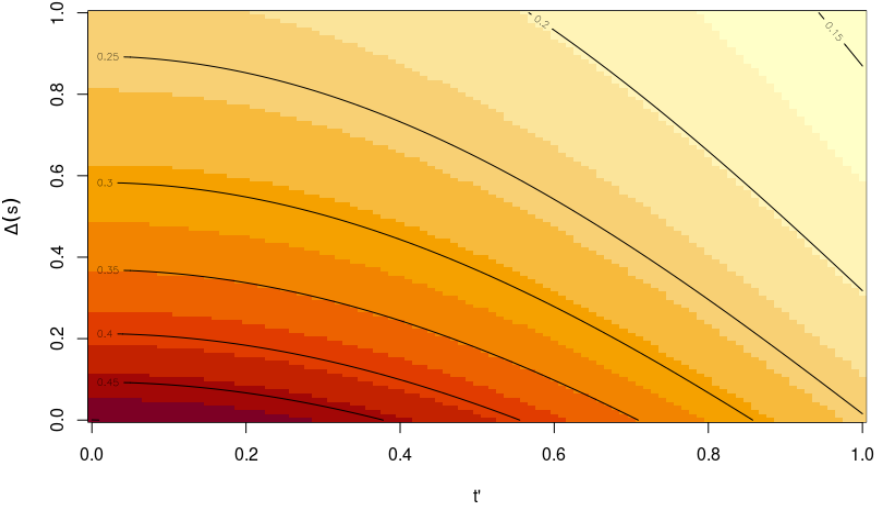
In this case

$$h(s,s',t,t') = \frac{\int_\Psi \int_\Xi w_k(s,\psi_k,t,\zeta_k) w_k(s',\psi_k,t',\zeta_k) p(\psi,\zeta) d\psi_k d\zeta_k}{\int_\Psi \int_\Xi w_k(s,\psi_k,t,\zeta_k) p(\psi,\zeta) d\psi_k d\zeta_k},$$

when considering γ and λ fixed.
In the specific case of the previous kernel, even when $\lambda = 0$

$$\begin{aligned} h(s,s',t,t') = \exp\left(-((s_1-s_1')^2 + (s_2-s_2')^2)\right) \\ \cdot \left(-\frac{\log(t') - \log(t'-1) - \log(t) - \log(t-1)}{\gamma^2(t-t')}\right), \end{aligned}$$

Non-separable kernels

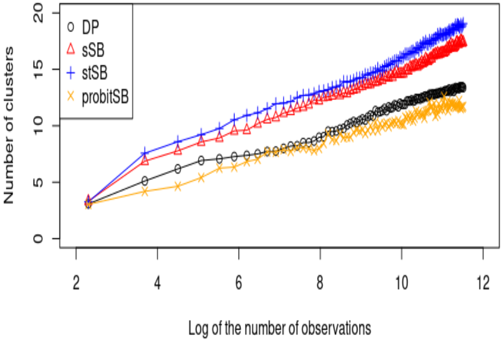


Testing for separability

The non-separable kernels proposed before allows for an automatic testing on the λ parameter, assuming a **spike-and-slab prior** [Mitchell and Beauchamp (1988), Andersen et al. (2017)]

$$\lambda \sim p_\lambda f(\lambda) + (1 - p_\lambda)\delta_0,$$

Clustering properties



Allowing atoms to depends on space and time

Atoms may also be allowed to vary by introducing a modification of the stick-breaking representation of each element

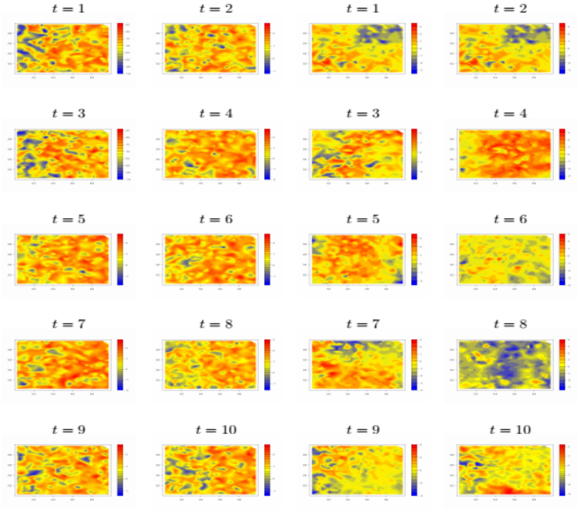
$$F_{s,t} = \sum_{k=1}^{\infty} \pi_k(s,t)\delta_{\theta_k(s,t)}$$

where $\theta_k(s,t)$ are independent stochastic processes with index set $\mathscr{S} \times \mathscr{T}$ and $F_{s,t,0}$ marginal distributions.
The atoms $\theta_k(s,t)$ can be generated from a Gaussian process with covariance

$$C(d(s,s'), d(t,t_0)) = C_s(d(s,s')) \times C_t(d(t,t_0)),$$

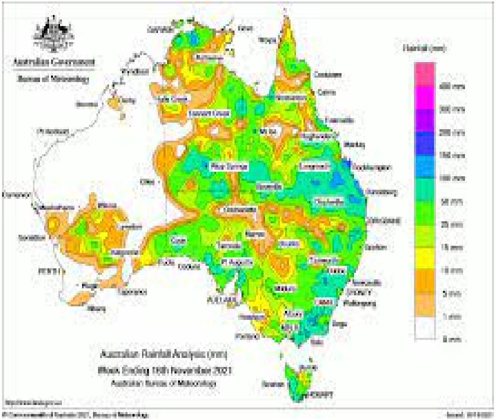
where C_s is a spatial covariance function and C_t is a temporal covariance function.

Example of spatio-temporal SB process



Modelling Rainfalls

Rainfalls in Australia are known to be highly variable [Nicholls et al. (1997)]

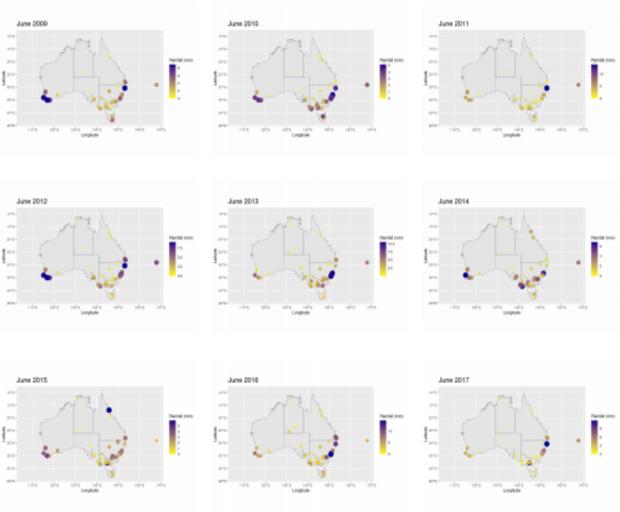


The dataset

The dataset contains

- 10 years of daily weather observations, from 31 October 2007 to 24 June 2017
- 49 locations across all Australia
- observations were drawn from weather stations
- the amount of rainfall recorded for the day in mm
- publicly available on the website of the Bureau of Meteorology of the Australian Government
- 140,793 data points
- **training dataset:** data from 2007 to 2016 (132,420 observations)
- **testing dataset:** data for 2017 (8,373 observations)

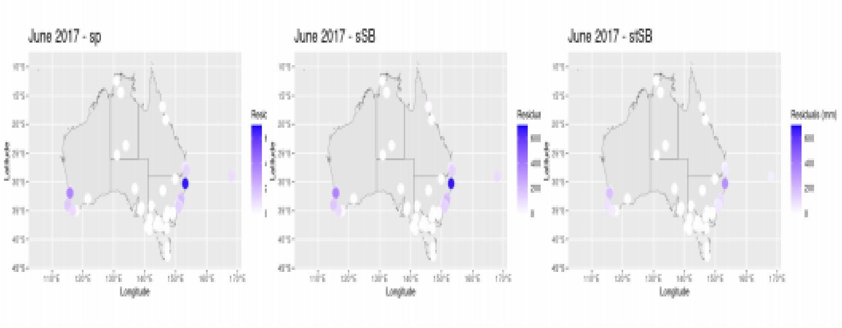
Modelling Rainfalls



Comparison between models

	ESPE	**DIC**	**WAIC**
sp	109.74	39,960,564	3,109,830
sSB	115.47	495,345	524,083
stSB	71.05	233,070	257,581

Prediction



June 2017 - sp
June 2017 - sSB
June 2017 - stSB
Longitude
Residuals (mm)

Conclusions

Modelling spatio-temporal phenomena is an important and challenging problem.

- the spatio-temporal SB avoids oversmoothing
- it can model the interaction between space and time and incorporate both spatial and temporal dependence in the definition of the stick-breaking weights.
- this allows to model nonstationary and non-normal models over time and space.
- this model avoids the need for space-time varying covariates as in [Hossain et al. (2013)] to model the weight components
- extensions to consider spatio-temporal atoms are straightforward

Where is seasonality?

Thank you!

Grazian, C. (2023+) "Spatio-temporal stick-breaking" (*work in progress*)

Grazian, C. (2023+) "A review on Bayesian model-based clustering" (*work in progress*)

Grazian, C. `https://github.com/bayesgra/spatiotempoSB`